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Integer valued functions on products

by

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Abstract. Let X and Y be two infinite subsets of $\mathbb{C}$ and f an entire function such that $f(xy) \in \mathbb{Z}$ for any $x \in X$ and $y \in Y$. Assuming some growth conditions on f (upper bound) and on the size of X and Y (lower bounds), we deduce that f must be a polynomial.

For an entire function $f : \mathbb{C} \to \mathbb{C}$ (i.e. a function which is analytic in $\mathbb{C}$), and for $R \geq 0$, we denote by $|f|_R$ the number $\sup_{|z|=R} |f(z)|$. We also denote by $D(0,R)$ the disc $\{z \in \mathbb{C}\,;\, |z| \leq R\}$.

Theorem . — *Let α, β, ϱ, c_1, c_2 be positive real numbers satisfying*

$$\frac{1}{\alpha} + \frac{1}{\beta} \leq \frac{1}{\varrho}.$$

There exists a constant $\eta > 0$ with the following property. Assume X and Y are subsets of $\mathbb{C}$ such that

$$\mathrm{Card}\big(X \cap D(0,R)\big) \geq c_1 R^{\alpha} \quad \textit{and} \quad \mathrm{Card}\big(Y \cap D(0,R)\big) \geq c_2 R^{\beta}$$

for any sufficiently large R. Further, let f be an entire function such that $f(xy) \in \mathbb{Z}$ for any $(x,y) \in X \times Y$. Furthermore, assume

$$\log |f|_R \leq \eta R^{\varrho} \tag{1.2}$$

for any sufficiently large R. Then f is a polynomial.

Reference

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