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Density measure of rational points on abelian varieties

by

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Abstract

Let $\mathcal{A}$ be a simple abelian variety of dimension g over $\mathbf{Q}$, and let ℓ be the rank of the Mordell-Weil group $\mathcal{A}(\mathbf{Q})$. Assume $\ell \geq 1$. A conjecture of Mazur asserts that the closure of $\mathcal{A}(\mathbf{Q})$ into $\mathcal{A}(\mathbf{R})$ for the real topology contains the neutral component $\mathcal{A}(\mathbf{R})^0$ of the origin. This is known only under the extra hypothesis $\ell \geq g^2 - g + 1$. We investigate here a quantitative refinement of this question: for each given positive H, the set of points in $\mathcal{A}(\mathbf{Q})$ of height $\leq H$ is finite, and we study how these points are distributed into $\mathcal{A}(\mathbf{R})^0$. More generally we consider an abelian variety $\mathcal{A}$ over a number field K embedded in $\mathbf{R}$, and a subgroup Γ of $\mathcal{A}(K)$ of sufficiently large rank. The effective result of density we obtain relies on an estimate of diophantine approximation, namely a lower bound for linear combinations of determinants involving abelian logarithms.

§1. Introduction

Let K be a number field with a given real embedding. Let V be a smooth variety over K. Denote by Z the closure, for the real topology, of $V(K)$ in $V(\mathbf{R})$. In his paper [9] on the topology of rational points, Mazur assumes that $K = \mathbf{Q}$ and that $V(\mathbf{Q})$ is Zariski dense; he asks whether Z is a union of connected components of $V(\mathbf{R})$. However an example is given in [2]of a smooth surface V over $\mathbf{Q}$, whose $\mathbf{Q}$-rational points are Zariski-dense, but such that the closure Z in $V(\mathbf{R})$ of the set of $\mathbf{Q}$-points is not a union of connected components.

We consider here a quantitative refinement. Assume that V is embedded, as a quasi-projective variety, into a projective space $\mathbf{P}_N$ over K. Denote by h the absolute logarithmic height (Weil height) on $\mathbf{P}_N(K)$. Further, choose a metric on $\mathbf{P}_N$. For each sufficiently large real number H, define $\eta_V(H)$ as follows:

$$\begin{aligned}\eta_V(H) = \inf\bigl\{\epsilon > 0\ ;&\\ &\text{for any } \zeta \in Z, \text{ there exists } \gamma \in V(K) \text{ with } \mathrm{h}(\gamma) \leq \log H \text{ and } \mathrm{dist}(\zeta,\gamma) \leq \epsilon\bigr\}.\end{aligned}$$

Since $V(K)$ is dense in Z,

$$\lim_{H\to\infty} \eta_V(H) = 0.$$

An upper bound for $\eta_V(H)$ can be considered as a measure of the density in Z of the rational points $V(K)$.

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