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Extrapolation with interpolation determinants

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Abstract

Let $f_1, \ldots, f_L$ be analytic functions and $\zeta_1, \ldots, \zeta_L$ be points where the functions are defined. Sharp upper bounds are known for the absolute value of the determinant of the $L \times L$ matrix

$$\Big(f_\lambda(\zeta_\mu)\Big)_{1\le\lambda,\mu\le L}.$$

These estimates play an important role in transcendental number theory. We give a report on this topic and we announce a new application to integer valued entire functions on a set of products.

1. Historical survey

In his proof of the transcendence of the number e, Ch. Hermite [He 1873] constructed explicitly polynomials $A_0, A_1, \ldots, A_n$ in $\mathbb{Q}[z]$ such that the function $A_0(z) + A_1(z)e^z + \cdots + A_n(z)e^{nz}$ has a zero of high multiplicity at the origin. This construction was then developed by Hermite's student H. Padé, and the so-called *Padé approximants* now play an important role in the theory of Diophantine approximation (see W. van Assche's lectures in these proceedings) as well as in other parts of mathematics.

The need for explicit formulae in Hermite's proof has been a long time an obstacle to further progress in transcendental number theory. In 1929, C.L. Siegel published his fundamental paper [Si 1929]; in the first part of it, he proves his well known result on integer points on algebraic curves (in particular there are only finitely many such points if the curve has positive genus). In the second part, he introduces E and G functions, and extends Hermite's method to these functions. In the proofs of these results in both parts, the main tool, which is emphasized in the introduction of the paper [Si 1929], arises from Thue's work [T 1908] on Diophantine approximation: it is the so-called *Dirichlet's box principle*, which is used to prove the following auxiliary result:

Lemma 1.1. *(Thue–Siegel) – Let*

$$\begin{aligned} y_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\ &\ \vdots \\ y_m &= a_{m1}x_1 + \cdots + a_{mn}x_n \end{aligned}$$

be m linear forms in n variables with rational integer coefficients. Assume $n > m$. If the absolute values of the mn coefficients a_{ij} are all bounded from above by a natural integer A, then the system of linear equations $y_1 = 0, \ldots, y_m = 0$ has a solution in rational integers $x_1, \ldots, x_n$, not all of which are 0*, but with absolute value less than*

$$1 + (nA)^{m/(n-m)}.$$

This device enabled Siegel to replace the explicit construction of Hermite by a much simpler argument. Given an analytic function f near the origin, if we want to ensure the existence of polynomials $A_0, A_1, \ldots, A_n$ in $\mathbb{C}[z]$, not all zero, where A_i has degree $\leq D_i$, $(0 \leq i \leq n)$, such that the function $A_0(z) + A_1(z)f(z) + \cdots + A_n(z)f(z)^n$ has a zero of multiplicity $\geq m$ at the origin, it is sufficient (by elementary linear algebra) to require $D_0 + \cdots + D_n + n \geq m$. Moreover, if the Taylor coefficients of f at the origin are all rational numbers, then $A_0, A_1, \ldots, A_n$ exist in $\mathbb{Z}[z]$. Now from lemma 1.1 one deduces the existence of such polynomials together with an upper bound for the absolute values of their coefficients. For instance, if, say, $D_0 + \cdots + D_n + n \geq 2m$, then, roughly speaking, this upper bound is of the order of magnitude of a common denominator of the first m coefficients in the Taylor expansion of f.

Siegel's construction of an auxiliary function has been quite influential in the theory of transcendental numbers. Before telling this story, we need first to speak on Pólya's work on integer valued entire functions [Po 1915], which led to Gel'fond's proof of the transcendence of e^π [Ge 1929].

In 1915, G. Pólya [Po 1915] proved that an entire function f in $\mathbb{C}$ which is not a polynomial and takes integer values (in $\mathbb{Z}$) at the points $0, 1, 2, \ldots$ satisfies

$$\limsup_{R\to\infty} \frac{\sqrt{R}}{2^R}|f|_R > 0,$$

where $|f|_R$ stands for $\sup_{|z|=R}|f(z)|$. Therefore

$$\limsup_{R\to\infty} \frac{1}{R}\log|f|_R \geq \log 2.$$

The stronger estimate

$$\limsup_{R\to\infty} 2^{-R}|f|_R > 0,$$

which was conjectured by Pólya, has been obtained by G.H. Hardy [Ha 1917]. Three years later, Pólya improved the estimate and reached the lower bound

$$\limsup_{R\to\infty} 2^{-R}|f|_R \geq 1,$$

which shows that 2^z is the transcendental entire function of least growth order which maps $\mathbb{N}$ into $\mathbb{Z}$. Further refinements are due to a number of authors (including G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross,...).

Pólya's proof involves the calculus of finite differences (discrete analog of differential equations): he writes $f(z)$ as an interpolation series

$$a_0 + a_1 z + a_2\frac{z(z-1)}{2} + \cdots + a_n\binom{z}{n} + \cdots,$$

where $\binom{z}{n}$ stands for the polynomial $z(z-1)\cdots(z-n+1)/n!$. Gel'fond's generalization [Ge 1929] dealt with entire functions f which take integer values in $\mathbb{Z}[i]$ at the Gaussian integers: $f(a+ib)\in\mathbb{Z}[i]$ for any $(a,b)\in\mathbb{Z}^2$. Such a function, he shows, either is a polynomial, or else grows at least like an exponential in R^2:

$$\limsup_{R\to\infty}\frac{1}{R^2}\log|f|_R\geq\gamma$$

where γ is a positive real number (an absolute constant, around 10^{-45}). The example of Weierstraß sigma function (which is the canonical product, vanishing on $\mathbb{Z}[i]$), shows that the constant γ is not greater than $\pi/2$. The exact value for γ has been obtained more than half a century later, by F. Gramain [Gr 1981]: $\gamma=\pi/(2e)$.

The connection with the transcendence of e^π arises through the function $e^{\pi z}$: if e^π were an algebraic number, then the function $e^{\pi z}$ would take algebraic values (in the field $\mathbb{Q}(e^\pi)$) at all points of $\mathbb{Z}[i]$.

Gel'fond's proof of the transcendence of $e^\pi=i^{-2i}$ was the first step towards a complete solution of Hilbert's seventh problem on the transcendence of α^β (for algebraic α and β with $\alpha\neq0$, $\log\alpha\neq0$ and $\beta\notin\mathbb{Q}$), by A.O. Gel'fond [Ge 1934] and Th. Schneider [Sc 1934]. Their proofs are different, but both involve the construction of an auxiliary function using lemma 1.1.

Until recently, most proofs of transcendental number theory (including Baker's work on lower bounds for linear forms in logarithms of algebraic numbers — see [Ba 1966]) involved the construction of an auxiliary function by means of Dirichlet's box principle. However, a few years ago, a new device has been introduced in the theory: *interpolation determinants*. Such determinants are related to *exact Lagrangian interpolation formulae* (see F. Calogero's lectures at this workshop). Here, they occur in a slightly different context.

One of the first occurrences of such determinants in the theory of diophantine approximation relates to the following problem of D.H. Lehmer [Le 1933]. For an algebraic number α, that is a root of an irreducible polynomial $f(z)=a_0z^d+a_1z^{d-1}+\cdots+a_d$ in $\mathbb{Z}[z]$, define

$$\mathrm{M}(\alpha)=|a_0|\prod_{i=1}^{d}\max\{1,|\alpha_i|\},$$

where $\alpha_1,\ldots,\alpha_d$ are the *complex conjugates* of α, namely the roots of f in $\mathbb{C}$:

$$f(z)=a_0(z-\alpha_1)\cdots(z-\alpha_d).$$

It is easy to check (Kronecker) that $\mathrm{M}(\alpha)=1$ if and only if α is either 0 or a root of unity. The smallest known value >1 for $\mathrm{M}(\alpha)$ is $1.17628\ldots$, which is the root of the polynomial

$$z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1.$$

Lehmer [Le 1933] asked whether, for each $\epsilon>0$, there exists an algebraic number α which satisfies $1<\mathrm{M}(\alpha)<1+\epsilon$. This question is still open. After the work of several mathematicians (including C.L. Siegel, A. Schinzel, H. Zassenhaus, C.J. Smyth, P.E. Blanksby, H.L. Montgomery, D. Boyd,. . .), methods from transcendental number theory (involving the construction of an auxiliary function) have been introduced in this context by C.L. Stewart [St 1978] and developed by E. Dobrowolski [Do 1979] who proved, for $\alpha\neq0$ of degree d which is not a root of unity,

$$\mathrm{M}(\alpha)>1+\frac{1}{1200}\left(\frac{\log\log d}{\log d}\right)^3.$$

A new proof of this estimate has been produced by D.C. Cantor and E.G. Straus [CS 1982], who replace the auxiliary function by a tricky determinant, which turns out to be an interpolation determinant.

Later, M. Laurent [Lau 1989] (see also [Pi 1993]) introduced interpolation determinants for proving the following transcendence result, due to S. Lang [La 1966], [La 1966] and K. Ramachandra [Ra 1968], [Ra 1969]:

Six exponentials theorem. – *Let $x_1,\ldots,x_d$ be $\mathbb{Q}$-linearly independent complex numbers and $y_1,\ldots,y_\ell$ be also $\mathbb{Q}$-linearly independent complex numbers. Assume $d\ell > d+\ell$. Then one at least of the $d\ell$ numbers*

$$e^{x_iy_j}, \qquad (1\le i\le d, \quad 1\le j\le \ell)$$

is transcendental.

The name *six exponentials theorem* comes from the fact that the relevant values for (d,ℓ) are $(2,3)$ or $(3,2)$. As explained in chapter 2 (p. 19–20) of [La 1966], this problem already occurs in the work of L. Alaoglu and P. Erdős [AE 1944] related to Ramanujan's superior highly composite numbers [R 1915] (see also [W 1987] and [W 1992] for further information on this subject), and Siegel apparently knew it already in 1944.

Here is a sketch of proof of the following irrationality statement:

- *Let $x_1,\ldots,x_d$ be $\mathbb{Q}$-linearly independent real numbers and $y_1,\ldots,y_\ell$ be also $\mathbb{Q}$-linearly independent real numbers. Assume $d\ell > d+\ell$. Then one at least of the $d\ell$ numbers*

  $$e^{x_iy_j}, \qquad (1\le i\le d, \quad 1\le j\le \ell)$$

  is irrational.

The idea of proof, using interpolation determinants, is as follows. Let N be a sufficiently large positive integer. Define $T=N^\ell$, $S=N^d$, $L=N^{d\ell}$. Consider the set of functions

$$\mathcal{F}=\left\{e^{(t_1x_1+\cdots+t_dx_d)z}\,;\,(t_1,\ldots,t_d)\in\mathbb{N}^d,\,0\le t_i<T,\,(1\le i\le d)\right\}$$

and the set of points

$$\mathcal{P}=\left\{s_1y_1+\cdots+s_\ell y_\ell\,;\,(s_1,\ldots,s_\ell)\in\mathbb{N}^\ell,\,0\le s_j<S,\,(1\le j\le \ell)\right\}.$$

Choose any ordering $\mathcal{F}=\{f_1,f_2,\ldots,f_L\}$ and $\mathcal{P}=\{\zeta_1,\zeta_2,\ldots,\zeta_L\}$ for each of these two sets. Consider the determinant Δ_L of the $L\times L$ matrix

$$\Bigl(f_\lambda(\zeta_\mu)\Bigr)_{1\le\lambda,\mu\le L}.$$

The set $\mathcal{F}$ is a *Chebyshev system* (see W. van Assche's lectures; see also Pólya's result in lemma 2.1 of chapter 2 in [W 1992]). That means that this determinant Δ_L does not vanish.

Assume now that the $d\ell$ numbers $e^{x_iy_j}$, $(1\le i\le d,\ 1\le j\le\ell)$ are rational; let $D\ge 1$ be a common denominator. Then $D^{LTS}\Delta_L\in\mathbb{Z}$, hence $\Delta_L\ge D^{-LTS}$. Now if we prove $|\Delta_L|<e^{-L^2}$, we shall reach a contradiction (for L large enough), thanks to the hypothesis $(1/d)+(1/\ell)<1$.

Here is the desired upper bound for the absolute value of the interpolation determinant.

Lemma 1.2. *(M. Laurent) – Let r and R be two real numbers with $0 < r \le R$, $f_1, \dots, f_L$ be functions of one complex variable, which are analytic in the disk $|z| \le R$ of $\mathbb{C}$, and let $\zeta_1, \dots, \zeta_L$ belong to the disk $|z| \le r$. Then the determinant*

$$\Delta = \det \begin{pmatrix} f_1(\zeta_1) & \dots & f_L(\zeta_1) \\ \vdots & \ddots & \vdots \\ f_1(\zeta_L) & \dots & f_L(\zeta_L) \end{pmatrix}$$

is bounded from above by

$$|\Delta| \le \left(\frac{R}{r}\right)^{-L(L-1)/2} L! \prod_{\lambda=1}^{L} |f_\lambda|_R.$$

The proof is an easy application of Schwarz lemma; see [Lau 1989], lemme 3, as well as [W 1992], chap. 2, lemma 2.2.

Such estimates are available in many other situations; in particular derivatives can be included, and functions of several variables can be considered. A rather general estimate, including further refinements (e.g. in relation with Baker's method) is given in [W 1997a], Proposition 5.1.

The above sketch of proof has a wide range of applications. Here is a sample.

M. Laurent [Lau 1994] succeeded to refine existing bounds for linear forms in two logarithms. In a later joint paper with M. Mignotte and Yu.V. Nesterenko [LMN 1995], they produce a very sharp explicit lower bound for $|\alpha_1^{b_1}\alpha_2^{b_2} - 1|$ for non zero algebraic numbers α_1, α_2 and rational integers b_1, b_2. This estimate is invaluable for several Diophantine problems, and for instance has been used in a number of papers solving explicitly (families of) Diophantine equations. A lower bound for the p-adic absolute value has also been achieved by M. Laurent and Y. Bugeaud [BuLau 1995].

This method also works for estimating $|\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1|$ with $n \ge 2$ [W 1992]; in the case $n = 1$ the method yields lower bounds for the height of algebraic numbers [MiW 1993], [Ma 1996a], [Ma 1996b], [V 1996], [BuMiNo 1995], [Am 1996], [Am 1997], [Bu 1997].

An other type of application arises in the work of P. Corvaja [Co 1992] related to the Thue-Siegel-Roth theorem on Diophantine approximation.

A new proof of Pólya's above mentioned result on integer valued entire functions has been obtained in [W 1993] by means of interpolation determinants; in this paper a method is derived in order to extrapolate with such determinants (see next section). Recently, interpolation determinants have been used by P. Philippon [Ph 1997] who proves very general results of transcendence and algebraic independence for values of entire functions of one variable, including most of known results as well as new ones.

Another kind of Diophantine approximation estimate for values of functions of several variables related to algebraic groups is given in [W 1997a]. The "duality" between Gel'fond's method and Schneider's one for the solution of Hilbert's seventh problem now reduces to a transposition of the interpolation matrix, using the identity

$$\left(\frac{d}{dz}\right)^{\sigma} \left(z^{\tau} e^{tz}\right)_{z=s} = \left(\frac{d}{dz}\right)^{\tau} \left(z^{\sigma} e^{sz}\right)_{z=t}.$$

The main result of [W 1997a] contains a theorem of Wüstholz which has been used by F. Beukers and J. Wolfart [BW 1988] in order to prove transcendence results for the values of hypergeometric functions; therefore the statements of [BW 1988] can now be proved by means of interpolation determinants. By the way, these attempts by Beukers and Wolfart to prove general transcendence results related to hypergeometric functions led them to discover new algebraic values, like

$$ {}_2F_1\bigl(1/12, 5/12, 1/2\,;\, 1323/1331\bigr) = \frac{3}{4}\sqrt[4]{11}. $$

2. Pólya's theorem with interpolation determinants

In this section we give a sketch of proof for the following weak form of Pólya's theorem:

- *If an entire function f satisfies $f(\mathbb{N}) \subset \mathbb{Z}$ and*

$$ \limsup_{R\to\infty} \frac{1}{R} \log |f|_R = 0, $$

then $f \in \mathbb{Q}[z]$.

Let T be a sufficiently large positive integer and T_0 a positive integer which is sufficiently large compared with T. Set $L = T_0T$, and consider the following set $\{f_1, \ldots, f_L\}$ of entire functions

$$ \binom{z}{\tau} f^t(z), \qquad (0 \le \tau < T_0,\, 0 \le t < T). $$

Introduce the matrix

$$ M = \Bigl(f_\lambda(n)\Bigr)_{\substack{1\le\lambda\le L\\ n\ge 0}} $$

with L rows and infinitely many columns. Our goal is to prove that the rank of M is $< L$. From this fact we shall deduce that there exist rational integers $a_1, \ldots, a_L$, not all of which are zero, such that the entire function $F = a_1f_1 + \cdots + a_Lf_L$ vanishes on $\mathbb{N}$. Our hypothesis on $|f|_R$, together with Schwarz lemma, yields $F = 0$. This means that $f_1, \ldots, f_L$ are linearly dependent over $\mathbb{Q}$, i. e. that f is an algebraic function, and since f is entire, it easily follows that f is a polynomial.

We first check that the rank of the square matrix constructed with the first L columns of M is $< L$. Indeed, the determinant Δ of this matrix, namely

$$ \Delta = \det\Bigl(f_\lambda(n)\Bigr)_{\substack{1\le\lambda\le L\\ 0\le n<L}}, $$

is a rational integer; now lemma 1.2 together with our hypothesis on $|f|_R$ easily imply $|\Delta| < 1$; therefore $\Delta = 0$.

Now we want to extrapolate, in order to prove by induction that for any $M \ge L$, the rank of the $L \times M$ matrix

$$ \Bigl(f_\lambda(n)\Bigr)_{\substack{1\le\lambda\le L\\ 0\le n<M}} $$

is $< L$. The key step is again a Schwarz lemma. We denote by $\mathfrak{S}_L$ the group of permutations of $\{1, \ldots, L\}$ (symmetric group of order $L!$).

Lemma 2.1. – *Let $F = (f_1, \ldots, f_L)$ be an analytic mapping from $\mathbb{C}$ to $\mathbb{C}^L$. Let $\zeta_0, \ldots, \zeta_N$ be pairwise distinct complex numbers, $0 = n_0 \le n_1 < n_2 < \cdots < n_L = N$ non-negative integers and $r_1, \ldots, r_L$, $E_1, \ldots, E_L$ positive real numbers with*

$$ E_\mu > 1 \qquad \textit{and} \qquad r_\mu \ge \max_{0\le i\le n_\mu} |\zeta_i| \qquad \textit{for} \qquad 1 \le \mu \le L; $$

define $R_\mu = 2E_\mu r_\mu$, $(1 \le \mu \le L)$. *Assume, for* $0 \le \nu < L$ *and* $n_\nu \le n < n_{\nu+1}$, *that the* $L \times (\nu+1)$ *matrix:*

$$\Big(F(\zeta_{n_1}), \ldots, F(\zeta_{n_\nu}), F(\zeta_n)\Big)$$

has rank $\le \nu$. *Then*

$$\left|\det\Big(f_\lambda(\zeta_{n_\mu})\Big)_{1\le\lambda,\mu\le L}\right| \le L! \left(\prod_{\mu=1}^{L} E_\mu^{-n_\mu}\right) \max_{\tau\in\mathfrak{S}_L} \prod_{\lambda=1}^{L} |f_\lambda|_{R_{\tau(\lambda)}}.$$

3. On integer valued functions on sets of products of complex numbers

Let X and Y be two infinite subsets of $\mathbb{C}$ and f an entire function such that $f(xy) \in \mathbb{Z}$ for any $x \in X$ and $y \in Y$. Assuming some growth condition on f, we deduce that f must be a polynomial.

We denote by $D(0,R)$ the disc $\{z \in \mathbb{C}\,;\, |z| \le R\}$. The following result is proved in [W 1997b].

Theorem 3.1. *Let* α, β, ϱ, c_1, c_2 *be positive real numbers satisfying*

$$\frac{1}{\alpha} + \frac{1}{\beta} \le \frac{1}{\varrho}.$$

There exists a constant $\eta > 0$ *with the following property. Assume* X *and* Y *are subsets of* $\mathbb{C}$ *such that*

$$\mathrm{Card}\big(X \cap D(0,R)\big) \ge c_1 R^\alpha$$

and

$$\mathrm{Card}\big(Y \cap D(0,R)\big) \ge c_2 R^\beta$$

for any sufficiently large R. *Further, let* f *be an entire function such that* $f(xy) \in \mathbb{Z}$ *for any* $(x,y) \in X \times Y$. *Furthermore, assume*

$$\log |f|_R \le \eta R^\varrho \tag{1.2}$$

for any sufficiently large R. *Then* f *is a polynomial.*

Example. Let $x_1, \ldots, x_d$ be $\mathbb{Q}$-linearly independent real numbers, and $y_1, \ldots, y_\ell$ be also $\mathbb{Q}$-linearly independent real numbers. We deduce from Theorem 3.1 that for $d\ell > d + \ell$, one at least of the $d\ell$ numbers $e^{x_i y_j}$, $(1 \le i \le d,\ 1 \le j \le \ell)$ is not in $\mathbb{Z}$ (compare with the six exponentials theorem).

The proof of this corollary runs as follows: assume that the $d\ell$ numbers $e^{x_i y_j}$ are all rational integers. Choose $f(z) = e^z$ and

$$X = \{t_1 x_1 + \cdots + t_d x_d\,;\, (t_1, \ldots, t_d) \in \mathbb{N}^d\} \quad \text{and} \quad Y = \{s_1 y_1 + \cdots + s_\ell y_\ell\,;\, (s_1, \ldots, s_\ell) \in \mathbb{N}^\ell\},$$

where $\mathbb{N}$ stands for the set of non-negative rational integers. We apply theorem 3.1 with $\alpha = d$, $\beta = \ell$, $c_1 = (|x_1| + \cdots + |x_d|)^{-1}$, $c_2 = (|y_1| + \cdots + |y_\ell|)^{-1}$, and any $\varrho > 1$.

The main tool in the proof of theorem 3.1 is lemma 2.1.

Concluding remarks

A large number of related problems have been studied and would deserve more attention. In the study of integer valued entire functions, derivatives can be considered. For instance the study of arithmetic values of solutions of differential equations is a wide and fruitful subject. Also special values of functions satisfying certain functional equations have been extensively studied. The existence of p-adic analogues as well as of q-analogues (work by Bundschuh, Gramain, Bézivin,...) is worth to be quoted. Another active area nowadays involves fields of finite characteristic, with Carlitz functions and Drindfeld modules.

Some variants of theorem 3.1 are available; for instance products xy may be replaced by sums $x+y$, but in this case exponential sums come into the picture, as shown by the example

$$\sum_{n=1}^{N} a_n e^{nz} \qquad \text{with } X = Y = \log(\mathbb{N}_{>0}).$$

Finally, again in connection with theorem 3.1, it would be interesting to answer the following question and solve the corresponding functional equation:

$\boxed{?}$ Which entire functions $f : \mathbb{C} \to \mathbb{C}$ satisfy the following property: *there exist two positive integers δ and λ such that the image of $\mathbb{C}^\delta \times \mathbb{C}^\lambda$ into $\mathbb{C}^{\delta\lambda}$ under the mapping*

$$(z_1, \ldots, z_\delta\, ;\, w_1, \ldots, w_\lambda) \longmapsto \bigl(f(z_h w_k)\bigr)_{1\le h\le\delta,\ 1\le k\le\lambda}$$

is not Zariski-dense?

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