The square root of 2, the Golden Ratio and the Fibonacci sequence

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Abstract

The square root of 2,

\[ \sqrt{2} = 1.414213562373095\ldots, \]

and the Golden ratio

\[ \Phi = \frac{1 + \sqrt{5}}{2} = 1.618033988749894\ldots \]

are two irrational numbers with many remarkable properties. The Fibonacci sequence

\[ 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233\ldots \]

occurs in many situations, in mathematics as well as in the real life. We review some of these properties.
Tablet YBC 7289: 1800 – 1600 BC

Babylonian clay tablet, accurate sexagesimal approximation to $\sqrt{2}$ to the equivalent of six decimal digits.

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.414212962962962\ldots$$

$$\sqrt{2} = 1.414213562373095048\ldots$$

https://en.wikipedia.org/wiki/YBC_7289
A4 format $21 \times 29.7$

ISO 216 International standard

$$\frac{297}{210} = \frac{99}{70} = 1.414285714285714285\ldots$$

$$\sqrt{2} = 1.414213562373095048\ldots$$
A, B, C formats

Large rectangle: sides $x, 1$; proportion $\frac{x}{1} = x$
Small rectangles: sides $1, \frac{x}{2}$; proportion $\frac{1}{x/2} = \frac{2}{x}$

$$x = \frac{2}{x}, \quad x^2 = 2.$$
The large rectangle and half of it are proportional.

Reference: Paul Gérardin

A format

The number \( \sqrt{2} \) is twice its inverse: \( \sqrt{2} = 2/\sqrt{2} \). Folding a rectangular piece of paper with sides in proportion \( \sqrt{2} \) yields a new rectangular piece of paper with sides in proportion \( \sqrt{2} \) again.

A0 is 118.9cm \( \times \) 84.1cm - area 1 m\(^2\).
B and C formats

B0 is $1 \text{m} \times 1.414 \text{m}$.
B7 (passeport) is $88 \text{mm} \times 125 \text{mm}$.

C0 is $917 \text{mm} \times 1297 \text{mm}$, approximately $\frac{1}{\sqrt{2}} \times 8\sqrt{8}$.
C6 : $114 \text{mm} \times 162 \text{mm}$
  enveloppe for a A6 paper $105 \text{mm} \times 148 \text{mm}$

Xerox machine : enlarging and reducing

<table>
<thead>
<tr>
<th>%</th>
<th>141%</th>
<th>119%</th>
<th>84%</th>
<th>71%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.41</td>
<td>1.19</td>
<td>0.84</td>
<td>0.71</td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>$1.4142$</td>
<td>$1.1892$</td>
<td>$0.8409$</td>
<td>$0.7071$</td>
</tr>
<tr>
<td></td>
<td>$\frac{4}{\sqrt{2}}$</td>
<td>$\frac{4}{2}$</td>
<td>$\frac{4}{\sqrt{2}}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
</tr>
</tbody>
</table>
Paper format A0, A1, A2,\ldots in cm

\[ x_1 = 100 \sqrt[4]{2} = 118.9, \quad x_2 = \frac{100}{\sqrt[4]{2}} = 84.1. \]

A0: \quad x_1 = 118.9 \quad x_2 = 84.1

A1: \quad x_2 = 84.1 \quad \frac{x_1}{2} = 59.4

A2: \quad \frac{x_1}{2} = 59.4 \quad \frac{x_2}{2} = 42

A3: \quad \frac{x_2}{2} = 42 \quad \frac{x_1}{4} = 29.7

A4: \quad \frac{x_1}{4} = 29.7 \quad \frac{x_2}{4} = 21

A5: \quad \frac{x_2}{4} = 21 \quad \frac{x_1}{8} = 14.8
Irrationality of $\sqrt{2}$

Assume $\sqrt{2} = \frac{a}{b}$. Then $a' = 2b - a$ and $b' = a - b$.

\[ a^2 = 2b^2, \quad a'^2 = 2b'^2, \quad a' < a, \quad b' < b. \]
Irrationality of $\sqrt{2}$ (again)

Assume $a^2 = 2b^2$. You have the same amount of green painting and yellow painting. Put two yellow squares of sides $b$ into a green square of side $a$. They overlap into a red square of side length $a' = 2b - a$. The green squares have a side length $b' = a - b$.

On the right image, you first paint the yellow part. The amount of yellow painting which is left enables you to paint either twice the red square, or once the red square and once both green squares. Hence the red area is the same as the green area: $a'^2 = 2b'^2$, with $a' < a$, $b' < b$. 
Irrationality of $\sqrt{3}$

Assume $a^2 = 3b^2$

Large equilateral triangle : side length $a$
Three red equilateral triangles : side length $b$
Three purple equilateral triangles : side length $b' = 2b - a$
White equilateral triangles : side length $a' = 2a - 3b$.

$0 < a' < a, \quad 0 < b' < b.$

Jean-Paul Delahaye, Cinq pépites mathématiques de John Conway.
In : Logique et Calcul, Pour La Science No° 515 / September 2020
https://www.pourlascience.fr/sr/logique-calcul/cinq-peeptes-mathematiques-de-john-conway-19963.php
Rectangle with proportion $\sqrt{2}$

One square plus 2 rectangles with proportion $1 + \sqrt{2}$:

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}, \quad 1 + \sqrt{2} = 2 + \frac{1}{1 + \sqrt{2}}.$$
Irrationality of $\sqrt{2}$: geometric proof

\[
\begin{align*}
\frac{99}{70} &= 1 + \frac{29}{70}, \\
\frac{70}{29} &= 2 + \frac{12}{29}, \\
\frac{29}{12} &= 2 + \frac{5}{12}, \\
\frac{12}{5} &= 2 + \frac{2}{5}, \\
\frac{5}{2} &= 2 + \frac{1}{2}.
\end{align*}
\]

\[
\frac{297}{210} = \frac{99}{70}.
\]
Continued fraction of $\sqrt{2}$

The number

$$\sqrt{2} = 1.41421356237309504880168872420 \ldots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}.$$

Hence

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{1 + \sqrt{2}}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{1 + \sqrt{2}}}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}}}.$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1, 2, 2, 2, 2, 2, \ldots] = [1, \overline{2}].$$
\[
\frac{297}{210} = 1 + \frac{29}{70}, \\
\frac{70}{29} = 2 + \frac{12}{29}, \\
\frac{29}{12} = 2 + \frac{5}{12}, \\
\frac{12}{5} = 2 + \frac{2}{5}, \\
\frac{5}{2} = 2 + \frac{1}{2}.
\]

Hence

\[
\frac{297}{210} = [1, 2, 2, 2, 2, 2].
\]
An interesting street number

The puzzle itself was about a street in the town of Louvain in Belgium, where houses are numbered consecutively. One of the house numbers had the peculiar property that the total of the numbers lower than it was exactly equal to the total of the numbers above it. Furthermore, the mysterious house number was greater than 50 but less than 500.

Prasanta Chandra Mahalanobis 1893 – 1972

Srinivasa Ramanujan 1887 – 1920

http://mathshistory.st-andrews.ac.uk/Biographies/Mahalanobis.html
Street number : examples

Examples :
- House number 6 in a street with 8 houses :

\[ 1 + 2 + 3 + 4 + 5 = 15, \quad 7 + 8 = 15. \]

- House number 35 in a street with 49 houses :

\[ 1 + 2 + 3 + \cdots + 34 = \frac{34 \times 35}{2} = 595, \]

\[ 36 + 37 + \cdots + 49 = \frac{49 \times 50}{2} - \frac{35 \times 36}{2} = 1225 - 630 = 595. \]

The puzzle requests the house number between 50 and 500.

Remark. Other solutions :
- House number 1 in a street with 1 house ;
Ramanujan : *if no banana is distributed to no student, will each student get a banana ?*
- House number 0 in a street with 0 house.
Let $m$ be the house number and $n$ the number of houses:

$$1 + 2 + 3 + \cdots + (m - 1) = (m + 1) + (m + 2) + \cdots + n.$$ 

$$\frac{m(m - 1)}{2} = \frac{n(n + 1)}{2} - \frac{m(m + 1)}{2}.$$ 

This is $2m^2 = n(n + 1)$. Complete the square on the right:

$$8m^2 = (2n + 1)^2 - 1.$$ 

Set $x = 2n + 1$, $y = 2m$. Then

$$x^2 - 2y^2 = 1.$$
\[ x^2 - 2y^2 = 1, \quad x = 2n + 1, \quad y = 2m \]

\[ x^2 - 2y^2 = (x - y\sqrt{2})(x + y\sqrt{2}). \]

Trivial solution : \( x = 1, \ y = 0, \ m = n = 0. \)

Non trivial solution : \( x_1 = 3, \ y_1 = 2. \) Other solutions \((x_\nu, y_\nu)\) with \( x_\nu - y_\nu \sqrt{2} = (3 - 2\sqrt{2})^\nu. \)

- \( \nu = 1, \ x_1 = 3, \ y_1 = 2, \ m = n = 1. \)
- \( \nu = 2, \)

\[ x_2 - y_2 \sqrt{2} = (3 - 2\sqrt{2})^2 = 17 - 12\sqrt{2}, \]

\[ x_2 = 17, \quad y_2 = 12, \quad n = 8, \quad m = 6. \]

- \( \nu = 3, \)

\[ x_3 - y_3 \sqrt{2} = (3 - 2\sqrt{2})^3 = 99 - 70\sqrt{2}, \]

\[ x_3 = 99, \quad y_3 = 70, \quad n = 49, \quad m = 35. \]

Recall : \( \frac{99}{70} = \frac{297}{210}. \)
Brahmagupta (628)

Brahmasphutasiddhanta: Solve in integers the equation

\[ x^2 - 92y^2 = 1 \]

Answer: \((x, y) = (1151, 120)\)

The continued fraction expansion of \(\sqrt{92}\) is

\[ \sqrt{92} = [9, 1, 1, 2, 4, 2, 1, 1, 18]. \]

Compute

\[ [9, 1, 1, 2, 4, 2, 1, 1] = \frac{1151}{120}. \]

Indeed \(1151^2 - 92 \cdot 120^2 = 1324801 - 1324800 = 1\).
Bhaskara II (12th Century)

*Lilavati*

*(Bijaganita, 1150)* \( x^2 - 61y^2 = 1 \)

A solution is:

\[
x = 1766319049, \\
y = 226153980.
\]

Cyclic method (Chakravala) of *Brahmagupta*.

\[
\sqrt{61} = [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14]
\]

\[
[7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 5] = \frac{1766319049}{226153980}
\]
Continued fraction of $\frac{x_\nu}{y_\nu}$

Trivial solution of $x^2 - 2y^2 = 1 : x_0 = 1, y_0 = 0$. First non trivial solution : $x_1 = 3, y_1 = 2$. We have

$$\frac{x_1}{y_1} = \frac{3}{2} = 1 + \frac{1}{2} = [1, 2].$$

Second solution : $x_2 = 17, y_2 = 12$

$$\frac{x_2}{y_2} = \frac{17}{12} = 1 + \frac{5}{12}, \quad \frac{12}{5} = 2 + \frac{2}{5}, \quad \frac{5}{2} = 2 + \frac{1}{2},$$

hence

$$\frac{17}{12} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = [1, 2, 2, 2].$$
Continued fraction of $\frac{x_3}{y_3}$

Third solution of $x^2 - 2y^2 = 1 : x_3 = 99, y_3 = 70.$

$$\frac{x_3}{y_3} = \frac{99}{70} = 1 + \frac{29}{70}, \quad \frac{70}{29} = 2 + \frac{12}{29}, \quad \frac{29}{12} = 2 + \frac{5}{12}$$

with

$$\frac{12}{5} = 2 + \frac{2}{5}, \quad \frac{5}{2} = 2 + \frac{1}{2}$$

hence

$$\frac{99}{70} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}}} = [1, 2, 2, 2, 2, 2].$$
Continued fraction of \( \frac{x_4}{y_4} \)

Fourth solution of \( x^2 - 2y^2 = 1 \)

\[
[1, 2, 2, 2, 2, 2, 2] = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}}} = \frac{577}{408}.
\]

\[577^2 - 2 \times 408^2 = 1, \quad 577 = 2 \times 288 + 1, \quad 408 = 2 \times 204.\]

Hence the solution to the puzzle is: the house number is 204 in a street with 288 houses:

\[
1 + 2 + 3 + 4 + 5 + \cdots + 203 = \frac{203 \times 204}{2} = 20706,
\]

\[
205 + 206 + \cdots + 288 = \frac{288 \times 289}{2} - \frac{204 \times 205}{2} = 20706.
\]
Balancing numbers

The next solution is \( m = 1189 > 500 \).

Sequence of balancing numbers (number of the house)
https://oeis.org/A001109

\[ 0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214 \ldots \]

This is a linear recurrence sequence

\[ u_{n+1} = 6u_n - u_{n-1} \]

with the initial conditions \( u_0 = 0, \ u_1 = 1 \).

The number of houses is https://oeis.org/A001108

\[ 0, 1, 8, 49, 288, 1681, 9800, 57121, 332928, 1940449, \ldots \]
OEIS

Neil J. A. Sloane’s encyclopaedia

http://oeis.org/A001597
First decimals of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.41421356237309504880168872420969807856967187537694807317667973
799073247846210703885038753432764157273501384623091229702492483
605585073721264412149709993583141322266592750559275579995050115
278206057147010955997160597027453459686201472851741864088919860
955232923048430871432145083976260362799525140798968725339654633
180882964062061525835239505474575028775996172983557522033753185
701135437460340849884716038689997069900481503054402779031645424
782306849293691862158057846311159666871301301561856898723723528
850926486124949771542183342042856860601468247207714358548741556
570696776537202264854470158580016207584749226572260020855844665
214583988939443709265918003113882464681570826301005948587040031
864803421948972782906410450726368813137398552561173220402450912
277002269411275736272804957381089675040183698683684507257993647
290607629969413804756548237289971803268024744206292691248590521
810044598421505911202494413417285314781058036033710773091828693
1471017111168391658172688941975871658215212822951848847 ...
First binary digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.011010100001001111001100111111100111100110010010000
100010110010111110110010011101011010110100100100010111110100
11111000111010110111101100000101110101000100100111011101101000
10011001111010100010111110101100100011010000011001100110011001
1000101101011111001110000110000100001110110101100010100
010110000111010100010110001111111100110011111110111001000011110
110110011100100001111011101010010111001000111001100110011001100
11110110100101001111000001110001011001111101001001001110101111101
0001001110100011001110010010001111011110010000100110001110100010101
11100011111001011100011001000011000010111110001100000010001110101
11100011010011110110010001011100010111110001100000010001110101
111000110011111110011001011100010111110001100000010001110101
0001000110011001000110010010100001011000110010100110010001100101
10111110001011100011100110011110101101001000111100011000111111101
0111011111101001111001110011001010010111000111001000000111111111111
000101011101100111001110001010100011111111000111111111111111111111111
00000101011101100101100000101110101010101011000010111111111111111111111111
Computation of decimals of $\sqrt{2}$

1542 decimals computed by hand by Horace Uhler in 1951

14000 decimals computed in 1967

1000000 decimals in 1971

$137 \cdot 10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

- Motivation: computation of $\pi$. 
Emile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques*,
  Palermo Rend. 27, 247-271 (1909).
  Jahrbuch Database
  http://www.emis.de/MATH/JFM/JFM.html

- *Sur les chiffres décimaux de \(\sqrt{2}\) et divers problèmes de probabilités en chaînes*,
  Zbl 0035.08302
Let $g \geq 2$ be an integer and $x$ a real irrational algebraic number. 
*The expansion in base $g$ of $x$ should satisfy some of the laws which are valid for almost all real numbers (with respect to Lebesgue’s measure).*

*Open problem.* Select one digit $c$ among \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}. Choose a real algebraic irrational number $\alpha$ like $\sqrt{2}$. Is it true that the digit $c$ occurs infinitely often in the decimal expansion of $\alpha$? It is conjectured that the answer is always yes. There is no example of $(c, \alpha)$ for which we can prove that it is true.
This is a nice rectangle
Golden rectangle

\[ \frac{\Phi}{1} = \frac{1}{\Phi - 1}, \quad \Phi^2 = \Phi + 1. \]
Irrationality of $\Phi$ and of $\sqrt{5}$

The number

$$\Phi = \frac{1 + \sqrt{5}}{2} = 1.618033988749894\ldots$$

satisfies

$$\Phi = 1 + \frac{1}{\Phi}.$$ 

Hence

$$\Phi = 1 + \frac{1}{1 + \frac{1}{\Phi}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\Phi}} \ldots}$$

If we start from a rectangle with the Golden ratio as proportion of sides lengths, at each step we get a square and a smaller rectangle with the same proportion for the sides lengths.

http://oeis.org/A001622
The Golden Ratio \( \frac{1 + \sqrt{5}}{2} = 1.618033988749894 \ldots \)
The diagonal of the pentagon and the diagonal of the octogon

The diagonal of the pentagon is $\Phi$  

The diagonal of the octogon is $1 + \sqrt{2}$
Nested roots

\[ \Phi^2 = 1 + \Phi. \]

\[ \Phi = \sqrt{1 + \Phi} \]

\[ = \sqrt{1 + \sqrt{1 + \Phi}} \]

\[ = \sqrt{1 + \sqrt{1 + \sqrt{1 + \Phi}}} \]

\[ = \ldots \]

\[ = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}} \]
Nested roots


\[
\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots} = 3}}}
\]

\[
\sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + \cdots} = 4}}}
\]

Srinivasa Ramanujan
1887 – 1920
Back to $\sqrt{2}$

$$(1 + \sqrt{2})^2 = 1 + 2(1 + \sqrt{2}).$$

$$1 + \sqrt{2} = \sqrt{1 + 2(1 + \sqrt{2})}$$

$$= \sqrt{1 + 2 \sqrt{1 + 2(1 + \sqrt{2})}}$$

$$= \sqrt{1 + 2 \sqrt{1 + 2 \sqrt{1 + 2 \sqrt{1 + 2 \sqrt{1 + \cdots}}}}}$$
Geometric series

\[ u_0 = 1, \quad u_{n+1} = 2u_n \]

How many ancestors do we have?

Sequence: 1, 2, 4, 8, 16 ...

\[ u_n = 2^n, \quad n \geq 0. \]
Bees genealogy

Male honeybees are born from unfertilized eggs. Female honeybees are born from fertilized eggs. Therefore males have only a mother, but females have both a mother and a father.
Genealogy of a male bee (bottom – up)

Number of bees:

1, 1, 2, 3, 5…

Number of females:

0, 1, 1, 2, 3…

Rule:

\[ u_{n+2} = u_{n+1} + u_n. \]
Bees genealogy $u_1 = 1, \ u_2 = 1, \ u_{n+2} = u_{n+1} + u_n$

Number of females at a given level = total population at the previous level
Number of males at a given level = number of females at the previous level

\[
\begin{align*}
3 + 5 &= 8 \\
2 + 3 &= 5 \\
1 + 2 &= 3 \\
1 + 1 &= 2 \\
0 + 1 &= 1 \\
1 + 0 &= 1
\end{align*}
\]
The Lamé Series

In 1844 the sequence

\[ 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots \]

was referred to as the Lamé series, because Gabriel Lamé used it to give an upper bound for the number of steps in the Euclidean algorithm for the gcd.

On a trip to Italy in 1876 Edouard Lucas found them in a copy of the Liber Abbaci of Leonardo da Pisa.
The Fibonacci sequence \((F_n)_{n \geq 0}\),

\[
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots
\]
is defined by

\[
F_0 = 0, \quad F_1 = 1,
\]

\[
F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n \geq 0.
\]

http://oeis.org/A000045
Leonardo Pisano (Fibonacci)

Guglielmo Bonacci : filius Bonacci or Fibonacci

travels around the mediterranean,

learns the techniques of Al-Khwarizmi

Liber Abbaci (1202)

https://commons.wikimedia.org/w/index.php?curid=720501
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, ... 

The Fibonacci sequence is available online

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

http://oeis.org/A000045
Fibonacci rabbits

Fibonacci considered the growth of a rabbit population.

A newly born pair of rabbits, a male and a female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was: how many pairs will there be in one year?

Answer: $F_{12} = 144$. 
Fibonacci’s rabbits

Modelization of a population

- First month
- Second month
- Third month
- Fourth month
- Fifth month
- Sixth month

Sequence: 1, 1, 2, 3, 5, 8, ...

Adult pairs

Young pairs
Modelization of a population of mice

Exponential sequence

- First month
- Second month
- Third month
- Fourth month

Number of pairs: 1, 2, 4, 8, ...
Fibonacci squares

http://mathforum.org/dr.math/faq/faq.golden.ratio.html
Geometric construction of the Fibonacci sequence
The Fibonacci numbers in nature

Ammonite (Nautilus shape)
Phyllotaxy

• Study of the position of leaves on a stem and the reason for them
• Number of petals of flowers: daisies, sunflowers, aster, chicory, asteraceae,…
• Spiral pattern to permit optimal exposure to sunlight
• Pine-cone, pineapple, Romanesco cawlfower, cactus
Leaf arrangements
• Université de Nice, Laboratoire Environnement Marin Littoral, Equipe d'Accueil "Gestion de la Biodiversité"

http://www.unice.fr/LEML/coursJDV/tp/tp3.htm
Phyllotaxy
Phyllotaxy

• J. Kepler (1611) uses the Fibonacci sequence in his study of the dodecahedron and the icosaedron, and then of the symmetry of order 5 of the flowers.

• Stéphane Douady and Yves Couder
  *Les spirales végétales*
Why are there so many occurrences of the Fibonacci numbers and of the Golden ratio in the nature?

According to Leonid Levin, objects with a small algorithmic Kolmogorov complexity (generated by a short program) occur more often than others.

Another example is given by Sierpinski triangles.

Reflections of a ray of light

Consider three parallel sheets of glass and a ray of light which crosses the first sheet. Each time it touches one of the sheets, it can cross it or reflect on it.

Denote by $p_n$ the number of different paths with the ray going out of the system after $n$ reflections.

$p_0 = 1,$

$p_1 = 2,$

$p_2 = 3,$

$p_3 = 5.$

In general, $p_n = F_{n+2}.$
Levels of energy of an electron of an atom of hydrogen

An atom of hydrogen can have three levels of energy, 0 at the ground level when it does not move, 1 or 2. At each step, it \textbf{alternatively} gains and looses some level of energy, either 1 or 2, without going sub 0 nor above 2. Let $\ell_n$ be the number of different possible scenarios for this electron after $n$ steps.

In general, $\ell_n = F_{n+2}$.

We have $\ell_0 = 1$ (initial state level 0)

$\ell_1 = 2$ : state 1 or 2, scenarios (ending with gain) 01 or 02.

$\ell_2 = 3$ : scenarios (ending with loss) 010, 021 or 020.

$\ell_3 = 5$ : scenarios (ending with gain) 0101, 0102, 0212, 0201 or 0202.
Rhythmic patterns

The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody. Several Indian scholars, Pingala (200 BC), Virahanka (c. 700 AD), Gopāla (c. 1135), and the Jain scholar Hemachandra (c. 1150), studied rhythmic patterns that are formed by concatenating one beat notes • and double beat notes ■.

one-beat note • : short syllable (ti in Morse Alphabet)
double beat note ■ : long syllable (ta ta in Morse)

1 beat, 1 pattern : •
2 beats, 2 patterns : •• and ■■
3 beats, 3 patterns : •••, •■■ and ■■•
4 beats, 5 patterns :

• • • •, ■■• •, •■■•, • •■■, ■■■■

n beats, $F_{n+1}$ patterns.
Fibonacci sequence and Golden Ratio

The developments

\[
[1], \quad [1, 1], \quad [1, 1, 1], \quad [1, 1, 1, 1], \quad [1, 1, 1, 1, 1], \quad [1, 1, 1, 1, 1, 1], \ldots
\]

are the quotients

\[
\begin{array}{cccccccc}
F_2 & F_3 & F_4 & F_5 & F_6 & F_7 \\
F_1 & F_2 & F_3 & F_4 & F_5 & F_6 \\
\| & \| & \| & \| & \| & \| \\
1 & 2 & 3 & 5 & 8 & 13 \\
\overline{1} & \overline{1} & \overline{2} & \overline{3} & \overline{5} & \overline{8}
\end{array}
\]

of consecutive Fibonacci numbers.

The development \([1, 1, 1, 1, 1, \ldots]\) is the continued fraction expansion of the Golden Ratio

\[
\Phi = \frac{1 + \sqrt{5}}{2} = \lim_{n \to \infty} \frac{F_{n+1}}{F_n} = 1.618033988749894\ldots
\]

which satisfies

\[
\Phi = 1 + \frac{1}{\Phi}.
\]
The Fibonacci sequence and the Golden ratio

For $n \geq 0$, the Fibonacci number $F_n$ is the nearest integer to

$$\frac{1}{\sqrt{5}} \Phi^n,$$

where $\Phi$ is the Golden Ratio:

$$\Phi = \lim_{n \to \infty} \frac{F_{n+1}}{F_n}.$$
Binet’s formula

For \( n \geq 0 \),

\[
F_n = \frac{\Phi^n - (-\Phi)^{-n}}{\sqrt{5}}
\]

\[
= \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}},
\]

\[\Phi = \frac{1 + \sqrt{5}}{2}, \quad -\Phi^{-1} = \frac{1 - \sqrt{5}}{2},\]

\[X^2 - X - 1 = (X - \Phi)(X + \Phi^{-1}).\]
The so-called Binet Formula

Formula of A. De Moivre (1718, 1730), Daniel Bernoulli (1726), L. Euler (1728, 1765), J.P.M. Binet (1843) : for $n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$
Generating series

A single series encodes all the Fibonacci sequence:

$$\sum_{n\geq 0} F_n X^n = X + X^2 + 2X^3 + 3X^4 + 5X^5 + \cdots + F_n X^n + \cdots$$

Fact: this series is the Taylor expansion of a rational fraction:

$$\sum_{n\geq 0} F_n X^n = \frac{X}{1 - X - X^2}.$$ 

Proof: the product

$$(X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots)(1 - X - X^2)$$

is a telescoping series

$$X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots$$
$$-X^2 - X^3 - 2X^4 - 3X^5 - 5X^6 - \cdots$$
$$-X^3 - X^4 - 2X^5 - 3X^6 - \cdots$$

$$= X.$$
Remark. The denominator $1 - X - X^2$ in the right hand side of

$$X + X^2 + 2X^3 + 3X^4 + \cdots + F_nX^n + \cdots = \frac{X}{1 - X - X^2}$$

is $X^2 f(X^{-1})$, where $f(X) = X^2 - X - 1$ is the irreducible polynomial of the Golden ratio $\Phi$. 

Homogeneous linear differential equation
Consider the homogeneous linear differential equation

\[ y'' - y' - y = 0. \]

If \( y = e^{\lambda x} \) is a solution, from \( y' = \lambda y \) and \( y'' = \lambda^2 y \) we deduce

\[ \lambda^2 - \lambda - 1 = 0. \]

The two roots of the polynomial \( X^2 - X - 1 \) are \( \Phi \) (the Golden ratio) and \( \Phi' \) with

\[ \Phi' = 1 - \Phi = -\frac{1}{\Phi}. \]

A basis of the space of solutions is given by the two functions \( e^{\Phi x} \) and \( e^{\Phi' x} \). Since (Binet’s formula)

\[ \sum_{n \geq 0} \frac{F_n x^n}{n!} = \frac{1}{\sqrt{5}} \left( e^{\Phi x} - e^{\Phi' x} \right), \]

this exponential generating series of the Fibonacci sequence is a solution of the differential equation.
Fibonacci and powers of matrices

The Fibonacci linear recurrence relation $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ can be written

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}.$$ 

By induction one deduces, for $n \geq 0$,

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

An equivalent formula is, for $n \geq 1$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$
The characteristic polynomial of the matrix

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \]

is

\[ \det(XI - A) = \det \begin{pmatrix} X & -1 \\ -1 & X - 1 \end{pmatrix} = X^2 - X - 1, \]

which is the irreducible polynomial of the Golden ratio \( \Phi \).
The Fibonacci sequence and the Golden ratio (continued)

For $n \geq 1$, $\Phi^n \in \mathbb{Z}[\Phi] = \mathbb{Z} + \mathbb{Z}\Phi$ is a linear combination of 1 and $\Phi$ with integer coefficients, namely

$$\Phi^n = F_{n-1} + F_n \Phi.$$ 

$\Phi = 0 + \Phi$  
$\Phi^2 = 1 + \Phi$  
$\Phi^3 = 1 + 2\Phi$  
$\Phi^4 = 2 + 3\Phi$  
$\Phi^5 = 3 + 5\Phi$  
$\Phi^6 = 5 + 8\Phi$  
$\Phi^7 = 8 + 13\Phi$  

...
The Fibonacci sequence satisfies a lot of very interesting properties. Four times a year, the *Fibonacci Quarterly* publishes an issue with new properties which have been discovered.
Narayana’s cows

Narayana was an Indian mathematician in the 14th century who proposed the following problem:

A cow produces one calf every year. Beginning in its fourth year each calf produces one calf at the beginning of each year. How many calves are there altogether after, for example, 17 years?
Narayana sequence

Narayana sequence is defined by the recurrence relation

\[ C_{n+3} = C_{n+2} + C_n \]

with the initial values \( C_0 = 2, \ C_1 = 3, \ C_2 = 4. \)

It starts with

\[ 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, \ldots \]

Real root of \( x^3 - x^2 - 1 \)

\[
\frac{3 \sqrt{\frac{29 + 3 \sqrt{93}}{2}}}{3} + \frac{3 \sqrt{\frac{29 - 3 \sqrt{93}}{2}}}{3} + 1 = 1.465571231876768\ldots
\]
Music:

http://www.pogus.com/21033.html

In working this out, **Tom Johnson** found a way to translate this into a composition called *Narayana’s Cows.*

**Music:** Tom Johnson  
**Saxophones:** Daniel Kientzy
Jean-Paul Allouche and Tom Johnson

http://www.math.jussieu.fr/~jean-paul.allouche/bibliorecente.html
Cows, music and morphisms

Jean-Paul Allouche and Tom Johnson

• Narayana’s Cows and Delayed Morphisms
  http://kalvos.org/johness1.html

• Finite automata and morphisms in assisted musical composition,
  http://www.tandfonline.com/doi/abs/10.1080/09298219508570676
Music and the Fibonacci sequence

- Dufay, XVème siècle
- Roland de Lassus
- Debussy, Bartok, Ravel, Webern
- Stockhausen
- Xenakis
- Tom Johnson *Automatic Music for six percussionists*
M.R. Schroeder.  
Number theory in science and communication:  
with applications in cryptography, physics, digital information, computing and self similarity  
Electric networks

• The resistance of a network in series

\[ R_1 + R_2. \]

• The resistance \( R \) of a network in parallel

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.
\]
Electric networks and continued fractions

The resistance $U$ of the circuit is given by

$$U = \frac{1}{S + \frac{1}{R + \frac{1}{T}}}$$
Decomposition of a square in squares

- For the network

\[
\begin{align*}
R_0 & \quad 1/S_1 \\
R_1 & \quad 1/S_2 \\
R_2 & \quad 1/S_3
\end{align*}
\]

the resistance is given by a continued fraction expansion

\[ [R_0, S_1, R_1, S_2, R_2, \ldots] \]

- Electric networks and continued fractions have been used to find the first solution to the problem of decomposing an integer square into a disjoint union of integer squares, all of which are distinct.
Squaring the square

There is a unique simple perfect square of order 21 (the lowest possible order), discovered in 1978 by A. J. W. Duijvestijn (Bouwkamp and Duijvestijn 1992). It is composed of 21 squares with total side length 112, and is illustrated above.
Applications of Diophantine Approximation


Further applications of Diophantine Approximation include equidistribution modulo 1, discrepancy, numerical integration, interpolation, approximate solutions to integral and differential equations.

http://www-history.mcs.st-and.ac.uk/Biographies/Hua.html
http://www-history.mcs.st-and.ac.uk/PictDisplay/Wang_Yuan.html
The square root of $\sqrt{2}$, the Golden Ratio
and the Fibonacci sequence

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