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On a problem of Mahler concerning the approximation of exponentials and logarithms

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Abstract. We first propose two conjectural estimates on diophantine approximation of logarithms of algebraic numbers. Next we discuss the state of the art and we give further partial results on this topic.

1. Two conjectures on diophantine approximation of logarithms of algebraic numbers

In 1953 K. Mahler [7] proved that for any sufficiently large positive integers a and b, the estimates

$$\|\log a\| \geq a^{-40\log\log a} \quad \text{and} \quad \|e^b\| \geq b^{-40b} \tag{1}$$

hold; here, $\|\cdot\|$ denotes the distance to the nearest integer: for $x \in \mathbb{R}$,

$$\|x\| = \min_{n\in\mathbb{Z}} |x-n|.$$

In the same paper [7], he remarks:

> *"The exponent* $40\log\log a$ *tends to infinity very slowly; the theorem is thus not excessively weak, the more so since one can easily show that*
>
> $$|\log a - b| < \frac{1}{a}$$

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for an infinite increasing sequence of positive integers a and suitable integers b."

(We have replaced Mahler's notation f and a by a and b, respectively for coherence with what follows).

In view of this remark we shall dub *Mahler's problem* the following open question:

(?) *Does there exist an absolute constant $c>0$ such that, for any positive integers a and b,*
$$|e^b - a| \geq a^{-c}\,?$$

Mahler's estimates (1) have been refined by MAHLER himself [8], M. MIGNOTTE [10] and F. WIELONSKY [19]: the exponent 40 can be replaced by 19.183.

Here we propose two generalizations of Mahler's problem. One common feature to our two conjectures is that we replace rational integers by algebraic numbers. However if, for simplicity, we restrict them to the special case of rational integers, then they deal with simultaneous approximation of logarithms of positive integers by rational integers. In higher dimension, there are two points of view: one takes either a hyperplane, or else a line. Our first conjecture is concerned with lower bounds for $|b_0+b_1\log a_1+\cdots+b_m\log a_m|$, which amounts to ask for lower bounds for $|e^{b_0}a_1^{b_1}\ldots a_m^{b_m}-1|$. We are back to the situation considered by Mahler in the special case $m=1$ and $b_m=-1$. Our second conjecture asks for lower bounds for $\max_{1\leq i\leq m}|b_i-\log a_i|$, or equivalently for $\max_{1\leq i\leq m}|e^{b_i}-a_i|$. Mahler's problem again corresponds to the case $m=1$. In both cases $a_1,\ldots,a_m$, $b_0,\ldots,b_m$ are positive rational integers.

Dealing more generally with algebraic numbers, we need to introduce a notion of height. Here we use Weil's absolute logarithmic height $\mathrm{h}(\alpha)$ (see [5, Chap. IV, §1] as well as [18]), which is related to Mahler's measure $\mathrm{M}(\alpha)$ by
$$\mathrm{h}(\alpha)=\frac{1}{d}\log \mathrm{M}(\alpha)$$
and
$$\mathrm{M}(\alpha)=\exp\left(\int_0^1 \log|f(e^{2i\pi t})|dt\right),$$

where $f \in \mathbb{Z}[X]$ is the minimal polynomial of α and d its degree. Another equivalent definition for $\mathrm{h}(\alpha)$ is given below (§3.3).

Before stating our two main conjectures, let us give a special case, which turns out to be the "intersection" of Conjectures 1 and 2 below: it is an extension of Mahler's problem where the rational integers a and b are replaced by algebraic numbers α and β.

Conjecture 0. *There exists a positive absolute constant c_0 with the following property. Let α and β be complex algebraic numbers and let $\lambda \in \mathbb{C}$ satisfy $e^\lambda = \alpha$. Define $D = [\mathbb{Q}(\alpha,\beta) : \mathbb{Q}]$. Further, let h be a positive number satisfying*

$$h \geq h(\alpha), \quad h \geq h(\beta), \quad h \geq \frac{1}{D}|\lambda| \quad \text{and } h \geq \frac{1}{D}.$$

Then

$$|\lambda - \beta| \geq \exp\{-c_0 D^2 h\}.$$

One may state this conjecture without introducing the letter λ: then the conclusion is a lower bound for $|e^\beta - \alpha|$, and the assumption $h \geq |\lambda|/D$ is replaced by $h \geq |\beta|/D$. It makes no difference, but for later purposes we find it more convenient to use logarithms.

The best known result in this direction is the following [11], which includes previous estimates of many authors; among them are K. Mahler, N. I. Fel'dman, P. L. Cijsouw, E. Reyssat, A. I. Galochkin and G. Diaz (for references, see [15], [4], Chap. 2 §4.4, [11] and [19]). For convenience we state a simpler version.*

- Let α and β be algebraic numbers and let $\lambda \in \mathbb{C}$ satisfy $\alpha = e^\lambda$. Define $D = [\mathbb{Q}(\alpha,\beta) : \mathbb{Q}]$. Let h_1 and h_2 be positive real numbers satisfying,

$$h_1 \geq h(\alpha), \quad h_1 \geq \frac{1}{D}|\lambda|, \quad h_1 \geq \frac{1}{D}$$

and

$$h_2 \geq h(\beta), \quad h_2 \geq \log(Dh_1), \quad h_2 \geq \log D, \quad h_2 \geq 1.$$

Then

$$|\lambda - \beta| \geq \exp\Big\{-2 \cdot 10^6 D^3 h_1 h_2 (\log D + 1)\Big\}. \tag{2}$$

*The main result in [11] involves a further parameter E which yields a sharper estimate when $|\lambda|/D$ is small compared with h_1.

To compare with Conjecture 0, we notice that from (2) we derive, under the assumptions of Conjecture 0,

$$|\lambda-\beta| \geq \exp\bigl\{-cD^3h(h+\log D+1)(\log D+1)\bigr\}$$

with an absolute constant c. This shows how far we are from Conjecture 0.

In spite of this weakness of the present state of the theory, we suggest two extensions of Conjecture 0 involving several logarithms of algebraic numbers. The common hypotheses for our two conjectures below are the following. We denote by $\lambda_1,\ldots,\lambda_m$ complex numbers such that the numbers $\alpha_i = e^{\lambda_i}$ $(1 \leq i \leq m)$ are algebraic. Further, let $\beta_0,\ldots,\beta_m$ be algebraic numbers. Let D denote the degree of the number field $\mathbb{Q}(\alpha_1,\ldots,\alpha_m,\beta_0,\ldots,\beta_m)$. Furthermore, let h be a positive number which satisfies

$$h \geq \max_{1\leq i\leq m} h(\alpha_i), \quad h \geq \max_{0\leq j\leq m} h(\beta_j), \quad h \geq \frac{1}{D}\max_{1\leq i\leq m}|\lambda_i| \quad \text{and } h \geq \frac{1}{D}.$$

Conjecture 1. *Assume that the number*

$$\Lambda = \beta_0+\beta_1\lambda_1+\cdots+\beta_m\lambda_m$$

is non zero. Then

$$|\Lambda| \geq \exp\bigl\{-c_1mD^2h\bigr\},$$

where c_1 is a positive absolute constant.

Conjecture 2. *Assume $\lambda_1,\ldots,\lambda_m$ are linearly independent over $\mathbb{Q}$. Then*

$$\sum_{i=1}^{m}|\lambda_i-\beta_i| \geq \exp\bigl\{-c_2mD^{1+(1/m)}h\bigr\},$$

with a positive absolute constant c_2.

Remark 1. Thanks to A.O. Gel'fond, A. Baker and others, a number of results have already been given in the direction of Conjecture 1. The best known estimates to date are those in [12], [16], [1] and [9]. Further, in the special case $m=2$, $\beta_0=0$, sharper numerical values for the constants are known [6]. However Conjecture 1 is much stronger than all known lower bounds:

– in terms of h: best known estimates involve h^{m+1} in place of h;

– in terms of D: so far, we have essentially D^{m+2} in place of D^2;
– in terms of m: the sharpest (conditional) estimates, due to E.M. Matveev [9], display c^m (with an absolute constant $c > 1$) in place of m.

On the other hand for concrete applications like those considered by K. Győry, a key point is often not to know sharp estimates in terms of the dependence in the different parameters, but to have non trivial lower bounds with small numerical values for the constants. From this point of view a result like [6], which deals only with the special case $m = 2$, $\beta_0 = 0$, plays an important role in many situations, in spite of the fact that the dependence in the height of the coefficients β_1, β_2 is not as sharp as other more general estimates from Gel'fond–Baker's method.

Remark 2. In case $D = 1$, $\beta_0 = 0$, sharper estimates than Conjecture 1 are suggested by Lang–Waldschmidt in [5], Introduction to Chapters X and XI. Clearly, our Conjectures 1 and 2 above are not the final word on this topic.

Remark 3. Assume $\lambda_1, \ldots, \lambda_m$ as well as D are fixed (which means that the absolute constants c_1 and c_2 are replaced by numbers which may depend on m, $\lambda_1, \ldots, \lambda_m$ and D). Then both conjectures are true: they follow for instance from (2). The same holds if $\beta_0, \ldots, \beta_m$ and D are fixed.

Remark 4. In the special case where $\lambda_1, \ldots, \lambda_m$ are fixed and $\beta_0, \ldots$ $\ldots, \beta_m$ are restricted to be rational numbers, Khinchine's Transference Principle (see [2], Chap. V) enables one to relate the two estimates provided by Conjecture 1 and Conjecture 2. It would be interesting to extend and generalize this transference principle so that one could relate the two conjectures in more general situations.

Remark 5. The following estimate has been obtained by N.I. Feld'man in 1960 (see [3], Th. 7.7 Chap. 7 §5); it is the sharpest know result in direction of Conjecture 2 when $\lambda_1, \ldots, \lambda_m$ are fixed:

- *Under the assumptions of Conjecture 2,*

$$\sum_{i=1}^{m} |\lambda_i - \beta_i| \geq \exp\{-cD^{2+(1/m)}(h + \log D + 1)(\log D + 1)^{-1}\}$$

with a positive constant c depending only on $\lambda_1, \ldots, \lambda_m$.

Theorem 8.1 in [14] enables one to remove the assumption that $\lambda_1, \dots, \lambda_m$ are fixed, but then yields the following weaker lower bound:

- *Under the assumptions of Conjecture 2,*

$$\sum_{i=1}^{m} |\lambda_i - \beta_i| \geq \exp\{-cD^{2+(1/m)} h(h + \log D + 1)(\log h + \log D + 1)^{1/m}\},$$

with a positive constant c depending only on m.

As a matter of fact, as in (2), Theorem 8.1 of [14] enables one to separate the contribution of the heights of α's and β's.

- *Under the assumptions of Conjecture 2, let h_1 and h_2 satisfy*

$$h_1 \geq \max_{1 \leq i \leq m} h(\alpha_i), \quad h_1 \geq \frac{1}{D} \max_{1 \leq i \leq m} |\lambda_i|, \quad h_1 \geq \frac{1}{D}$$

and

$$h_2 \geq \max_{0 \leq j \leq m} h(\beta_j), \quad h_2 \geq \log\log(3Dh_1), \quad h_2 \geq \log D.$$

Then

$$\sum_{i=1}^{m} |\lambda_i - \beta_i| \geq \exp\{-cD^{2+(1/m)} h_1 h_2 (\log h_1 + \log h_2 + 2\log D + 1)^{1/m}\}, \tag{3}$$

with a positive constant c depending only on m.

Again, Theorem 8.1 of [14] is more precise (it involves the famous parameter E).

In case $m = 1$ the estimate (3) gives a lower bound with

$$D^3 h_1 h_2 (\log h_1 + \log h_2 + 2\log D + 1),$$

while (2) replaces the factor $(\log h_1 + \log h_2 + 2\log D + 1)$ by $\log D + 1$. The explanation of this difference is that the proof in [11] involves the so-called Fel'dman's polynomials, while the proof in [14] does not.

Remark 6. A discussion of relations between Conjecture 2 and algebraic independence is given in [18], starting from [14].

Remark 7. One might propose more general conjectures involving simultaneous linear forms in logarithms. Such extensions of our conjectures are also suggested by the general transference principles in [2]. In this direction a partial result is given in [13].

Remark 8. We deal here with complex algebraic numbers, which means that we consider only Archimedean absolute values. The ultrametric situation would be also worth of interest and deserves to be investigated.

2. Simultaneous approximation of logarithms of algebraic numbers

Our goal is to give partial results in the direction of Conjecture 2. Hence we work with several algebraic numbers β (and as many logarithms of algebraic numbers λ), but we put them into a matrix B. Our estimates will be sharper when the rank of B is small.

We need a definition:

Definition. An $m\times n$ matrix $\mathrm{L}=(\lambda_{ij})_{\substack{1\le i\le m\\1\le j\le n}}$ satisfies the *linear independence condition* if, for any non zero tuple $\underline{t}=(t_1,\dots,t_m)$ in $\mathbb{Z}^m$ and any non zero tuple $\underline{s}=(s_1,\dots,s_n)$ in $\mathbb{Z}^n$, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}t_is_j\lambda_{ij}\neq 0.$$

This assumption is much stronger than what is actually needed in the proof, but it is one of the simplest ways of giving a sufficient condition for our main results to hold.

Theorem 1. *Let m, n and r be positive rational integers. Define*

$$\theta=\frac{r(m+n)}{mn}.$$

There exists a positive constant c_1 with the following property. Let B be an $m\times n$ matrix of rank $\le r$ with coefficients β_{ij} in a number field K. For $1\le i\le m$ and $1\le j\le n$, let λ_{ij} be a complex number such that the number $\alpha_{ij}=e^{\lambda_{ij}}$ belongs to $K^\times$ and such that the $m\times n$ matrix $\mathrm{L}=(\lambda_{ij})_{\substack{1\le i\le m\\1\le j\le n}}$ satisfies the linear independence condition. Define $D=[K:\mathbb{Q}]$. Let h_1 and h_2 be positive real numbers satisfying the following conditions:

$$h_1\ge \mathrm{h}(\alpha_{ij}),\quad h_1\ge\frac{1}{D}|\lambda_{ij}|,\qquad h_1\ge\frac{1}{D}$$

and

$$h_2 \geq \mathrm{h}(\beta_{ij}), \quad h_2 \geq \log(Dh_1), \quad h_2 \geq \log D, \quad h_2 \geq 1$$

for $1 \leq i \leq m$ *and* $1 \leq j \leq n$. *Then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left|\lambda_{ij} - \beta_{ij}\right| \geq e^{-c_1\Phi_1}$$

where

$$\Phi_1 = \begin{cases} Dh_1(Dh_2)^{\theta} & \text{if } Dh_1 \geq (Dh_2)^{1-\theta}, \\ (Dh_1)^{1/(1-\theta)} & \text{if } Dh_1 < (Dh_2)^{1-\theta}. \end{cases} \tag{4}$$

Remark 1. One could also state the conclusion with the same lower bound for

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left|e^{\beta_{ij}} - \alpha_{ij}\right|.$$

Remark 2. Theorem 1 is a variant of Theorem 10.1 in [14]. The main differences are the following.

In [14], the numbers λ_{ij} are fixed (which means that the final estimate is not explicited in terms of h_1).

The second difference is that in [14] the parameter r is the rank of the matrix L. Lemma 1 below shows that our hypothesis, dealing with the rank of the matrix B, is less restrictive.

The third difference is that in [14], the linear independence condition is much weaker than here; but the cost is that the estimate is slightly weaker in the complex case, where $D^{1+\theta}h_2^{\theta}$ is replaced by $D^{1+\theta}h_2^{1+\theta}(\log D)^{-1-\theta}$. However it is pointed out p. 424 of [14] that the conclusion can be reached with $D^{1+\theta}h_2^{\theta}(\log D)^{-\theta}$ in the special case where all λ_{ij} are real number. It would be interesting to get the sharper estimate without this extra condition.

Fourthly, the negative power of $\log D$ which occurs in [14] could be included also in our estimate by introducing a parameter E (see Remark 5 below).

Finally our estimate is sharper than Theorem 10.1 of [14] in case $Dh_1 < (Dh_2)^{1-\theta}$.

Remark 3. In the special case $n = 1$, we have $r = 1$, $\theta = 1 + (1/m)$ and the lower bound (4) is slightly weaker than (3): according to (3), in the estimate

$$D^{2+(1/m)} h_1 h_2^{1+(1/m)},$$

given by (4), one factor $h_2^{1/m}$ can be replaced by

$$\bigl(\log(eD^2 h_1 h_2)\bigr)^{1/m}.$$

Similarly for $n = 1$ (by symmetry). Hence Theorem 1 is already known when $\min\{m, n\} = 1$.

Remark 4. One should stress that (4) is not the sharpest result one can prove. Firstly the linear independence condition on the matrix L can be weakened. Secondly the same method enables one to split the dependence of the different α_{ij} (see Theorem 14.20 of [18]). Thirdly a further parameter E can be introduced (see [11], [17] and [18], Chap. 14 for instance – our statement here corresponds to $E = e$).

Remark 5. In case $Dh_1 < (Dh_2)^{1-\theta}$, the number Φ_1 does not depend on h_2: in fact one does not use the assumption that the numbers β_{ij} are algebraic! Only the rank r of the matrix comes into the picture. This follows from the next result.

Theorem 2. *Let m, n and r be positive rational integers with $mn > r(m+n)$. Define*

$$\kappa = \frac{mn}{mn - r(m+n)}.$$

There exists a positive constant c_2 with the following property. Let $\mathrm{L} = (\lambda_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$ be a matrix, whose entries are logarithms of algebraic numbers, which satisfies the linear independence condition. Let K be a number field containing the algebraic numbers $\alpha_{ij} = e^{\lambda_{ij}}$ $(1 \le i \le m,\ 1 \le j \le n)$. Define $D = [K : \mathbb{Q}]$. Let h be a positive real number satisfying

$$h \ge \mathrm{h}(\alpha_{ij}), \quad h \ge \frac{1}{D}|\lambda_{ij}| \quad \text{and } h \ge \frac{1}{D}$$

for $1 \le i \le m$ and $1 \le j \le n$. Then for any $m \times n$ matrix $M = (x_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$ of rank $\le r$ with complex coefficients we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\bigl|\lambda_{ij} - x_{ij}\bigr| \ge e^{-c_2\Phi_2}$$

where

$$\Phi_2 = (Dh)^{\kappa}.$$

Since $\kappa(1-\theta) = 1$, Theorem 2 yields the special case of Theorem 1 where $Dh_1 < (Dh_2)^{1-\theta}$ (cf. Remark 5 above).

3. Proofs

Before proving the theorems, we first deduce (2) from Theorem 4 in [11] and (3) from Theorem 8.1 in [14].

The following piece of notation will be convenient: for n and S positive integers,

$$\begin{aligned}\mathbb{Z}^n[S] &= [-S, S]^n \cap \mathbb{Z}^n \\ &= \bigl\{\underline{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n,\ \max_{1\le j\le n} |s_j| \le S\bigr\}.\end{aligned}$$

This is a finite set with $(2S+1)^n$ elements.

3.1. Proof of (2)

We use Theorem 4 of [11] with $E = e$, $\log A = eh_1$, and we use the estimates

$$\mathrm{h}(\beta) + \log\max\{1, eh_1\} + \log D + 1 \le 4h_2 \quad \text{and } 4e\cdot 105\,500 < 2\cdot 10^6.$$

3.2. Proof of (3)

We use Theorem 8.1 of [14] with $E = e$, $\log A = eh_1$, $B' = 3D^2h_1h_2$ and $\log B = 2h_2$. We may assume without loss of generality that h_2 is sufficiently large with respect to m. The assumption $B \ge D\log B'$ of [14] is satisfied: indeed the conditions $h_2 \ge \log\log(3Dh_1)$ and $h_2 \ge \log D$ imply $h_2 \ge \log\log(3D^2h_1h_2)$.

We need to check

$$s_1\beta_1 + \cdots + s_m\beta_m \ne 0 \quad \text{for } \underline{s} \in \mathbb{Z}^m[S] \setminus \{0\}$$

with

$$S = \bigl(c_1 D\log B'\bigr)^{1/m}.$$

Assume on the contrary $s_1\beta_1 + \cdots + s_m\beta_m = 0$. Then

$$|s_1\lambda_1 + \cdots + s_m\lambda_m| \leq mS \max_{1\leq i\leq m} |\lambda_i - \beta_i|.$$

Since $\lambda_1, \dots, \lambda_m$ are linearly independent, we may use Liouville's inequality (see for instance [18], Chap. 3) to derive

$$|s_1\lambda_1 + \cdots + s_m\lambda_m| \geq 2^{-D} e^{-mDSh_1}.$$

In this case one deduces a stronger lower bound than (3), with

$$cD^{2+(1/m)}h_2 \quad \text{replaced by } c'D^{1+(1/m)}.$$

3.3. Auxiliary results

The proof of the theorems will require a few preliminary lemmas.

Lemma 1. *Let* $\mathrm{B} = \big(\beta_{ij}\big)_{\substack{i\leq m\\ 1\leq j\leq n}}$ *be a matrix whose entries are algebraic numbers in a field of degree* D *and let* $\mathrm{L} = \big(\lambda_{ij}\big)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ *be a matrix of the same size with complex coefficients. Assume*

$$\mathrm{rank}(\mathrm{B}) > \mathrm{rank}(\mathrm{L}).$$

Let $B \geq 2$ *satisfy*

$$\log B \geq \max_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \mathrm{h}(\beta_{ij}).$$

Then

$$\max_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} |\lambda_{ij} - \beta_{ij}| \geq n^{-nD} B^{-n(n+1)D}.$$

Proof. Without loss of generality we may assume that B is a square regular $n \times n$ matrix. By assumption $\det(\mathrm{L}) = 0$.

In case $n = 1$ we write $\mathrm{B} = \big(\beta\big)$, $\mathrm{A} = \big(\lambda\big)$ where $\beta \neq 0$ and $\lambda = 0$. Liouville's inequality ([18], Chap. 3) yields

$$|\lambda - \beta| = |\beta| \geq B^{-D}.$$

Suppose $n \geq 2$. We may assume

$$\max_{1\leq i,j\leq n} |\lambda_{ij} - \beta_{ij}| \leq \frac{D\log B}{(n-1)B^D},$$

otherwise the conclusion is plain. Since

$$|\beta_{ij}| \le B^D \quad \text{and} \quad B^{D/(n-1)} \ge 1 + \frac{D}{n-1}\log B,$$

we deduce

$$\max_{1\le i,j\le n} \max\{|\lambda_{ij}|\ ,\ |\beta_{ij}|\} \le B^{nD/(n-1)}.$$

The polynomial $\det\bigl(X_{ij}\bigr)$ is homogeneous of degree n and length $n!$; therefore (see Lemma 13.10 of [18])

$$|\Delta| = |\Delta - \det(\mathrm{L})| \le n\cdot n!\bigl(\max_{1\le i,j\le n}\max\{|\lambda_{ij}|,\ |\beta_{ij}|\}\bigr)^{n-1}\max_{1\le i,j\le n}|\lambda_{ij}-\beta_{ij}|.$$

On the other hand the determinant Δ of B is a non zero algebraic number of degree $\le D$. We use Liouville's inequality again. Now we consider $\det\bigl(X_{ij}\bigr)$ as a polynomial of degree 1 in each of the n^2 variables:

$$|\Delta| \ge (n!)^{D-1}B^{-n^2D}.$$

Finally we conclude the proof of Lemma 1 by means of the estimate
$n\cdot n! \le n^n$. $\square$

Lemma 1 shows that the assumption rank(B) $\le r$ of Theorem 1 is weaker than the condition rank(L) $= r$ of Theorem 10.1 in [14]. For the proof of Theorem 1 there is no loss of generality to assume rank(B) $= r$ and rank(L) $\ge r$.

In the next auxiliary result we use the notion of absolute logarithmic height on a projective space $\mathbb{P}_N(K)$, when K is a number field ([18], Chap. 3): for $(\gamma_0:\cdots:\gamma_N)\in\mathbb{P}_N(K)$,

$$\mathrm{h}(\gamma_0:\cdots:\gamma_N) = \frac{1}{D}\sum_{v\in M_K} D_v \log\max\{|\gamma_0|_v,\ldots,|\gamma_N|_v\},$$

where $D=[K:\mathbb{Q}]$, M_K is the set of normalized absolute values of K, and for $v\in M_K$, D_v is the local degree. The normalization of the absolute values is done in such a way the for $N=1$ we have $\mathrm{h}(\alpha)=\mathrm{h}(1:\alpha)$.

Here is a simple property of this height. Let N and M be positive integers and $\vartheta_1,\ldots,\vartheta_N$, $\theta_1,\ldots,\theta_M$ algebraic numbers. Then

$$\mathrm{h}(1:\vartheta_1:\cdots:\vartheta_N:\theta_1:\cdots:\theta_M) \le \mathrm{h}(1:\vartheta_1:\cdots:\vartheta_N)+\mathrm{h}(1:\theta_1:\cdots:\theta_M).$$

One deduces that for algebraic numbers $\vartheta_0, \ldots, \vartheta_N$, not all of which are zero, we have

$$\mathrm{h}(\vartheta_0 : \cdots : \vartheta_N) \leq \sum_{i=0}^{N} \mathrm{h}(\vartheta_i). \tag{5}$$

Let K be a number field and B be an $m \times n$ matrix of rank r whose entries are in K. There exist two matrices B' and B'', of size $m \times r$ and $r \times n$, respectively, such that $\mathrm{B} = \mathrm{B}'\mathrm{B}''$. We show how to control the heights of the entries of B' and B'' in terms of the heights of the entries of B (notice that the proof of Theorem 10.1 in [14] avoids such estimate).

We write

$$\mathrm{B} = \bigl(\beta_{ij}\bigr)_{\substack{1 \leq i \leq m,\\ 1 \leq j \leq n}} \quad \mathrm{B}' = \bigl(\beta'_{i\varrho}\bigr)_{\substack{1 \leq i \leq m,\\ 1 \leq \varrho \leq r}} \quad \mathrm{B}'' = \bigl(\beta''_{\varrho j}\bigr)_{\substack{1 \leq \varrho \leq r\\ 1 \leq j \leq n}}$$

and we denote by $\underline{\beta}'_1, \ldots, \underline{\beta}'_m$ the m rows of B' and by $\underline{\beta}''_1, \ldots, \underline{\beta}''_n$ the n columns of B''. Then

$$\beta_{ij} = \underline{\beta}'_i \cdot \underline{\beta}''_j \qquad (1 \leq i \leq m,\ 1 \leq j \leq n),$$

where the dot $\cdot$ denotes the scalar product in K^r.

Lemma 2. *Let $\bigl(\beta_{ij}\bigr)_{\substack{1 \leq i \leq m\\ 1 \leq j \leq n}}$ be an $m \times n$ matrix of rank r with entries in a number field K. Define*

$$B = \exp\Bigl\{\max_{\substack{1 \leq i \leq m\\ 1 \leq j \leq n}} \mathrm{h}(\beta_{ij}\Bigr\}.$$

Then there exist elements

$$\underline{\beta}'_i = (\beta'_{i1}, \ldots, \beta'_{ir}) \quad (1 \leq i \leq m) \quad \text{and} \quad \underline{\beta}''_j = (\beta''_{1j}, \ldots, \beta''_{rj}) \quad (1 \leq j \leq n),$$

in K^r such that

$$\beta_{ij} = \sum_{\varrho=1}^{r} \beta'_{i\varrho}\beta''_{\varrho j} \quad (1 \leq i \leq m,\ 1 \leq j \leq n)$$

and such that, for $1 \leq \varrho \leq r$, we have

$$\mathrm{h}(1 : \beta'_{1\varrho} : \cdots : \beta'_{m\varrho}) \leq m \log B$$

and

$$\mathrm{h}(1:\beta''_{\varrho 1}:\cdots:\beta''_{\varrho n}) \le rn\log B + \log(r!). \tag{6}$$

PROOF. We may assume without loss of generality that the matrix $(\beta_{i\varrho})_{1\le i,\varrho\le r}$ has rank r. Let Δ be its determinant. We first take $\beta'_{i\varrho}=\beta_{i\varrho}$ $(1\le i\le m,\ 1\le\varrho\le r)$, so that, by (5),

$$\mathrm{h}(1:\beta'_{1\varrho}:\cdots:\beta'_{m\varrho}) \le m\log B \quad (1\le\varrho\le r).$$

Next, using Kronecker's symbol, we set

$$\beta''_{\varrho j}=\delta_{\varrho j} \quad \text{for } 1\le \varrho, j\le r.$$

Finally we define $\beta''_{\varrho j}$ for $1\le\varrho\le r$, $r<j\le n$ as the unique solution of the system

$$\beta_{ij}=\sum_{\varrho=1}^{r}\beta'_{i\varrho}\beta''_{\varrho j} \quad (1\le i\le m,\ r<j\le n).$$

Then for $1\le\varrho\le r$ we have

$$(1:\beta''_{\varrho,r+1}:\cdots:\beta''_{\varrho n}) = (\Delta:\Delta_{\varrho,r+1}:\cdots:\Delta_{\varrho n}), \tag{7}$$

where, for $1\le\varrho\le r$ and $r<j\le n$, $\Delta_{\varrho j}$ is (up to sign) the determinant of the $r\times r$ matrix deduced from the $r\times(r+1)$ matrix

$$\begin{pmatrix}\beta_{11} & \dots & \beta_{1r} & \beta_{1j}\\ \vdots & \ddots & \vdots & \vdots\\ \beta_{r1} & \dots & \beta_{rr} & \beta_{rj}\end{pmatrix}$$

by deleting the ϱ-th column. From (7) one deduces (6). This completes the proof of Lemma 2. □

We need another auxiliary result:

Lemma 3. *Let* $\mathrm{L}=(\lambda_{ij})_{\substack{1\le i\le m\\1\le j\le n}}$ *be an* $m\times n$ *matrix of complex numbers which satisfies the linear independence condition. Define* $\alpha_{ij}=e^{\lambda_{ij}}$ *for* $i=1,\dots,m$ *and* $j=1,\dots,n$.

1) *Consider the set*

$$E = \left\{ (\underline{t}, \underline{s}) \in \mathbb{Z}^m \times \mathbb{Z}^n;\ \prod_{i=1}^m \prod_{j=1}^n \alpha_{ij}^{t_i s_j} = 1 \right\}.$$

For each $\underline{s} \in \mathbb{Z}^n \setminus \{0\}$,

$$\{ \underline{t} \in \mathbb{Z}^m;\ (\underline{t}, \underline{s}) \in E \}$$

is a subgroup of $\mathbb{Z}^m$ *of rank* ≤ 1, *and similarly, for each* $\underline{t} \in \mathbb{Z}^m \setminus \{0\}$,

$$\{ \underline{s} \in \mathbb{Z}^n;\ (\underline{t}, \underline{s}) \in E \}$$

is a subgroup of $\mathbb{Z}^n$ *of rank* ≤ 1.

2) *Fix* $\underline{t} \in \mathbb{Z}^m \setminus \{0\}$. *For each positive integer* S, *the set*

$$\left\{ \prod_{i=1}^m \prod_{j=1}^n \alpha_{ij}^{t_i s_j};\ \underline{s} \in \mathbb{Z}^n[S] \right\} \subset \mathbb{C}^\times$$

has at least $(2S+1)^{n-1}$ *elements.*

Proof. For the proof of 1), fix $\underline{s} \in \mathbb{Z}^n \setminus \{0\}$ and assume $\underline{t}'$ and $\underline{t}''$ in $\mathbb{Z}^m$ are such that $(\underline{t}', \underline{s}) \in E$ and $(\underline{t}'', \underline{s}) \in E$. Taking logarithms we find two rational integers k' and k'' such that

$$\sum_{i=1}^m \sum_{j=1}^n t_i' s_j \lambda_{ij} = 2k'\pi\sqrt{-1} \quad \text{and} \quad \sum_{i=1}^m \sum_{j=1}^n t_i'' s_j \lambda_{ij} = 2k''\pi\sqrt{-1}.$$

Eliminating $2\pi\sqrt{-1}$ one gets

$$\sum_{i=1}^m \sum_{j=1}^n (k' t_i'' - k'' t_i') s_j \lambda_{ij} = 0.$$

Using the linear independence condition on the matrix L one deduces that $\underline{t}'$ and $\underline{t}''$ are linearly dependent over $\mathbb{Z}$, which proves the first part of 1). The second part of 1) follows by symmetry.

Now fix $\underline{t} \in \mathbb{Z}^m \setminus \{0\}$ and define a mapping ψ from the finite set $\mathbb{Z}^n[S]$ to $\mathbb{C}^\times$ by

$$\psi(\underline{s}) = \prod_{i=1}^m \prod_{j=1}^n \alpha_{ij}^{t_i s_j}.$$

If $\underline{s}'$ and $\underline{s}''$ in $\mathbb{Z}^n[S]$ satisfy $\psi(\underline{s}')=\psi(\underline{s}'')$, then $(\underline{s}'-\underline{s}'',\underline{t})\in E$. From the first part of the lemma we deduce that, for each $\underline{s}_0\in\mathbb{Z}^n[S]$, the set $\underline{s}-\underline{s}_0$, for $\underline{s}$ ranging over the set of elements in $\mathbb{Z}^n[S]$ for which $\psi(\underline{s})=\psi(\underline{s}_0)$, does not contain two linearly independent elements. Hence the set

$$\bigl\{\underline{s}\in\mathbb{Z}^n[S];\ \psi(\underline{s})=\psi(\underline{s}_0)\bigr\}$$

has at most $2S+1$ elements. Since $\mathbb{Z}^n[S]$ has $(2S+1)^n$ elements, the conclusion of part 2) of Lemma 3 follows by a simple counting argument (Lemma 7.8 of [18]). $\square$

3.4. Proof of Theorem 1

As pointed out earlier Theorem 1 in case $Dh_1<(Dh_2)^{1-\theta}$ is a consequence of Theorem 2 which will be proved in §3.5. In this section we assume $Dh_1\ge(Dh_2)^{1-\theta}$ and we prove Theorem 1 with $\Phi_1=Dh_1(Dh_2)^{\theta}$.

The proof of Theorem 1 is similar to the proof of Theorem 10.1 in [14]. Our main tool is Theorem 2.1 of [17]. We do not repeat this statement here, but we check the hypotheses. For this purpose we need to introduce some notation. We set

$$d_0=r,\quad d_1=m,\quad d_2=0,\quad d=r+m,$$

and we consider the algebraic group $G=G_0\times G_1$ with $G_0=\mathbb{G}_{\mathrm{a}}^r$ and $G_1=\mathbb{G}_{\mathrm{m}}^m$.

There is no loss of generality to assume that the matrix B has rank r (since the conclusion is weaker when r is larger). Hence we may use Lemma 2 and introduce the matrix

$$\mathrm{M}=\begin{pmatrix} & & & \beta''_{11} & \cdots & \beta''_{1n}\\ & \mathrm{I}_r & & \vdots & \ddots & \vdots\\ & & & \beta''_{r1} & \cdots & \beta''_{rn}\\ \beta'_{11} & \cdots & \beta'_{1r} & \lambda_{11} & \cdots & \lambda_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \beta'_{m1} & \cdots & \beta'_{mr} & \lambda_{m1} & \cdots & \lambda_{mn}\end{pmatrix}.$$

Define $\ell_0=r$ and let $\underline{w}_1,\dots,\underline{w}_{\ell_0}$ denote the first r columns of M, viewed as elements in K^{r+m}:

$$\underline{w}_k=(\delta_{1k},\dots,\delta_{rk},\beta'_{1k},\dots,\beta'_{mk})\quad(1\le k\le r)$$

(with Kronecker's diagonal symbol δ). The K-vector space they span, namely $W = K\underline{w}_1 + \cdots + K\underline{w}_r \subset K^d$, has dimension r.

Denote by $\underline{\eta}_1, \ldots, \underline{\eta}_n$ the last n columns of M, viewed as elements in $\mathbb{C}^{r+m}$:

$$\underline{\eta}_j = (\beta''_{1j}, \ldots, \beta''_{rj}, \lambda_{1j}, \ldots, \lambda_{mj}) \quad (1 \leq j \leq n).$$

Hence for $1 \leq j \leq n$ the point

$$\underline{\gamma}_j = \exp_G \underline{\eta}_j = (\beta''_{1j}, \ldots, \beta''_{rj}, \alpha_{1j}, \ldots, \alpha_{mj})$$

lies in $G(K) = K^r \times (K^\times)^m$.

For $\underline{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n$, define an element $\underline{\eta}_{\underline{s}}$ in $\mathbb{C}^d$ by

$$\begin{aligned}\underline{\eta}_{\underline{s}} &= s_1\eta_1 + \cdots + s_n\eta_n \\ &= \left(\sum_{j=1}^n s_j\beta''_{1j}, \ldots, \sum_{j=1}^n s_j\beta''_{rj}, \sum_{j=1}^n s_j\lambda_{1j}, \ldots, \sum_{j=1}^n s_j\lambda_{mj}\right).\end{aligned}$$

Again the point

$$\gamma_{\underline{s}} = \exp_G \underline{\eta}_{\underline{s}} = \left(\sum_{j=1}^n s_j\beta''_{1j}, \ldots, \sum_{j=1}^n s_j\beta''_{rj}, \prod_{j=1}^n \alpha_{1j}^{s_j}, \ldots, \prod_{j=1}^n \alpha_{mj}^{s_j}\right)$$

lies in $G(K)$. We denote by

$$\gamma_{\underline{s}}^{(1)} = \left(\prod_{j=1}^n \alpha_{1j}^{s_j}, \ldots, \prod_{j=1}^n \alpha_{mj}^{s_j}\right) \in (K^\times)^m$$

the projection of $\gamma_{\underline{s}}$ on $G_1(K)$.

Next put $\underline{w}'_k = \underline{w}_k \quad (1 \leq k \leq r)$ and, for $1 \leq j \leq n$,

$$\underline{\eta}'_j = (\beta''_{1j}, \ldots, \beta''_{rj}, \beta_{1j}, \ldots, \beta_{mj}) \in K^{m+1},$$

so that $\underline{w}'_1, \ldots, \underline{w}'_r$, $\underline{\eta}'_1, \ldots, \underline{\eta}'_n$ are the column vectors of the matrix

$$\mathrm{M}' = \begin{pmatrix} \mathrm{I}_r & \mathrm{B}'' \\ \mathrm{B}' & \mathrm{B} \end{pmatrix}.$$

Further, for $\underline{s} \in \mathbb{Z}^n$, set

$$\underline{\eta}'_{\underline{s}} = s_1\eta'_1 + \cdots + s_n\eta'_n.$$

Consider the vector subspaces

$$\mathcal{W}' = \mathbb{C}\underline{w}'_1 + \cdots + \mathbb{C}\underline{w}'_r \quad \text{and} \quad \mathcal{V}' = \mathbb{C}\underline{\eta}'_1 + \cdots + \mathbb{C}\underline{\eta}'_n$$

of $\mathbb{C}^d$. Since

$$\mathrm{M}' = \begin{pmatrix} \mathrm{I}_r \\ \mathrm{B}' \end{pmatrix} \cdot \begin{pmatrix} \mathrm{I}_r & \mathrm{B}'' \end{pmatrix},$$

the matrix M' has rank r, and it follows that $\mathcal{V}'$ and $\mathcal{W}' + \mathcal{V}'$ have dimension r. We set $r_1 = r_2 = 0$ and $r_3 = r$.

Theorem 2.1 of [17] is completely explicit, it would not be difficult to derive an explicit value for the constant c in Theorem 1 in terms of m and n only; but we shall only show it exists. We denote by c_0 a sufficiently large constant which depend only on m and n. Without loss of generality we may assume that both Dh_1 and h_2 are sufficiently large compared with c_0.

We set

$$S = \left[(c_0^3 D h_2)^{r/n} \right] \quad \text{and} \quad M = (2S+1)^n,$$

where the bracket denotes the integral part. Define

$$\Sigma = \left\{ \gamma_{\underline{s}};\ \underline{s} \in \mathbb{Z}^n[S] \right\} \subset G(K).$$

We shall order the elements of $\mathbb{Z}^n[S]$:

$$\mathbb{Z}^n[S] = \left\{ \underline{s}^{(1)}, \ldots, \underline{s}^{(M)} \right\}.$$

Put $B_1 = B_2 = e^{c_0 h_2}$. The estimates

$$\mathrm{h}\left(1 : \sum_{j=1}^n s_j^{(1)} \beta''_{hj} : \cdots : \sum_{j=1}^n s_j^{(M)} \beta''_{hj} \right) \le \log B_1 \quad (1 \le h \le r)$$

and

$$\mathrm{h}(1 : \beta'_{1k} : \cdots : \beta'_{mk}) \le \log B_2 \quad (1 \le k \le r)$$

follow from Lemma 2 thanks to the conditions $h_2 \ge 1$ and $h_2 \ge \log D$.

Next we set

$$A_1 = \cdots = A_n = \exp\{c_0 S h_1\}, \quad E = e.$$

Thanks to the definition of h_1, we have, for $1 \le i \le m$,

$$\frac{e}{D} \le \log A_i, \quad \mathrm{h}\Biggl(\prod_{j=1}^{n} \alpha_{ij}^{s_j}\Biggr) \le \log A_i \quad \text{and} \quad \frac{e}{D}\Biggl|\sum_{j=1}^{n} s_j\lambda_{ij}\Biggr| \le \log A_i.$$

Then define

$$T = \Bigl[(c_0^2 D h_2)^{r/m}\Bigr], \quad V = c_0^{3+4\theta}\Phi_1, \quad U = V/c_0,$$

$$T_0 = S_0 = \left[\frac{U}{c_0 D h_2}\right], \quad T_1 = \cdots = T_m = T, \quad S_1 = \cdots = S_n = S.$$

The inequalities

$$DT_0 \log B_1 \le U, \quad DS_0 \log B_2 \le U \quad \text{and} \quad \sum_{i=1}^{m} DT_i \log A_i \le U$$

are easy to check. The integers $T_0, \dots, T_m$ and $S_0, \dots, S_n$ are all ≥ 1. We have $U > c_0 D(\log D + 1)$ and

$$\binom{T_0 + r}{r}(T+1)^m > 4V^r.$$

It will be useful to notice that we also have

$$S_0^r(2S+1)^n > c_0 T_0^r T^m. \tag{8}$$

Finally the inequality

$$B_2 \ge T_0 + mT + dS_0$$

is satisfied thanks to the conditions $h_2 \ge \log(Dh_1)$ and $h_2 \ge \log D$.

Assume now

$$|\lambda_{ij} - \beta_{ij}| \le e^{-V}$$

for $1 \le i \le m$ and $1 \le j \le n$. Then all hypotheses of Theorem 2.1 of [17] are satisfied. Hence we obtain an algebraic subgroup $G^* = G_0^* \times G_1^*$ of G, distinct from G, such that

$$S_0^{\ell_0^*} M^* \mathcal{H}(G^*; \underline{T}) \le \frac{(r+m)!}{r!} T_0^r T^m \tag{9}$$

where

$$\ell_0^* = \dim_K W^*, \quad W^* = \frac{W + T_{G^*}(K)}{T_{G^*}(K)},$$

$$M^* = \mathrm{Card}(\Sigma^*), \quad \Sigma^* = \frac{\Sigma + G^*(K)}{G^*(K)}.$$

Define $d_0^* = \dim(G_0/G_0^*)$ and $d^* = \dim(G/G^*)$. Since $\mathcal{H}(G^*; \underline{T}) \geq T_0^{r-d_0^*}$, we deduce from (8) and (9)

$$S_0^{\ell_0^*} M^* < S_0^{d_0^*}(2S+1)^n. \tag{10}$$

We claim $\ell_0^* \geq d_0^*$. Indeed, consider the diagram

$$\begin{array}{ccc} \mathbb{C}^d & \xrightarrow{\pi_0} & \mathbb{C}^r \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g_0} \\ \mathbb{C}^{d^*} & \xrightarrow{\pi_0^*} & \mathbb{C}^{d_0^*} \end{array}$$

where

$$\pi_0 : \mathbb{C}^d \to \mathbb{C}^r \quad \text{and} \quad \pi_0 : \mathbb{C}^{d^*} \to \mathbb{C}^{d_0^*}$$

denote the projections with kernels

$$\{0\} \times \mathbb{C}^m \quad \text{and} \quad \{0\} \times T_{G_1^*}(K),$$

respectively, and

$$g : \mathbb{C}^d \to \mathbb{C}^{d^*} \quad \text{and} \quad g_0 : \mathbb{C}^r \to \mathbb{C}^{d_0^*}$$

denote the projections

$$T_G(K) \to T_G(K)/T_{G^*}(K) \simeq T_{G/G^*}(K)$$

and

$$T_{G_0}(K) \to T_{G_0}(K)/T_{G_0^*}(K) \simeq T_{G_0/G_0^*}(K)$$

respectively.

We have $W^* = g(W)$ and $\pi_0(W) = \mathbb{C}^r$. Since g_0 is surjective we deduce $\pi_0^*(W^*) = \mathbb{C}^{d_0^*}$, hence

$$\ell_0^* = \dim W^* \geq \dim \pi_0^*(W^*) = d_0^*.$$

Combining the inequality $\ell_0^* \geq d_0^*$ with (10) we deduce

$$M^* < (2S+1)^n.$$

Therefore $\dim G_1^* > 0$. Let Σ_1 denotes the projection of Σ on G_1:

$$\Sigma_1 = \left\{ \left(\prod_{j=1}^{n} \alpha_{1j}^{s_j}, \ldots, \prod_{j=1}^{n} \alpha_{mj}^{s_j} \right);\ \underline{s} \in \mathbb{Z}^n[S] \right\} = \{ \underline{\gamma}_{\underline{s}^{(1)}}^{(1)}, \ldots, \underline{\gamma}_{\underline{s}^{(M)}}^{(1)} \}.$$

For each $\underline{s}' \neq \underline{s}''$ in $\mathbb{Z}^n[S]$ such that $\underline{\gamma}_{\underline{s}'}^{(1)} / \underline{\gamma}_{\underline{s}''}^{(1)} \in G_1^*(K)$, and for each hyperplane of $T_{G^*}(K)$ containing $T_{G_1^*}(K)$ of equation $t_1 z_1 + \cdots + t_m z_m = 0$, we get a relation

$$\prod_{i=1}^{m} \prod_{j=1}^{n} \alpha_{ij}^{t_i s_j} = 1$$

with $\underline{s} = \underline{s}' - \underline{s}''$. Using the linear independence condition on the matrix L, we deduce from Lemma 3, part 1), that G_1^* has codimension 1 in G_1; hence

$$\mathcal{H}(G^*; \underline{T}) \geq \frac{(r+m-1)!}{r!} T_0^{r-d_0^*} T^{m-1}. \tag{11}$$

Next from part 2) of Lemma 3 we deduce that the set

$$\Sigma_1^* = \frac{\Sigma_1 + G_1^*(K)}{G_1^*(K)}$$

has at least $(2S+1)^{n-1}$ elements. Hence

$$M^* = \mathrm{Card}(\Sigma) \geq \mathrm{Card}(\Sigma_1^*) \geq (2S+1)^{n-1}. \tag{12}$$

If $mn \geq m+n$ the estimates (9), (11) and (12) are not compatible. This contradiction concludes the proof of Theorem 1 in the case $\max\{m,n\} > 1$ and $Dh_1 \geq (Dh_2)^{1-\theta}$. Finally, as we have seen in Remark 3 of §2, Theorem 1 is already known in case either $m = 1$ or $n = 1$.

3.5. Proof of Theorem 2

We start with the easy case where all entries x_{ij} of M are zero: in this special case Liouville's inequality gives

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\lambda_{ij}| \geq 2^{-D} e^{-Dh}.$$

Next we remark that we may, without loss of generality, replace the number r by the actual rank of the matrix M.

Thanks to the hypothesis $mn > r(m+n)$, there exist positive real numbers γ_u, γ_t and γ_s satisfying

$$\gamma_u > \gamma_t + \gamma_s \quad \text{and} \quad r\gamma_u < m\gamma_t < n\gamma_s.$$

For instance

$$\gamma_u = 1, \quad \gamma_t = \frac{r}{m} + \frac{1}{2m^2n}, \quad \gamma_s = \frac{r}{n} + \frac{1}{mn^2}$$

is an admissible choice.

Next let c_0 be a sufficiently large integer. How large it should be can be explicitly written in terms of m, n, r, γ_u, γ_t and γ_s.

We shall apply Theorem 2.1 of [17] with $d_0 = \ell_0 = 0$, $d = d_1 = m$, $d_2 = 0$, $G = \mathbb{G}_{\mathrm{m}}^m$, $r_3 = r$, $r_1 = r_2 = 0$,

$$\underline{\eta}_j = (\lambda_{ij})_{1\le i\le m}, \qquad \underline{\eta}'_j = (x_{ij})_{1\le i\le m} \qquad (1 \le j \le n).$$

Since $d_0 = \ell_0 = 0$ we set $T_0 = S_0 = 0$. Therefore the parameters B_1 and B_2 will play no role, but for completenes we set

$$B_1 = B_2 = mn(Dh)^{mn}.$$

We also define $E = e$,

$$U = c_0^{\gamma_u}(Dh)^{\kappa}, \qquad V = (12m+9)U,$$

$$T_1 = \cdots = T_m = T, \qquad S_1 = \cdots = S_n = S,$$

where

$$T = \left[c_0^{\gamma_t}(Dh)^{r\kappa/m}\right], \quad S = \left[c_0^{\gamma_s}(Dh)^{r\kappa/n}\right].$$

Define $A_1 = \cdots = A_m$ by

$$\log A_i = \frac{1}{em} c_0^{\gamma_u - \gamma_t - \gamma_s} Sh \quad (1 \le i \le m).$$

The condition $\gamma_t + \gamma_s < \gamma_u$ enables us to check

$$\sum_{j=1}^{n} s_j \mathrm{h}(\alpha_{ij}) \le \log A_i \quad \text{and} \quad \sum_{j=1}^{n} s_j |\lambda_{ij}| \le \frac{D}{E} \log A_i$$

for $1 \le i \le m$ and for any $\underline{s} \in \mathbb{Z}^n[S]$. Moreover, from the very definition of κ we deduce

$$r\kappa\left(\frac{1}{m}+\frac{1}{n}\right)+1=\kappa,$$

and this yields

$$D\sum_{i=1}^{m} T_i \log A_i \le U.$$

Define

$$\Sigma=\left\{\left(\alpha_{11}^{s_1}\dots\alpha_{1n}^{s_n},\dots,\alpha_{m1}^{s_1}\dots\alpha_{mn}^{s_n}\right)\in (K^\times)^m;\ \underline{s}\in\mathbb{Z}^n[S]\right\}.$$

From the condition $m\gamma_t > r\gamma_u$ one deduces

$$(2T+1)^m > 2V^r.$$

Assume that the conclusion of Theorem 2 does not hold for $c = c_0^{\gamma_u+1}$. Then the hypotheses of Theorem 2.1 of [17] are satisfied, and we deduce that there exists a connected algebraic subgroup G^* of G, distinct from G, which is incompletely defined by polynomials of multidegrees $\le \underline{T}$ where $\underline{T}$ stands for the m-tuple $(T,\dots,T)$, such that

$$M^*\mathcal{H}(G^*;\underline{T}) \le m!\,T^m, \qquad \text{where} \qquad M^* = \mathrm{Card}\left(\frac{\Sigma+G^*(K)}{G^*(K)}\right).$$

Since $m\gamma_t < n\gamma_s$, we have

$$m!\,T^m < (2S+1)^n,$$

and since $\mathcal{H}(G^*;\underline{T}) \ge 1$, we deduce

$$M^* < (2S+1)^n.$$

Hence $\Sigma[2]\cap G^*(K) \ne \{e\}$. Therefore there exist $\underline{s}\in\mathbb{Z}^n[2S]\setminus\{0\}$ and $\underline{t}\in\mathbb{Z}^m[T]\setminus\{0\}$ with

$$\sum_{i=1}^{m}\sum_{j=1}^{n} t_i s_j \lambda_{ij} \in 2\pi\sqrt{-1}\mathbb{Z}.$$

Let us check, by contradiction, that G^* has codimension 1. We already know $G^* \neq G$. If the codimension of G^* were ≥ 2, we would have two linearly independent elements $\underline{t}'$ and $\underline{t}''$ in $\mathbb{Z}^m[T]$ such that the two numbers

$$a' = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{m} \sum_{j=1}^{n} t_i' s_j \lambda_{ij} \quad \text{and} \quad a'' = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{m} \sum_{j=1}^{n} t_i'' s_j \lambda_{ij}$$

are in $\mathbb{Z}$. Notice that

$$\max\{|a'|, |a''|\} \leq mnTSDh.$$

We eliminate $2\pi\sqrt{-1}$: set $\underline{t} = a''\underline{t}' - a'\underline{t}''$, so that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} t_i s_j \lambda_{ij} = 0$$

and

$$0 < |\underline{t}| \leq 2mnT^2SDh < (2mnTSDh)^2 < U^2.$$

This is not compatible with our hypothesis that the matrix L_{mn} satisfies the linear independence condition.

Hence G^* has codimension 1 in G. Therefore

$$\mathcal{H}(G^*; \underline{T}) \geq T^{m-1} \quad \text{and consequently} \quad M^* \leq m!\,T.$$

On the other hand a similar argument shows that any $\underline{s}'$, $\underline{s}''$ in $\mathbb{Z}^n[2S]$ for which

$$\sum_{i=1}^{m} \sum_{j=1}^{n} t_i s_j' \lambda_{ij} \in 2\pi\sqrt{-1}\mathbb{Z} \quad \text{and} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} t_i s_j'' \lambda_{ij} \in 2\pi\sqrt{-1}\mathbb{Z}$$

are linearly dependent over $\mathbb{Z}$. From Lemma 7.8 of [18] we deduce

$$M^* \geq S^{n-1}.$$

Therefore

$$S^{n-1} \leq m!\,T.$$

This is not compatible with the hypotheses $mn > r(m+n)$ and $r \geq 1$. This final contradiction completes the proof of Theorem 2.

References

[1] A. Baker and Gisbert Wüstholz, Logarithmic forms and group varieties, *J. reine Angew. Math.* **442** (1993), 19–62.

[2] J. W. S. Cassels, An Introduction to diophantine approximation, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, *Cambridge University Press, New York*, 1957, Reprint of the 1957 edition: *Hafner Publishing Co., New York*, 1972.

[3] N. I. Fel'dman, Hilbert's seventh problem, *Moskov. Gos. Univ., Moscow*, 1982. (in *Russian*)

[4] N. I. Fel'dman and Yuri V. Nesterenko, Number theory, IV: Transcendental Numbers, Encyclopaedia of Mathematical Sciences, 44, *Springer-Verlag, Berlin*, 1998.

[5] S. Lang, Elliptic curves: Diophantine analysis, Grundlehren der Mathematischen Wissenschaften, 231, *Springer-Verlag, Berlin – New York*, 1978.

[6] M. Laurent, M. Mignotte and Y. Nesterenko, Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory* **55** no. 2 (1995), 285–321.

[7] K. Mahler, On the approximation of logarithms of algebraic numbers, *Philos. Trans. Roy. Soc. London. Ser. A.* **245** (1953), 371–398.

[8] K. Mahler, Applications of some formulae by Hermite to the approximation of exponentials and logarithms, *Math. Ann.* **168** (1967), 200–227.

[9] E. M. Matveev, Explicit lower estimates for rational homogeneous linear forms in logarithms of algebraic numbers, *Izv. Akad. Nauk SSSR. Ser. Mat.* **62** no. 4 (1998), 81–136; Engl. transl.: *Izvestiya Mathematics* **62** no. 4 (1998), 723–772.

[10] M. Mignotte, Approximations rationnelles de π et quelques autres nombres, Journées Arithmétiques, Bull. Soc. Math. France, Mém., vol. 37, *Soc. Math. France, Paris*, 1974, 121–132.

[11] Y. V. Nesterenko and Michel Waldschmidt, On the approximation of the values of exponential function and logarithm by algebraic numbers, Diophantine approximations (Yu. V. Nesterenko, ed.), Proceedings of papers dedicated to the memory of Prof. N. I. Fel'dman, *Centre for applied research under Mech.-Math. Faculty of MSU, Moscow*, 1996, 23–42. (in *Russian*)

[12] P. Philippon and M. Waldschmidt, Lower bounds for linear forms in logarithms, New advances in transcendence theory (Durham, 1986), *Cambridge Univ. Press, Cambridge – New York*, 1988, 280–312.

[13] P. Philippon and M. Waldschmidt, Formes linéaires de logarithmes simultanées sur les groupes algébriques commutatifs, Séminaire de Théorie des Nombres, Paris 1986–87,, Progr. Math., 75, *Birkhäuser Boston, Boston, MA*, 1988, 313–347.

[14] D. Roy and M. Waldschmidt, Simultaneous approximation and algebraic independence, *The Ramanujan Journal* **1** Fasc. 4 (1997), 379–430.

[15] M. Waldschmidt, Simultaneous approximation of numbers connected with the exponential function, *J. Austral. Math. Soc.* **25** (1978), 466–478.

[16] M. Waldschmidt, Minorations de combinaisons linéaires de logarithmes de nombres algébriques, *Canad. J. Math.* **45** no. 1 (1993), 176–224.

[17] M. Waldschmidt, Approximation diophantienne dans les groupes algébriques commutatifs – (I): Une version effective du théorème du sous-groupe algébrique, *J. reine Angew. Math.* **493** (1997), 61–113.

[18] M. Waldschmidt, Diophantine Approximation on Linear Algebraic Groups, Transcendence Properties of the Exponential Function in Several Variables, *Springer Verlag* (*to appear*).

[19] F. Wielonsky, Hermite–Padé approximants to exponential functions and an inequality of Mahler, *J. Number Theory* **74** no. 2, 230–249.

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