Integer valued entire functions on Cartesian products

Michel Waldschmidt

À Andrzej Schinzel pour son 60-ème anniversaire

Abstract. Our goal is to study entire functions of n complex variables which take integer values (in $\mathbb{Z}$) on a Cartesian product $X_1 \times \ldots \times X_n \subset \mathbb{C}^n$, where $X_1, \ldots, X_n$ are subsets of $\mathbb{C}$. Our main result states that several such functions which satisfy some growth condition are then linearly dependent, provided that each set X_ν has sufficiently many points.

One application concerns integer valued entire functions of a single variable on a set of products $x_1 x_2 \quad (x_1 \in X_1,\ x_2 \in X_2)$: to such a function f we associate the function of two complex variables $(z, w) \mapsto f(zw)$. This question had already been considered in previous papers by the author by means of Laurent's interpolation determinants. Here we use an auxiliary function.

Introducing the function $(z, w) \mapsto f(z + w)$ enables us to consider also integer valued entire functions on a set of sums $x_1 + x_2 \quad (x_1 \in X_1,\ x_2 \in X_2)$.

Finally we shall consider also integer valued entire functions on geometrical progressions $q^n \quad (n \geq 1)$.

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1. Linear dependence of integer valued entire functions

We investigate the arithmetic nature of the values of several functions $\varphi_1, \ldots, \varphi_L$ at the points of a Cartesian product. For simplicity we assume that the functions are entire, and that the values are rational integers. Under suitable assumptions we show that these functions are then linearly dependent over $\mathbb{Q}$.

Notice that if we have only one integer valued function f, then we find further ones by considering $f^2, f^3, \ldots$ More generally, if we start with finitely many functions $f_1, \ldots, f_d$, then we construct new ones by considering monomials $f_1^{t_1} \cdots f_d^{t_d}$ $((t_1, \ldots, t_d) \in \mathbb{N}^d)$. This is how our criterion of linear dependence (Theorem 1.1) will give rise to criteria of algebraic dependence (see §2). This remark has often been used. However there is another way of constructing several functions starting from a single one: if, say, f is an entire function of a single variable and X, Y two subsets of $\mathbb{C}$ such that

$$f(xy) \in \mathbb{Z} \quad \text{for any} \quad x \in X,\ y \in Y,$$

then for any positive integers δ and λ, and for any $\delta\lambda$-tuple $(t_{hk})_{\substack{1\le h\le\delta\\1\le k\le\lambda}} \in \mathbb{N}^{\delta\lambda}$, the function $f(z_h w_k)^{t_{hk}}$ takes values in $\mathbb{Z}$ at the points of the Cartesian product $X^\delta \times Y^\lambda \subset \mathbb{C}^{\delta+\lambda}$. This remark yields a new proof of the Six Exponentials Theorem; we provide the details only for the proof of the weaker statement that one at least among six numbers of the form $e^{x_i y_j}$ is not an integer (see Corollary 2.7), but the same proof produces transcendence results.

The results we get are close to our previous ones in [17], but the proofs are different. We compare both approaches. We also use the main result of [17] in order to study integer valued entire functions of a single variable at points of the form $x + y$ where each of x and y runs over a subset of $\mathbb{C}$.

a) Statement of the criterion

For $R \geq 0$, we denote by $D(0, R)$ the closed complex disc of center 0 and radius R:

$$D(0,R) = \bigl\{z \in \mathbb{C};\ |z| \leq R\bigr\}.$$

Theorem 1.1. *Let n be a positive integer, $r_1, \ldots, r_n$, $R_1, \ldots, R_n$ and ψ—maps from $\mathbb{N}$ into the set of positive real numbers. We assume, for $1 \leq \nu \leq n$ and for all sufficiently large integer N,*

$$r_\nu(N) \leq r_\nu(N+1) \leq R_\nu(N) \quad \textit{and} \quad \frac{R_\nu(N)^2 + r_\nu(N+1)^2}{2R_\nu(N) r_\nu(N+1)} \geq e^{1/\psi(N)}. \tag{1.1}$$

We also assume that the function $N/\psi(N)$ is increasing[1)]*, and that*

$$\liminf_{N\to\infty} \frac{N}{\psi(N)\log N} > 6n. \tag{1.2}$$

There exists a positive real number L_0 with the following property. Let $L \geq L_0$ be an integer, $\varphi_1, \ldots, \varphi_L$ entire functions in $\mathbb{C}^n$ and $X_1, \ldots, X_n$ subsets of $\mathbb{C}$. Assume, for $1 \leq \lambda \leq L$,

$$\varphi_\lambda(X_1 \times \ldots \times X_n) \subset \mathbb{Z}.$$

Assume also that for any integer $N \geq (L/4)^{1/n}$, the two following estimates hold:

$$\mathrm{Card}\bigl(X_\nu \cap D\bigl(0, r_\nu(N)\bigr)\bigr) \geq N \qquad (1 \leq \nu \leq n) \tag{1.3}$$

and

$$\log \max_{1\le\lambda\le L} \sup\bigl\{|\varphi_\lambda(\underline{z})|\,;\ \underline{z} = (z_1, \ldots, z_n),\ |z_\nu| \leq R_\nu(N) \quad (1 \leq \nu \leq n)\bigr\} \leq \frac{N}{2\psi(N)}. \tag{1.4}$$

Then the functions $\varphi_1, \ldots, \varphi_L$ are linearly dependent over $\mathbb{Q}$.

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1) We use the word "increasing" where some other authors would say "non-decreasing".

Remark. Several generalizations of this result are possible:

- Derivatives can be included.
- One can extend the result to meromorphic functions in place of entire functions: in this case one only needs to add suitable hypotheses which ensure that the points are not too close to the sets of zeroes of the denominators of the meromorphic functions (see for instance [14]).
- Integer values can be replaced by algebraic values: in this case one only needs to add suitable hypotheses on the growth of the height of these values.
- Other sets than Cartesian products can be considered, provided that we know a Schwarz' Lemma for such sets.
- One can replace the assumption that the functions take integer or algebraic values by the weaker assumption that the values are closed to integers or to algebraic numbers (see [8]). In order to do so, one needs either an interpolation formula (which is known only for certain subsets of $\mathbb{C}^n$) or else a zero estimate (which is available only for certain sets of functions).
- Finally p-adic analogs are also worth of study.

b) Auxiliary lemmas

We recall two auxiliary results: the first one is the well-known Thue-Siegel Lemma whose proof involves Dirichlet's box principle; we use here the original formulation of Siegel [12] which in fact is sharper than some more recent versions. The second tool is a Schwarz' Lemma for Cartesian products for the proof of which we refer to [15].

Lemma 1.2. *Let N and M be two positive integers with $N > M$. Let $a_{\nu\mu}$ $(1 \le \nu \le N,\ 1 \le \mu \le M)$ be rational integers, and let A be a positive integer satisfying*

$$A \ge \max_{1 \le \mu \le M} \sum_{\nu=1}^{N} |a_{\nu\mu}|.$$

There there exist rational integers $x_1, \dots, x_N$ such that

$$\sum_{\nu=1}^{N} a_{\nu\mu} x_\nu = 0 \qquad (1 \le \mu \le M)$$

and

$$0 < \max_{1 \le \nu \le N} |x_\nu| \le A^{M/(N-M)}.$$

Proof. See [12]. □

Lemma 1.3. *Let n, N be positive integers and E, $r_1,\dots,r_n$, $R_1,\dots,R_n$ positive real numbers. Assume $E\ge 1$ and, for $1\le\nu\le n$,*

$$R_\nu \ge r_\nu \quad \textit{and} \quad \frac{R_\nu^2+r_\nu^2}{2R_\nu r_\nu}\ge E.$$

For $1\le\nu\le n$, let X_ν be a subset of $D(0,r_\nu)$, with at least N elements. Further, let F be an entire function in $\mathbb{C}^n$ which vanishes on the Cartesian product $X_1\times\ldots\times X_n$. Define

$$|F|_{\underline{r}} = \sup\bigl\{|F(\underline{z})|\,;\,|z_\nu| = r_\nu \quad (1\le\nu\le n)\bigr\}$$

and

$$|F|_{\underline{R}} = \sup\bigl\{|F(\underline{z})|\,;\,|z_\nu| = R_\nu \quad (1\le\nu\le n)\bigr\}.$$

Then

$$|F|_{\underline{r}} \le E^{-N}|F|_{\underline{R}}.$$

Proof. Lemma 1.3 is obtained by introducing Blaschke products (see for instance Lemma 2.2 of [17]) into the proof of Proposition 7.2.1 of [15]. Notice that the statement is homogeneous: replacing F by

$$\tilde F(z_1,\dots,z_n) = F(r_1z_1,\dots,r_nz_n),$$

and, for $1\le\nu\le n$, X_ν by $\tilde X_\nu=\bigl\{x/r_\nu\,;\,x\in X_\nu\bigr\}$ and R_ν by $\tilde R_\nu = R_\nu/r_\nu$, we may assume $r_1=\ldots=r_n=1$. Also, replacing $R_1,\dots,R_n$ by $R=E+\sqrt{E^2-1}$, we may assume $R_1=\ldots=R_n=R$ and $R^2+1=2ER$. □

Remark. For simplicity we did not include multiplicities. For the record, here is the well known Schwarz' Lemma in one variable:

> *Let F be an entire function of a single variable. For $r>0$, denote by $n(F,r)$ the number of zeros (counting multiplicities) of F in the closed disc of radius r. Then for $0<r\le R$ we have*
>
> $$|F|_r \le \left(\frac{R^2+r^2}{2Rr}\right)^{-n(F,r)}|F|_R.$$

We shall use the following consequence:

Lemma 1.4. *Let F be a non-constant entire function of one variable. Then there exists $r_0\ge 0$ such that, for any $\theta>1$ and any $r>r_0$, we have*

$$n(F,r)\log\left(\frac{\theta^2+1}{2\theta}\right) < \log|F|_{\theta r}.$$

Proof. Since F is not constant, there exists $r_0 \geq 0$ be such that $|F|_{r_0} \geq 1$. Then for any $r > r_0$, we have

$$1 < |F|_r \leq \left(\frac{\theta^2+1}{2\theta}\right)^{-n(F,r)} |F|_{\theta r}. \qquad \square$$

Remark. A variant of Lemma 1.4 may also be proved by means of Jensen's formula. See Lemma 2.3.4 p. 193 of [3].

c) Schneider's method with an auxiliary function

In this section we prove Theorem 1.1.

Define $N_0 = \left[(L/4)^{1/n}\right] + 1$. Using hypothesis (1.3) we select, for $1 \leq \nu \leq n$ and $N \geq N_0$, a subset $X_\nu(N)$ of $X_\nu \cap D\big(0, r_\nu(N)\big)$ such that

$$\mathrm{Card}\big(X_\nu(N)\big) = N.$$

For any $N \geq N_0$, we define $\underline{X}(N) = X_1(N) \times \ldots \times X_n(N) \subset \mathbb{C}^n$.

The first step of the proof is the construction of an auxiliary function: we apply lemma 1.2 in order to solve the system of equations

$$\sum_{\lambda=1}^{L} p_\lambda \varphi_\lambda(\underline{x}) = 0 \qquad \text{for all} \quad \underline{x} = (x_1, \ldots, x_n) \in \underline{X}(N_0), \tag{1.5}$$

where the unknowns $p_1, \ldots, p_L$ are rational integers. The coefficients $\varphi_\lambda(\underline{x})$ of this system of equations are rational integers. We have L unknowns, and N_0^n equations, with

$$\frac{N_0^n}{L - N_0^n} < \frac{1}{2}.$$

From (1.4) we deduce, for $x \in \underline{X}(N_0)$,

$$\sum_{\lambda=1}^{L} |\varphi_\lambda(\underline{x})| \leq L \cdot \max_{1 \leq \lambda \leq L} |\varphi_\lambda(\underline{x})| \leq L e^{N_0/(2\psi(N_0))}.$$

Using (1.2) we obtain (recall that L is sufficiently large)

$$N_0/\psi(N_0) > 6 \log L.$$

Lemma 1.2 shows that there exists a solution $(p_1, \ldots, p_L) \in \mathbb{Z}^L \setminus \{0\}$ to (1.5) with

$$\max_{1 \leq \lambda \leq L} |p_\lambda| \leq L^{1/2} e^{N_0/(4\psi(N_0))},$$

hence

$$\sum_{\lambda=1}^{L} |p_\lambda| < e^{N_0/(2\psi(N_0))}. \tag{1.6}$$

The second step of the proof is an inductive argument. Define

$$F = \sum_{\lambda=1}^{L} p_\lambda \varphi_\lambda.$$

We shall now prove, for any integer $N \geq N_0$,

$$(\mathrm{i})_N \qquad F(\underline{x}) = 0 \quad \text{for all} \quad \underline{x} = (x_1, \ldots, x_n) \in \underline{X}(N),$$

and

$$(\mathrm{ii})_N \qquad \log\sup\bigl\{|F(\underline{z})|\,;\, |z_\nu| \leq r_\nu(N+1),\, (1 \leq \nu \leq n)\bigr\} < -\frac{N}{2\psi(N)} + \frac{N_0}{2\psi(N_0)}.$$

Our construction of F shows that $(\mathrm{i})_{N_0}$ is true. Since the set $X_\nu(N+1)$ is contained in the disc of center 0 and radius $r_\nu(N+1)$, since $F(\underline{x})$ is a rational integer for $\underline{x} \in \underline{X}(N+1)$, and since the right hand side of $(\mathrm{ii})_N$ is ≤ 0, we readily see that property $(\mathrm{ii})_N$ implies $(\mathrm{i})_{N+1}$. Further, if property $(\mathrm{ii})_N$ holds for any $N \geq N_0$, since the right hand side of $(\mathrm{ii})_N$ tends to $-\infty$ as N tends to ∞, we deduce $F = 0$.

Therefore the proof of Theorem 1.1 will be completed once we have checked that $(\mathrm{i})_N$ implies $(\mathrm{ii})_N$. We assume that $(\mathrm{i})_N$ holds for some $N \geq N_0$, and we are going to check

$$|F(\underline{z})| < e^{-N/(2\psi(N))} \cdot e^{N_0/(2\psi(N_0))}$$

for all $\underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ with $|z_\nu| \leq r_\nu(N+1)$ for $1 \leq \nu \leq n$. This will prove that $(\mathrm{ii})_N$ holds.

We use Lemma 1.3 with

$$E = e^{1/\psi(N)}, \quad r_\nu = r_\nu(N+1), \quad R_\nu = R_\nu(N) \qquad (1 \leq \nu \leq n).$$

The conditions

$$R_\nu > r_\nu \quad \text{and} \quad \frac{R_\nu^2 + r_\nu^2}{2R_\nu r_\nu} \geq E \qquad \text{for} \quad 1 \leq \nu \leq n$$

follow from the assumptions of Theorem 1.1. Therefore, with the notations of Lemma 1.3, we get

$$\log|F|_{\underline{r}} \leq -N\log E + \log|F|_{\underline{R}} \leq -\frac{N}{\psi(N)} + \log|F|_{\underline{R}}.$$

Notice that the left hand side of $(\mathrm{ii})_N$ is nothing else that $\log|F|_{\underline{r}}$.

On the other hand from (1.4) we derive

$$\log|\varphi_\lambda|_{\underline{R}} \leq \frac{N}{2\psi(N)},$$

and therefore (1.6) yields the bounds

$$\log|F|_{\underline{R}} \leq \log\left(\sum_{\lambda=1}^{L} |p_\lambda|\right) + \log\max_{1\leq\lambda\leq L} |\varphi_\lambda|_{\underline{R}} < \frac{N_0}{2\psi(N_0)} + \frac{N}{2\psi(N)}.$$

Hence

$$\log|F|_{\underline{r}} < \frac{N_0}{2\psi(N_0)} - \frac{N}{2\psi(N)},$$

which is the desired estimate. □

2. Algebraic dependence of integer valued entire functions

Applying Theorem 1.1 to a set of functions $f_1^{t_1} \cdots f_d^{t_d}$ yields a criterion for algebraic independence. The special case $n = 1$ provides a simple statement. An example is a variant of Pólya's Theorem on integer valued entire functions.

a) Consequence of the criterion for linear dependence

Corollary 2.1. *Let n, d be two positive integers and $c, \theta_1, \ldots, \theta_n, \mu_1, \ldots, \mu_d$ be positive real numbers satisfying*

$$\mu_1 + \ldots + \mu_d \leq d - n. \tag{2.1}$$

There exists a positive real number η with the following property. Let $f_1, \ldots, f_d$ be entire functions in $\mathbb{C}^n$, and let $X_1, \ldots, X_n$ be subsets of $\mathbb{C}$ such that

$$f_i(X_1 \times \ldots \times X_n) \subset \mathbb{Z} \qquad (1 \leq i \leq d).$$

We assume that for all sufficiently large R, the two following estimates hold:

$$\operatorname{Card}\bigl(X_\nu \cap D(0, R^{\theta_\nu})\bigr) \geq cR \qquad (1 \leq \nu \leq n) \tag{2.2}$$

and, for $1 \leq i \leq d$,

$$\log \sup\bigl\{|f_i(\underline{z})| ; \underline{z} = (z_1, \ldots, z_n), |z_\nu| \leq R^{\theta_\nu}, (1 \leq \nu \leq n)\bigr\} \leq \eta R^{\mu_i}. \tag{2.3}$$

Then the functions $f_1, \ldots, f_d$ are algebraically dependent over $\mathbb{Q}$.

Proof. We first check that there is no loss of generality to assume $\mu_i < 1$ for $1 \leq i \leq d$. Indeed, assume that Corollary 2.1 is proved in this special case. Consider the subset I of $\{1, \ldots, d\}$ such that $\mu_i < 1$ for $i \in I$ and $\mu_i \geq 1$ for $i \notin I$. Then from (2.1) we deduce

$$\sum_{i \in I} \mu_i \leq \sum_{i=1}^{d} \mu_i - d + |I| \leq |I| - n.$$

Therefore we can apply (the special case of) Corollary 2.1 to the set of functions $\{f_i \,;\, i \in I\}$, with d replaced by $|I|$.

We also notice that assumption (2.1) implies $d > n$, hence $d \geq 2$.

Define, for $1 \leq \nu \leq n$ and for sufficiently large N,

$$r_\nu(N) = (N/c)^{\theta_\nu} \quad \text{and} \quad R_\nu(N) = 6 r_\nu(N).$$

We now check the hypotheses of Theorem 1.1 with $\psi(N) = 1$. Notice that $r_\nu(N)$ and $R_\nu(N)$ have been chosen so that, for sufficiently large N, the estimates (1.1) are satisfied. By assumption there exists $R_0 \geq 1$ such that (2.2) and (2.3) hold for $R \geq R_0$. Define $L_0 = 4(cR_0)^n$. Let L and N be two integers with $L \geq L_0$ and $N \geq (L/4)^{1/n}$. Since $N \geq cR_0$, we may apply (2.2) with $R = N/c$ in order to check (1.3).

Define positive integers $T_1, \dots, T_d$ by

$$T_i = \left[L^{(1-\mu_i)/(d-\mu_1-\dots-\mu_d)}\right] + 1 \qquad (1 \le i \le d),$$

so that $T_1 \cdots T_d \ge L$. Hence we can choose $\{\varphi_1, \dots, \varphi_L\}$ as a subset of

$$\left\{f_1^{t_1} \cdots f_d^{t_d}\,;\, 0 \le t_i < T_i \quad (1 \le i \le d)\right\}$$

with pairwise distinct $(t_1, \dots, t_d)$.

Define

$$\theta_* = \min\{\theta_1, \dots, \theta_n\}.$$

For $1 \le \nu \le n$ we have

$$R_0^{\theta_\nu} \le R_\nu(N) = 6(N/c)^{\theta_\nu} \le \left(6^{1/\theta_*}c^{-1}N\right)^{\theta_\nu}.$$

For $x \in \mathbb{R}$, define $\log^+ x = \log\max\{x, 1\}$. Using (2.3) with $R = 6^{1/\theta_*}N/c$ we deduce that for $1 \le i \le d$ and for any $\underline{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ satisfying $|z_\nu| \le R_\nu(N)$ $(1 \le \nu \le n)$, we have

$$\log^+ |f_i(\underline{z})| \le \eta_i N^{\mu_i} \quad \text{with} \quad \eta_i = \eta 6^{\mu_i/\theta_*}c^{-\mu_i}.$$

Therefore, for the same $\underline{z}$, we deduce from (2.1) and from the estimate

$$T_i \le 2L^{(1-\mu_i)/(d-\mu_1-\dots-\mu_d)} \qquad (1 \le i \le d),$$

that we have

$$\log \max_{1\le\lambda\le L} |\varphi_\lambda(\underline{z})| \le \sum_{i=1}^d T_i \log^+ |f_i|_{R_\nu(N)} \le 2N \sum_{i=1}^d \eta_i 4^{(1-\mu_i)/(d-\mu_1-\dots-\mu_d)}.$$

Assuming η sufficiently small, so that

$$\max_{1\le i\le d}\left(4d\eta 6^{\mu_i/\theta_*}c^{-\mu_i}\right)^{d-\mu_1-\dots-\mu_d} 4^{1-\mu_i} \le 1,$$

we deduce

$$\log \max_{1\le\lambda\le L} |\varphi_\lambda(\underline{z})| \le N/2$$

and (1.4) follows.

Therefore we may use Theorem 1.1: the functions $\varphi_1, \dots, \varphi_L$ are linearly dependent over $\mathbb{Q}$, hence the functions $f_1, \dots, f_d$ are algebraically dependent over $\mathbb{Q}$. This completes the proof of Corollary 2.1. $\square$

b) Application to entire functions of a single variable

Here is a consequence of Corollary 2.1 in the case $n = 1$.

Corollary 2.2. *Let $d \ge 2$ be an integer, c, ℓ, $\varrho_1, \dots, \varrho_d$, positive real numbers satisfying*

$$\varrho_1 + \dots + \varrho_d \le (d-1)\ell.$$

There exists a positive real number η with the following property. Let $f_1,\ldots,f_d$ be entire functions in $\mathbb{C}$, and let X be a subset of $\mathbb{C}$ such that

$$f_i(X)\subset\mathbb{Z}\qquad(1\le i\le d).$$

We assume that for all sufficiently large R, the two following estimates hold:

$$\mathrm{Card}\bigl(X\cap D(0,R)\bigr)\ge cR^{\ell},$$

and

$$\log|f_i|_R\ \le\eta R^{\varrho_i}\qquad(1\le i\le d).$$

Then the functions $f_1,\ldots,f_d$ are algebraically dependent over $\mathbb{Q}$.

Proof. In Corollary 2.1 take $n=1$, $\theta_1=1/\ell$ and $\mu_i=\varrho_i/\ell$ $(1\le i\le d)$. □

Remark. Corollary 2.2 also follows from Théorème 2.1 of [6]. It is interesting to compare it with Schneider's Satz II in [10] and with [13] Théorème 2.2.1: a weaker result follows from these two references, where the assumption $\varrho_1+\ldots+\varrho_d\le(d-1)\ell$ needs to be replaced by the stronger one $\varrho_1+\ldots+\varrho_d<(d-1)\ell$. However Satz II in [10] allows derivatives, while Théorème 2.2.1 in [13] allows algebraic values in place of rational integers. Obviously it is possible to generalize Corollary 2.2 so that it includes not only these two statements, but also Satz III in [10], and the later transcendence criteria by Th. Schneider in [11], by S. Lang in [7], and by K. Ramachandra in [9].

Here is another corollary to Theorem 1.1, dealing again with integer valued entire functions: it is a variant of Pólya's Theorem (see [14]):

Corollary 2.3. *For any $\vartheta\ge 1/2$ there exists a positive constant η with the following property. Let f be an entire function of one complex variable and X an infinite subset of $\mathbb{Z}$ such that $f(X)\subset\mathbb{Z}$. Assume, for all sufficiently large real number r,*

$$\mathrm{Card}\bigl(X\cap D(0,r)\bigr)\ge r/\vartheta$$

and

$$\log|f|_r\le\eta r.$$

Then f is a polynomial.

Proof. The proof involves the binomial functions

$$\binom{z}{0}=1\qquad\text{and}\qquad\binom{z}{h}=\frac{z(z-1)\cdots(z-h+1)}{h!}\qquad\text{for}\quad h\ge 1.$$

For $R>0$, $H>0$, $|z|\le R$ and $0\le h\le H$, the estimate

$$\left|\binom{z}{h}\right|\le e^H\left(\frac{R}{H}+1\right)^H \tag{2.4}$$

follows from the inequalities

$$e^H \geq \frac{H^h}{h!} \quad \text{and} \quad \frac{(R+H)^h}{h!} \leq e^H \left(\frac{R}{H}+1\right)^H.$$

Choose positive real numbers h_0, k_0 and ℓ_0, which are subject to the following conditions:

$$h_0 k_0 \geq 4, \tag{2.5}$$

$$h_0 + h_0 \log\left(\frac{\ell_0}{h_0}+1\right) + h_0 \log(2\vartheta) + \frac{1}{2}\log 2 \leq \frac{1}{4}\log \ell_0 \tag{2.6}$$

and

$$\eta k_0 \ell_0 \vartheta \leq \frac{1}{4}\log \ell_0. \tag{2.7}$$

For instance fix $h_0 < 1/4$, define $k_0 = 4/h_0$ so that (2.5) holds, next select ℓ_0 sufficiently large so that (2.6) is satisfied, and finally assume η is sufficiently small in order that (2.7) is true.

Let L_0 be a sufficiently large integer, L an integer $\geq L_0$ and N an integer $\geq L/4$. From (2.5) we deduce that the set $\bigl\{(h,k)\,;\, 1 \leq h \leq h_0 N,\, 0 \leq k < k_0\bigr\}$ contains at least L elements. Let $\{\varphi_1, \ldots, \varphi_L\}$ be any subset of the set of functions

$$\binom{z}{h} f(z)^k \qquad (1 \leq h \leq h_0 N, \quad 0 \leq k < k_0).$$

We are going to apply Theorem 1.1 with $n = 1$, $X = X_1$,

$$r_1(N) = \vartheta N, \quad R_1(N) = \ell_0 r_1(N), \quad \psi(N) = \frac{1}{\log(\ell_0/2)}.$$

Condition (1.3) follows from

$$\mathrm{Card}\bigl(X \cap D(0, \vartheta N)\bigr) \geq N.$$

Next from (2.4) we deduce

$$\max_{1 \leq \lambda \leq L} \log |\varphi_\lambda|_{R_1(N)} \leq h_0 N + h_0 N \log\left(\frac{\ell_0 \vartheta}{h_0}+1\right) + \eta k_0 \ell_0 \vartheta N.$$

Using (2.6) together with $\vartheta \geq 1/2$, we obtain

$$h_0 + h_0 \log\left(\frac{\ell_0\vartheta}{h_0}+1\right) \leq h_0 + h_0 \log\left(\frac{\ell_0}{h_0}+1\right) + h_0 \log(2\vartheta) \leq \frac{1}{4\psi(N)} - \frac{1}{4}\log 2$$

while from (2.7) we deduce

$$\eta k_0 \ell_0 \vartheta \leq \frac{1}{4\psi(N)} + \frac{1}{4}\log 2.$$

Therefore (1.4) holds. It follows that the two functions z and $f(z)$ are algebraically dependent over $\mathbb{Q}$, and since f is entire, we may conclude $f \in \mathbb{Q}[z]$. $\square$

c) Cartesian products

We consider a single function φ of n variables taking integer values at the points of a Cartesian product $X_1 \times \ldots \times X_n \subset \mathbb{C}^n$.

Corollary 2.4. *Let c, $\alpha_1, \ldots, \alpha_n$ and μ be positive real numbers with $\mu < 1$. There exists a positive real number η with the following property. Let φ be an entire function in $\mathbb{C}^n$, and let $X_1, \ldots, X_n$ be subsets of $\mathbb{C}$ such that*

$$\varphi(X_1 \times \ldots \times X_n) \subset \mathbb{Z}.$$

Assume that for all sufficiently large R, the following estimates hold:

$$\mathrm{Card}\bigl(X_\nu \cap D(0,R)\bigr) \ge cR^{\alpha_\nu} \qquad (1 \le \nu \le n)$$

and

$$\log\sup\bigl\{|\varphi(z_1,\ldots,z_n)|\,;\,|z_\nu| \le R^{1/\alpha_\nu}\ (1 \le \nu \le n)\bigr\} \le \eta R^{\mu}.$$

Let $\delta_1, \ldots, \delta_n$ be positive integers satisfying

$$\mu \le 1 - \frac{\delta_1 + \ldots + \delta_n}{\delta_1 \cdots \delta_n}.$$

Then the $\delta_1 \cdots \delta_n$ functions $\varphi(z^{(1)}_{h_1}, \ldots, z^{(n)}_{h_n})$ $(1 \le h_\nu \le \delta_\nu,\ 1 \le \nu \le n)$ of $\delta_1 + \ldots + \delta_n$ variables $z^{(1)}_1, \ldots, z^{(1)}_{\delta_1}, \ldots z^{(n)}_1, \ldots, z^{(n)}_{\delta_n}$ are algebraically dependent over $\mathbb{Q}$.

Remark. The assumptions imply $n \ge 2$.

Proof. Use Corollary 2.1 with $d = \delta_1 \cdots \delta_n$ for the functions of $\delta_1 + \ldots + \delta_n$ variables

$$\varphi(z^{(1)}_{h_1}, \ldots, z^{(n)}_{h_n})$$

with

$$\theta_i = \frac{1}{\alpha_\nu} \qquad (\delta_1 + \ldots + \delta_{\nu-1} < i \le \delta_1 + \ldots + \delta_\nu, \quad 1 \le \nu \le n),$$

and $\mu_1 = \ldots = \mu_d = \mu$, so that

$$\mu_1 + \ldots + \mu_d = d\mu = \delta_1 \cdots \delta_n \mu \le d - (\delta_1 + \ldots + \delta_n). \qquad \square$$

Remark. It is instructive to compare the case $n = 2$ of Corollary 2.4 with Theorem 3.1 below (which is Proposition 4.1 in [17]). The assumptions of both statements are similar (however Theorem 3.1 allows $\mu = 1$), but the conclusions are substantially different. Also the methods of proofs are quite different.

d) Integer valued functions on sets of products

We apply Corollary 2.4 with $n=2$ to a function of the form $\varphi(z,w)=f(zw)$ in order to get a result concerning the values of a function of a single variable f at a set of products xy, where x and y run over the sets X,Y respectively.

Corollary 2.5. *Let c_0, c_1, c_2, α, β and ϱ be positive real numbers with*

$$\frac{1}{\alpha}+\frac{1}{\beta}<\frac{1}{\varrho},$$

f an entire function in $\mathbb{C}$ and X, Y subsets of $\mathbb{C}$ such that

$$f(xy)\in\mathbb{Z}\quad \textit{for any}\quad (x,y)\in X\times Y.$$

Assume, for all sufficiently large R,

$$\log|f|_R\le c_0R^{\varrho}$$

and

$$\mathrm{Card}\bigl(X\cap D(0,R)\bigr)\ge c_1R^{\alpha},\qquad \mathrm{Card}\bigl(Y\cap D(0,R)\bigr)\ge c_2R^{\beta}.$$

Then f is a polynomial.

Remark. Again it is interesting to compare Corollary 2.5 with Corollary 3.2 below (which is Theorem 1.1 in [17]). In Corollary 3.2, the main assumption is only $(1/\alpha)+(1/\beta)\le 1/\varrho$ (but c_0 is replaced by a sufficiently small η).

Proof. Select a real number ϱ_1 in the range

$$\varrho<\varrho_1<\frac{\alpha\beta}{\alpha+\beta}.$$

Let δ and λ be two positive integers satisfying

$$\frac{\varrho_1}{\alpha}+\frac{\varrho_1}{\beta}+\frac{1}{\delta}+\frac{1}{\lambda}\le 1.$$

We apply Corollary 2.4 with

$$n=2,\quad X_1=X,\quad X_2=Y,\quad \alpha_1=\alpha,\quad \alpha_2=\beta,\quad c=\min\{c_1,c_2\},$$
$$\delta_1=\delta,\quad \delta_2=\lambda,\quad \mu=\frac{\varrho_1}{\alpha}+\frac{\varrho_1}{\beta}.$$

We deduce that the $\delta\lambda$ functions $f(z_hw_k)$ $(1\le h\le\delta, 1\le k\le\lambda)$ are algebraically dependent over $\mathbb{C}$. Corollary 2.5 now follows from the next lemma. □

Lemma 2.6 (J.-P. Bézivin). *Let δ and λ be positive integers and f an entire function of one variable. Assume that the $\delta\lambda$ functions $f(z_hw_k)$ $(1\le h\le\delta$, $1\le k\le\lambda)$ of $\delta+\lambda$ variables $z_1,\dots,z_\delta$, $w_1,\dots,w_\lambda$ are algebraically dependent over $\mathbb{C}(z_1,\dots,z_\delta,w_1,\dots,w_\lambda)$. Then f is a polynomial.*

Proof. We prove this result by induction on $\delta+\lambda$. For $\delta=\lambda=1$, the hypothesis is that the three functions z, w and $f(zw)$ of two variables z and w are algebraically dependent. Hence f is an algebraic function, and since f is entire, f is a polynomial.

We now assume $\delta+\lambda>1$. By symmetry we may assume $\delta>1$. Let P be a polynomial in $\delta+\lambda+\delta\lambda$ variables $X_1,\dots,X_\delta$, $Y_1,\dots,Y_\lambda$ and Z_{hk} $(1\le h\le\delta$, $1\le k\le\lambda)$ which vanishes (as a function of $\delta+\lambda$ variables) when one substitutes to the variables respectively $z_1,\dots,z_\delta$, $w_1,\dots,w_\lambda$ and $f(z_hw_k)$. Assume f is not a polynomial; we want to prove $P=0$.

We write P as a polynomial in $X_1,Z_{11},\dots,Z_{1\lambda}$, with coefficients which are polynomials in the other variables:

$$P=\sum_i\sum_{j_1}\cdots\sum_{j_\lambda}B_{ij}(X,Y,Z)X_1^iZ_{11}^{j_1}\cdots Z_{1\lambda}^{j_\lambda}.$$

where $B_{ij}(X,Y,Z)$ stands for

$$B_{ij_1\dots j_\lambda}(X_2,\dots,X_\delta,Y_1,\dots,Y_\lambda,Z_{21},\dots,Z_{2\lambda},\dots,Z_{\delta1},\dots,Z_{\delta\lambda}).$$

For each tuple $(ij)=(i,j_1,\dots,j_\lambda)$, define a function g_{ij} of $\delta+\lambda-1$ variables $z_2,\dots,z_\delta$, $w_1,\dots,w_\lambda$ by

$$\begin{aligned}&g_{ij}(z_2,\dots,z_\delta,w_1,\dots,w_\lambda)\\&\quad=B_{ij}\big(z_2,\dots,z_\delta,w_1,\dots,w_\lambda,f(z_2w_1),\dots,f(z_2w_\lambda),\dots,f(z_\delta w_1),\dots,f(z_\delta w_\lambda)\big).\end{aligned}$$

From our assumption

$$P\big(z_1,\dots,z_\delta,w_1,\dots,w_\lambda,f(z_1w_1),\dots,f(z_1w_\lambda),\dots,f(z_\delta w_1),\dots,f(z_\delta w_\lambda)\big)=0$$

we deduce

$$\sum_i\sum_{j_1}\cdots\sum_{j_\lambda}g_{ij}(z_2,\dots,z_\delta,w_1,\dots,w_\lambda)z_1^if(z_1w_1)^{j_1}\cdots f(z_1w_\lambda)^{j_\lambda}=0. \tag{2.8}$$

For $j\ge 0$, consider the Taylor expansion of $f(z)^j$ at the origin:

$$f(z)^j=\sum_{n\ge0}a_{nj}z^n.$$

For any $n\ge0$, the coefficient of z_1^n in the Taylor expansion of the left hand side of (2.8) vanishes:

$$\sum_{n_1+\dots+n_\lambda+i=n}\ \sum_{j_1}\cdots\sum_{j_\lambda}g_{ij}(z_2,\dots,z_\delta,w_1,\dots,w_\lambda)a_{n_1j_1}\cdots a_{n_\lambda j_\lambda}w_1^{n_1}\cdots w_\lambda^{n_\lambda}=0.$$

We use our induction hypothesis: since f is not a polynomial, this is a trivial relation between $z_2,\dots,z_\delta$, $w_1,\dots,w_\lambda$ and $f(z_hw_k)$ $(2\le h\le\delta$, $1\le k\le\lambda)$. This means that for each $n\ge0$ the polynomial

$$\sum_{n_1+\dots+n_\lambda+i=n}\ \sum_{j_1}\cdots\sum_{j_\lambda}B_{ij}a_{n_1j_1}\cdots a_{n_\lambda j_\lambda}Y_1^{n_1}\cdots Y_\lambda^{n_\lambda}$$

in $X_2,\dots,X_\delta$, $Y_1,\dots,Y_\lambda$ and Z_{hk} $(2\le h\le\delta$, $1\le k\le\lambda)$ vanishes. We substitute $(w_1,\dots,w_\lambda)$ to $(Y_1,\dots,Y_\lambda)$, multiply by z_1^n and sum for $n\ge0$: for any complex tuple $(z_1, w_1, \ldots, w_\lambda)$, the polynomial

$$P(z_1, X_2, \ldots, X_\delta, w_1, \ldots, w_\lambda, f(z_1w_1), \ldots, f(z_1w_\lambda), Z_{21}, \ldots, Z_{\delta\lambda})$$

vanishes. According to the case $\delta = 1$ (which we take for granted thanks to the induction hypothesis) this implies $P = 0$. $\square$

From Corollary 2.5 we deduce the following special case of the Six Exponentials Theorem:

Corollary 2.7. *Let $\{x_1, \ldots, x_d\}$ and $\{y_1, \ldots, y_\ell\}$ be two sets of $\mathbb{Q}$-linearly independent real numbers, with $d\ell > d + \ell$. Then one at least of the $d\ell$ numbers $e^{x_iy_j}$ $(1 \le i \le d,\ 1 \le j \le \ell)$ is not in $\mathbb{Z}$.*

Proof. Assume that the $d\ell$ numbers $e^{x_iy_j}$ are all rational integers. Choose $f(z) = e^z$,

$$X = \{t_1x_1 + \ldots + t_dx_d\,;\,(t_1, \ldots, t_d) \in \mathbb{N}^d\}$$

and

$$Y = \{s_1y_1 + \ldots + s_\ell y_\ell\,;\,(s_1, \ldots, s_\ell) \in \mathbb{N}^\ell\}.$$

Then $e^{xy} \in \mathbb{Z}$ for any $x \in X$ and $y \in Y$. Using the assumption $d\ell > d+\ell$, we may apply Corollary 2.5 with $\varrho = 1$, $c_0 = 1$, $\alpha = d$, $\beta = \ell$,

$$c_1 = (|x_1| + \ldots + |x_d|)^{-d}, \qquad c_2 = (|y_1| + \ldots + |y_\ell|)^{-\ell}.$$

Since the exponential function is not a polynomial, we reach the expected contradiction. $\square$

3. Schneider's method with interpolation determinants

We compare the preceding results with those in [17]. Further, we also consider integer valued functions on a set of sums.

a) Cartesian products (again)

The following result is Proposition 4.1 of [17].

Theorem 3.1. *Consider two sequences $(x_0, x_1, x_2, \ldots)$ and $(y_0, y_1, y_2, \ldots)$ of pairwise distinct complex numbers with*

$$|x_0| \le |x_1| \le |x_2| \le \ldots, \quad \lim_{t\to\infty} |x_t| \to \infty,$$

$$|y_0| \le |y_1| \le |y_2| \le \ldots, \quad \lim_{s\to\infty} |y_s| \to \infty.$$

Let $1 < \theta_0 \le \theta_1 \le \theta_2 \le \ldots$ *be an increasing sequence of real numbers such that, if we set* $E_n = (\theta_n^2 + 1)/(2\theta_n)$, *then the sequence* $(n \log E_n)$ *is increasing. Let* φ *be an entire function in* $\mathbb{C}^2$ *satisfying*

$$\varphi(x_t, y_s) \in \mathbb{Z} \quad \text{for any } t \ge 0 \text{ and any } s \ge 0.$$

Define

$$M(t,s) = \log \sup\bigl\{ |\varphi(u,v)|;\ |u| = \theta_t |x_t|,\ |v| = \theta_s |y_s| \bigr\}.$$

Assume that for any sufficiently large integer L, *any sequence* $t_1 < t_2 < \ldots < t_L$ *of non-negative integers, any sequence* $s_1 < s_2 < \ldots < s_L$ *of non-negative integers, and any* $\tau \in \mathfrak{S}_L$,

$$\log(L!) + \sum_{\lambda=1}^{L} M\bigl(t_\lambda, s_{\tau(\lambda)}\bigr) < \max\Biggl\{ \sum_{\lambda=1}^{L} t_\lambda \log E_{t_\lambda},\ \sum_{\mu=1}^{L} s_\mu \log E_{s_\mu} \Biggr\}. \tag{3.1}$$

Then the $\mathbb{Q}$*-vector space* V *spanned by the functions*

$$w \longmapsto \varphi(x_t, w) \qquad (t \ge 0)$$

is finite dimensional, the same holds for the $\mathbb{Q}$*-vector space* W *spanned by the functions*

$$z \longmapsto \varphi(z, y_s) \qquad (s \ge 0),$$

and $\dim V = \dim W$.

b) Integer valued entire functions of a single variable

As it was already observed in [17], Corollary 2.7 also follows from Theorem 3.1. We now recall Theorem 1.1 of [17] (see also Theorem 3.1 in [16]).

Corollary 3.2. *Let* c_1, c_2, α, β *and* ϱ *be positive real numbers with*

$$\frac{1}{\alpha} + \frac{1}{\beta} \le \frac{1}{\varrho}. \tag{3.2}$$

There exists a positive real number η *with the following property. Let* f *be an entire function in* $\mathbb{C}$, *and let* X, Y *be subsets of* $\mathbb{C}$ *such that*

$$f(xy) \in \mathbb{Z} \quad \text{for any} \quad (x,y) \in X \times Y.$$

Assume, for all sufficiently large R,

$$\mathrm{Card}\bigl(X \cap D(0,R)\bigr) \ge c_1 R^\alpha, \qquad \mathrm{Card}\bigl(Y \cap D(0,R)\bigr) \ge c_2 R^\beta,$$

and

$$\log |f|_R \le \eta R^\varrho.$$

Then f *is a polynomial.*

It was shown in [17] (equation (5.2)) that

$$\eta = \frac{1}{20} c_1^{\varrho/\alpha} c_2^{\varrho/\beta} \min\left\{1, \frac{1}{\varrho^2}\right\}$$

is an admissible value.

A much stronger statement than Corollary 3.2 might be true, with (3.2) replaced by $\varrho \le \max\{\alpha, \beta\}$. To stay on the safe side, we only propose:

Conjecture 3.3. *Corollary* 3.2 *holds for any* $\eta > 0$.

A consequence of Conjecture 3.3 is the following special case of the Four Exponentials Conjecture (which is still open—see [16]):

Let a *and* b *be two positive integers which are multiplicatively independent, namely*

$$a^m b^n = 1 \quad \text{with } m \text{ and } n \text{ in } \mathbb{Z} \text{ implies} \quad m = n = 0.$$

Let t *be a real number. If* a^t *and* b^t *are both integers, then* $t \in \mathbb{Q}$.

Remark. Denote by $\overline{\mathbb{Q}}$ the set of algebraic numbers (algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$). Assuming $X \subset \overline{\mathbb{Q}}$ and taking derivatives would enable one to prove a variant of Corollary 3.2 which contains the Theorem of Gel'fond-Schneider on Hilbert's seventh problem. Moreover, assuming also $Y \subset \overline{\mathbb{Q}}$, one can get a generalization of Hermite-Lindemann's Theorem on the transcendence of e^α.

We now study a similar question for sums in place of products.

Corollary 3.4. *Given positive real numbers* ϱ *and* c*, there exists a positive constant* η *with the following property. Let* X*,* Y *be subsets of* $\mathbb{C}$ *and* f *an entire function of a single complex variable such that*

$$f(x+y) \in \mathbb{Z} \quad \text{for any} \quad (x,y) \in X \times Y.$$

Assume, for all sufficiently large R*,*

$$\operatorname{Card}\big(X \cap D(0,R)\big) \ge cR^\varrho, \qquad \operatorname{Card}\big(Y \cap D(0,R)\big) \ge cR^\varrho \tag{3.3}$$

and

$$\log|f|_R \le \eta R^\varrho.$$

Then f *is an exponential polynomial.*

Remark. We shall prove that Corollary 3.4 holds with

$$\eta = \frac{ec}{5 \cdot 2^{\varrho+3}} \min\left\{1, \frac{1}{\varrho^2}\right\}. \tag{3.4}$$

This formula is homogeneous, as it should, under the transformation $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^\times$ (in other terms there is no loss of generality to assume $c = 1$).

Applying Corollary 3.4 with $X = Y = \mathbb{N}\gamma_1 + \ldots + \mathbb{N}\gamma_\ell$ yields

Corollary 3.5. *Let $\gamma_1, \ldots, \gamma_\ell$ be complex numbers which are linearly independent over $\mathbb{Q}$. There exists a positive constant $\eta > 0$ with the following property. Let f be an entire function such that*

$$f(s_1\gamma_1 + \ldots + s_\ell\gamma_\ell) \in \mathbb{Z} \quad \textit{for any} \quad (s_1, \ldots, s_\ell) \in \mathbb{N}^\ell.$$

Assume, for all sufficiently large R,

$$\log |f|_R \leq \eta R^\ell.$$

Then f is an exponential polynomial.

Exponential polynomials satisfying the assumptions of Corollary 3.5 do occur: let $a_1, \ldots, a_\ell$ be multiplicatively independent positive integers with $\ell \geq 2$; define $\gamma_j = \log a_j$; then any polynomial in e^z satisfies the hypotheses of Corollary 3.5. Another simple example is $f(z) = z$ with $\ell = 1$ and $\gamma_1 = 1$.

There is no analog of Conjecture 3.3 for sets of sums: a first example with $\ell = 1$ is $f(z) = g(z)\sin(\pi z/\gamma_1)$ where g is any entire function of order < 1 which is not a polynomial. Other examples with $\ell = 2$ are given by Weierstraß sigma functions (i.e. canonical products associated with discrete lattices in $\mathbb{C}$).

Nevertheless a sharper estimate than (3.4) would have interesting consequences, in connection with the following open problem (which is again a special case of the Four Exponentials Conjecture):

Conjecture 3.6. *Let a and b be multiplicatively independent positive integers and c a positive integer. Then*

$$(\log a)(\log b) \neq (\log c)^2.$$

From Gel'fond-Schneider's Theorem, it follows that an equivalent formulation is:

> *Let $a \geq 2$ be a positive integer and $t > 0$ a real number. Assume that a^t and a^{t^2} are both integers. Then t is rational.*

Here is the connection with Corollary 3.5: take $f(z) = e^{z^2}$, $\varrho = 2$, $\ell = 2$, $\gamma_1 = \sqrt{\log a}$, $\gamma_2 = \sqrt{\log b}$. Unfortunately (3.4) gives a value of η which is < 1.

Another connection between integer valued functions on sets of sums on one hand, and the Six Exponentials Theorem on the other, involves the double exponential function e^{e^z}, together with the sets,

$$X = \bigl\{\log(t_1x_1 + \ldots + t_dx_d) + 2i\pi\tau\,;\, (t_1, \ldots, t_d) \in \mathbb{N}^d \setminus \{0\},\, \tau \in \mathbb{Z}\bigr\}$$

and

$$Y = \bigl\{\log(s_1y_1 + \ldots + s_\ell y_\ell) + 2i\pi\sigma\,;\, (s_1, \ldots, s_\ell) \in \mathbb{N}^\ell \setminus \{0\},\, \sigma \in \mathbb{Z}\bigr\}.$$

Indeed, one deduces from Theorem 3.1 the following variant of Corollary 3.4, which is another generalization of Corollary 2.7.

Corollary 3.7. *Let c_0, c_1, c_2, α, β and ϱ be positive real numbers with*

$$\frac{1}{\alpha}+\frac{1}{\beta}<\frac{1}{\varrho},$$

f an entire function in $\mathbb{C}$ and X, Y subsets of $\mathbb{C}$ such that

$$f(x+y)\in\mathbb{Z}\quad\text{for any}\quad(x,y)\in X\times Y.$$

Assume, for all sufficiently large R,

$$\log|f|_R\le c_0e^{\varrho R}$$

and

$$\operatorname{Card}\bigl(X\cap D(0,R)\bigr)\ge c_1e^{\alpha R},\qquad\operatorname{Card}\bigl(Y\cap D(0,R)\bigr)\ge c_2e^{\beta R}.$$

Then f is an exponential polynomial.

Remark. Corollary 2.5 can be deduced from Corollary 3.7 as follows: if f, X, Y satisfy the assumptions of Corollary 2.5, one checks that

$$\tilde f(z)=f(e^z),\quad\tilde X=\bigl\{\tilde x\in\mathbb{C};\ e^{\tilde x}\in X\bigr\},\quad\tilde Y=\bigl\{\tilde y\in\mathbb{C};\ e^{\tilde y}\in Y\bigr\}$$

satisfy the assumptions of Corollary 3.7 (there is no loss of generality to assume that $X\cap D(0,1)$ is empty). Therefore, according to Corollary 3.7, $\tilde f$ is an exponential polynomial. The conclusion then follows from the next lemma:

Lemma 3.8. *Let f be an entire function and let $\lambda\in\mathbb{C}$, $\lambda\ne0$. Assume that the function $f(e^{\lambda z})$ is an exponential polynomial. Then f is a polynomial.*

Proof. This follows from Lemma 3.1 of [5]. Though the proof is easy, we reproduce it here. The function $\tilde f(z)=f(e^{\lambda z})$ is periodic with period $\omega=2i\pi/\lambda$. Write

$$\tilde f(z)=a_1(z)e^{w_1z}+\ldots+a_s(z)e^{w_sz}$$

with $s\ge0$, non-zero polynomials $a_1,\ldots,a_s$ in $\mathbb{C}[z]$ and pairwise distinct complex numbers $w_1,\ldots,w_s$. Since the functions $z\mapsto e^{w_jz}$ $(1\le j\le s)$ are linearly independent over $\mathbb{C}[z]$, we deduce from the periodicity of $\tilde f$:

$$a_j(z)=e^{w_j\omega}a_j(z+\omega)\qquad(1\le j\le s).$$

Therefore each $w_j\omega/2i\pi=w_j/\lambda$ is a rational integer, say $m_j\in\mathbb{Z}$, and each $a_j(z)$ is a constant, say $a_j\in\mathbb{C}$. Define $d=\min\{m_j\,;\,1\le j\le k,\}$ so that $P(z)=a_1z^{m_1-d}+\ldots+a_sz^{m_s-d}$ is a polynomial with $P(0)\ne0$. Now

$$\tilde f(z)=e^{\lambda dz}P(e^{\lambda z}),$$

and therefore

$$f(z)=z^dP(z).$$

Since f is analytic at the origin, we conclude $d\ge0$ and $f\in\mathbb{C}[z]$. □

c) Proof of Corollary 3.4

We may assume f is not a constant. Since $f(X+Y)\subset\mathbb{Z}$, it follows that X and Y are both discrete subsets of $\mathbb{C}$. We order them with increasing modulus

$$X=\{x_0,x_1,\dots,x_t,\dots\}\quad\text{and}\quad Y=\{y_0,y_1,\dots,y_s,\dots\}$$

with

$$|x_0|\le|x_1|\le\dots\le|x_t|\le\dots\quad\text{and}\quad|y_0|\le|y_1|\le\dots\le|y_s|\le\dots$$

so that $|x_t|\to\infty$ as $t\to\infty$, and also $\lim_{s\to\infty}|y_s|=\infty$. From (3.3) we deduce, for sufficiently large t and s,

$$|x_t|\le c^{-1/\varrho}t^{1/\varrho}\quad\text{and}\quad|y_s|\le c^{-1/\varrho}s^{1/\varrho}.$$

Our first goal is to prove that the vector space V spanned over $\mathbb{Q}$ by the functions $w\mapsto f(x+w)$ $(x\in X)$ has finite dimension. We shall use Theorem 3.1. We take the sequence $(\theta_0,\theta_1,\dots)$ constant equal to $\theta=e^{\min\{1,1/\varrho\}}$, so that

$$\theta^{-\varrho}\log\left(\frac{\theta^2+1}{2\theta}\right)>\frac{e}{20}\min\left\{1,\frac{1}{\varrho^2}\right\}.$$

We assume η satisfies

$$2^{\varrho+1}\eta<c\theta^{-\varrho}\log\left(\frac{\theta^2+1}{2\theta}\right)$$

(compare with (3.4)). Define

$$R(t,s)=\theta(|x_t|+|y_s|)$$

and

$$M(t,s)=\log\sup\{|f(u+v)|;\ |u|=\theta|x_t|,\ |v|=\theta|y_s|\}.$$

For $|u|=\theta|x_t|$ and $|v|=\theta|y_s|$ we have $|u+v|\le R(t,s)$, hence

$$M(t,s)\le\log|f|_{R(t,s)}\le\eta R(t,s)^\varrho$$

as soon as t and s are sufficiently large. Since

$$R(t,s)\le\theta c^{-1/\varrho}(t^{1/\varrho}+s^{1/\varrho})\le 2\theta c^{-1/\varrho}\max\{t\,;\,s\}^{1/\varrho},$$

we have

$$R(t,s)^\varrho\le(2\theta)^\varrho c^{-1}\max\{t\,;\,s\}\le(2\theta)^\varrho c^{-1}(t+s)$$

and we deduce

$$M(t,s)\le\eta(2\theta)^\varrho c^{-1}(t+s)$$

provided that t and s are sufficiently large. However one easily checks that the same estimate holds as soon as $t+s$ is sufficiently large, say $t+s>c_3$. Define

$$c_4=\sum_{t+s\le c_3}M(t,s).$$

Then, for any $\tau \in \mathfrak{S}_L$,

$$\sum_{\lambda=1}^{L} M\big(t_\lambda, s_{\tau(\lambda)}\big) \leq c_4 + \eta(2\theta)^{\varrho} c^{-1} \left(\sum_{\lambda=1}^{L} t_\lambda + \sum_{\mu=1}^{L} s_\mu\right)$$
$$\leq c_4 + 2\eta(2\theta)^{\varrho} c^{-1} \max\left\{\sum_{\lambda=1}^{L} t_\lambda \,;\, \sum_{\mu=1}^{L} s_\mu\right\}.$$

Since the number

$$\epsilon = \log\left(\frac{\theta^2+1}{2\theta}\right) - 2\eta(2\theta)^{\varrho} c^{-1}$$

is positive, for L sufficiently large, we may bound $\log(L!)$ by

$$\log(L!) \leq \frac{\epsilon}{4} L(L-1) \leq \frac{\epsilon}{2} \sum_{\lambda=1}^{L} t_\lambda.$$

Therefore (3.1) is satisfied, and we deduce from Theorem 3.1 that V has finite dimension.

Denote by $\mathcal{V}$ the vector space spanned over $\mathbb{C}$ by the functions $w \mapsto f(x+w)$. Of course $\mathcal{V}$ has also finite dimension. Let $f(\xi_1+w), \ldots, f(\xi_n+w)$ be a basis of $\mathcal{V}$. For each $x \in X$ there exist complex numbers $c_i(x)$ $(1 \leq i \leq n)$ such that

$$f(x+w) = c_1(x) f(\xi_1+w) + \ldots + c_n(x) f(\xi_n+w).$$

Since the functions $f(\xi_1+w), \ldots, f(\xi_n+w)$ are linearly independent over $\mathbb{C}$, there exist complex numbers $\zeta_1, \ldots, \zeta_n$ such that the matrix $\big(f(\xi_i+\zeta_j)\big)_{1\leq i,j\leq n}$ is non-singular. Therefore there exist complex numbers γ_{ij} $(1 \leq i, j \leq n)$ such that

$$c_i(x) = \gamma_{i1} f(x+\zeta_1) + \ldots + \gamma_{in} f(x+\zeta_n).$$

We deduce

$$f(x+w) = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} f(x+\zeta_j) f(\xi_i+w).$$

Fix $w \in \mathbb{C}$. The function

$$F : z \longmapsto f(z+w) - \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} f(z+\zeta_j) f(\xi_i+w)$$

vanishes on X. We apply Lemma 1.4: if F does not vanish identically, then for any sufficiently large r,

$$n(F,r) \log\left(\frac{\theta^2+1}{2\theta}\right) < \log|F|_{\theta r}.$$

Put

$$c_1 = 1 + \sum_{i=1}^{n} \sum_{j=1}^{n} |\gamma_{ij} f(\xi_i+w)| \quad \text{and} \quad c_2 = \max\big\{|w|, |\zeta_1|, \ldots, |\zeta_n|\big\},$$

so that, for any $R > 0$,

$$|F|_R \le c_1 |f|_{R+c_2}.$$

We apply this estimate with $R = \theta r$, and r sufficiently large so that

$$\eta(\theta + c_2 r^{-1})^\varrho + r^{-\varrho} \log c_1 < c \log\left(\frac{\theta^2+1}{2\theta}\right).$$

Therefore

$$n(F, r) < c r^\varrho,$$

which is not compatible with (3.3).

Hence f satisfies the functional equation

$$f(z+w) = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} f(z+\zeta_j) f(\xi_i + w).$$

We take derivatives with respect to z, and then set $z = 0$. It follows that the $n+1$ functions $f, f', f'', \dots, f^{(n)}$ belong to $\mathcal{V}$, hence they are linearly dependent, and f satisfies an homogeneous linear differential equation with constant coefficients. Therefore f is an exponential polynomial. □

The proof of Corollary 3.7 is similar and is left to the reader (Hint: select θ in the range

$$1 < \theta < \frac{\alpha\beta}{\varrho(\alpha+\beta)}$$

and apply Theorem 3.1 with $\theta_n = \theta$ for all $n \ge 0$).

d) On a Theorem of Gel'fond

A Theorem of A.O. Gel'fond states that *for $\beta \in \mathbb{Z}$ with $|\beta| > 1$, an entire function f which satisfies $f(\beta^n) \in \mathbb{Z}$ for any $n \in \mathbb{N}$ and*

$$|f|_R < \epsilon(R) \exp\left(\frac{1}{4} \frac{(\log R)^2}{\log|\beta|}\right)$$

for all sufficiently large R, with $\epsilon(R) \to 0$ for $R \to \infty$, is a polynomial (see [4]; see also [1]). A similar result, without the assumption $\beta \in \mathbb{Z}$, is due to F. Gramain: *for $q \in \mathbb{C}$ with $|q| > 1$, an entire function f which satisfies $f(q^n) \in \mathbb{Z}$ for any $n \in \mathbb{N}$ and*

$$\log|f|_R < \eta \frac{(\log R)^2}{\log|q|}$$

for all sufficiently large R with $\eta < 0.079730\dots$ is a polynomial (as pointed out by F. Gramain, this is in fact what is proved in [5], Proposition 4.3). Using Gramain's method, P. Bundschuh [2] produced another proof of Gel'fond's result on entire functions f satisfying $f(\beta^n) \in \mathbb{Z}$ when $\beta \ge 2$ is an integer, but he needs the assumption

$$\log|f|_R < \eta \frac{(\log R)^2}{\log \beta}$$

with $\eta = 1/13.657\ldots$

Following a suggestion of J.-P. Bézivin, we deduce from Theorem 3.1 the following weaker form of Gramain's result.

Corollary 3.9. *Let $q \in \mathbb{C}$ satisfy $|q| > 1$ and let f be an entire function such that $f(q^n) \in \mathbb{Z}$ for any $n \in \mathbb{N}$. Assume that there exists $\eta < 1/16$ such that, for all sufficiently large R*

$$\log|f|_R < \eta \frac{(\log R)^2}{\log|q|}. \tag{3.5}$$

Then f is a polynomial.

Proof. We first deduce from Corollary 3.5 the weaker assertion with $\eta < 1/16$ replaced by $\eta < \eta_0$ where $\eta_0 > 0$ may depend on q. Choose a complex logarithm $\log q$ of q and consider the entire function $\tilde{f}(t) = f(q^t)$, which is periodic with period $2i\pi/\log q$. Put $\ell = 2$, $\gamma_1 = 1$, $\gamma_2 = 2i\pi/\log q$. From (3.5) we deduce

$$\log|\tilde{f}|_R < \eta \frac{|\log q|^2}{\log|q|} R^2.$$

Assuming η is sufficiently small (depending on q), Corollary 3.5 then shows that the entire function $\tilde{f}$ is an exponential polynomial. From Lemma 3.8 (with $\lambda = \log q$) we deduce $f \in \mathbb{C}[z]$.

In order to check that assumption (3.5) with any $\eta < 1/16$ is sufficient, we shall apply Theorem 3.1 to the function $\varphi(u,v) = f(uv)$ with $x_t = q^t$, $y_s = q^s$ and $\theta_n = |q|^n$ for $n \geq 1$ and $\theta_0 = |q|$.

One easily checks that the sequence $E_n = (\theta_n^2+1)/(2\theta_n)$ is increasing. Moreover, for sufficiently large n, say for $n \geq n_0$, we have $\log E_n \geq (1-\epsilon)n\log|q|$.

Let $R_0 > 0$ be such that (3.5) holds for $R \geq R_0$. Define $K = \log|f|_{R_0}$. Further, select ϵ in the range $0 < \epsilon < 1 - 16\eta$.

Let L_0 be sufficiently large so that, for any $L \geq L_0$,

$$L(K + \log L) + (1-\epsilon)(\log|q|)\sum_{n=1}^{n_0} n^2 < (1-\epsilon-16\eta)(\log|q|)\sum_{n=1}^{L} n^2. \tag{3.6}$$

For $|u| = \theta_t|x_t|$ and $|v| = \theta_s|y_s|$, we have $|uv| = |q|^{2(t+s)}$ and

$$\log|\varphi(u,v)| = \log|f(uv)| \leq \max\left\{K\,,\,\eta\frac{(\log R)^2}{\log|q|}\right\} \quad \text{with} \quad R = |q|^{2(t+s)}.$$

We deduce the estimate, valid for all $t \geq 0$ and $s \geq 0$:

$$M(t,s) \leq 4\eta(t+s)^2\log|q| + K.$$

Let $L \geq L_0$ be a positive integer, $t_1 < t_2 < \ldots < t_L$, and $s_1 < s_2 < \ldots < s_L$ two sequences of non-negative integers and $\tau \in \mathfrak{S}_L$. Using the estimate (in the Euclidean space $\mathbb{R}^L$)

$$(x_1 + y_1)^2 + \ldots + (x_L + y_L)^2 \leq 4 \max\{x_1^2 + \ldots + x_L^2\,,\, y_1^2 + \ldots + y_L^2\}$$

which holds for any real numbers $x_1, \ldots, x_L, y_1, \ldots, y_L$, we deduce

$$\begin{aligned}\log(L!) + \sum_{\lambda=1}^{L} M\big(t_\lambda, s_{\tau(\lambda)}\big) &\leq L \log L + \sum_{\lambda=1}^{L} \Big(4\eta(t_\lambda + s_{\tau(\lambda)})^2 \log|q| + K\Big) \\ &\leq L(K + \log L) + 16\eta(\log|q|) \max\Big\{\sum_{\lambda=1}^{L} t_\lambda^2\,;\, \sum_{\mu=1}^{L} s_\mu^2\Big\}.\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{\lambda=1}^{L} t_\lambda \log E_{t_\lambda} &\geq (1-\epsilon)(\log|q|) \sum_{\substack{\lambda=1 \\ t_\lambda \geq n_0}}^{L} t_\lambda^2 \\ &\geq (1-\epsilon)(\log|q|) \sum_{\lambda=1}^{L} t_\lambda^2 - (1-\epsilon)(\log|q|) \sum_{n=1}^{n_0} n^2.\end{aligned}$$

Using (3.6), we deduce that (3.1) holds. Hence the functions $f(q^n z)$ $(n \geq 0)$ span a $\mathbb{Q}$-vector space of finite dimension. This implies that f is solution of a functional equation

$$f(q^n z) = a_1 f(q^{n-1} z) + \ldots + a_{n-1} f(qz) + a_n f(z).$$

The conclusion that f is a polynomial can now be achieved by three different ways: either one considers the growth of $R \mapsto |f|_R$, or (like P. Bundschuh in [2]) one looks at the Taylor expansion of f at the origin, or else, following [5], one studies the periodic function $f(q^z)$ which satisfies a functional equation. $\square$

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