On the numbers e^e, e^{e^2} and e^{π^2}

by

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Abstract. We give two measures of simultaneous approximation by algebraic numbers, the first one for the triple (e, e^e, e^{e^2}) and the second one for (π, e, e^{π^2}). We deduce from these measures two transcendence results which had been proved in the early 70's by W.D. Brownawell and the author.

Introduction

In 1949, A.O. Gel'fond introduced a new method for algebraic independence, which enabled him to prove that the two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent. At the same time, he proved that one at least of the three numbers e^e, e^{e^2}, e^{e^3} is transcendental (see [G] Chap. III, §4).

At the end of his book [S] on transcendental numbers, Th. Schneider suggested that one at least of the two numbers e^e, e^{e^2} is transcendental; this was the last of a list of eight problems, and the first to be solved, in 1973, by W.D. Brownawell [B] and M. Waldschmidt [W 1], independently and simultaneously. For this result they shared the Distinguished Award of the Hardy-Ramanujan Society in 1986. Another consequence of their main result is that one at least of the two following statements holds true:

(i) *The numbers e and π are algebraically independent*

(ii) *The number e^{π^2} is transcendental*

Our goal is to shed a new light on these results. It is hoped that our approach will yield further progress towards a solution of the following open problems:

(?) *Two at least of the three numbers e, e^e, e^{e^2} are algebraically independent.*

(?) *Two at least of the three numbers π, e, e^{π^2} are algebraically independent.*

Further conjectures are as follows:

(?) *Each of the numbers e^e, e^{e^2}, e^{π^2} is transcendental*

(?) *The numbers e and π are algebraically independent*

We conclude this note by showing how stronger statements are consequences of Schanuel's conjecture.

1. Heights

Let γ be a complex algebraic number. The *minimal polynomial* of γ over $\mathbb{Z}$ is the unique polynomial
$$f(X) = a_0X^d + a_1X^{d-1} + \cdots + a_d \in \mathbb{Z}[X]$$
which vanishes at the point γ, is irreducible in the factorial ring $\mathbb{Z}[X]$ and has leading coefficient $a_0 > 0$. The integer $d = \deg f$ is the *degree* of γ, denoted by $[\mathbb{Q}(\gamma) : \mathbb{Q}]$. The *usual height* $\mathrm{H}(\gamma)$ of γ is defined by
$$\mathrm{H}(\gamma) = \max\{|a_0|, \ldots, |a_d|\}.$$
It will be convenient to use also the so-called *Mahler's measure* of γ, which can be defined in three equivalent ways. The first one is
$$\mathrm{M}(\gamma) = \exp\left(\int_0^1 \log|f(e^{2i\pi t})|dt\right).$$
For the second one, let $\gamma_1, \ldots, \gamma_d$ denote the complex roots of f, so that
$$f(X) = a_0\prod_{i=1}^d (X - \gamma_i).$$
Then, according to Jensen's formula, we have
$$\mathrm{M}(\gamma) = |a_0|\prod_{i=1}^d \max\{1, |\gamma_i|\}.$$
For the third one, let K be a number field (that is a subfield of $\mathbb{C}$ which is a $\mathbb{Q}$-vector space of finite dimension $[K : \mathbb{Q}]$) containing γ, and let M_K be the set of (normalized) absolute values of K. Then
$$\mathrm{M}(\gamma) = \prod_{v\in M_K} \max\{1, |\gamma|_v\}^{[K_v:\mathbb{Q}_v]}$$

where K_v is the completion of K for the absolute value v and $\mathbb{Q}_v$ the topological closure of $\mathbb{Q}$ in K_v and $[K_v : \mathbb{Q}_v]$ the local degree.

Mahler's measure is related to the usual height by

$$2^{-d}\mathrm{H}(\gamma) \le \mathrm{M}(\gamma) \le \sqrt{d+1}\cdot \mathrm{H}(\gamma).$$

From this point of view it does not make too much difference to use H or M, but one should be careful that d denotes the exact degree of γ, not an upper bound. We shall deal below with algebraic numbers of degree d bounded by some parameter D.

Definition. For an algebraic number γ of degree d and Mahler's measure $\mathrm{M}(\gamma)$, we define the *absolute logarithmic height* $\mathrm{h}(\gamma)$ by

$$\mathrm{h}(\gamma) = \frac{1}{d}\log \mathrm{M}(\gamma).$$

2. Simultaneous Approximation

We state two results dealing with simultaneous Diophantine approximation. Both of them are consequences of the main result in [W 2]. Details of the proof will appear in the forthcoming book [W 3].

2.1. Simultaneous Approximation to e, e^e and e^{e^2}

Theorem 1. *There exists a positive absolute constant c_1 such that, if γ_0, γ_1, γ_2 are algebraic numbers in a field of degree D, then*

$$|e-\gamma_0|+|e^e-\gamma_1|+|e^{e^2}-\gamma_2| > \exp\bigl\{-c_1D^2(h_0+h_1+h_2)^{1/2}(h_1+h_2)^{1/2}(h_0+\log D)(\log D)^{-1}\bigr\}$$

where $h_i = \max\bigl\{e, \mathrm{h}(\gamma_i)\bigr\}$ $(i=0,1,2)$.

2.2. Simultaneous Approximation to π, e and e^{π^2}

Theorem 2. *There exists a positive absolute constant c_2 such that, if γ_0, γ_1, γ_2 are algebraic numbers in a field of degree D, then*

$$|\pi-\gamma_0|+|e-\gamma_1|+|e^{\pi^2}-\gamma_2| > \exp\bigl\{-c_2D^2\bigl(h_0+\log(Dh_1h_2)\bigr)h_1^{1/2}h_2^{1/2}(\log D)^{-1}\bigr\}$$

where $h_i = \max\bigl\{e, \mathrm{h}(\gamma_i)\bigr\}$ $(i=0,1,2)$.

3. Transcendence Criterion

3.1. Algebraic Approximations to a Given Transcendental Number

The following result is Théorème 3.2 of [R-W 1]; see also Theorem 1.1 of [R-W 2].

Theorem 3. *Let $\theta \in \mathbb{C}$ be a complex number. The two following conditions are equivalent:*
(i) The number θ is transcendental.
(ii) For any real number $h \ge 10^7$, there are infinitely many integers $d \ge 1$ for which there exists an algebraic number γ of degree d and absolute logarithmic height $\mathrm{h}(\gamma) \le h$ which satisfies

$$0 < |\theta-\gamma| \le \exp(-10^{-7}hd^2).$$

Notice that the proof of $(ii) \Rightarrow (i)$ is an easy consequence of Liouville's inequality.

3.2. Application to e^e and e^{e^2}

Corollary to Theorem 1. *One at least of the two numbers e^e, e^{e^2} is transcendental.*

Proof of the corollary. Assume that the two numbers e^e and e^{e^2} are algebraic, say γ_1 and γ_2. Then, according to Theorem 1, there exists a constant $c_3 > 1$ such that, for any algebraic number γ of degree $\le D$ and height $\mathrm{h}(\gamma) \le h$ with $h \ge e$,

$$|e-\gamma| > \exp\bigl\{-c_3D^2h^{1/2}(h+\log D)(\log D)^{-1}\bigr\}.$$

We now use Theorem 3 for $\theta = e$ with $h = 10^{15}c_3^2$ and derive a contradiction. □

4. Algebraic Independence

4.1. Simultaneous Approximation

The proof of the following result is given in [R-W 1], Théorème 3.1, as a consequence of Theorem 3 (see also [R-W 2] Corollary 1.2).

Corollary to Theorem 3. *Let $\theta_1,\dots,\theta_m$ be complex numbers such that the field $\mathbb{Q}(\theta_1,\dots,\theta_m)$ has transcendence degree 1 over $\mathbb{Q}$. There exists a constant $c>0$ such that, for any real number $h \ge c$, there are infinitely many integers D for which there exists a tuple $(\gamma_1,\dots,\gamma_m)$ of algebraic numbers satisfying*

$$[\mathbb{Q}(\gamma_1,\dots,\gamma_m):\mathbb{Q}] \le D, \quad \max_{1\le i\le m}\mathrm{h}(\gamma_i) \le h$$

and

$$\max_{1\le i\le m}\{|\theta_i-\gamma_i|\} \le \exp(-c^{-1}hD^2).$$

4.2. Application to π, e and e^{π^2}

Corollary to Theorem 2. *One at least of the two following statements is true:*
(*i*) *The numbers e and π are algebraically independent.*
(*ii*) *The number e^{π^2} is transcendental.*

Remark. This corollary can be stated in an equivalent way as follows:

For any non constant polynomial $P\in\mathbb{Z}[X]$, the complex number

$$e^{\pi^2}+iP(e,\pi)$$

is transcendental.

The idea behind this remark originates in [R].

Proof of the corollary. Assume that the number e^{π^2} is algebraic. Theorem 2 with $\gamma_2=e^{\pi^2}$ shows that there exists a constant $c_4>0$ such that, for any pair (γ_0,γ_1) of algebraic numbers, if we set

$$D=[\mathbb{Q}(\gamma_0,\gamma_1):\mathbb{Q}]\quad\text{and}\quad h=\max\bigl\{e,\mathrm{h}(\gamma_0),\mathrm{h}(\gamma_1)\bigr\},$$

then

$$|\pi-\gamma_0|+|e-\gamma_1|>\exp\bigl\{-c_4D^2\bigl(h+\log D\bigr)h^{1/2}(\log D)^{-1}\bigr\}.$$

Therefore we deduce from the Corollary to Theorem 3 that the field $\mathbb{Q}(\pi,e)$ has transcendence degree 2. □

5. Schanuel's Conjecture

The following conjecture is stated in [L] Chap. III p.30:

Schanuel's Conjecture. *Let $x_1,\ldots,x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then, among the $2n$ numbers*

$$x_1,\ldots,x_n,e^{x_1},\ldots,e^{x_n},$$

at least n are algebraically independent.

Let us deduce from Schanuel's Conjecture the following statement (which is an open problem):

(?) *The 7 numbers*

$$e,\ \pi,\ e^e,\ e^{e^2},\ e^{\pi^2},\ 2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{4}}$$

are algebraically independent.

We shall use Schanuel's conjecture twice. We start with the numbers 1, $\log 2$ and $i\pi$ which are linearly independent over $\mathbb{Q}$ because $\log 2$ is irrational. Therefore, according to Schanuel's conjecture, three at least of the numbers

$$1,\ \log 2,\ i\pi,\qquad e,\ 2,\ -1$$

are algebraically independent. This means that the three numbers $\log 2$, π and e are algebraically independent. Therefore the 8 numbers

$$1,\ i\pi,\ \pi^2,\ e,\ e^2,\ \log 2,\ \sqrt[3]{2}\log 2,\ \sqrt[3]{4}\log 2$$

are $\mathbb{Q}$-linearly independent. Again, Schanuel's conjecture implies that 8 at least of the numbers

$$1,\ i\pi,\ \pi^2,\ e,\ e^2,\ \log 2,\ \sqrt[3]{2}\log 2,\ \sqrt[3]{4}\log 2,\qquad e,\ -1,\ e^{\pi^2},\ e^e,\ e^{e^2},\ 2,\ 2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{4}}$$

are algebraically independent, and this means that the 8 numbers

$$e,\ \pi,\ e^e,\ e^{e^2},\ e^{\pi^2},\ 2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{4}},\ \log 2$$

are algebraically independent.

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