CHAPTER 14

Algebraic Independence in Algebraic Groups. Part II: Large Transcendence Degrees

1. Introduction

This chapter is a continuation of chapter 13 on small transcendence degrees.

Our first goal is to introduce conjectures. We are not looking for the most general ones (see [**And2**]): we only propose some open problems which might be easier to prove, given the currently available methods.

Our next purpose is to present a proof of a result of algebraic independence which shows that some fields have a "large transcendence degree": these fields are generated by numbers of the form $\alpha^{\beta}, \alpha^{\beta^2}, \ldots$, when α and β are algebraic. We do not give the proof of the best known result on this topic (due to G. Diaz [**Dia2**]), but only of a weaker statement for which the arguments may look more transparent. We shall explain how to use the transcendence criterion (chapter 8) and the zero estimate (chapter 11) together with an auxiliary function.

2. Conjectures

2.1. Commutative Algebraic Groups. We keep the following notation already introduced in chapter 13. Let K be a subfield of $\mathbf{C}$ of transcendence degree t over $\mathbf{Q}$ and G a commutative connected algebraic group of dimension d defined over K. Assume $G = G_0 \times G_1 \times G_2$ where $G_0 = \mathbf{G}_{\mathrm{a}}^{d_0}$, $G_1 = \mathbf{G}_{\mathrm{m}}^{d_1}$ and the dimension d_2 of G_2 satisfies $d = d_0 + d_1 + d_2$. Denote by T_G the tangent space at the origin of the algebraic group G. The set $T_G(K)$ of K-rational points of T_G is a K-vector space of dimension d, and the set $T_G(\mathbf{C})$ of complex points of T_G is the Lie algebra of the Lie group $G(\mathbf{C})$. Let $\exp_G : T_G(\mathbf{C}) \longrightarrow G(\mathbf{C})$ denote the exponential map of $G(\mathbf{C})$. Let V be a subspace of $T_G(\mathbf{C})$ of dimension $n < d$ such that $\exp_G V$ is Zariski dense in $G(\mathbf{C})$. Let $Y = \mathbf{Z}\eta_1 + \cdots + \mathbf{Z}\eta_\ell$ be a finitely generated subgroup of V of rank $\ell \geq 1$ such that $\exp_G Y \subset G(K)$.

A $\mathbf{C}$-vector subspace W of $T_G(\mathbf{C})$ is *defined over K* if it is spanned by a K-vector subspace of $T_G(K)$.

2.1.1. *One Parameter Subgroups.* We first assume $n = 1$ — this is the situation which is considered in chapter 13.

CONJECTURE 2.1. *Assume $d\ell > \ell + d_1 + 2d_2$. Then*

$$t > \frac{d\ell}{\ell + d_1 + 2d_2} - 1.$$

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Example. Theorem 2.1 in chapter 13 proves the following special case of Conjecture 2.1 concerning small transcendence degrees:

$$d\ell \geq 2(\ell + d_1 + 2d_2) \Longrightarrow t \geq 2.$$

Conjecture 2.1 is related with Schneider's solution of Hilbert's seventh problem. Here is a conjecture related with Gel'fond's solution of the same problem.

Conjecture 2.2. *Assume further that V is defined over K. Then*

$$t > \frac{(d-1)\ell}{\ell + d_1 + 2d_2}.$$

Example. A consequence of Conjecture 2.2 for small transcendence degree is

$$(d-1)\ell \geq \ell + d_1 + 2d_2 \Longrightarrow t \geq 2.$$

The state of the art on this question is described in chapter 13, § 2: the result is proved in the case $d_0 = 0$, while for $d_0 = 1$ the conclusion $t \geq 2$ is reached only under the stronger assumption $(d-1)\ell > \ell + d_1 + 2d_2$.

Conjectures 2.1 and 2.2 in the special case of a linear algebraic group reduce to the following one, which includes also the Four Exponentials Conjecture:

Conjecture 2.3. *Let $x_1, \ldots, x_d$ be complex numbers which are $\mathbf{Q}$-linearly independent, $y_1, \ldots, y_\ell$ be also $\mathbf{Q}$-linearly independent complex numbers and K a subfield of $\mathbf{C}$ which contains the $d\ell$ numbers $e^{x_i y_j}$ $(1 \leq i \leq d,\ 1 \leq j \leq \ell)$. Assume $d \geq 2$ and $\ell \geq 2$. Denote by t the transcendence degree over $\mathbf{Q}$ of K. Then*

$$t > \frac{d\ell}{\ell + d} - 1.$$

Moreover the transcendence degree t_1 of the field $K_1 = K(x_1, \ldots, x_d)$ is bounded from below by

$$t_1 > \frac{(d-1)\ell}{\ell + d}$$

and the transcendence degree t_2 of the field $K_2 = K_1(y_1, \ldots, y_\ell)$ by

$$t_2 > \frac{d\ell}{\ell + d}.$$

2.1.2. *Several Parameters Subgroups.* We consider now the general case $n \geq 1$. We define

$$\mu = \mu(Y, V) = \min_{V' \neq V} \left\{ \frac{\mathrm{rank}_{\mathbf{Z}}(Y/Y \cap V')}{\dim_{\mathbf{C}}(V/V')} \right\}$$

where V' runs over the set of $\mathbf{C}$-vector subspaces of V of dimension $< n$. Hence $\mu \leq \ell/n$ with $\ell = \mathrm{rank}_{\mathbf{Z}} Y$. On the other hand the condition $\mu > 0$ means that V is the $\mathbf{C}$-vector space spanned by Y; in this case $\mu \geq 1$. For a subgroup Y of rank $\ell \geq n$ of V, the condition $\mu(Y, V) = \ell/n$ is not a very strong assumption; roughly speaking, it only means that no set of $\mathbf{Q}$-linearly independent points of Y is contained in a subspace of V of too small a dimension.

Here is the extension of Conjecture 2.1 to n variables:

Conjecture 2.4. *Assume $d\mu > n\mu + d_1 + 2d_2$. Then*

$$t > \frac{d\mu}{n\mu + d_1 + 2d_2} - 1.$$

Conjecture 2.4 corresponds to Schneider's method in several variables. We can extend Conjecture 2.2 as follows: *if V is defined over K and $\mu > 0$, then*

$$t > \frac{(d-n)\mu}{n\mu + d_1 + 2d_2}.$$

This corresponds to Gel'fond's method in several variables. However there are intermediate situations which may be compared with Baker's method:

CONJECTURE 2.5. *Assume that there exists a $\mathbf{C}$-vector subspace W of $T_G(\mathbf{C})$, of dimension $n' < d$, which is defined over K and contains Y. Assume also $\mu > 0$. Then*

$$t > \frac{(d-n')\mu}{n\mu + d_1 + 2d_2}.$$

REMARK. Several variations are interesting. In particular one may expect refinements of the conjectural estimates for the transcendence degree when Y contains periods of $\exp_G$. For this purpose it is useful to introduce a new parameter $\kappa = \mathrm{rank}_{\mathbf{Z}}\big(Y \cap \ker \exp_G\big)$ (see [**Wal7**], §16).

2.2. Results: Large Transcendence Degree for the Values of the Exponential Function in One Variable. From now on we restrict the discussion to one parameter subgroups of linear algebraic groups.

Partial results are known concerning the above conjectures, under a so called "Technical Hypothesis". For simplicity, we shall use only the following assumption, which is a measure of linear independence. One should stress that most known results actually involve a much weaker hypothesis, while much sharper estimates are valid for concrete applications. Therefore this condition is in fact not too strong.

DEFINITION 2.6. We shall say that a set $\{u_1, \dots, u_n\}$ of $\mathbf{Q}$-linearly independent complex numbers *satisfies the Technical Hypothesis* `(T.H.)` if, for any $\epsilon > 0$, there exists a positive number H_0 such that, for any $H \ge H_0$ and n-tuple $(h_1, \dots, h_n)$ of rational integers satisfying $0 < \max\{|h_1|, \dots, |h_n|\} \le H$, the inequality

$$|h_1u_1 + \cdots + h_nu_n| \ge \exp\big\{-H^\epsilon\big\}$$

holds.

Here is the main result of G. Diaz in [**Dia2**] :

THEOREM 2.7. *Let $x_1, \dots, x_d$ be complex numbers which are $\mathbf{Q}$-linearly independent and satisfy* `(T.H.)` *and $y_1, \dots, y_\ell$ be also complex numbers which are $\mathbf{Q}$-linearly independent and satisfy* `(T.H.)`. *Let K a subfield of $\mathbf{C}$ which contains the $d\ell$ numbers $e^{x_iy_j}$ $(1 \le i \le d,\ 1 \le j \le \ell)$. Assume $d\ell > \ell + d$ and denote by t the transcendence degree over $\mathbf{Q}$ of K. Then*

$$t > \frac{d\ell}{\ell + d} - 1.$$

Moreover the transcendence degree t_1 of the field $K_1 = K(x_1, \dots, x_d)$ is bounded from below by

$$t_1 > \frac{(d-1)\ell}{\ell + d},$$

and the transcendence degree t_2 of the field $K_2 = K_1(y_1, \dots, y_\ell)$ by

$$t_2 \ge \frac{d\ell}{\ell + d}.$$

Remark. For $t \in \mathbf{Z}$ and $x \in \mathbf{R}$, we have $t > x - 1$ if and only if $t \geq [x]$. For instance the conclusions for t and t_1 can be written

$$t \geq \left[\frac{d\ell}{\ell + d}\right] \quad \text{and} \quad t_1 \geq \left[\frac{d(\ell+1)}{\ell + d}\right].$$

In some cases G. Diaz [**Dia1**] succeeds to prove also

$$t_2 > \frac{d\ell}{\ell + d}, \quad \text{that is} \quad t_2 \geq \left[\frac{d\ell}{\ell + d}\right] + 1.$$

The lower bound for t_1 yields the following partial answer to the Gel'fond-Schneider problem on the algebraic independence of α^{β^i}:

Corollary 2.8. *Let α be a non zero complex algebraic number,* $\log\alpha$ *a non zero logarithm of α and β an algebraic number of degree $d \geq 2$. For $z \in \mathbf{C}$, define $\alpha^z := \exp(z \log\alpha)$. Then*

$$\mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}\big(\alpha^{\beta}, \ldots, \alpha^{\beta^{d-1}}\big) \geq \left[\frac{d+1}{2}\right].$$

Proof. Take $\ell = d$ and define

$$x_i = \beta^{i-1} \quad (1 \leq i \leq d) \qquad \text{and} \qquad y_j = \beta^{j-1}\log\alpha \quad (1 \leq j \leq d). \quad \square$$

As pointed out in [**Nes7**], A.O. Gel'fond already anounced Corollary 2.8 in 1948 [**Gel1**], but his papers [**Gel2**] and [**Gel3**] contain a proof only of the small transcendence degree result: *for $d \geq 3$, the transcendence degree is at least* 2 (see chapter 13). Hence Corollary 2.8, claimed by Gel'fond, has been proved by Diaz only 40 years later.

2.3. Historical Sketch. We refer to [**FN**], Chapter 6 for a survey on *Gel'fond's method for algebraic independence and its recent developments* together with plenty of references. See also [**Wal5**] and [**Wal11**].

The first step is due to A.O. Gel'fond at the end of the 40's [**Gel1**], [**Gel2**], who proved results of small transcendence degree under a technical hypothesis `(T.H.)` for $x_1, \ldots, x_d$ as well as $y_1, \ldots, y_\ell$. This technical hypothesis was removed by R. Tijdeman 1970 [**Tij2**], by means of a sharper analytic zero estimate for exponential polynomials [**Tij1**].

Here is the state of our knowledge concerning "small transcendence degree" for the values of the exponential function in one variable:

Theorem 2.9. *Let $x_1, \ldots, x_d$ be complex numbers which are $\mathbf{Q}$-linearly independent and $y_1, \ldots, y_\ell$ be also complex numbers which are $\mathbf{Q}$-linearly independent. Denote by t the transcendence degree of the field K generated over $\mathbf{Q}$ by the $d\ell$ numbers $e^{x_i y_j}$ $(1 \leq i \leq d,\ 1 \leq j \leq \ell)$, by t_1 the transcendence degree of the field $K_1 = K(x_1, \ldots, x_d)$ and by t_2 the transcendence degree of the field $K_2 = K_1(y_1, \ldots, y_\ell)$. Then*

$$d\ell \geq 2(\ell + d) \Rightarrow t \geq 2$$
$$d\ell \geq d + 2\ell \Rightarrow t_1 \geq 2$$
$$d\ell > \ell + d \Rightarrow t_2 \geq 2$$

Moreover, if $d = \ell = 2$, and if the two numbers $e^{x_1 y_1}$ and $e^{x_1 y_2}$ are algebraic, then $t_2 \geq 2$.

Further developments are described in W.D. Brownawell's survey [**Bro2**]. These are the only known results which do not require a Technical Hypothesis.

The first results on large transcendence degree are due to G.V. Chudnovsky [**Chu2**] who proved, under the assumptions of Theorem 2.7, the lower bound $2^t \geq d\ell/(\ell+d)$. Chudnovsky's method has been worked out by P. Warkentin, P. Philippon, E. Reyssat, R. Endell, W.D. Brownawell and Yu.V. Nesterenko (see [**Wal5**]). P. Philippon introduced the trick of *redundant variables* (which is a variant of Landau's method, that is an homogeneity argument which we are going to describe in §3.2). By means of a sharp criterion of his own [**PPh2**](cf. chapter 8), he then succeeded to replace 2^t by $t+1$ in Chudnovsky's result and to prove, under `(T.H.)`, the inequalities

$$t \geq \frac{d\ell}{\ell+d} - 1, \quad t_1 \geq \frac{(d-1)\ell}{\ell+d} \quad \text{and} \quad t_2 \geq \frac{d\ell}{\ell+d}.$$

Finally G. Diaz [**Dia2**] obtained Theorem 2.7, which is the sharpest known result to date, apart from a weakening of the technical hypothesis which arises from the work of W.D. Brownawell [**Bro5**], and also apart from the quantitative aspect of the subject (see [**Ably**], [**Jab**], [**Del**], [**Cav**] and [**FN**]). Notice also that in [**Nes7**] Yu.V. Nesterenko removed the use of Philippon's criterion from the proof of Theorem 2.7.

3. Proofs

3.1. Statement. We shall prove the following result:

THEOREM 3.1. *Let α be a positive real number, $\alpha \neq 1$. Let β be a real algebraic number of degree $d \geq 2$. Then the transcendence degree t of the field $\mathbf{Q}\big(\alpha, \alpha^{\beta}, \dots, \alpha^{\beta^{d-1}}\big)$ over $\mathbf{Q}$ satisfies*

$$t \geq \frac{d}{2} - 1$$

REMARK. The proof of this result will involve several variables. As we shall see, the same method restricted to functions of a single variable yields only the weaker estimate

$$t \geq \frac{d}{4} - \frac{1}{2}.$$

The refinement to $(d/2) - 1$ will come from Philippon's "redundant variables".

We are going to use Schneider's method for the torus $\mathbf{G}_{\mathrm{m}}^d$. If we were using either Gel'fond's method (i.e. including derivatives) for $\mathbf{G}_{\mathrm{m}}^d$, or else Schneider's method for the product $\mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{m}}^d$, we would get $t \geq (d-1)/2$. In order to reach $t > (d-1)/2$, Diaz [**Dia2**] replaces the auxiliary polynomial $P \in \mathbf{Z}[X_1, \dots, X_d]$ by a polynomial with coefficients in the ring generated over $\mathbf{Z}$ by the numbers $e^{x_i y_j}$. Finally the restrictions $\beta \in \mathbf{R}$ and $\alpha > 0$ provide a slight simplification, but is it an easy exercise to remove them.

3.2. Tools. Let $\underline{x}_1, \dots, \underline{x}_d$ be elements in $\mathbf{C}^n$ and also $\underline{y}_1, \dots, \underline{y}_\ell$ elements in $\mathbf{C}^n$. Denote by $\underline{x}_i \underline{y}_j$ the standard scalar product in $\mathbf{C}^n$. Let K be a subfield of $\mathbf{C}$ of transcendence degree t over $\mathbf{Q}$ containing all numbers $e^{\underline{x}_i \underline{y}_j}$.

<u>Goal</u>: Under "suitable assumptions", we want to prove

$$t \geq \frac{d\ell + \ell + d}{(n+1)(\ell+d)} - 1.$$

Here is how Philippon's redundant variables occur. For a positive integer k, we take the k-th Cartesian powers and we replace n by kn, d by kd, ℓ by $k\ell$: for large k, we deduce

$$t \geq \frac{d\ell}{n(\ell+d)} - 1.$$

Of course one shall need to check that the above mentioned "suitable assumptions" are satisfied for Cartesian products.

3.2.1. *Criterion of Algebraic Independence.* Our first tool is the following special case of Philippon's criterion for algebraic independence [**PPh2**] and chapter 8, § 1. We denote by $\mathrm{H}(P)$ the *usual height of a polynomial* P with, say, complex coefficients, that is the maximum absolute value of its coefficients.

Proposition 3.2. *Let $a \geq 1$ be a real number and $\theta = (\theta_1, \dots, \theta_q)$ an element in $\mathbf{C}^q$. There exists a positive number C having the following property. Assume that for all sufficiently large integer N, there exist a positive integer $m = m(N) \geq 1$ and polynomials $Q_{N1}, \dots, Q_{Nm}$ in $\mathbf{Z}[X_1, \dots, X_q]$ with*

$$\max_{1\leq j\leq m} \deg Q_{Nj} \leq N, \qquad \max_{1\leq j\leq m} \mathrm{H}(Q_{Nj}) \leq e^N$$

and

$$\max_{1\leq j\leq m} |Q_{Nj}(\theta_1, \dots, \theta_q)| \leq e^{-CN^a}, \tag{114}$$

such that the polynomials $Q_{N1}, \dots, Q_{Nm}$ have no common zero in the domain

$$\left\{ z \in \mathbf{C}^q\,;\ \max_{1\leq i\leq q} |z_i - \theta_i| \leq e^{-3CN^a} \right\}.$$

Then the transcendence degree t of the field $\mathbf{Q}(\theta_1, \dots, \theta_q)$ over $\mathbf{Q}$ satisfies

$$t > a - 1.$$

3.2.2. *Auxiliary Function.* For $r > 0$, and for an entire function φ of n variables, we denote by $|\varphi|_r$ the number

$$|\varphi|_r = \sup \left\{ |\varphi(\underline{z})|\,;\ \underline{z} = (z_1, \dots, z_n) \in \mathbf{C}^n,\ \max_{1\leq i\leq n} |z_i| \leq r \right\}.$$

Proposition 3.3. *Let M be a positive integer, r, Δ, U be positive real numbers and $\varphi_1, \dots, \varphi_M$ entire functions in $\mathbf{C}^n$. Assume*

$$(8U)^{n+1} \leq M\Delta, \qquad \Delta \leq U \tag{115}$$

and

$$\sum_{\mu=1}^{M} |\varphi_\mu|_{er} \leq e^U.$$

Then there exists rational integers $p_1, \dots, p_M$ in $\mathbf{Z}$ with

$$0 < \max\bigl\{|p_1|, \dots, |p_M|\bigr\} \leq e^\Delta$$

such that the function $F = p_1\varphi_1 + \cdots + p_M\varphi_M$ satisfies

$$|F|_r \leq e^{-U}.$$

Sketch of proof. Let $T \in \mathbf{Z}$ satisfy $4U \leq T < 4U+1$. Use Dirichlet's pigeonhole principle to solve the following system of inequalities involving M unknowns $p_1, \dots, p_M$ in $\mathbf{Z}$:

$$\sum_{\|\tau\|<T} \frac{r^{\|\tau\|}}{\tau!} \cdot \left| \frac{d^\tau}{dz^\tau} F(0) \right| \leq \frac{1}{2} e^{-U}$$

for $F = p_1\varphi_1 + \cdots + p_M\varphi_M$. Next use an interpolation formula

$$|F|_r \leq \big(1+\sqrt{T}\big)e^{-T}|F|_{er} + \sum_{\|\tau\|<T} \frac{r^{\|\tau\|}}{\tau!} \cdot \left| \frac{d^\tau}{dz^\tau} F(0) \right|.$$

□

(For more details, see [**Wal4**]).

3.2.3. *Zeros Estimate.*
a) *Degree of hypersurfaces and condition* `(Z.E.)`

Notation. For Σ a finite subset of $\mathbf{C}^m$, define $\omega(\Sigma)$ as the smallest total degree of a non zero polynomial in $\mathbf{C}[z_1, \dots, z_m]$ which vanishes on Σ:

$$\omega(\Sigma) = \min\big\{\deg P\,;\, P \in \mathbf{C}[z_1, \dots, z_m],\ P \neq 0,\ P(\sigma) = 0 \text{ for any } \sigma \in \Sigma\big\}.$$

Definition 3.4. Let $X = \mathbf{Z}\underline{x}_1 + \cdots + \mathbf{Z}\underline{x}_d$ and $Y = \mathbf{Z}\underline{y}_1 + \cdots + \mathbf{Z}\underline{y}_\ell$ be two finitely generated subgroups of $\mathbf{C}^n$ of rank d and ℓ respectively. Denote by $\theta_1, \dots, \theta_q$ the distinct elements of the set

$$\big\{e^{\underline{x}_i \underline{y}_j}\,;\, 1 \leq i \leq d,\ 1 \leq j \leq \ell\big\}.$$

We shall say that (X, Y) *satisfies the condition* `(Z.E.)` if for each $\eta > 0$ there exists a positive real number $R_0 > 0$ with the following property: for any positive integer $R \geq R_0$ and any q-tuple $(\mu_1, \dots, \mu_q)$ of complex numbers satisfying

$$\max_{1\leq h\leq q} \big|\theta_h - \mu_h\big| < e^{-R^\eta},$$

if we set $\mu_{ij} = \mu_h$ for $e^{\underline{x}_i \underline{y}_j} = \theta_h$ and

$$\Sigma = \left\{ \left(\prod_{j=1}^{\ell} \mu_{ij}^{r_j} \right)_{1\leq i\leq d}\,;\, 0 \leq r_j < R\ (1 \leq j \leq \ell) \right\} \subset (\mathbf{C}^\times)^d,$$

then

$$\omega(\Sigma) \geq (R/d)^{\ell/d}.$$

One should remark that this condition `(Z.E.)` is not only about (X, Y) but rather concerns $(\underline{x}_1, \dots, \underline{x}_d)$ and $(\underline{y}_1, \dots, \underline{y}_\ell)$. However in our situation the choice of the bases of these $\mathbf{Z}$-modules will be clear from the context.

We shall check that the condition `(Z.E.)` holds for $n = 1$ when the two tuples of real numbers $(x_1, \dots, x_d)$ and $(y_1, \dots, y_\ell)$ satisfy `(T.H.)`. Next we show that this condition `(Z.E.)` is stable under Cartesian products.

b) *Consequence of the Technical Hypothesis*

LEMMA 3.5. *Let $x_1, \dots, x_d$ in $\mathbf{R}$ satisfy* (T.H.)*, and let $y_1, \dots, y_\ell$ in $\mathbf{R}$ also satisfy* (T.H.)*. For each $\eta_0 > 0$ there exists $L_0 > 0$ and $R_0 > 0$ with the following property. Let $L \geq L_0$ and $R \geq R_0$ be positive integers and let μ_{ij} be complex numbers ($1 \leq i \leq d$, $1 \leq j \leq \ell$) satisfying*

$$\max_{\substack{1\leq i\leq d\\ 1\leq j\leq \ell}} \left| e^{x_i y_j} - \mu_{ij} \right| < e^{-(LR)^{\eta_0}}.$$

Then for any $\underline{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbf{Z}^d$ and $\underline{r} = (r_1, \dots, r_\ell) \in \mathbf{Z}^\ell$ with

$$0 < \max_{1\leq i\leq d} |\lambda_i| \leq L \quad \text{and} \quad 0 < \max_{1\leq j\leq \ell} |r_j| \leq R$$

we have

$$\prod_{i=1}^{d} \prod_{j=1}^{\ell} \mu_{ij}^{\lambda_i r_j} \neq 1.$$

PROOF. We start the proof with the following remark. Let v and w be two complex numbers which satisfy $|we^{-v} - 1| \leq 1/2$. Set $z = v + \log(we^{-v})$, where log denotes the principal branch of the logarithm. Then $e^z = w$ and $|z - v| \leq 2|we^{-v} - 1|$.

Therefore for $1 \leq i \leq d$ and $1 \leq j \leq \ell$ we can find a complex logarithm z_{ij} of μ_{ij} such that

$$\max_{\substack{1\leq i\leq d\\ 1\leq j\leq \ell}} |x_i y_j - z_{ij}| < e^{-(1/2)(LR)^{\eta_0}}.$$

Suppose

$$\prod_{i=1}^{d} \prod_{j=1}^{\ell} \mu_{ij}^{\lambda_i r_j} = 1.$$

Since the imaginary part of z_{ij} has absolute value $< e^{-(1/2)(LR)^{\eta_0}}$, and since, for sufficiently large L and R,

$$d\ell LR e^{-(1/2)(LR)^{\eta_0}} < 2\pi,$$

the absolute value of the imaginary part of the number

$$\sum_{i=1}^{d} \sum_{j=1}^{\ell} \lambda_i r_j z_{ij}$$

is $< 2\pi$. Therefore

$$\sum_{i=1}^{d} \sum_{j=1}^{\ell} \lambda_i r_j z_{ij} = 0.$$

We deduce

$$\begin{aligned}
\left| \sum_{i=1}^{d} \lambda_i x_i \right| \left| \sum_{j=1}^{\ell} r_j y_j \right| &= \left| \sum_{i=1}^{d} \sum_{j=1}^{\ell} \lambda_i r_j x_i y_j \right| \\
&\leq \sum_{i=1}^{d} \sum_{j=1}^{\ell} \lambda_i r_j |x_i y_j - z_{ij}| \\
&\leq d\ell LR e^{-(1/2)(LR)^{\eta_0}} < e^{-(1/3)(LR)^{\eta_0}}
\end{aligned}$$

for L and R large enough. Now, for L and R sufficiently large, the left hand side is bounded from below by $e^{-L^{\eta_0}-R^{\eta_0}}$, and we get the desired contradiction as soon as $3(L^{\eta_0}+R^{\eta_0})<(LR)^{\eta_0}$. ◻

c)*Single Variable: $n=1$*

Applying Philippon's zero estimate (theorem 5.1 from chapter 11), we now prove:

PROPOSITION 3.6. *Let $x_1,\dots,x_d$ in $\mathbf{R}$ satisfy* (T.H.)*, and let $y_1,\dots,y_\ell$ in $\mathbf{R}$ also satisfy* (T.H.)*. Let $X=\mathbf{Z}\underline{x}_1+\cdots+\mathbf{Z}\underline{x}_d$ and $Y=\mathbf{Z}\underline{y}_1+\cdots+\mathbf{Z}\underline{y}_\ell$. Then (X,Y) satisfies the condition* (Z.E.).

PROOF. Let η be a positive real number, R a sufficiently large positive real number and $\mu_1,\dots,\mu_q$ complex numbers satisfying

$$\max_{1\le h\le q}\bigl|\theta_h-\mu_h\bigr|<e^{-R^\eta}.$$

For $1\le j\le\ell$ and $1\le i\le d$, define μ_{ij} by $\mu_{ij}=\mu_h$ where $h\in\{1,\dots,q\}$ is the index for which $e^{x_iy_j}=\theta_h$. Moreover, for $\underline{r}=(r_1,\dots,r_\ell)\in\mathbf{Z}^\ell$ and $1\le i\le d$, define

$$\mu_{i\underline{r}}=\prod_{j=1}^{\ell}\mu_{ij}^{r_j}.$$

Let

$$\Sigma=\Bigl\{\bigl(\mu_{1\underline{r}},\dots,\mu_{d\underline{r}}\bigr)\,;\,0\le r_j<R\ (1\le j\le\ell)\Bigr\}\subset(\mathbf{C}^\times)^d.$$

Assume $\omega(\Sigma)<(R/d)^{\ell/d}$, there exists a non zero polynomial $P\in\mathbf{C}[X_1,\dots,X_d]$ of total degree $D<(R/d)^{\ell/d}$ such that the R^ℓ numbers

$$P(\mu_{1\underline{r}},\dots,\mu_{d\underline{r}})\qquad(0\le r_j<R,\ 1\le j\le\ell)$$

vanish. We use theorem 5.1 from chapter 11 with $G=\mathbf{G}_{\mathrm{m}}^d$, $W=0$, $c=1$,

$$\Sigma_0=\bigl\{\bigl(\mu_{1\underline{r}},\dots,\mu_{d\underline{r}}\bigr)\,;\,0\le r_j<R/d\,(1\le j\le\ell)\bigr\}\subset\Sigma\subset(\mathbf{C}^\times)^d.$$

Since P vanishes on the set

$$\Sigma\supset\Sigma_0(d)=\bigl\{\sigma_1\cdots\sigma_d\,;\,(\sigma_1,\dots,\sigma_d)\in\Sigma^d\bigr\},$$

we deduce that there exists a connected algebraic subgroup H of G, of dimension $<d$, such that

$$\mathrm{Card}\bigl((\Sigma_0+H)/H\bigr)\mathcal{H}(H,D)\le\mathcal{H}(G,D).$$

Here $\mathcal{H}(G,D)=D^d$. Moreover H is contained in a hypersurface of equation

$$X_1^{\lambda_1}\cdots X_d^{\lambda_d}=1,$$

where $(\lambda_1,\dots,\lambda_d)\in\mathbf{Z}^d$ satisfies $0<\max_{1\le i\le d}|\lambda_i|\le D$.

Define $\eta_0=\eta d/(\ell+d)$ and $L=D$, so that

$$(LR)^{\eta_0}\le R^\eta.$$

Using the condition (T.H.), we may apply Lemma 3.5: we get an injective mapping from the set

$$\bigl\{(r_1,\dots,r_\ell)\,;\,0\le r_j<R/d\ (1\le j\le\ell)\bigr\}$$

into $(\Sigma_0+H)/H$ by mapping $(r_1,\dots,r_\ell)$ onto the image of $(\mu_{1\underline{r}},\dots,\mu_{d\underline{r}})$ in $(\Sigma_0+H)/H$. Therefore

$$\mathrm{Card}\bigl((\Sigma_0+H)/H\bigr)\ge(R/d)^\ell>D^d,$$

which yields the desired contradiction. □

d) *Cartesian Products*

The next lemma is well known (but we include the easy proof).

LEMMA 3.7. *Let $\Sigma_1, \ldots, \Sigma_k$ be finite subsets of $\mathbf{C}^n$. Then*

$$\omega(\Sigma_1 \times \cdots \times \Sigma_k) = \min_{1 \leq h \leq k} \omega(\Sigma_h).$$

PROOF. The lower bound

$$\omega(\Sigma_1 \times \cdots \times \Sigma_k) \leq \min_{1 \leq h \leq k} \omega(\Sigma_h).$$

is easy: if $h_0 \in \{1, \ldots, k\}$ satisfies $\omega(\Sigma_{h_0}) = \min_{1 \leq h \leq k} \omega(\Sigma_h)$ and if $P \in \mathbf{C}[\underline{z}]$ is a non zero polynomial in n variables of total degree $\omega(\Sigma_{h_0})$ which vanishes on Σ_{h_0}, then the image of P in $\mathbf{C}[\underline{z}_1, \ldots, \underline{z}_k]$ under the morphism $\mathbf{C}[\underline{z}] \longrightarrow \mathbf{C}[\underline{z}_1, \ldots, \underline{z}_k]$ which maps $\underline{z}$ onto $\underline{z}_{h_0}$ is a non zero polynomial in nk variables of total degree $\omega(\Sigma_{h_0})$ which vanishes on $\Sigma_1 \times \cdots \times \Sigma_k$.

We prove the upper bound

$$\omega(\Sigma_1 \times \cdots \times \Sigma_k) \geq \min_{1 \leq h \leq k} \omega(\Sigma_h).$$

by induction on k. For $k = 1$ there is nothing to prove. Let $k \geq 2$. Assume that the result is true for $k - 1$. Let $P \in \mathbf{C}[\underline{z}_1, \ldots, \underline{z}_k]$ be a polynomial of total degree $< \min_{1 \leq h \leq k} \omega(\Sigma_h)$ which vanishes on $\Sigma_1 \times \cdots \times \Sigma_k$. For each $\underline{z} \in \mathbf{C}^n$, the polynomial $P_{\underline{z}} = P(\underline{z}_1, \ldots, \underline{z}_{k-1}, \underline{z}) \in \mathbf{C}[\underline{z}_1, \ldots, \underline{z}_{k-1}]$ has total degree $< \min_{1 \leq h \leq k-1} \omega(\Sigma_h)$ and vanishes on $\Sigma_1 \times \cdots \times \Sigma_{k-1}$. By the induction hypothesis $P_{\underline{z}} = 0$. Hence for each $(\underline{z}_1, \ldots, \underline{z}_{k-1}) \in \mathbf{C}^{n(k-1)}$ the polynomial $P(\underline{z}_1, \ldots, \underline{z}_{k-1}, \underline{z}) \in \mathbf{C}[\underline{z}]$ has total degree $< \omega(\Sigma_k)$ and vanishes on Σ_k. From the definition of $\omega(\Sigma_k)$ it follows that this polynomial is 0, hence $P = 0$. □

PROPOSITION 3.8. *Let $X = \mathbf{Z}\underline{x}_1 + \cdots + \mathbf{Z}\underline{x}_d$ and $Y = \mathbf{Z}\underline{y}_1 + \cdots + \mathbf{Z}\underline{y}_\ell$ be two finitely generated subgroups of $\mathbf{C}^n$ of rank d and ℓ respectively such that (X, Y) satisfies the condition* (Z.E.). *Let k be a positive integer. Define X^k and Y^k in $\mathbf{C}^{nk}$ by*

$$X^k = \sum_{h=1}^{k} \sum_{i=1}^{d} \mathbf{Z}\underline{x}_{hi} \quad \text{and} \quad Y^k = \sum_{h=1}^{k} \sum_{j=1}^{\ell} \mathbf{Z}\underline{y}_{hj}$$

where, for $1 \leq h \leq k$,

$$\underline{x}_{hi} = (\delta_{h1}\underline{x}_i, \ldots, \delta_{hk}\underline{x}_i) \in \mathbf{C}^{nk} \qquad (1 \leq i \leq d)$$

and

$$\underline{y}_{hj} = (\delta_{h1}\underline{y}_j, \ldots, \delta_{hk}\underline{y}_j) \in \mathbf{C}^{nk} \qquad (1 \leq j \leq \ell),$$

($\delta_{h,m}$ is Kronecker's symbol). Then (X^k, Y^k) satisfies (Z.E.).

PROOF. This is a consequence of Lemma 3.7. □

3.3. Proof of Theorem 3.1. We proceed in four steps.

• Step 0: *Data* We start with two finitely generated subgroups $X = \mathbf{Z}\underline{x}_1 + \cdots + \mathbf{Z}\underline{x}_d$ and $Y = \mathbf{Z}\underline{y}_1 + \cdots + \mathbf{Z}\underline{y}_\ell$ of $\mathbf{C}^n$, of rank d and ℓ respectively, such that (X, Y) satisfies `(Z.E.)`. We denote by $\{\theta_1, \ldots, \theta_q\}$ the set $\{e^{\underline{x}_i \underline{y}_j}\,;\, 1 \le i \le d,\ 1 \le j \le \ell\}$, by K the field $\mathbf{Q}(\theta_1, \ldots, \theta_q)$ and by t be the transcendence degree of K over $\mathbf{Q}$.

Define

$$\eta = \frac{d\ell + \ell + d}{(n+1)(\ell + d)}.$$

Our first goal is to check

$$t \ge \eta - 1.$$

We shall prove that the inequality $t > a - 1$ holds for any $a < \eta$, which will yield the desired conclusion. There is obviously no loss of generality to assume $\eta > 1$, that is $d\ell > n(\ell + d)$.

• Step 1: *Choice of parameters*

We fix $1 \le a < \eta$ and we denote by c_1, c_2, c_3 suitable positive real numbers. An admissible choice is to select first a “large constant” c_0 (sufficiently large with respect to the previous data), and then to define $c_1 = 1$, $c_2 = 1/c_0$, $c_3 = 1/c_0^d$.

Next let N be a sufficiently large positive integer (that is, large with respect to c_0). Define

$$R = \left[c_1 N^{d/(\ell+d)}\right], \quad L = \left[c_2 N^{\ell/(\ell+d)}\right], \quad U = \left[c_3 N^{\eta}\right].$$

• Step 2: *Construction of an auxiliary function*

We show that there exists a non zero polynomial $P \in \mathbf{Z}[X_1, \ldots, X_d]$ of degree $\le L$ in each X_i $\quad (1 \le i \le d)$ and usual height $\le e^{N/2}$, such that the exponential sum in n variables

$$F(\underline{z}) = P(e^{\underline{x}_1 \underline{z}}, \ldots, e^{\underline{x}_d \underline{z}}) \qquad (\underline{z} \in \mathbf{C}^n)$$

satisfies

$$|F(r_1 \underline{y}_1 + \cdots + r_\ell \underline{y}_\ell)| \le e^{-U}$$

for all $(r_1, \ldots, r_\ell) \in \mathbf{Z}^\ell$ with $0 \le r_j < R \quad (1 \le j \le \ell)$.

The construction of this auxiliary function rests on Proposition 3.3: set

$$r = R(|\underline{y}_1| + \cdots + |\underline{y}_\ell|), \qquad \Delta = N$$

and

$$\{\varphi_1, \ldots, \varphi_M\} = \{e^{(\lambda_1 \underline{x}_1 + \cdots + \lambda_d \underline{x}_d)\underline{z}}\,;\, 0 \le \lambda_i < L \ (1 \le i \le d)\},$$

so that $M = L^d$. The main condition (3.2) follows from the bound

$$(8U)^{n+1} \le L^d N/2,$$

which holds as soon as

$$2(8c_3)^{n+1} < c_2^d.$$

Notice also that the inequality

$$\sum_{\mu=1}^{L} |\varphi_\mu|_{er} < e^U$$

is satisfied, since the condition $\eta > 1$ implies

$$\log L + c_4 LR < U,$$

where

$$c_4 = e(|\underline{x}_1| + \cdots + |\underline{x}_d|)(|\underline{y}_1| + \cdots + |\underline{y}_\ell|).$$

• Step 3: *Using the criterion for algebraic independence and the zero estimate*

For each $\underline{r} \in \mathbf{Z}^\ell$ with $0 \le r_j < R \quad (1 \le j \le \ell)$ we define

$$Q_{N\underline{r}}(X_{11}, \dots, X_{d\ell}) = P\left(\prod_{j=1}^{\ell} X_{1j}^{r_j}, \dots, \prod_{j=1}^{\ell} X_{dj}^{r_j}\right).$$

This is a polynomial in $d\ell$ variables, with rational integer coefficients, total degree $\le LR$ and usual height $\le e^{N/2}$, which satisfies

$$Q_{N\underline{r}}(e^{\underline{x}_1\underline{y}_1}, \dots, e^{\underline{x}_d\underline{y}_\ell}) = F(r_1\underline{y}_1 + \cdots + r_\ell\underline{y}_\ell).$$

This number can be written as

$$Q_{N\underline{r}}(e^{\underline{x}_1\underline{y}_1}, \dots, e^{\underline{x}_d\underline{y}_\ell}) = \widetilde{Q}_{N\underline{r}}(\underline{\theta}),$$

for some polynomial $\widetilde{Q}_{N\underline{r}} \in \mathbf{Z}[X_1, \dots, X_r]$ of total degree $\le N$ and usual height $\le e^N$. We are going to use Proposition 3.2 with $n = d\ell$ for these polynomials and the point $\underline{\theta} = (\theta_1, \dots, \theta_q)$. The required upper bound (3.1), which reads

$$\max_{\underline{r}} |\widetilde{Q}_{N\underline{r}}(\underline{\theta})| \le e^{-CN^a},$$

where C is the constant in Proposition 3.2 associated to $\underline{\theta}$ and a, follows from step 2.

We now check that the polynomials $Q_{N\underline{r}}$ have no common zero in the domain

$$\left\{\underline{\mu} = (\mu_1, \dots, \mu_q) \in \mathbf{C}^q \,;\, \max_{1\le h\le q} |\theta_h - \mu_h| \le e^{-3CN^a}\right\}.$$

Choose $\underline{\mu} = (\mu_1, \dots, \mu_q) \in \mathbf{C}^q$ with

$$\max_{1\le h\le q} |\theta_h - \mu_h| \le e^{-3CN^a}.$$

Since c_1 and c_2 satisfy

$$c_1^\ell > d^{\ell+d}c_2^d,$$

we have

$$R^\ell > d^{\ell+d}L^d,$$

and the total degree of P is $\le dL < (R/d)^{\ell/d}$. From condition `(Z.E.)` we deduce that there exists $\underline{r} \in \mathbf{Z}^\ell$ with $0 \le r_j < R \quad (1 \le j \le \ell)$ such that, if we set

$$\mu_{i\underline{r}} = \prod_{j=1}^{\ell} \mu_{ij}^{r_j} \qquad (1 \le i \le d),$$

then the number

$$\widetilde{Q}_{N\underline{r}}(\underline{\theta}) = P(\mu_{1\underline{r}}, \dots, \mu_{d\underline{r}})$$

does not vanish.

• Step 4: *Conclusion of the proof*

Since we have checked all hypotheses of Proposition 3.2, we deduce

$$t \ge \frac{d\ell + \ell + d}{(n+1)(\ell+d)} - 1.$$

As explained before, using redundant variables together with Proposition 3.8, we obtain

$$t \ge \frac{d\ell}{n(\ell+d)} - 1.$$

We apply this result to the case $n = 1$, writing x for $\underline{x}$ and y for $\underline{y}$. In order to complete the proof of Theorem 3.1, we take $\ell = d$,

$$x_i = \beta^{i-1} \quad (1 \leq i \leq d), \qquad y_j = \beta^{j-1} \log \alpha \quad (1 \leq j \leq \ell),$$

so that

$$e^{x_i y_j} = \alpha^{\beta^{i+j-2}} \qquad \text{for } 1 \leq i \leq d \text{ and } 1 \leq j \leq \ell.$$

In this case `(T.H.)` for $x_1, \ldots, x_d$ as well as for $y_1, \ldots, y_\ell$ is satisfied by Liouville's inequality: there exists $L_0 > 0$ so that, for any $(\lambda_1, \ldots, \lambda_d) \in \mathbf{Z}^d \setminus \{0\}$, we have

$$|\lambda_1 + \lambda_2 \beta + \cdots + \lambda_d \beta^{d-1}| \geq L^{-c} \geq e^{-L^\epsilon},$$

where L stands for $\max\{|\lambda_1|, \ldots, |\lambda_d|, L_0\}$. $\square$