Transcendence and algebraic independence of Liouville-like numbers

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Abstract - *We investigate the arithmetic nature of complex numbers which are well approximated by algebraic numbers. In particular we construct such algebraically independent numbers.*

1 - Introduction

It has been shown recently [2], [3], [4], [5] that complex numbers which generate a field of "small" transcendence degree can be simultaneously well approximated by algebraic numbers. This enables one to produce proofs of algebraic independence: given complex numbers, if we can establish a lower bound for their approximation by algebraic numbers, then we can deduce a lower bound for the transcendence degree of the field they generate. Here is such a precise statement ([4] Théorème 3.1 and [5], Corollary 1.2).

THEOREM 1 *Let $\theta_1, \ldots, \theta_m$ be complex numbers such that*

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(\theta_1, \ldots, \theta_m) \leq 1.$$

There exists a constant $c > 0$ with the following property. For any $h \geq c$, and for infinitely many positive integers D, there exists a m-tuple $(\gamma_1, \ldots, \gamma_m)$ of algebraic numbers satisfying

$$[\mathbb{Q}(\gamma_1, \ldots, \gamma_m) : \mathbb{Q}] \leq D, \quad \max_{1\leq i\leq m} \mathrm{h}(\gamma_i) \leq h,$$

and

$$\max_{1\leq i\leq m} |\theta_i - \gamma_i| \leq e^{-D^2h/c}.$$

A natural question is whether the converse holds. Here we show that the answer is negative: our main goal is to construct algebraically independent complex numbers which admit good simultaneous approximation by algebraic numbers of bounded height. These numbers are of the form α^{ξ_i} where α is an algebraic number (in fact we choose $\alpha = 2$) and ξ_i are Liouville-type numbers, that is real numbers which have very good (simultaneous) rational approximations.

First we investigate the transcendence of such numbers. In section 2 we deduce from Liouville's inequality the transcendence of α^ξ under rather strong conditions on the approximations of ξ. Next in section 3 we obtain the transcendence of such numbers assuming only that ξ is a Liouville number; the proof rests on Baker-Fel'dman's measure of linear independence for logarithms of algebraic numbers. The main result is Theorem 2 in section 4 where we produce algebraically independent numbers having good simultaneous algebraic approximations.

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2 - Consequence of a Liouville-Type Transcendence Criterion

The *usual height* $\mathrm{H}(\gamma)$ of an algebraic number γ is the maximum absolute value of the coefficients of its minimal polynomial over $\mathbb{Z}$. For instance the usual height of a rational number p/q with $\gcd(p,q) = 1$ and $q > 0$ is

$$\mathrm{H}(p/q) = \max\{|p|, q\}.$$

PROPOSITION 1 *Let θ be a complex number. Assume that for any $\kappa > 0$, there exists an algebraic number γ of degree $[\mathbb{Q}(\gamma) : \mathbb{Q}] = d$ and usual height $\mathrm{H}(\gamma) = H$ satisfying*

$$0 < |\theta - \gamma| < \exp\Big\{-\kappa \max\{d, \log H\}\Big\}.$$

Then θ is transcendental.

The statement of Proposition 1 involves the usual height $\mathrm{H}(\gamma)$, but it is also convenient to introduce the *absolute logarithmic height* $\mathrm{h}(\gamma)$ which we

now define. Let γ be an algebraic number. Write the minimal polynomial of γ over $\mathbb{Z}$ as
$$a_0(X-\gamma_1)\cdots(X-\gamma_d)$$
where $a_0>0$ is the leading term, $d=[\mathbb{Q}(\gamma):\mathbb{Q}]$ is the degree of γ and $\gamma_1,\ldots,\gamma_d$ are the conjugates of γ. Then *Mahler's measure of* γ is defined by
$$\mathrm{M}(\gamma)=a_0\prod_{i=1}^{d}\max\{1,|\gamma_i|\}$$
and the *absolute logarithmic height* of γ is
$$\mathrm{h}(\gamma)=\frac{1}{d}\log\mathrm{M}(\gamma).$$
For instance the absolute logarithmic height of a rational number $p/q\in\mathbb{Q}$ is
$$\mathrm{h}(p/q)=\log\max\{|p|,q\}=\log\mathrm{M}(p/q)=\log\mathrm{H}(p/q).$$
In general, for an algebraic number γ of degree d, we have
$$\mathrm{H}(\gamma)\le 2^d e^{d\mathrm{h}(\gamma)}\quad\text{and}\quad e^{d\mathrm{h}(\gamma)}\le\sqrt{d+1}\mathrm{H}(\gamma).$$
Therefore one can replace the hypothesis of Proposition 1 by the following one: *for any* $\kappa>0$*, there exists an algebraic number* γ *of degree* $[\mathbb{Q}(\gamma):\mathbb{Q}]=d$ *and absolute logarithmic height* $\mathrm{h}(\gamma)=h$ *satisfying*
$$0<|\theta-\gamma|<\exp\Bigl\{-\kappa d\max\{1,h\}\Bigr\}.$$

Proof of Proposition 1. Assume θ is algebraic. Define $d_0=[\mathbb{Q}(\theta):\mathbb{Q}]$. According to Liouville's estimate (for instance [1] Chap I, § 1.7, Th. 1.5), for any algebraic number $\gamma\ne\theta$ of degree $[\mathbb{Q}(\gamma):\mathbb{Q}]=d$ and usual height $\mathrm{H}(\gamma)=H$, we have
$$|\theta-\gamma|\ge 2^{-D+1}e^{-D\mathrm{h}(\theta)}e^{-D\mathrm{h}(\gamma)}$$
where $D=[\mathbb{Q}(\gamma,\theta):\mathbb{Q}]$ is bounded by $D\le d_0d$. Since $d\mathrm{h}(\gamma)\le d+\log H$, we deduce
$$|\theta-\gamma|\ge e^{-\kappa\max\{d,\log H\}}$$
where $\kappa=d_0(3+\mathrm{h}(\theta))$. This completes the proof of Proposition 1. □

We deduce from Proposition 1 the following result:

(*) *Let* ξ *be a real number. Assume that for any* $\kappa>0$*, there exists a rational number* p/q *satisfying*
$$0<\left|\xi-\frac{p}{q}\right|<e^{-\kappa q}.$$
Then, for any non zero algebraic number α *and for any non zero logarithm* $\log\alpha$ *of* α*, the number*
$$\alpha^\xi=e^{\xi\log\alpha}$$
is transcendental.

We shall check the hypotheses of Proposition 1 for the number $\theta=\alpha^\xi$. Fix first $\kappa>0$ and then κ' with
$$\kappa'\ge 2d_0\kappa\Bigl(1+(1+|\xi|)\mathrm{h}(\alpha)\Bigr),\quad e^{\kappa'}\ge|\log\alpha|\quad\text{and}\quad e^{\kappa'/2}\ge 2|\log\alpha||\theta|,$$
where $d_0=[\mathbb{Q}(\alpha):\mathbb{Q}]$. We apply the hypothesis of (*) with κ replaced by κ': there exists a rational number p/q satisfying
$$0<\left|\xi-\frac{p}{q}\right|<e^{-\kappa' q}.$$
Define
$$\gamma=\alpha^{p/q}=\exp\{(p/q)\log\alpha\}.$$
Since $\gamma^q\in\mathbb{Q}(\alpha)$, the degree $d=[\mathbb{Q}(\gamma):\mathbb{Q}]$ of γ is bounded by $d\le qd_0$. From
$$\mathrm{h}(\gamma)=\frac{|p|}{q}\mathrm{h}(\alpha)\le(|\xi|+1)\mathrm{h}(\alpha),$$
we deduce an upper bound for the usual height $H=\mathrm{H}(\gamma)$ of γ:
$$\log H\le qd_0(\mathrm{h}(\gamma)+\log 2)\le qd_0\Bigl(1+(|\xi|+1)\mathrm{h}(\alpha)\Bigr).$$
Since $e^{\kappa'}\ge|\log\alpha|$, we can use the estimate
$$0<|e^z-1|\le 2|z|\qquad\text{for}\quad 0<|z|\le 1$$
with
$$z=(\xi-\frac{p}{q})\log\alpha$$
and get
$$0<|\theta-\gamma|\le 2|\theta\log\alpha|\cdot\left|\xi-\frac{p}{q}\right|.$$

From the choice of κ' we conclude

$$0 < |\theta - \gamma| < e^{-\kappa' q/2} \leq \exp\Big\{-\kappa \max\{d, \log H\}\Big\}.$$

Therefore the assumptions of Proposition 1 are satisfied and θ is transcendental. ∎

3 - Linear Forms in Logarithms of Liouville Numbers.

The proof of Proposition 1 was elementary, using only Liouville's inequality. We now improve (*) by means of a measure of linear independence for two logarithms of algebraic numbers.

Let us recall the definition: *A Liouville number* is a real number ξ with the following property: for any $\kappa > 0$ there exists a rational number p/q with $q \geq 2$ such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^\kappa}.$$

Any Liouville number is transcendental ([1], Chap. I, § 1.6). We want to prove:

$\binom{*}{*}$ *If ξ is a Liouville number, then for any non zero algebraic number α and any non zero logarithm of α, the number α^ξ is transcendental.*

Indeed take a non zero algebraic number α' and fix a value $\log\alpha'$ of its logarithm. For a sufficiently large κ, let p/q a rational number such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^\kappa}.$$

For sufficiently large κ we have $p\log\alpha \neq q\log\alpha'$. Indeed, this is trivial if $(\log\alpha')/(\log\alpha)$ is irrational (!), and it is also trivial if $\xi = (\log\alpha')/(\log\alpha)$ because $\xi \neq p/q$. In the remaining case where

$$\frac{\log\alpha'}{\log\alpha} = \frac{p_0}{q_0} \neq \xi$$

with $p_0/q_0 \in \mathbb{Q}$, then for $\kappa \geq \kappa_0$ with κ_0 defined by

$$\frac{1}{2^{\kappa_0}} = \left|\xi - \frac{p_0}{q_0}\right|,$$

we have $p/q \neq p_0/q_0$.

According to Th. 4.9 in [1], Chap.4, § 1, we have

$$|p\log\alpha - q\log\alpha'| \geq \max\{|p|, q\}^{-c}$$

where $c > 0$ depends only on $\log\alpha$ and $\log\alpha'$. One easily deduces $\xi\log\alpha \neq \log\alpha'$, which proves $\binom{*}{*}$.

The same argument yields further related results. Indeed, from measures of linear independence of logarithms of algebraic numbers (see for instance [6] Th. 1.1) one can deduce the transcendence of numbers of the form

$$\log(\alpha\xi), \quad e^{\beta\xi}, \quad \alpha^{\beta\xi}, \quad (\alpha\xi)^\beta, \quad \xi\log\alpha, \quad \frac{\log\alpha}{\log\xi}$$

when ξ is a Liouville number and α, β are non zero algebraic numbers with $\log\alpha \neq 0$. More generally, when $a_1, \ldots, a_n$, $b_0, b_1, \ldots, b_n$ are complex numbers which are simultaneously very well approximated by algebraic numbers, then

$$b_1\log a_1 + \cdots + b_m \log a_m \quad \text{and} \quad e^{b_0}a_1^{b_1}\cdots a_m^{b_m}$$

are transcendental.

It follows that such numbers as a^b, $\log a_1/\log a_2$ are transcendental when a, a_1, a_2 and b are Liouville-like numbers.

Here is an example:

PROPOSITION 2 *Let $\ell_1, \ldots, \ell_n$ be complex numbers such that, for $1 \leq i \leq n$, the number $\alpha_i = e^{\ell_i}$ is algebraic. Let $b_0, b_1, \ldots, b_n$ be complex numbers. Assume that for any $\kappa > 0$, there exist algebraic numbers $\beta_0, \beta_1, \ldots, \beta_n$ such that*

$$\beta_0 + \beta_1\ell_1 + \cdots + \beta_n\ell_n \neq 0$$

and

$$\sum_{j=0}^n |b_j - \beta_j| < H^{-\kappa D^{n+2}},$$

where

$$D = [\mathbb{Q}(\beta_0, \ldots, \beta_n) : \mathbb{Q}]$$

and

$$H = \max\{e, \mathrm{H}(\beta_0), \ldots, \mathrm{H}(\beta_n)\}.$$

Then

$$b_0 + b_1\ell_1 + \cdots + b_n\ell_n \neq 0.$$

Proof. According to [6] Th. 1.1, we have

$$|\beta_0+\beta_1\ell_1+\cdots+\beta_n\ell_n|\geq H^{-cD^{n+2}}$$

for some constant $c>0$ which depends only on $\ell_1,\ldots,\ell_n$. Proposition 2 easily follows. □

Example. *Let $\xi_0,\ldots,\xi_m$ be complex numbers. Assume either the number ξ_0 does not vanish, or else the numbers $1,\xi_1,\ldots,\xi_m$ are linearly independent over $\mathbb{Q}$. Let $\log\alpha_1,\ldots,\log\alpha_m$ be logarithms of non zero algebraic numbers. Assume*

$$(\log\alpha_1,\ldots,\log\alpha_m)\neq(0,\ldots,0).$$

Assume further that for any $\kappa>0$, there exists a tuple $(\gamma_0,\ldots,\gamma_m)$ of algebraic numbers such that

$$\max_{0\leq j\leq m}|\xi_j-\gamma_j|<H^{-\kappa D^{m+4}}$$

where

$$D=[\mathbb{Q}(\gamma_0,\ldots,\gamma_m):\mathbb{Q}]\quad\text{and}\quad H=\max\{e,\max_{0\leq j\leq m}\mathrm{H}(\gamma_j)\}.$$

Then the number

$$e^{\xi_0}\alpha_1^{\xi_1}\cdots\alpha_m^{\xi_m}$$

is transcendental.

Proof. Assume $e^{\xi_0}\alpha_1^{\xi_1}\cdots\alpha_m^{\xi_m}$ is algebraic. Define

$$\log\alpha_{m+1}=\xi_0+\xi_1\log\alpha_1+\cdots\xi_m\log\alpha_m.$$

Let $(\ell_1,\ldots,\ell_n)$ be any basis of the $\mathbb{Q}$-vector subspace of $\mathbb{C}$ spanned by $\log\alpha_1,\ldots,\log\alpha_{m+1}$. For $1\leq i\leq n$ and $1\leq j\leq m+1$, let $c_{ji}\in\mathbb{Q}$ be the component of $\log\alpha_j$ in the basis $(\ell_1,\ldots,\ell_n)$:

$$\log\alpha_j=\sum_{i=1}^n c_{ji}\ell_i\qquad(1\leq j\leq m+1).$$

Further, let $b_0=\xi_0$ and

$$b_i=-c_{m+1,i}+\sum_{j=1}^m c_{ji}\xi_j\quad(1\leq i\leq n).$$

From the definition of $\log\alpha_{m+1}$ we deduce

$$b_0+b_1\ell_1+\cdots+b_n\ell_n=0.$$

Define $\beta_0=\gamma_0$ and

$$\beta_i=-c_{m+1,i}+\sum_{j=1}^m c_{ji}\gamma_j\quad(1\leq i\leq n).$$

There is a constant $c>0$ which depends only on m and the c_{ij}'s such that

$$\max_{0\leq i\leq n}\mathrm{H}(\beta_i)\leq H^{cD}.$$

Indeed we have

$$\mathrm{h}(\gamma_j)\leq\frac{1}{d_j}\log\mathrm{H}(\gamma_j)+\frac{1}{2d_j}\log(d_j+1)\leq 2\log H$$

where $d_j=[\mathbb{Q}(\gamma_j):\mathbb{Q}]$, and for $1\leq i\leq n$

$$\mathrm{h}(\beta_i)\leq m\max_{1\leq j\leq m}\mathrm{h}(\gamma_j)+\log(m+1)+\sum_{j=1}^{m+1}\mathrm{h}(c_{ji}).$$

From the estimate

$$\mathrm{H}(\beta_i)\leq 2^De^{D\mathrm{h}(\beta_i)}$$

our claim on the existence of c follows.

Using Proposition 2 with H replaced by H^{cD} and m by n, we deduce that for infinitely many $\kappa>0$, the tuple $(\gamma_0,\ldots,\gamma_m)$ satisfies

$$\gamma_0+\sum_{i=1}^n\beta_i\ell_i=0.$$

Since $\ell_1,\ldots,\ell_n$ are $\mathbb{Q}$-linearly independent, Baker's Theorem implies $\beta_0=\beta_1=\cdots=\beta_n=0$:

$$\gamma_0=0\quad\text{and}\quad\sum_{j=1}^m c_{ji}\gamma_j=c_{m+1,i}\quad\text{for}\quad 1\leq i\leq n.$$

Recall that the tuple $(\gamma_0,\ldots,\gamma_m)$ depends on κ. These relations hold for infinitely many κ, hence

$$\xi_0=0\quad\text{and}\quad\sum_{j=1}^m c_{ji}\xi_j=c_{m+1,i}\quad\text{for}\quad 1\leq i\leq n.$$

From $\xi_0 = 0$ the numbers $1, \xi_1, \ldots, \xi_m$ are linearly independent over $\mathbb{Q}$, hence $c_{ji} = 0$ for $1 \le i \le n$ and $1 \le j \le m+1$, which implies $\log \alpha_j = 0$ for $1 \le j \le m+1$. This is not compatible with our assumptions, hence the number $e^{\xi_0}\alpha_1^{\xi_1}\cdots\alpha_m^{\xi_m}$ cannot be algebraic. □

We refer to [6] § 6 for further statements related to simultaneous algebraic approximation, including results of algebraic independence.

4 - Algebraically Independent Numbers.

Here is the main result of this paper.

THEOREM 2 *Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $\varphi(D)/D \to \infty$ as D tends to infinity. Let $\Phi : \mathbb{N} \to \mathbb{N}$ be another positive valued increasing function such that*

$$\Phi(n+1) \ge 2\varphi(2^{n\Phi(n)})$$

for all $n \ge 0$. For $a \in \mathbb{R}$ in the range $0 \le a < 1/2$, define

$$\xi_a = \sum_{n\ge 1}[n^a]2^{-\Phi(n)} \quad \text{and} \quad \theta_a = 2^{\xi_a}.$$

Then the family $\{\theta_a\,;\, 0 \le a < 1/2\}$ is algebraically independent. Moreover, for any $m \ge 1$ and any tuple $(a_1, \ldots, a_m)$ of real numbers in the range $[0, 1/2)$, there exist infinitely many integers $D > 0$ having the following property: there exist algebraic numbers $\gamma_1, \ldots, \gamma_m$ of absolute logarithmic height ≤ 1 such that

$$[\mathbb{Q}(\gamma_1, \ldots, \gamma_m) : \mathbb{Q}] \le D \quad \text{and} \quad \sum_{i=1}^{m} |\theta_{a_i} - \gamma_i| \le e^{-\varphi(D)}.$$

Proof. We start with the second part of the conclusion, which is the existence of good simultaneous algebraic approximations. Fix a sufficiently large integer n. We shall check the desired property with $D = 2^{m\Phi(n)}$. Define, for $1 \le i \le m$,

$$r_i = \sum_{\nu=1}^{n}[\nu^{a_i}]2^{-\Phi(\nu)}, \qquad \gamma_i = 2^{r_i}.$$

Further, set $q = 2^{\Phi(n)}$. Since the numbers

$$p_i = qr_i = \sum_{\nu=1}^{n}[\nu^{a_i}]2^{\Phi(n)-\Phi(\nu)} \quad (1 \le i \le m)$$

are positive rational integers, we have

$$\gamma_i^q = 2^{p_i} \in \mathbb{Z},$$

and therefore

$$[\mathbb{Q}(\gamma_i) : \mathbb{Q}] \le q \quad \text{for } 1 \le i \le m.$$

The upper bound

$$[\mathbb{Q}(\gamma_1, \ldots, \gamma_m) : \mathbb{Q}] \le q^m = D$$

plainly follows.

We estimate $r_1, \ldots, r_m$ from above as follows. Since φ is strictly increasing we have

$$\varphi(n) \ge n \quad \text{for any} \quad n \ge 0.$$

We deduce

$$\Phi(1) \ge 2, \quad \Phi(2) \ge 8, \quad \Phi(3) \ge 2^{17}, \ldots$$

The inequality $\Phi(\nu) \ge 4\nu$ for $\nu \ge 2$ is sufficient to get

$$r_i \le \sum_{\nu=1}^{n}[\nu^{1/2}]2^{-\Phi(\nu)} \le \frac{1}{4} + \sum_{\nu=2}^{n}\sqrt{\nu}\cdot 2^{-4\nu} < \frac{1}{3}.$$

The estimates

$$\mathrm{h}(\gamma_i) = r_i \log 2 < 1 \quad (1 \le i \le m)$$

plainly follow.

Recall that n is sufficiently large. Since $\Phi(\nu) > \nu + \Phi(n+1)$ for $\nu > n+1$ we have

$$\xi_{a_i} - r_i = \sum_{\nu=n+1}^{\infty}[\nu^{a_i}]2^{-\Phi(\nu)} \le 2\sqrt{n}2^{-\Phi(n+1)}.$$

Using once more the inequality

$$|e^z - 1| \le 2|z| \qquad \text{for} \quad 0 < |z| \le 1$$

with $z = (\xi_{a_i} - r_i)\log 2$ we obtain

$$0 < \theta_{a_i} - \gamma_i = \theta_{a_i}(1 - 2^{r_i - \xi_{a_i}}) \le 2\theta_{a_i}(\log 2)(\xi_{a_i} - r_i).$$

From the upper bound

$$\theta_{a_i} \le \frac{1}{\log 2}$$

we deduce

$$\sum_{i=1}^{m} |\theta_{a_i} - \gamma_i| \le 4m\sqrt{n}2^{-\Phi(n+1)}.$$

We claim that the right hand side is bounded by $e^{-\varphi(D)}$. Indeed, since $D = 2^{m\Phi(n)}$, we have $\Phi(n+1) \ge 2\varphi(D)$ and therefore it suffices to check

$$(2\log 2 - 1)\varphi(D) > 2\log 2 + \log m + \frac{1}{2}\log n.$$

This inequality plainly follows from the conditions on φ and Φ.

Now the algebraic independence proof. The fact that the family

$$\{\theta_a\,;\, 0 \le a < 1/2\}$$

is algebraically free means that for any $m \ge 1$, any tuple $(a_1, \dots, a_m)$ of pairwise distinct real numbers in the range $[0, 1/2)$, and any non zero polynomial $P \in \mathbb{Z}[X_1, \dots, X_m]$, the number $P(\theta_{a_1}, \dots, \theta_{a_m})$ does not vanish. Obviously there is no loss of generality to assume $a_1 < a_2 \dots < a_m < 1/2$ and $a_1 = 0$.

Denote by $\deg P$ the total degree of P (which may be assumed to be ≥ 1). We start with the following observation:

- *For any tuple $(b_1, \dots, b_m)$ of positive integers satisfying*

  $$b_i > b_{i-1}\deg P \quad \text{for } 2 \le i \le m,$$

  and for any tuple $(\lambda_1, \dots, \lambda_m)$ of non zero complex numbers, the polynomial

  $$Q(X) = P(\lambda_1 X^{b_1}, \dots, \lambda_m X^{b_m})$$

  does not vanish and has degree $\le b_m \deg P$.

Indeed if we write

$$P(X_1, \dots, X_m) = \sum_{\underline{i}\in\mathbb{N}^m} p_{\underline{i}} X_1^{i_1}\cdots X_m^{i_m}$$

and if $\underline{j} \in \mathbb{N}^m$ is the maximal index for the lexicographic ordering on $(j_m, j_{m-1}, \dots, j_1)$ such that $p_{\underline{j}} \ne 0$, then the degree of Q is

$$b_1 j_1 + \cdots + b_m j_m \le b_m(j_1 + \cdots + j_m) \le b_m \deg P$$

and its leading coefficient is

$$p_{\underline{j}}\lambda_1^{j_1}\cdots\lambda_m^{j_m}.$$

Let L be an integer with $L \ge \deg P$. Our second observation is:

- *There exist infinitely many integers N for which the following holds:*

  $$[n^{a_i}] = [N^{a_i}]$$

  for $1 \le i \le m$ and for any n in the range $N \le n \le N + L[N^{a_m}]$.

Here is a proof of this claim. Fix a sufficiently large integer N_0. From the inequalities

$$([N_0^{a_i}] + 2)^{1/a_i} > (N_0^{a_i} + 1)^{1/a_i} > N_0 + \sqrt{N_0}$$

we deduce that for any n in the range $N_0 \le n \le N_0 + \sqrt{N_0}$, the number $[n^{a_i}]$ is either $[N_0^{a_i}]$ or else $[N_0^{a_i}]+1$. For $1 \le i \le m$ denote by M_i the largest integer in the range $N_0 \le n \le N_0 + \sqrt{N_0}$ such that $[M_i^{a_i}] = [N_0^{a_i}]$. Since $a_m < 1/2$, there exists a subinterval $N \le n \le N + L[N^{a_m}]$ of the interval $N_0 \le n \le N_0 + \sqrt{N_0}$ which does not contain any of $M_1, \dots, M_m$. Then N satisfies the required property.

This completes the proof of our second observation and enables us to fix such a sufficiently large integer N . Further, we set $M = N + L[N^{a_m}]$. For any n in the range $N \le n \le M$ and any index i with $1 \le i \le m$, we define

$$q^{(n)} = 2^{\Phi(n)}, \quad r_i^{(n)} = \sum_{\nu=1}^{n} [\nu^{a_i}]2^{-\Phi(\nu)},$$

$$p_i^{(n)} = q^{(n)} r_i^{(n)} \quad \text{and} \quad \gamma_i^{(n)} = 2^{r_i^{(n)}}.$$

For $N \le n < M$ and $1 \le i \le m$ we have

$$r_i^{(n+1)} - r_i^{(n)} = [(n+1)^{a_i}]2^{-\Phi(n+1)} = b_i 2^{-\Phi(n+1)}$$

with $b_i = [N^{a_i}]$. Since $a_1 = 0$, we have $b_1 = 1$ and we deduce, for $N \le n < M$,

$$\frac{\gamma_i^{(n+1)}}{\gamma_i^{(n)}} = 2^{r_i^{(n+1)} - r_i^{(n)}} = \left(\frac{\gamma_1^{(n+1)}}{\gamma_1^{(n)}}\right)^{b_i}.$$

Therefore

$$\frac{\gamma_i^{(n+1)}}{(\gamma_1^{(n+1)})^{b_i}} = \frac{\gamma_i^{(n)}}{(\gamma_1^{(n)})^{b_i}} \quad \text{for } N \le n < M \text{ and } 1 \le i \le m .$$

We write these relations

$$\gamma_i^{(n)} = \lambda_i (\gamma_1^{(n)})^{b_i} \quad (N \le n \le M), \quad \text{where} \quad \lambda_i = \frac{\gamma_i^{(N)}}{(\gamma_1^{(N)})^{b_i}}.$$

The point is that λ_i does not depend on n in the interval $N \le n \le M$. From the conditions $0 = a_1 < a_2 < \cdots < a_m$ we derive

$$\frac{b_i}{b_{i-1}} = \frac{[N^{a_i}]}{[N^{a_{i-1}}]} > \frac{1}{2} N^{a_i - a_{i-1}} > \deg P \quad \text{for } 2 \le i \le m.$$

Hence, according to our first observation, the polynomial

$$Q(X) = P(\lambda_1 X^{b_1}, \ldots, \lambda_m X^{b_m}) \in \mathbb{C}[X]$$

does not vanish and has degree $\le b_m \deg P \le M - N$. Since

$$r_1^{(N)} < r_1^{(N+1)} < \cdots < r_1^{(M)},$$

the numbers $\gamma_1^{(n)}$, $(N \le n \le M)$, are pairwise distinct, hence one at least of the numbers $Q(\gamma_1^{(n)})$ $(N \le n \le M)$ does not vanish. We fix an integer n in this range such that $Q(\gamma_1^{(n)}) \ne 0$ and we write γ_i in place of $\gamma_i^{(n)}$, so that

$$P(\gamma_1, \ldots, \gamma_m) = Q(\gamma_1) \ne 0.$$

From the estimate

$$\sum_{i=1}^{m} |\theta_{a_i} - \gamma_i| \le e^{-\varphi(D)},$$

we deduce

$$|P(\theta_{a_1}, \ldots, \theta_{a_m}) - P(\gamma_1, \ldots, \gamma_m)| < e^{-\frac{1}{2}\varphi(D)}.$$

On the other hand Liouville's estimate yields

$$|P(\gamma_1, \ldots, \gamma_m)| > 2e^{-\frac{1}{2}\varphi(D)}.$$

Hence

$$|P(\theta_{a_1}, \ldots, \theta_{a_m})| > e^{-\frac{1}{2}\varphi(D)}.$$

We conclude $P(\theta_{a_1}, \ldots, \theta_{a_m}) \ne 0$, which completes the proof of Theorem 2. □

Final comments. We include two clever remarks from David Masser.

- (A) In Theorem 2, the condition $a < 1/2$ can be relaxed to $a < 1$. The key remark is that the polynomials $Q(X)$ in the proof have $O(1)$ terms independent of the b's. Therefore, by Descartes's Rules of Signs, they have $O(1)$ real zeroes. So any large interval $N \le n \le N + C$ will suffice for the "zero estimate". It only remains to check that there are arbitrarily long sequences of consecutive positive integers on which the functions $[n^{a_1}], \ldots, [n^{a_m}]$ are simultaneously constant.

  Here is the proof of this last fact. It suffices to avoid the "jump points" $\nu \ge 2$ satisfying $[(\nu-1)^a] \ne [\nu^a]$ for $a = a_1, \ldots, a_m$. Here $[n^a]$ jumps by at least 1. If n ranges over $\{1, 2, \ldots, X\}$, then $[n^a]$ starts with 1 and ends with $[X^a]$; so the number of jump points is at most $[X^a] - 1 \le X^a$. Thus altogether there are at most $X^{a_1} + \cdots + X^{a_m}$ jump points $\nu \le X$. Because this is $o(X)$, the result follows.

- (B) Since the height of each γ_i in Theorem 2 is bounded (by 1) and since the set of tuples $(\gamma_1, \ldots, \gamma_m)$ is infinite, one deduces that the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_m)$ is not bounded. So there are infinitely many values of D for which the conclusion holds with

  $$[\mathbb{Q}(\gamma_1, \ldots, \gamma_m) : \mathbb{Q}] = D.$$

  Another small modification allows one to make the D's explicit: for sufficiently large n define $D_n = 2^{\Phi(n)}$. Since the γ_i's lie in $K = \mathbb{Q}(2^{1/q})$, with $q = D_n$ the upper bound

  $$[\mathbb{Q}(\gamma_1, \ldots, \gamma_m) : \mathbb{Q}] \le D_n$$

  still holds. And one can easily restrict to n's with say $[n^{a_1}]$ odd: e.g. start with an interval where $[n^{a_1}]$ is odd and constant, and then find a large subinterval containing none of the jump points of the other functions. It follows that $r_1 = p_1/q$ with p_1 odd, so $\mathbb{Q}(\gamma_1) = K$, which has degree exactly $q = D_n$. So

$$[\mathbb{Q}(\gamma_1, \ldots, \gamma_m) : \mathbb{Q}] = D_n.$$

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