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Polylogarithms

and

Multiple Zeta Values

by

Michel WALDSCHMIDT

http://www.math.jussieu.fr/~miw/articles/ps/saskatoon.ps

Special Values of the Riemann Zeta Function

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots \qquad s \geq 2.$$

L. Euler:

$$\zeta(2m) = -\frac{1}{2} \cdot \frac{(2\pi i)^{2m} B_{2m}}{(2m)!} \quad \text{for} \quad m \geq 1$$

with

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} \cdot t^k, \quad |t| < 2\pi.$$

In particular $\zeta(2m)\pi^{-2m} \in \mathbb{Q}$ for $m \geq 1$.

F. Lindemann: π *is transcendental.*

R. Apéry: $\zeta(3) \notin \mathbb{Q}$.

T. Rivoal: *Infinitely many* $\zeta(a)$ *with* a *odd* ≥ 5 *are irrational.*

K. Ball + T. Rivoal: *The* $\mathbb{Q}$*-vector subspace of* $\mathbb{R}$ *spanned by*

$$\zeta(3), \zeta(5), \cdots, \zeta(a)$$

has dimension

$$\geq \frac{\log a}{1+\log 2}\bigl(1+o(1)\bigr).$$

T. Rivoal, W. Zudilin: *One at least of the* 9 *numbers*

$$\zeta(5), \zeta(7), \cdots, \zeta(21)$$

is irrational.

Expected: Euler's relations are the only algebraic relation among the numbers

$$\zeta(2), \zeta(3), \zeta(4), \ldots, \zeta(s), \ldots$$

Conjecture. *The numbers*

$$\pi,\ \zeta(3),\ \zeta(5), \ldots, \zeta(2n+1), \ldots$$

algebraically independent.

In other terms:
For $n \geq 0$ *and* $P \in \mathbb{Q}[X_0, X_1, \ldots, X_n] \setminus \{0\}$,

$$P\bigl(\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)\bigr) \neq 0.$$

Definition. *For* $\underline{s} = (s_1, \ldots, s_k)$ *in* $\mathbb{Z}^k$ *with* $s_1 \geq 2$ *and* $s_j \geq 1$ $(2 \leq j \leq k)$*,*

$$\zeta(s_1, \ldots, s_k) := \sum_{n_1 > \cdots > n_k \geq 1} n_1^{-s_1} \cdots n_k^{-s_k}.$$

Examples of algebraic relations

[1]
$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$$

Proof.

$$\sum_{n\geq 1} \frac{1}{n^2} \sum_{m \geq 1} \frac{1}{m^2} = \sum_{n>m\geq 1} \frac{1}{n^2 m^2} + \sum_{m>n\geq 1} \frac{1}{n^2 m^2} + \sum_{n \geq 1} \frac{1}{n^4}.$$

[2]
$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1)$$

Corollary. $\zeta(3,1) = \frac{1}{4}\zeta(4)$.

[*3*]
$$\zeta(3) = \zeta(2,1)$$

`EZ-Face` by J. Borwein, P. Lisonek and P. Irvine

`http://www.cecm.sfu.ca/projects/EZFace/index.html`

Conjectures of Zagier, Drinfeld, Kontsevich, Goncharov, Hoffman, Broadhurst, Cartier…

For $p \geq 2$ let $\mathcal{Z}_p$ denote the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by the 2^{p-2} elements $\zeta(\underline{s})$ for $\underline{s} = (s_1, \ldots, s_k)$ of *weight* $s_1 + \cdots + s_k = p$ and $s_1 \geq 2$. Let d_p be the dimension of $\mathcal{Z}_p$.

$$\begin{aligned}
\mathcal{Z}_0 &= \mathbb{Q} && \text{with } \zeta(\underline{s}) := 1 \text{ for } k = 0\\
\mathcal{Z}_1 &= \{0\}\\
\mathcal{Z}_2 &= \langle \zeta(2) \rangle_{\mathbb{Q}}\\
\mathcal{Z}_3 &= \langle \zeta(3) \rangle_{\mathbb{Q}} && \text{since } \zeta(2,1) = \zeta(3)\\
\mathcal{Z}_4 &= \langle \zeta(4) \rangle_{\mathbb{Q}} = \langle \zeta(2)^2 \rangle_{\mathbb{Q}} = \langle \pi^4 \rangle_{\mathbb{Q}}
\end{aligned}$$

since

$$\zeta(3,1) = \frac{1}{4}\zeta(4), \quad \zeta(2,2) = \frac{3}{4}\zeta(4), \quad \zeta(2,1,1) = \zeta(4) = \frac{2}{5}\zeta(2)^2.$$

Hence

$$d_0 = 1, \quad d_1 = 0, \quad d_2 = d_3 = d_4 = 1.$$

$$\mathcal{Z}_5 = \langle \zeta(2)\zeta(3), \zeta(5) \rangle_{\mathbb{Q}}, \qquad \text{hence } d_5 \le 2.$$

Proof.

$$\begin{aligned} \zeta(2,1,1,1) &= \zeta(5), \\ \zeta(3,1,1) = \zeta(4,1) &= 2\zeta(5) - \zeta(2)\zeta(3), \\ \zeta(2,1,2) = \zeta(2,3) &= \frac{9}{2}\zeta(5) - 2\zeta(2)\zeta(3), \\ \zeta(2,2,1) = \zeta(3,2) &= 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5). \end{aligned}$$

Conjecture (D. Zagier). *We have*

$$d_p = d_{p-2} + d_{p-3} \quad \textit{for} \quad p \ge 4.$$

Equivalent formulation:

$$\sum_{p \ge 0} d_p X^p = \frac{1}{1 - X^2 - X^3}.$$

Conjecture (M. Hoffman). *For $p \geq 1$, a basis for the $\mathbb{Q}$-vector space $\mathcal{Z}_p$ is given by the numbers $\zeta(\underline{s})$ for $(s_1, \ldots, s_k) \in \mathbb{Z}^k$ $(k \geq 1)$ satisfying*

$$s_1 + \cdots + s_k = p \quad \text{and} \quad s_j \in \{2, 3\}.$$

The subalgebra $\mathcal{Z}$ of $\mathbb{R}$ generated by all $\zeta(\underline{s})$ is graded for the weight:

$$\mathcal{Z}_p \mathcal{Z}_{p'} \subset \mathcal{Z}_{p+p'}.$$

The *length* k defines a filtration on $\mathcal{Z}$. Denote by $\mathcal{F}^k \mathcal{Z}_p$ the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by all $\zeta(\underline{s})$ of weight p and length k. Further, let d_{pk} denote the dimension of $\mathcal{F}^k \mathcal{Z}_p / \mathcal{F}^{k-1} \mathcal{Z}_p$.

Conjecture (D. Broadhurst). *We have*

$$\left(\sum_{p \geq 0} \sum_{k \geq 0} d_{pk} X^p Y^k \right)^{-1} = 1 - X^2 - X^3 + \frac{XY^2(1 - Y^2)}{(1 + X^2)(1 - Y^6)}.$$

Conjecture (A.B. Goncharov). *As a $\mathbb{Q}$-algebra, $\mathcal{Z}$ is the direct sum of $\mathcal{Z}_p$ for $p \geq 0$.*

Example. Any $\zeta(\underline{s})$ with $\underline{s}$ of weight $p \leq 12$ is a homogeneous polynomial in the following 11 MZV:

$k \backslash p$	2	3	5	7	8	9	10	11	12
1	$\zeta(2)$	$\zeta(3)$	$\zeta(5)$	$\zeta(7)$		$\zeta(9)$		$\zeta(11)$	
2					$\zeta(6,2)$		$\zeta(8,2)$		$\zeta(10,2)$
3								$\zeta(8,2,1)$	
4									$\zeta(8,2,1,1)$

Classical polylogarithms

$$\mathrm{Li}_s(z) := \sum_{n\geq 1} \frac{z^n}{n^s} \qquad \text{for} \qquad |z| < 1.$$

Recursive definition:

$$\mathrm{Li}_1(z) = \sum_{n\geq 1} \frac{z^n}{n} = -\log(1-z),$$

$$z\frac{d}{dz}\mathrm{Li}_s(z) = \mathrm{Li}_{s-1}(z) \quad (s \geq 2)$$

with $\mathrm{Li}_s(0) = 0$.

For $s \geq 2$, $\mathrm{Li}_s(1) = \zeta(s)$.

Multiple polylogarithms in one variable

$$\mathrm{Li}_{\underline{s}}(z) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}},$$

for $\underline{s} = (s_1, \ldots, s_k)$ with $s_j \geq 1$ $(1 \leq j \leq k)$.

$$\zeta(\underline{s}) = \mathrm{Li}_{\underline{s}}(1) \quad \text{for} \quad s_1 \geq 2.$$

Recursive definition:

$$z\frac{d}{dz}\mathrm{Li}_{(s_1,\ldots,s_k)}(z) = \mathrm{Li}_{(s_1-1,s_2,\ldots,s_k)}(z) \qquad \text{if } s_1 \geq 2$$

$$(1-z)\frac{d}{dz}\mathrm{Li}_{(1,s_2,\ldots,s_k)}(z) = \mathrm{Li}_{(s_2,\ldots,s_k)}(z) \qquad \text{if } s_1 = 1.$$

Initial conditions $\mathrm{Li}_{\underline{s}}(0) = 0$.

Proof:

$$\sum_{n_1 > n_2} z^{n_1 - 1} = \frac{z^{n_2}}{1-z}\cdot$$

Noncommutative Polynomials

Let $X := \{x_0, x_1\}$ denote an alphabet with two letters. Let X^* denote the set of words on X.

The linear combinations of words with rational coefficients

$$\sum_u c_u u,$$

where $\{c_u \; ; \; u \in X^*\}$ is a set of rational numbers with finite support, is the non-commutative ring

$$\mathfrak{H} := \mathbb{Q}\langle x_0, x_1 \rangle.$$

The product is concatenation, the unit is the empty word e.

An example of such a polynomial is

$$3e - 4x_0 + 5x_1 - x_0x_1 + x_1x_0 + 2x_0x_1x_0^2.$$

For $s \geq 1$ define $x_s := x_0^{s-1} x_1$. For $\underline{s} = (s_1, \ldots, s_k) \in \mathbb{Z}^k$ with $s_j \geq 1$ define $y_{\underline{s}} := x_{s_1} \cdots x_{s_k}$. Hence

$$y_{\underline{s}} = x_0^{s_1-1} x_1 x_0^{s_2-1} x_1 \cdots x_0^{s_k-1} x_1.$$

For instance

$$y_{2,3} = x_0 x_1 x_0^2 x_1.$$

The number of letters in $y_{\underline{s}}$ is the weight p of $\underline{s}$, while the numbers of x_1 is the length k.

The set of words $y_{\underline{s}}$ with $\underline{s} = (s_1, \ldots, s_k)$ is nothing else than the set $X^* x_1$ of words which end with x_1. Define

$$\widehat{\mathrm{Li}}_{y_{\underline{s}}}(z) := \mathrm{Li}_{\underline{s}}(z)$$

for such a word.

Further define $\hat{\zeta}(y_{\underline{s}}) := \zeta(\underline{s})$ if $s_1 \geq 2$. Hence $\hat{\zeta}(w)$ is defined for any word $w \in x_0 X^* x_1$ which starts with x_0 and ends with x_1. Furthermore

$$\hat{\zeta}(w) = \widehat{\mathrm{Li}}_w(1) \quad \text{for} \quad w \in x_0 X^* x_1.$$

For a polynomial $w = \sum_u c_u u$ define

$$\widehat{\mathrm{Li}}_w(z) := \sum_u c_u \widehat{\mathrm{Li}}_u(z)$$

if each u with $u \notin \{e\} \cup X^* x_1$ has $c_u = 0$. Hence $\widehat{\mathrm{Li}}_w(z)$ is defined for

$$w \in \boxed{\mathfrak{H}^1 := \mathbb{Q}e + \mathfrak{H}x_1}$$

Similarly set

$$\hat{\zeta}(w) := \sum_u c_u \hat{\zeta}(u)$$

if each u with $u \notin \{e\} \cup x_0 X^* x_1$ has $c_u = 0$. For

$$w \in \boxed{\mathfrak{H}^0 := \mathbb{Q}e + x_0 \mathfrak{H} x_1}$$

we have

$$\hat{\zeta}(w) = \widehat{\mathrm{Li}}_w(1).$$

Hence $\hat{\zeta} : \mathfrak{H}^0 \to \mathbb{R}$ is a $\mathbb{Q}$-linear map.

Definition. *Shuffle product of two words in X^*: element in $\mathfrak{H}$ defined inductively by:*

$$e ш u = u ш e = u \quad \textit{for } u \textit{ and } u' \textit{ in } X^*,$$

and

$$(x_i u) ш (x_j v) = x_i \big(u ш (x_j v)\big) + x_j \big((x_i u) ш v\big)$$

for u, v in X^* and i, j in $\{0,1\}$.

Examples. For i_1, i_2, j, j_1, j_2 in $\{0,1\}$,

$$(x_{i_1} x_{i_2}) ш x_j = x_{i_1} x_{i_2} x_j + x_{i_1} x_j x_{i_2} + x_j x_{i_1} x_{i_2}.$$

$$\begin{aligned}(x_{i_1} x_{i_2}) ш (x_{j_1} x_{j_2}) = & x_{i_1} x_{i_2} x_{j_1} x_{j_2} + x_{i_1} x_{j_1} x_{i_2} x_{j_2} + x_{i_1} x_{j_1} x_{j_2} x_{i_2} + \\ & x_{j_1} x_{i_1} x_{i_2} x_{j_2} + x_{j_1} x_{i_1} x_{j_2} x_{i_2} + x_{j_1} x_{j_2} x_{i_1} x_{i_2}.\end{aligned}$$

Extending by distributivity with respect to the addition to $\mathfrak{H}$ defines commutative and associative algebras

$$\boxed{\mathfrak{H}^0_ш \subset \mathfrak{H}^1_ш \subset \mathfrak{H}_ш}$$

Radford's Theorem: $\mathfrak{H}_{ш}^0$, $\mathfrak{H}_{ш}^1$ and $\mathfrak{H}_{ш}$ are (commutative) polynomials algebras on the set of Lyndon words.
Consequence:

$$\mathfrak{H}_{ш}^1 = \mathfrak{H}_{ш}^0[x_1], \qquad \mathfrak{H}_{ш} = \mathfrak{H}_{ш}^1[x_0] = \mathfrak{H}_{ш}^0[x_0, x_1].$$

Proposition. *For u and u' in $\mathfrak{H}_{ш}^1$,*

$$\widehat{\mathrm{Li}}_u(z)\widehat{\mathrm{Li}}_{u'}(z) = \widehat{\mathrm{Li}}_{u ш u'}(z).$$

Consequence. *For u and u' in $\mathfrak{H}_{ш}^0$,*

$$\hat{\zeta}(u)\hat{\zeta}(u') = \hat{\zeta}(u ш u').$$

Proposition.

$$\begin{cases} z\dfrac{d}{dz}\widehat{\mathrm{Li}}_{x_0 u}(z) & = \widehat{\mathrm{Li}}_u(z) \\ (1-z)\dfrac{d}{dz}\widehat{\mathrm{Li}}_{x_1 u}(z) & = \widehat{\mathrm{Li}}_u(z) \end{cases}$$

Notation.

$$\omega_0(z) := \frac{dz}{z}, \qquad \omega_1(z) := \frac{dz}{1-z}.$$

Definition. *Define* $\widehat{\mathrm{Li}}_w(z)$ *for any* $w \in X^*$ *as follows.*

$$\mathrm{Li}_e(z) := 1, \qquad \widehat{\mathrm{Li}}_{x_0^s}(z) := \frac{1}{s!}(\log z)^s \quad \textit{for} \quad s \geq 1$$

and, for $j \in \{0,1\}$,

$$\widehat{\mathrm{Li}}_{x_j u}(z) := \int_0^z \widehat{\mathrm{Li}}_u(z)\omega_j(z)$$

if the word $x_j u$ *contains a letter* x_1.

Generating series.

$$\widehat{\mathrm{Li}}(z) := \sum_{w \in X^*} \widehat{\mathrm{Li}}_w(z) w.$$

Knizhnik-Zamolodchikov Differential Equation

$$\boxed{\frac{d}{dz}\widehat{\mathrm{Li}}(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)\widehat{\mathrm{Li}}(z).}$$

Initial condition: $z \mapsto \widehat{\mathrm{Li}}(z) z^{-x_0}$ is holomorphic near 0 in

$\mathbb{C} \setminus \{(-\infty, 0] \cup [1, \infty)\}$:

Theorem (H.N. Minh and M. Petitot). *The functions* $\widehat{\mathrm{Li}}_w(z)$, *for* w *in* X^* *are linearly independent over* $\mathbb{C}$.

Drinfeld associator:

$$\Phi_{KZ} = \sum_{w \in X^*} \hat{\zeta}(w) w \in \mathbb{C}\langle\langle x_0, x_1 \rangle\rangle.$$