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Extrapolation with interpolation determinants

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Abstract

Let $f_1, \ldots, f_L$ be analytic functions and $\zeta_1, \ldots, \zeta_L$ be points where the functions are defined. Sharp upper bounds are known for the absolute value of the determinant of the $L \times L$ matrix

$$\Bigl(f_\lambda(\zeta_\mu) \Bigr)_{1 \le \lambda, \mu \le L}.$$

These estimates play an important role in transcendental number theory. We give a report on this topic and we announce a new application to integer valued entire functions on a set of products.

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