On the approximation of the values of exponential function and logarithm by algebraic numbers

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Dedicated to the memory of Professor N.I. Feldman

§1. Introduction

According to the theorem of Lindemann for any non-zero number θ both of numbers θ and e^θ can not be algebraic. For any algebraic numbers α and β the expression $|e^\theta-\alpha|+|\theta-\beta|$ does not vanish. How small can this expression be? The answer should obviously depend on the three following parameters: the heights $h(\alpha)$, $h(\beta)$ of the algebraic numbers, the degree of the number field $\mathbb{D}=\mathbb{Q}(\alpha,\beta)$. Here, we denote by $h(\alpha)$ the *absolute logarithmic Weil height* of α: when the minimal polynomial of α is $P(x)=a_0x^d+\cdots+a_d\in\mathbb{Z}[x]$, and its complex conjugates are $\alpha_1,\dots,\alpha_d$:

$$a_0x^d+\cdots+a_d=a_0(x-\alpha_1)\cdots(x-\alpha_d),$$

then the absolute logarithmic Weil height $h(\alpha)$ of α is defined by

$$h(\alpha)=\frac{1}{d}\left(\log|a_0|+\sum_{i=1}^{d}\log\max\{1,|\alpha_i|\}\right)$$

while

$$L(\alpha)=L(P)=\sum_{i=0}^{d}|a_i|$$

is the length of the number α and of the polynomial P. It is possible to prove that

$$h(\alpha)\le d^{-1}\cdot\log L(\alpha) \tag{1.1}$$

(see [F 1982], Lemma 8.2).

In the next Theorem it will be more convenient to have a parameter E, which will be choosen separately in each special situation.

Theorem 1 (Main Theorem). *Let $\theta\in\mathbb{C}$, $\theta\ne0$, and α,β be algebraic numbers; define $\mathbb{K}=\mathbb{Q}(\alpha,\beta)$ and $D=[\mathbb{K}:\mathbb{Q}]$. Let A, B and E be positive real numbers with $E\ge e$ satisfying*

$$\log A\ge\max\bigl(h(\alpha),D^{-1}\bigr),\quad \log B\ge h(\beta).$$

Then

$$|e^\theta-\alpha|+|\theta-\beta|\ge\exp\Bigl(-211D\bigl(\log B+\log\log A+4\log D+2\log(E|\theta|_+)+10\bigr)$$
$$\cdot\bigl(D\log A+2E|\theta|+6\log E\bigr)\cdot\bigl(3.3D\log(D+2)+\log E\bigr)\cdot(\log E)^{-2}\Bigr),$$

where $|\theta|_+=\max(1,|\theta|)$.

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From the inequality of our Main Theorem we deduce transcendence measures for several numbers: π, $\log 2$, e and more generally $\log\alpha$ and e^β (for algebraic numbers α and β, $\alpha\ne1$, $\beta\ne0$). A transcendence measure of a transcendental complex number θ is a lower bound for $|P(\theta)|$, when $P\in\mathbb{Z}[x]$ is a non-zero polynomial, in terms of the degree of P and of the length of P. For deducing the estimates of the measure of transcendence we need the following assertion, connecting the measure of transcendence and the measure of approximation by algebraic numbers.

Lemma 1. *Let $\theta\in\mathbb{C}$. Assume that for any algebraic number ξ with $\deg\xi=d$ and $L(\xi)=L$, the inequality*

$$|\theta-\xi|\ge e^{-d\varphi(d,L)}$$

holds, where $\varphi(x,y)$ is an increasing function of all arguments. Then for any non-zero polynomial $P\in\mathbb{Z}[x]$ with $\deg P=N$ and $L(P)=M$, we have

$$|P(\theta)|\ge e^{-d\varphi(N,2^NM)}\cdot\bigl(4L\sqrt{N}\bigr)^{-N}.$$

Proof. See, for example, [F 1982], Lemma 3.7.

Theorem 2. 1) *Let ξ be a real algebraic number, $d=\deg\xi$, $L(\xi)\le L$, $L\ge3$. Then*

$$|\pi-\xi|\ge\exp\bigl\{-1.2\cdot10^6d\cdot(\log L+d\log d)\cdot(1+\log d)\bigr\}.$$

2) *If $P\in\mathbb{Z}[x]$, $P\ne0$, $\deg P\le d$, $L(P)\le L$, and $L\ge3$, then*

$$\bigl|P(\pi)\bigr|\ge\exp\bigl\{-2\cdot10^6d\cdot(\log L+d\log d)\cdot(1+\log d)\bigr\}.$$

For the proof of the first assertion we choose $\theta=\pi i$, $\alpha=-1$, $\beta=i\xi$, $E=e^2$, $\log A=D^{-1}$, $\log B=h(\xi)=h(\beta)$ and note that $D\le2d$. Since

$$6.6d\log(2d+2)+\log E<11.2d(1+\log d),$$
$$d\bigl(h(\xi)+3\log(2d)+2\log\pi+14\bigr)\le17(\log L+d\log d),$$
$$1+2E|\theta|+6\log E\le59.5$$

we derive the assertion 1).

The second assertion follows from the first one and Lemma 1.

By the same way can be proved

Theorem 3. 1) *Let ξ be a real algebraic number with $d=\deg\xi$, $L(\xi)\le L$ and $L\ge3$. Then*

$$|\log2-\xi|\ge\exp\bigl\{-151000\cdot d^2\cdot(\log L+d\log d)\cdot(1+\log d)^{-1}\bigr\}.$$

2) *If $P\in\mathbb{Z}[x]$, $P\ne0$, $\deg P\le d$, $L(P)\le L$, and $L\ge3$, then*

$$\bigl|P(\log2)\bigr|\ge\exp\bigl\{-2.6\cdot10^5d^2\cdot(\log L+d\log d)\cdot(1+\log d)^{-1}\bigr\}.$$

For the proof of the first assertion we choose $\theta = \log 2$, $\alpha = 2$, $\beta = \xi$, $E = eD$, $A = e$, $\log B = h(\beta)$. In this case $D = d$. We deduce

$$d\big(h(\xi) + 4\log d + 12\big) \le 13(\log L + d\log d),$$
$$3.3d\log(d+2) + \log(ed) < 5d(1+\log d),$$
$$d + 2E|\theta| + 6\log E \le 11d.$$

Therefore the first inequality of the Theorem 3 holds. The second one follows from the first and Lemma 1.

Theorem 4. 1) *Let ξ be a real algebraic number with $d = \deg\xi$, $L(\xi) \le L$, $L \ge 3$. Then*

$$|e - \xi| \ge \exp\big\{-76000 \cdot d^2 \cdot (\log L + d)\big\}.$$

2) *If $P \in \mathbb{Z}[x]$, $P \ne 0$, $\deg P \le d$, $L(P) \le L$, and $L \ge 3$, then*

$$\big|P(e)\big| \ge \exp\big\{-1.3 \cdot 10^5 \cdot d^2 \cdot (\log L + d)\big\}.$$

For the proof of this theorem we take $\theta = 1$, $\alpha = \xi$, $\beta = 1$, $\log A = 1 + d^{-1}\log L$, $B = 1$, $E = ed\log A$, $D = d$. The desired estimates follow from the inequalities

$$3\log\log A + 6\log d + 12 \le 9(1 + \log D + \log\log A) = 9\log E,$$
$$3.3d\log(d+2) + \log E \le \tfrac{10}{3}d\log E,$$
$$d\log A + 2E|\theta| + 6\log E \le 12(d + \log L).$$

Taking $\theta = \beta$ or $\theta = \log\alpha$ for any determination of the logarithm of α we can prove a lower bound for $|e^\beta - \alpha|$ and $|\log\alpha - \beta|$.

Theorem 5. *Let α and β be algebraic numbers; define $\mathbb{K} = \mathbb{Q}(\alpha,\beta)$ and $D = [\mathbb{K} : \mathbb{Q}]$. Let A and E be positive real numbers satisfying $E \ge e$ and*

$$\log A \ge \max\big(h(\alpha)\,,\ D^{-1}\log E\,,\ D^{-1}|\beta|E\big).$$

1) If $\beta \ne 0$, then

$$|e^\beta - \alpha| \ge \exp\Big(-105500 \cdot D^2\log A \cdot \big(h(\beta) + \log_+\log A + \log D + \log E\big) \cdot \big(D\log D + \log E\big) \cdot (\log E)^{-2}\Big),$$

where $\log_+ x = \log\max(1,x)$.

2) If $\alpha \ne 0$, and if $\log\alpha$ is any non-zero determination of the logarithm of α, then

$$|\beta - \log\alpha| \ge \exp\Big(-105500 \cdot D^2\log A \cdot \big(h(\beta) + \log_+\log A + \log D + \log E\big) \cdot \big(D\log D + \log E\big) \cdot (\log E)^{-2}\Big).$$

1) For the proof of the first assertion we choose $\theta = \beta$ and we use the estimates

$$h(\beta) + \log\log A + 4\log D + 2\log(E|\beta|_+) + 10 \le 12\big(h(\beta) + \log_+\log A + \log D + \log E\big),$$

$$D\log A + 2E|\beta| + 6\log E \le 9D\log A$$

and

$$9 \cdot 12\big(3.3D\log(D+2) + \log E\big) \le 500(D\log D + \log E).$$

2) The proof of the second assertion is essentially the same, with the choice $\theta = \log\alpha$, using the estimate $|\theta| \le |\beta| + |\beta - \theta|$.

Theorem 6. 1) *Let α be an algebraic number, $\alpha \ne 0, 1$. Then there exists a constant $\gamma_1 > 0$, depending only on α and the determination of the logarithm of α such that if $P \in \mathbb{Z}[x]$, $P \not\equiv 0$, $\deg P \le d$, $L(P) \le L$, then*

$$\big|P(\log\alpha)\big| \ge \exp\big\{-\gamma_1 d^2 \cdot (\log L + d\log d) \cdot (1 + \log d)^{-1}\big\}.$$

2) *Let β be an algebraic number, $\beta \ne 0$. Then there exists a constant γ_2, depending only on β, such that if $P \in \mathbb{Z}[x]$, $P \not\equiv 0$, $\deg P \le d$, $L(P) \le L$, then*

$$\big|P(e^\beta)\big| \ge \exp\big\{-\gamma_2 d^2 \cdot (\log L + d)\big\}.$$

Theorem 6 follows from Theorem 5 with the help of Lemma 1.

There are plenty of results like our Theorems 2–6. The first transcendence measure for the number e goes back to Borel in 1899 [Bo 1899]. Early results on this subject, including works by Popken (1929) and Mahler (1932), are quoted in [FS 1967]. We point out here that, without explicit computation of the constants in the bounds, Theorem 2 was proved for the first time by N.I. Feldman, [F 1951, 1960] and Theorem 6 by P.L. Cijsouw [C 1974]. The main theorem of [D 1993] provides a lower bound for $|e^\beta - \alpha|$; the conclusion is that either the estimate of our theorem 5 holds with the constant 105500 replaced by 10^{11}, or else

$$|e^\beta - \alpha| \ge e^{-10^{11} dDh(\beta)} \quad \text{with} \quad d = [\mathbb{Q}(\beta) : \mathbb{Q}].$$

Further references are given in [W 1978] and [D 1993], as well as in Feld'man's papers which are listed below.

For the proof of the Main Theorem we use M. Laurent's method of interpolation determinants, which enables us to avoid the construction of the auxiliary function and also to avoid the extrapolation, to derive good constants in lower bounds. The organization of this paper is as follows: in Section 2 we prove a variant of the zero estimate of [LMN 1993]; Section 3 is devoted to analytic estimates for Laurent's interpolation determinants. An important tool in our proof is the use of binomial polynomials §4. Next, in §5, we provide an arithmetic lower bound for non-zero algebraic numbers (Liouville's inequality). The proof of the Main Theorem is completed in Section 6.

§2. Multiplicity estimate

The proof of Hermite-Lindemann Theorem involves the complex analytic functions z and $e^{\beta z}$; for $P \in \mathbb{C}[X,Y]$, the derivative $(d/dz)F$ of the function

$$F(z) = P(z, e^{\beta z})$$

is a polynomial in z and $e^{\beta z}$, which we call δP:

$$(d/dz)P(z, e^z) = \delta P(z, e^z).$$

It is plain that δ is the derivative operator $\frac{\partial}{\partial X} + \beta Y \frac{\partial}{\partial Y}$. Hence we can define δ on $\mathbb{K}[X,Y]$ by

$$\delta = \frac{\partial}{\partial X} + \beta Y \frac{\partial}{\partial Y},$$

when $\mathbb{K}$ is any field containing β. In this paper we work with a field $\mathbb{K}$ of zero characteristic.

Here is our multiplicity estimate.

Lemma 2. *Let $\mathbb{K}$ be a field of zero characteristic, β a non-zero element of $\mathbb{K}$, and let D_0, D_1, S and M be positive integers satisfying*

$$SM > (D_0 + M)(D_1 + 1). \tag{2.1}$$

Let $(\xi_1, \eta_1), \ldots, (\xi_M, \eta_M)$ be elements in $\mathbb{K} \times \mathbb{K}^*$ with $\xi_1, \ldots, \xi_M$ pairwise distinct. Then there is no non-zero polynomial $P \in \mathbb{K}[X,Y]$, of degree $\le D_0$ in X and of degree $\le D_1$ in Y which satisfies

$$\delta^\sigma P(\xi_\mu, \eta_\mu) = 0 \quad \text{for } 1 \le \mu \le M \text{ and } 0 \le \sigma < S. \tag{2.2}$$

The proof is essentially the same as the proof of the zero estimate in [LMN 1995]: we shall eliminate Y using $D_1 + 1$ derivatives, and get a polynomial in X which vanishes at ξ_j with multiplicity at least $S - D_1$.

Proof. Let us suppose that a polynomial P satisfies all the conditions of the lemma, equalities (2.2) and $P \ne 0$. We assume, as we may without loss of generality, that Y does not divide the polynomial P, and also that P has degree ≥ 1 with respect to Y. Let us define the numbers $k_0 = 0 < k_1 < \ldots < k_n \le D_1$ by the conditions

$$P(X,Y) = \sum_{i=0}^{n} Q_i(X) Y^{k_i},$$
$$Q_i(X) = b_i X^{m_i} + \cdots \in \mathbb{K}[X], \quad b_i \ne 0, \quad i = 0, \ldots, n.$$

For $0 \le \sigma \le n$, we consider the polynomials

$$\delta^\sigma P(X,Y) = \sum_{i=0}^{n} Q_{\sigma i}(X) \cdot Y^{k_i}, \tag{2.3}$$

where

$$Q_{\sigma i}(X) = \sum_{j=0}^{\sigma} \binom{\sigma}{j} Q_i^{(\sigma - j)}(X)(\beta k_i)^j = b_i (\beta k_i)^\sigma \cdot X^{m_i} + \cdots.$$

It follows from this representation that the determinant

$$\begin{aligned}\Delta(X) &= \det\big(Q_{\sigma i}(X)\big)_{0 \le i, \sigma \le n} = \det\big(b_i(\beta k_i)^\sigma \cdot X^{m_i} + \cdots\big)_{0 \le i, \sigma \le n} \\ &= b_0 \ldots b_n \beta^{n(n+1)/2} \cdot B \cdot X^{m_0 + \cdots + m_n} + \cdots,\end{aligned}$$

where B is a Vandermonde determinant constructed from the numbers $k_0, \ldots, k_n$, hence $B \ne 0$. Now from (2.3) we derive

$$\Delta(X) = \sum_{\sigma=0}^{n} \Delta_\sigma(X,Y) \cdot \delta^\sigma P(X,Y), \quad \Delta_\sigma(X,Y) \in \mathbb{K}[X,Y],$$

and for any $\tau \in \mathbb{Z}$, $0 \le \tau < S - n$, with some $c_{\tau,j,\sigma} \in \mathbb{K}$,

$$\Delta^{(\tau)}(\xi_j) = \sum_{\sigma=0}^{n+\tau} c_{\tau,j,\sigma} \cdot \delta^\sigma P(\xi_j, \eta_j) = 0, \quad j = 1, \ldots, M.$$

Since $n \le D_1$ and $\deg \Delta(X) = m_0 + \cdots + m_n \le (n+1)D_0 \le D_0(D_1+1)$, we deduce

$$(S - n)M \le \deg \Delta(X) \le D_0(D_1 + 1),$$

and $SM \le D_0(D_1+1) + nM \le (D_0 + M)(D_1 + 1)$. This contradicts to the condition (2.1) and completes the proof of lemma 2.

§3. Analytic upper bound

We prove an upper bound for the absolute value of some interpolation determinants; this estimate is a variant of some of Laurent's results in [L 1989] and [L 1993].

Lemma 3. *Let L be a positive integer, E, M, S, and ε be positive real numbers with*

$$0 < \varepsilon < E^{-L}.$$

For $1 \le \lambda \le L$, let $b_{\lambda 1}, \ldots, b_{\lambda L}$ be complex numbers, $\varphi_\lambda(z)$ be a complex integral functions of one variable; further, for $1 \le \mu \le L$, let ζ_μ be a complex number and σ_μ be a non-negative integer, $0 \le \sigma_\mu \le S$. Assume that for $1 \le \lambda \le L$ and $1 \le \mu \le L$ we have

$$\log |b_{\lambda\mu}| \le M, \quad \log \max_{z \le E} \big|\varphi_\lambda^{(\sigma_\mu)}(z\zeta_\mu)\big| \le M.$$

Then the logarithm of the absolute value of the determinant

$$\mathcal{D} = \det\big\|\varphi_\lambda^{(\sigma_\mu)}(\zeta_\mu) + \varepsilon b_{\lambda\mu}\big\|_{1 \le \lambda, \mu \le L}$$

is bounded by

$$L^{-1} \cdot \log |\mathcal{D}| \le -\frac{L}{2} \cdot \log E + M + S \log E + \log(2LE).$$

Proof. Let us define

$$a_{\lambda\mu}(z) = \varphi_\lambda^{(\sigma_\mu)}(z\zeta_\mu) \quad \text{and} \quad \mathcal{D}(z) = \det\big\|a_{\lambda\mu}(z) + \varepsilon b_{\lambda\mu}\big\|_{1\le\lambda,\mu\le L}.$$

Then

$$\mathcal{D}(z) = \sum_{I\subset\{1,\ldots,L\}} \varepsilon^{L-|I|}\cdot \mathcal{D}_I(z), \tag{3.1}$$

where

$$\mathcal{D}_I(z) = \det\big\|c_{\lambda\mu}(z)\big\| \quad \text{and} \quad c_{\lambda\mu}(z) = \begin{cases} a_{\lambda\mu}(z), & \text{if } \lambda\in I,\\ b_{\lambda\mu}, & \text{if } \lambda\notin I.\end{cases}$$

We claim that the function of one variable $\mathcal{D}_I(z)$ has a zero at the origin of multiplicity

$$\ge \frac{|I|\cdot(|I|-1)}{2} - \sigma_1 - \ldots - \sigma_L.$$

The determinant $\mathcal{D}_I(z)$ is a linear combination with constant coefficients of determinants $\mathcal{D}_{I,J}(z) = \det\|a_{\lambda\mu}(z)\|_{\lambda\in I,\mu\in J}$, where J runs all subsets of $\{1,\ldots,L\}$ with condition $|J|=|I|$. For the proof of our claim it is sufficient to prove the inequality

$$\operatorname{ord}\mathcal{D}_{I,J}(z) \ge \frac{|I|\cdot(|I|-1)}{2} - \sigma_1 - \ldots - \sigma_L.$$

By multilinearity we reduce the proof of this last inequality to the special case $\varphi_\lambda(z) = z^{n_\lambda}$ for some $n_\lambda\in\mathbb{N}$, $\lambda\in I$. In this special case

$$\begin{aligned}\mathcal{D}_{I,J}(z) &= \det\left(\binom{n_\lambda}{\sigma_\mu}\cdot\sigma_\mu!\cdot(z\zeta_\mu)^{n_\lambda-\sigma_\mu}\right)_{\lambda\in I,\mu\in J}\\ &= z^{\sum_{\lambda\in I} n_\lambda - \sum_{\mu\in J}\sigma_\mu}\cdot\det\left(\binom{n_\lambda}{\sigma_\mu}\cdot\sigma_\mu!\cdot\zeta_\mu^{n_\lambda-\sigma_\mu}\right)_{\lambda\in I,\mu\in J},\end{aligned}$$

where the binomial coefficient $\binom{n_\lambda}{\sigma_\mu}$ means 0 if $\sigma_\mu > n_\lambda$. If the right hand side is not identically zero, then the numbers n_λ, $\lambda\in I$, are pairwise distinct, and then the right hand side has a zero at the origin of multiplicity

$$\ge \frac{|I|\cdot(|I|-1)}{2} - \sum_{\mu\in J}\sigma_\mu \ge \frac{|I|\cdot(|I|-1)}{2} - \sigma_1 - \ldots - \sigma_L.$$

Our claim on the order of vanishing of $\mathcal{D}_I(z)$ at the origin easily follows.

By means of the Schwarz lemma we conclude

$$\begin{aligned}\log|\mathcal{D}_I(1)| &\le -\left(\frac{|I|\cdot(|I|-1)}{2} - \sigma_1 - \ldots - \sigma_L\right)\log E + \log\max_{z\le E}\big|\mathcal{D}_I(z)\big|\\ &\le -\left(\frac{|I|\cdot(|I|-1)}{2} - \sigma_1 - \ldots - \sigma_L\right)\log E + L\log L + M|I| + M(L-|I|)\\ &\le -\frac{|I|^2}{2}\cdot\log E + \frac{|I|}{2}\log E + ML + L\log L + SL\cdot\log E.\end{aligned}$$

We derive now from (3.1)

$$\begin{aligned}\log|\mathcal{D}| = \log|\mathcal{D}(1)| &\le L\log 2 - L^2\log E + ML + L\log L + SL\cdot\log E\\ &\quad + \max_{I\subset\{1,\ldots,L\}}\left(-\frac{|I|^2}{2}\cdot\log E + \big(L+\tfrac12\big)(\log E)\cdot|I|\right).\end{aligned}$$

The polynomial

$$-\frac{\log E}{2}\cdot t^2 + \big(L+\tfrac12\big)(\log E)\cdot t$$

is an increasing function in the interval $1\le t\le L$. Then we have

$$\begin{aligned}L^{-1}\cdot\log|\mathcal{D}| &\le -\frac{L}{2}\log E + \big(L+\tfrac12\big)\log E + \log(2L) - L\log E + M + S\log E\\ &< -\frac{L}{2}\cdot\log E + M + S\log E + \log(2LE).\end{aligned}$$

This completes the proof of Lemma 3.

§4. Binomial polynomials

When N, H be a non-negative integers, and z a complex number, let us define $\Delta(z,0,H) = 1$, and

$$\Delta(z,N,H) = \left(\frac{z(z+1)\cdots(z+H-1)}{H!}\right)^q\cdot\frac{z(z+1)\cdots(z+r-1)}{r!}, \tag{4.1}$$

where

$$N = qH + r, \quad 1\le r\le H.$$

For u a non-negative integer, we write $\Delta^{(u)}(z,N,H)$ for the derivative $(d/dz)^u\Delta(z,N,H)$.

The first idea of eliminating the factorials from the derivatives of auxiliary functions with the help of such polynomials was introduced (in the case $H=N$) by Feldman in [F 1960, a,b] for the improvement of estimates of the measure of transcendence of π and logarithms of algebraic numbers. Later ([F 1968]) this was one of his key tools in order to achieve a best possible dependence of the estimate in terms of the heights of the coefficients β_i in lower bounds for linear combinations $\beta_0+\beta_1\log\alpha_1+\cdots+\beta_n\log\alpha_n$; in turn, such an estimate has dramatic consequences, especially the first effective improvement to Liouville's inequality. The introduction of polynomials of this kind in the case $r=H$ and $H<N$ into the transcendence theory is due to A. Baker [Ba 1972], who improved in this way the dependence of lower bounds for linear forms in logarithms in terms of the heights of the α_i. The polynomials (4.1) were introduced in [M 1994] where a more general assertion than the next Lemma 4 is proved.

For each positive integer k and real number a, we denote by $\nu(k)$ the least common multiple of $1,2,\ldots,k$ and by $[a]$ the integer part of a.

Lemma 4. *Let $N\ge1$, $H\ge1$, $\sigma\ge0$ and x be integers. Define $d_\sigma = \nu(H)^\sigma$. Then*

$$d_\sigma\cdot\Delta^{(u)}(x,N,H)\in\mathbb{Z}, \quad 0\le u\le\sigma,$$

and

$$\log d_\sigma < \tfrac{107}{103}\cdot \sigma H, \tag{4.2}$$

$$\sum_{u=0}^{\sigma}\binom{\sigma}{u}\cdot\left|\Delta^{(u)}(x,N,H)\right| < \sigma^\sigma\cdot e^{N+H}\left(1+\frac{|x|}{H}\right)^N. \tag{4.3}$$

Proof. Let p be a prime number and $b_1\le b_2\le\ldots\le b_N$ be integers. For any integer $k>0$ we denote r_k the number of b_i which are multiple of p^k. Then

$$\mathrm{ord}_p(b_1\cdots b_N) = r_1 + r_2 + \cdots .$$

If we delete any u numbers from $b_1,\ldots,b_N$ and if $b_{j_1},\ldots,b_{j_{N-u}}$ denote the remaining $N-u$ numbers, we derive

$$\mathrm{ord}_p(b_{j_1},\ldots,b_{j_{N-u}}) \ge \sum_{k\ge 1}\max(r_k-u,0).$$

We define now the numbers b_j as the N factors in the product

$$\big(x(x+1)\cdots(x+H-1)\big)^q\cdot x(x+1)\cdots(x+r-1).$$

In this case

$$r_k \ge q\left[\frac{H}{p^k}\right]+\left[\frac{r}{p^k}\right],\quad k\ge 1.$$

Now from the identity

$$\Delta^{(u)}(z,N,H) = u!\cdot\Delta(z,N,H)\cdot\sum(z+j_1)^{-1}\cdots(z+j_u)^{-1}, \tag{4.4}$$

where summation is taken over all sets $\{j_1,\ldots,j_u\}$ such that the polynomial $(z+j_1)\cdots(z+j_u)$ divides $\Delta(z,N,H)$, we see that

$$\mathrm{ord}_p\Big(d_\sigma\cdot\Delta^{(u)}(x,N,H)\Big) \ge \sigma\cdot\left[\frac{\log H}{\log p}\right] - \sum_{p^k\le H}\left(q\left[\frac{H}{p^k}\right]+\left[\frac{r}{p^k}\right]\right)$$
$$+\sum_{k\ge 1}\max\left(q\left[\frac{H}{p^k}\right]+\left[\frac{r}{p^k}\right]-u,0\right) \ge \sum_{p^k\le H}\max\left(\sigma-u,\sigma-q\left[\frac{H}{p^k}\right]-\left[\frac{r}{p^k}\right]\right)\ge 0.$$

This proves the assertion $d_\sigma\cdot\Delta^{(u)}(x,N,H)\in\mathbb{Z}$, $0\le u\le\sigma$, $x\in\mathbb{Z}$.

The estimate (4.1) follows from the inequality $\log\nu(k)\le\frac{107}{103}\cdot k$ (see for instance [Y 1989] Lemma 2.3 p. 127).

By the identity (4.4) we see

$$\begin{aligned}\sum_{u=0}^{\sigma}\binom{\sigma}{u}\cdot|\Delta^{(u)}(x,N,H)| &\le \sum_{u=0}^{\sigma}\binom{\sigma}{u}\cdot\binom{N}{u}\cdot u!\cdot\big(|x|+H-1\big)^{N-u}(H!)^{-q}(r!)^{-1}\\ &\le \sigma^\sigma\cdot\sum_{u=0}^{N}\binom{N}{u}\big(|x|+H-1\big)^{N-u}(H!)^{-q}(r!)^{-1}\\ &\le \sigma^\sigma\big(|x|+H\big)^N\left(\frac{H^H}{H!}\right)^q\left(\frac{H^r}{r!}\right)H^{-N}\\ &\le \sigma^\sigma\cdot e^{N+H}\cdot\left(1+\frac{|x|}{H}\right)^N.\end{aligned}$$

This completes the proof of Lemma 4.

§5. Liouville's inequality

For the next result, we use the notion of *length* of a polynomial $f\in\mathbb{C}[X_1,\ldots,X_n]$ (see §1).

Lemma 5 (Liouville's inequality). *Let $\Bbbk$ be a subfield of $\mathbb{C}$ which is a finite extension of $\mathbb{Q}$ of degree D. Further let $\alpha_1,\ldots,\alpha_n$ be elements in $\Bbbk$. Furthermore let f be a polynomial in $\Bbbk[X_1,\ldots,X_n]$, with coefficients in $\mathbb{Z}$, of degree at most N_i with respect to X_i, and which does not vanish at the point $(\alpha_1,\ldots,\alpha_n)$. Then*

$$\log\left|f(\alpha_1,\ldots,\alpha_n)\right| \ge -(D'-1)\cdot\log L(f) - D'\sum_{i=1}^{n}N_i h(\alpha_i),$$

where

$$D' = \begin{cases} D/2 & \text{if } \Bbbk \text{ is not a real field,}\\ D & \text{if } \Bbbk \text{ is a real field.}\end{cases}$$

Proof. See [F 1982], Lemma 9.2.

§6. Proof of the Main Theorem

Let us suppose that under the conditions of the Main Theorem the inequality

$$|e^\theta-\alpha|+|\theta-\beta| < E^{-211DUVW}, \tag{6.1}$$

holds, with

$$U=\frac{3.3D\log(D+2)+\log E}{\log E},\quad V=\frac{2E|\theta|+D\log A+6\log E}{\log E},$$
$$W=\frac{\log B+\log\log A+4\log D+2\log(E|\theta|_+)+10}{\log E}.$$

Note that $U\ge 1$, $V\ge 6$, $W\ge 2$.

a) *Step one: Constuction of a non-zero determinant $\mathcal{D}$.*

The proof of the Main theorem involves complex analytic functions in one variable $\Delta(z,\tau,H)e^{\theta tz}$, for non negative integers τ, H and t; the derivative of order σ of this function at the point $s\in\mathbb{Z}$, $s\geq 0$, is

$$\gamma_{\tau t}^{\sigma s}=\left(\frac{d}{dz}\right)^{\sigma}\left(\Delta(z,\tau,H)e^{\theta tz}\right)\Big|_{z=s}=\sum_{k=0}^{\min(\tau,\sigma)}\frac{\sigma!}{(\sigma-k)!k!}\cdot\Delta^{(k)}(s,\tau,H)\cdot(t\theta)^{\sigma-k}\cdot e^{\theta ts}. \quad (6.2)$$

We choose parameters T, T_1, S, S_1 and H:

$$S=[10.5UV],\quad S_1=[12DW+0.5],$$

$$T=[20.2DVW],\quad T_1=[4.2U+0.5],\quad H=[1.5W\log E]$$

and restrict ourselves to the ranges

$$0\leq\tau\leq T,\quad |t|\leq T_1,\quad 0\leq\sigma\leq S,\quad |s|\leq S_1.$$

Replacing the numbers e^{θ} by α and θ by β in (6.2), we find an algebraic number

$$\sum_{k=0}^{\min(\tau,\sigma)}\frac{\sigma!}{(\sigma-k)!k!}\cdot\Delta^{(k)}(s,\tau,H)\cdot(t\beta)^{\sigma-k}\cdot\alpha^{ts},$$

which will be a good approximation to $\gamma_{\tau t}^{\sigma s}$. According to Lemma 4 the number

$$a_{\tau t}^{\sigma s}=d_{\sigma}\cdot\sum_{k=0}^{\min(\tau,\sigma)}\frac{\sigma!}{(\sigma-k)!k!}\cdot\Delta^{(k)}(s,\tau,H)\cdot(t\beta)^{\sigma-k}\cdot\alpha^{ts}, \quad (6.3)$$

will be an polynomial in α, α^{-1}, β with integer coefficients.

We also define $L=(T+1)(2T_1+1)$, which is the number of (τ,t).

Lemma 6. *There exists a set $\{(\sigma_\mu,s_\mu);1\leq\mu\leq L\}$ of elements in $\mathbb{Z}\times\mathbb{Z}$ with $0\leq\sigma_\mu\leq S$ and $0\leq s_\mu\leq S_1$ with the property that the determinant of the $L\times L$ matrix*

$$\mathcal{D}=\det\left\|a_{\tau}^{\sigma_\mu}{}_{t}^{s_\mu}\right\|,\quad 0\leq\tau\leq T,\quad |t|\leq T_1,\quad 1\leq\mu\leq L,$$

does not vanish.

Proof. Let $\mathbb{C}[X,Y,Y^{-1}]$ be the ring of polynomials in X, Y, Y^{-1} and let δ be the derivative operator on $\mathbb{C}[X,Y,Y^{-1}]$ defined by

$$\delta=\tfrac{\partial}{\partial X}+\beta Y\tfrac{\partial}{\partial Y}.$$

Then

$$\delta^{\sigma}\left(\Delta(X,\tau,H)Y^{t}\right)=\sum_{k=0}^{\min(\tau,\sigma)}\frac{\sigma!}{(\sigma-k)!k!}\cdot\Delta^{(k)}(X,\tau,H)\cdot(t\beta)^{\sigma-k}\cdot Y^{t}$$

and

$$a_{\tau t}^{\sigma s}=d_{\sigma}\delta^{\sigma}\left(\Delta(X,\tau,H)Y^{t}\right)\Big|_{(X,Y)=(s,\alpha^{s})}.$$

Let us suppose that the rank of the matrix

$$\|a_{\tau t}^{\sigma s}\|,\quad 0\leq\tau\leq T,\quad |t|\leq T_1,\quad 0\leq\sigma\leq S,\quad |s|\leq S_1,$$

is less than L. Then there exist complex numbers $c_{\tau t}$, $0\leq\tau\leq T$, $|t|\leq T_1$, not all zero, such that the polynomial

$$R(X,Y)=\sum_{(\tau,t)}c_{\tau t}\Delta(X,\tau,H)Y^{t}\in\mathbb{C}[X,Y,Y^{-1}]$$

is not 0 and satisfies

$$\delta^{\sigma}R(X,Y)\Big|_{(X,Y)=(s,\alpha^{s})}=0,\quad 0\leq\sigma\leq S,\quad |s|\leq S_1.$$

But this contradicts Lemma 2 with $P(X,Y)=Y^{T_1}R(X,Y)$, $D_0=T$, $D_1=2T_1$, $M=2S_1+1$, $\xi_s=s$, $\eta_s=\alpha^{s}$ and S changed to $S+1$: indeed, from the inequalities

$$2S_1+1\geq 24DW,\quad 2T_1+1\leq 10.4U,\quad S+1\geq 10.5UV,\quad T\leq 20.2DVW,$$

$V\geq 6$ and

$$\begin{aligned}\frac{(T+2S_1+1)(2T_1+1)}{(S+1)(2S_1+1)}=\frac{2T_1+1}{S+1}\cdot\left(\frac{T}{2S_1+1}+1\right)&<\frac{10.4}{10.5V}\cdot\left(\frac{20.2V}{24}+1\right)\\&\leq\frac{104}{105}\cdot\left(\frac{101}{120}+\frac{1}{6}\right)<1\end{aligned}$$

we derive $P=0$. This completes the proof of Lemma 6.

b) *upper bound for* $|\mathcal{D}|$.

We plan to use Lemma 3 with λ replaced by (τ,t), for the L functions

$$f_{\tau t}(z)=\Delta(z,\tau,H)\cdot e^{\theta tz},\quad 0\leq\tau\leq T,\quad |t|\leq T_1,$$

with the points $\zeta_\mu=s_\mu$, $1\leq\mu\leq L$, with $\varepsilon=E^{-211DUVW}$ and $b_{\tau t\mu}$, instead $b_{\lambda\mu}$ in Lemma 3, defined by

$$d_{\sigma}^{-1}\cdot a_{\tau}^{\sigma_\mu}{}_{t}^{s_\mu}=\gamma_{\tau}^{\sigma_\mu}{}_{t}^{s_\mu}+\varepsilon\cdot b_{\tau t\mu}.$$

The estimates

$$T+1\leq 20.2DVW+1\leq\left(20.2+\tfrac{1}{12}\right)DVW,\quad T_1+\tfrac{1}{2}\leq 5.2U,$$

means that

$$L=(T+1)(2T_1+1)<211DUVW \quad (6.4)$$

and $\varepsilon<E^{-L}$.

From Lemma 4 and (6.2) we deduce

$$\max_{|z|\le E}\left|f_{\tau t}^{(\sigma)}(sz)\right| \le \max_{|z|\le E}\sum_{k=0}^{\min(\tau,\sigma)}\frac{\sigma!}{(\sigma-k)!k!}\left|\Delta^{(k)}(sz,\tau,H)\right|\cdot|t\theta|^{\sigma-k}e^{|\theta zts|}$$
$$\le S^S\cdot e^{H+T}\cdot\left(1+\frac{ES_1}{H}\right)^T\cdot\big(|\theta|_+T_1\big)^S\cdot e^{|\theta|ET_1S_1}\le e^M,$$

where

$$M=S\log S+H+T+T\log\left(1+\frac{ES_1}{H}\right)+S\log\big(E|\theta|_+T_1\big)+E|\theta|S_1T_1+S_1T_1.$$

It follows from (6.1) that

$$\max\big(|\beta|,|\theta|\big)\le|\theta|_+(1+(2S)^{-1}),$$
$$\max\big(|\alpha|,|\alpha|^{-1},|e^{\theta}|,|e^{-\theta}|\big)\le e^{|\theta|}\cdot\big(1+(2S_1T_1+2)^{-1}\big),$$

and for any integer k and ℓ with $0\le k\le S$ and $|\ell|\le S_1T_1$,

$$|\beta^k\alpha^\ell-\theta^ke^{\theta\ell}|\le|\beta|^k\cdot|\alpha^\ell-e^{\theta\ell}|+|e^{\theta}|^\ell\cdot|\beta^k-\theta^k|$$
$$\le\varepsilon\cdot|\theta|_+^k\cdot e^{(|\ell|+1)|\theta|}\cdot\left(1+\frac{1}{2S}\right)^k\cdot\left(1+\frac{1}{2S_1T_1+2}\right)^{|\ell|+1}\cdot\max(|\ell|,k)$$
$$\le\varepsilon e|\theta|_+^S\cdot e^{2S_1T_1|\theta|}\cdot\max(S,S_1T_1).$$

Now we use the inequalities

$$eS\le e^S\le E^S,\quad eS_1T_1\le e^{S_1T_1}$$

and we write σ and s in place of σ_μ and s_μ. From (6.2), (6.3) and Lemma 4 we derive

$$\varepsilon\cdot|b_{\tau t\mu}|=|d_\sigma^{-1}\cdot a_{\tau t}^{\sigma s}-\gamma_{\tau t}^{\sigma s}|$$
$$\le\sum_{k=0}^{\min(\tau,\sigma)}\frac{\sigma!}{(\sigma-k)!k!}\cdot\left|\Delta^{(k)}(s,\tau,H)\right|\cdot|t|^{\sigma-k}\cdot|\beta^{\sigma-k}\alpha^{ts}-\theta^{\sigma-k}e^{\theta ts}|$$
$$\le S^S\cdot e^{H+T+1}\cdot\left(1+\frac{S_1}{H}\right)^T\cdot T_1^S\cdot|\theta|_+^Se^{2|\theta|T_1S_1}\max(S,S_1T_1)\cdot\varepsilon\le\varepsilon\cdot e^M.$$

Since $\log d_\sigma\le\frac{107}{103}SH$ we deduce from Lemma 3

$$\frac{1}{L}\cdot\log|\mathcal{D}|\le-\frac{L}{2}\cdot\log E+\tfrac{107}{103}SH+S\log S+H+T+T\log\left(1+\frac{ES_1}{H}\right)$$
$$+S\log\big(E|\theta|_+T_1\big)+|\theta|ES_1T_1+S_1T_1+S\log E+\log(2LE).\qquad(6.5)$$

c) *lower bound for* $|\mathcal{D}|$.

Let us define the polynomials

$$q_{\tau t}^{\sigma s}(X,Y)=d_\sigma\cdot\sum_{k=0}^{\min(\tau,\sigma)}\frac{\sigma!}{(\sigma-k)!k!}\cdot\Delta^{(k)}(s,\tau,H)\cdot(tX)^{\sigma-k}\cdot Y^{ts}\in\mathbb{Z}[X,Y,Y^{-1}],$$

and

$$R(X,Y)=\det\left\|q_{\tau\ t}^{\sigma_\mu s_\mu}(X,Y)\right\|,\quad 0\le\tau\le T,\quad|t|\le T_1,\quad 1\le\mu\le L.$$

It follows from (6.3) that $a_{\tau t}^{\sigma s}=q_{\tau t}^{\sigma s}(\beta,\alpha)$ and $\mathcal{D}=R(\beta,\alpha)$. From the inequalities

$$\deg_X q_{\tau t}^{\sigma s}(X,Y)\le S,\quad\deg_Y q_{\tau t}^{\sigma s}(X,Y)\le|t|S_1,\quad\deg_{Y^{-1}}q_{\tau t}^{\sigma s}(X,Y)\le|t|S_1$$

we derive

$$\deg_X R(X,Y)\le LS,$$
$$\deg_Y R(X,Y)\le\tfrac14LS_1(T_1+0.5),\quad\deg_{Y^{-1}}R(X,Y)\le\tfrac14LS_1(T_1+0.5).$$

It follows from Lemma 4 that

$$L\big(q_{\tau t}^{\sigma s}(X,Y)\big)\le\exp\big(\tfrac{107}{103}SH\big)\cdot S^S\cdot e^{H+T}\cdot\left(1+\frac{S_1}{H}\right)^T\cdot T_1^S$$

and

$$L\big(R(X,Y)\big)\le L^L\cdot\left(\exp\big(\tfrac{107}{103}SH\big)\cdot S^S\cdot e^{H+T}\cdot\left(1+\frac{S_1}{H}\right)^T\cdot T_1^S\right)^L.$$

Next from Lemma 5 we derive

$$\frac{1}{L}\cdot\log|\mathcal{D}|\ge-(D-1)\Bigg(\log L+\tfrac{107}{103}SH+S\log S+H+T+T\log\left(1+\frac{S_1}{H}\right)$$
$$+S\log T_1\Bigg)-DS\log B-\tfrac12DS_1(T_1+0.5)\log A.\qquad(6.6)$$

d) *End of the proof of Theorem 1.*

Let us compare the upper bound (6.5) for $|\mathcal{D}|$ and the lower bound (6.6). We derive

$$\frac{L}{2}\log E\le\tfrac12S_1(T_1+0.5)\Big(D\log A+2E|\theta|+2\Big)+\left(DT\log\left(1+\frac{S_1}{H}\right)+DT+T\log E\right)$$
$$+DS\Big(\log B+\log S+\log\big(E|\theta|_+T_1\big)\Big)+DH+DSH\tfrac{107}{103}+S\log E+\log(2E)+D\log L.$$

Using the definition of all parameters we deduce

$$\tfrac12S_1(T_1+0.5)\Big(D\log A+2E|\theta|+2\Big)\le0.5\cdot12.25\cdot5.2DUVW\log E=31.85DUVW\log E,\qquad(6.7)$$

$$\log\left(1+\frac{S_1}{H}\right) \le \log\left(1+12.25\frac{D}{\log E}\right) \le \log 13.25 + \log D \le 2.6 + \log D,$$

$$\begin{aligned} DT\log\left(1+\frac{S_1}{H}\right) + DT + T\log E &\le 20.2(D^2VW\log D + 3.6D^2VW + DVW\log E) \\ &\le 20.2DVW\big(\log E + 3.3D\log(D+2)\big) \le 20.2DUVW\log E, \end{aligned} \tag{6.8}$$

$$U \le 1+3.3D\log(D+2) \le 4.7D^{3/2}, \quad V \le 9.8E|\theta|_+ D\log A, \tag{6.9}$$

$$\begin{aligned} \log\big(ST_1E|\theta|_+\big) &\le \log\big(50U^2VE|\theta|_+\big) \le \log\log A + 4\log D + 2\log\big(E|\theta|_+\big) + 10, \\ DS\Big(\log B + \log S + \log\big(E|\theta|_+T_1\big)\Big) &\le 10.5DUV\Big(\log B + \log\log A + 4\log D \\ &+2\log\big(E|\theta|_+\big) + 10\Big) \le 10.5DUVW\log E \end{aligned} \tag{6.10}$$

The estimates (6.4) and (6.9) mean that

$$\begin{aligned} \log L &\le \log(211DUVW) \le 10 + 3.5\log D + \log\big(E|\theta|_+\big) + \log\log A + \log W \\ &\le W\log E + \log W \le 1.4W\log E \le 0.24UVW\log E. \end{aligned}$$

With the help of this estimate and inequalities

$$\begin{gathered} DSH \le 15.75DUVW\log E, \quad DH \le 1.5DW\log E \le 0.25DUVW\log E, \\ S\log E \le 10.5UV\log E \le 5.25DUVW\log E, \quad \log(2E) \le 2\log E \le \tfrac{1}{6}DUVW\log E, \end{gathered}$$

we derive

$$\begin{aligned} &DH + \tfrac{107}{103}DSH + S\log E + \log(2E) + D\log L \\ &\quad \le \big(15.75\cdot\tfrac{107}{103} + 5.25 + 0.49 + \tfrac{1}{6}\big)DUVW\log E < 22.28DUVW\log E. \end{aligned} \tag{6.11}$$

Finally from (6.6)–(6.11) we deduce

$$\begin{aligned} \frac{L}{2}\log E &\le (31.85 + 20.2 + 10.5 + 22.28)DUVW\log E \\ &= 84.83DUVW\log E < \frac{(T+1)(2T_1+1)}{2}\cdot\log E = \frac{L}{2}\log E. \end{aligned}$$

This contradiction means that (6.1) is wrong and completes the proof of the Theorem 1.

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