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Syntaxic identities among harmonic series and automata

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`http://www.math.jussieu.fr/~miw/articles/pdf/graz.pdf`

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I Cartesian Product of Automata and Shuffle

Alphabet : X **Free monoid** : X^* (words)

Automaton associated with a word $x_1x_2\cdots x_n \in X^*$:

$$\Longrightarrow \boxed{1} \xrightarrow{x_1} \boxed{2} \xrightarrow{x_2} \boxed{3} \xrightarrow{x_3} \cdots \xrightarrow{x_n} \boxed{n+1} \Longrightarrow$$

Series associated with the automaton

$$\begin{tikzcd} \arrow[r, Rightarrow] & \boxed{1} \arrow[l, Rightarrow] \arrow[r, bend right, "a"'] & \boxed{2} \arrow[l, bend right, "b"'] \end{tikzcd}$$

is

$$S_1 = e + ab + (ab)^2 + \cdots + (ab)^n + \cdots = (ab)^\star,$$

The automata associated with a and b:

$$\Longrightarrow \boxed{1} \xrightarrow{a} \boxed{2} \Longrightarrow \qquad \Longrightarrow \boxed{A} \xrightarrow{b} \boxed{B} \Longrightarrow$$

have Cartesian product

$$\begin{tikzcd} \arrow[r, Rightarrow] & \boxed{1A} \arrow[r, "a"] \arrow[d, "b"'] & \boxed{2A} \arrow[d, "b"] & \\ & \boxed{1B} \arrow[r, "a"'] & \boxed{2B} \arrow[r, Rightarrow] & {} \end{tikzcd}$$

associated with

$$a ш b = ab + ba.$$

Cartesian product of

$$\Longrightarrow \boxed{A_0} \xrightarrow{x_1} \boxed{A_1} \xrightarrow{x_2} \boxed{A_2} \xrightarrow{x_3} \cdots \xrightarrow{x_n} \boxed{A_n} \Longrightarrow$$

and

$$\Longrightarrow \boxed{B_0} \xrightarrow{y_1} \boxed{B_1} \xrightarrow{y_2} \boxed{B_2} \xrightarrow{y_3} \cdots \xrightarrow{y_k} \boxed{B_k} \Longrightarrow$$

$$\begin{tikzcd}
\Longrightarrow \boxed{A_0B_0} \arrow[r, "x_1"] \arrow[d, "y_1"] & \boxed{A_1B_0} \arrow[r, "x_2"] \arrow[d, "y_1"] & \cdots \arrow[r, "x_n"] & \boxed{A_nB_0} \arrow[d, "y_1"] \\
\boxed{A_0B_1} \arrow[r, "x_1"] \arrow[d, "y_2"] & \boxed{A_1B_1} \arrow[r, "x_2"] \arrow[d, "y_2"] & \cdots \arrow[r, "x_n"] & \boxed{A_nB_1} \arrow[d, "y_2"] \\
\vdots \arrow[d, "y_k"] & \vdots \arrow[d, "y_k"] & & \vdots \arrow[d, "y_k"] \\
\boxed{A_0B_k} \arrow[r, "x_1"] & \boxed{A_1B_k} \arrow[r, "x_2"] & \cdots \arrow[r, "x_n"] & \boxed{A_nB_k} \Longrightarrow
\end{tikzcd}$$

Shuffle Product

$$x_1x_2\cdots x_n ш y_1y_2\cdots y_k$$

Definition by induction

$$xu ш yv = x(u ш yv) + y(xu ш v)$$

Lemma. *The following syntaxic identity holds:*

$$(x_0x_1)^\star ш (-x_0x_1)^\star = (-4x_0^2x_1^2)^\star.$$

Proof. The series associated to the automaton

$$\begin{tikzcd}
\Longleftarrow \atop \Longrightarrow \boxed{1} \arrow[r, shift right, "x_0"'] & \boxed{2} \arrow[l, shift right, "x_1"']
\end{tikzcd}$$

is

$$S_1 = e + x_0x_1 + (x_0x_1)^2 + \cdots + (x_0x_1)^n + \cdots = (x_0x_1)^\star,$$

while the series associated to

$$\begin{tikzcd}
\Longleftarrow \atop \Longrightarrow \boxed{A} \arrow[r, shift right, "-x_0"'] & \boxed{B} \arrow[l, shift right, "x_1"']
\end{tikzcd}$$

is

$$S_A = e - x_0x_1 + (x_0x_1)^2 + \cdots + (-x_0x_1)^n + \cdots = (-x_0x_1)^\star.$$

The following automaton is the Cartesian product of the automata associated with S_1 and S_A:

$$\begin{tikzcd}
\Longleftarrow \atop \Longrightarrow \boxed{1A} \arrow[r, shift right, "x_0"'] \arrow[d, shift right, "-x_0"'] & \boxed{2A} \arrow[l, shift right, "x_1"'] \arrow[d, shift right, "-x_0"'] \\
\boxed{1B} \arrow[u, shift right, "x_1"'] \arrow[r, shift right, "x_0"'] & \boxed{2B} \arrow[l, shift right, "x_1"'] \arrow[u, shift right, "x_1"']
\end{tikzcd}$$

One computes the associated series $S_{1A} = S_1 ш S_A$ by solving a system of linear (noncommutative) equations as follows. Define also S_{1B}, S_{2A} and S_{2B} as the series of labels of the paths starting at the corresponding state and ending at a terminal state. Then

$$\begin{aligned}
S_{1A} &= e - x_0S_{1B} + x_0S_{2A},\\
S_{1B} &= x_1S_{1A} + x_0S_{2B},\\
S_{2A} &= x_1S_{1A} - x_0S_{2B},\\
S_{2B} &= x_1S_{1B} + x_1S_{2A}.
\end{aligned}$$

One deduces

$$S_{1A} = e - x_0(S_{1B} - S_{2A}), \quad S_{1B} - S_{2A} = -2x_0S_{2B},$$

$$S_{2B} = x_1(S_{1B} + S_{2A}), \quad S_{1B} + S_{2A} = 2x_1S_{1A}$$

and therefore

$$S_{1A} = e + 4x_0^2x_1^2S_{1A},$$

Other syntaxic identities:

$$a^* ш (x_1 \cdots x_n) = a^* x_1 a^* x_2 \cdots a^* x_n a^*$$

$$a ш (bc)^* = (bc)^*(a + bac)(bc)^*, \qquad a ш w^* = a + w^*(a ш w) w^*$$

$$(a+b)^* = a^* ш b^*$$

$$(1+a) ш a^* = (a^*)^2$$

$$a^* ш (ab)^* = (a + aa^*b)^* = (2a + ab - a^2)^*(1-a)$$

II Polylogarithms

$s \in \mathbb{Z}$, $s \geq 1$
$z \in \mathbb{C}$, $|z| < 1$

$$\mathrm{Li}_s(z) = \sum_{n \geq 1} \frac{z^n}{n^s}.$$

For $s \geq 2$, $\zeta(s) = \mathrm{Li}_s(1)$.

Differential equations:

$$\frac{d}{dz}\mathrm{Li}_s(z) = \frac{1}{z}\mathrm{Li}_{s-1}(z) \qquad (s \geq 2),$$
$$\frac{d}{dz}\mathrm{Li}_1(z) = \frac{1}{1-z} \qquad (s = 1).$$

Initial conditions $\mathrm{Li}_s(0) = 0$.

$$\mathrm{Li}_1(z) = \sum_{n \geq 1} \frac{z^n}{n} = -\log(1-z) = \int_0^z \frac{dt}{1-t},$$

$$\mathrm{Li}_2(z) = \int_0^z \mathrm{Li}_1(t)\frac{dt}{t} = \int_0^z \frac{dt}{t}\int_0^t \frac{du}{1-u},$$

and by induction, for $s \geq 2$,

$$\mathrm{Li}_s(z) = \int_0^z \mathrm{Li}_{s-1}(t)\frac{dt}{t} = \int_0^z \frac{dt_1}{t_1}\int_0^{t_1} \frac{dt_2}{t_2} \cdots \int_0^{t_{s-2}} \frac{dt_{s-1}}{t_{s-1}} \int_0^{t_{s-1}} \frac{dt_s}{1-t_s}.$$

III Chen Iterated Integrals

$$\int_0^z \varphi_1 \cdots \varphi_k := \int_0^z \varphi_1 \cdots \varphi_k = \int_0^z \varphi_1(t) \int_0^t \varphi_2 \cdots \varphi_k.$$

$$\mathrm{Li}_s(z) = \int_0^z \omega_0^{s-1}\omega_1,$$

with

$$\omega_0 = \frac{dt}{t}, \qquad \omega_1 = \frac{dt}{1-t}$$

Product of iterated integrals:

Let $\varphi_1,\dots,\varphi_n$, $\psi_1,\dots,\psi_k$ be differential forms with $n\ge 0$ and $k\ge 0$. Then

$$\int_0^z \varphi_1\cdots\varphi_n \int_0^z \psi_1\cdots\psi_k = \int_0^z \varphi_1\cdots\varphi_n ш \psi_1\cdots\psi_k.$$

Proof. Decompose the Cartesian product

$$\{\underline{t}\in\mathbb{R}^n\ ;\ z\ge t_1\ge\cdots\ge t_n\ge 0\}\times\{\underline{u}\in\mathbb{R}^k\ ;\ z\ge u_1\ge\cdots\ge u_k\ge 0\}$$

into a disjoint union of simplices (up to sets of zero measure)

$$\{\underline{v}\in\mathbb{R}^{n+k}\ ;\ z\ge v_1\ge\cdots\ge v_{n+k}\ge 0\}.$$

We consider the product of two polylogarithms:

$$\mathrm{Li}_s(z)\mathrm{Li}_{s'}(z)=\int_0^z\omega_s\int_0^z\omega_{s'}$$

where

$$\omega_s=\omega_0^{s-1}\omega_1.$$

We need more general polylogarithms related to $\omega_s ш \omega_{s'}$ involving

$$\omega_{\underline{s}}=\omega_{s_1}\cdots\omega_{s_k}=\omega_0^{s_1-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1.$$

for $\underline{s}=(s_1,\dots,s_k)$.

IV Multiple Polylogarithms in One Variable

For k, $s_1,\dots,s_k$ be positive integers and $z\in\mathbb{C}$, $|z|<1$, define

$$\mathrm{Li}_{\underline{s}}(z)=\sum_{n_1>n_2>\cdots>n_k\ge 1}\frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}}.$$

In case $s_1\ge 2$, set $\zeta(\underline{s})=\mathrm{Li}_{\underline{s}}(1)$:

$$\zeta(\underline{s})=\sum_{n_1>n_2>\cdots>n_k\ge 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}.$$

For $k=1$ one recovers the usual $\mathrm{Li}_s(z)$ and $\zeta(s)$.

Inductive definition:

$$\frac{d}{dz}\mathrm{Li}_{(s_1,\dots,s_k)}(z)=\frac{1}{z}\mathrm{Li}_{(s_1-1,s_2,\dots,s_k)}(z)\qquad(s_1\ge 2)$$
$$\frac{d}{dz}\mathrm{Li}_{(1,s_2,\dots,s_k)}(z)=\frac{1}{1-z}\mathrm{Li}_{(s_2,\dots,s_k)}(z)\qquad(s_1=1).$$

Recall

$$\omega_{\underline{s}}=\omega_0^{s_1-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1.$$

Hence

$$\mathrm{Li}_{\underline{s}}(z)=\int_0^z\omega_{\underline{s}}.$$

Product of multiple polylogarithms

Example.
$$\omega_1 ш \omega_0\omega_1 = 2\omega_0\omega_1^2 + \omega_1\omega_0\omega_1$$
$$\begin{aligned}\mathrm{Li}_1(z)\mathrm{Li}_2(z) &= \int_0^z \omega_1 \int_0^z \omega_0\omega_1 = \int_0^z \omega_1 ш \omega_0\omega_1\\ &= 2\int_0^z \omega_0\omega_1^2 + \int_0^z \omega_1\omega_0\omega_1\\ &= 2\mathrm{Li}_{(2,1)}(z) + \mathrm{Li}_{(1,2)}(z).\end{aligned}$$

$$\omega_0\omega_1 ш \omega_0\omega_1 = 4\omega_0^2\omega_1^2 + 2\omega_0\omega_1\omega_0\omega_1$$
$$\mathrm{Li}_2(z)^2 = 4\mathrm{Li}_{(3,1)}(z) + 2\mathrm{Li}_{(2,2)}(z).$$
$$\zeta(2)^2 = 4\zeta(3,1) + 2\zeta(2,2).$$

Example. For any $n \geq 1$,
$$\mathrm{Li}_{\{1\}_n}(z) = \frac{1}{n!}\big(-\log(1-z)\big)^n$$
where
$$\{a\}_n = (a, a, \dots, a) \text{ with } n \text{ occurences of } a.$$

Proof.
$$\omega_1^{n-1} ш \omega_1 = n\omega_1^n$$
$$\mathrm{Li}_{\{1\}_{n-1}}(z)\mathrm{Li}_1(z) = n\mathrm{Li}_{\{1\}_n}(z)$$
hence
$$\omega_1^{шn} = n!\omega_1^n$$
$$\big(\mathrm{Li}_1(z)\big)^n = n!\mathrm{Li}_{\{1\}_n}(z).$$

The Algebra $\mathfrak{H} = \mathbb{Q}\langle x_0, x_1\rangle$ and the Subalgebras $\mathfrak{H}^1 = \mathbb{Q}e + \mathfrak{H}x_1$ and $\mathfrak{H}^0 = \mathbb{Q}e + x_0\mathfrak{H}x_1$.

Define $y_s = x_0^{s-1}x_1$ for $s \geq 1$, $Y = \{y_1, y_2, y_3 \dots\}$ and
$$y_{\underline{s}} = y_{s_1}\cdots y_{s_k} = x_0^{s_1-1}x_1\cdots x_0^{s_k-1}x_1$$
so that $Y^* = \{y_{\underline{s}}\ ;\ \underline{s} = (s_1, \dots, s_k)\}$.
Let $x = x_{\epsilon_1}\cdots x_{\epsilon_p} \in X^*x_1$ be a word in x_0 and x_1 which ends with x_1: each ϵ_i is 0 or 1 and $\epsilon_p = 1$. If k is the number of x_1, we define $s_1, \dots, s_k$ by
$$x = x_0^{s_1-1}x_1\cdots x_0^{s_k-1}x_1.$$
Hence
$$\mathfrak{H}^1 = \mathbb{Q}e + \mathfrak{H}x_1 = \mathbb{Q}\langle Y\rangle.$$

For $\underline{s} = (s_1, \dots, s_k)$, set
$$\widehat{\mathrm{Li}}_{y_{\underline{s}}}(z) = \mathrm{Li}_{\underline{s}}(z).$$
This defines $\widehat{\mathrm{Li}}_x(z)$ when $x \in X^*x_1$:
$$\widehat{\mathrm{Li}}_x(z) = \int_0^z \omega_{\epsilon_1}\cdots\omega_{\epsilon_p}$$
when $x = x_{\epsilon_1}\cdots x_{\epsilon_p}$, where each ϵ_i is 0 or 1, and $\epsilon_p = 1$.
The subalgebra $\mathfrak{H}^0$ spanned by $x_0X^*x_1$ (set of words starting with x_0 and ending with x_1) is
$$\mathfrak{H}^0 = \mathbb{Q}e + x_0\mathfrak{H}x_1 = \mathbb{Q}\langle y_2, y_3, \dots\rangle.$$
For $\underline{s} = (s_1, \dots, s_k)$ with $s_1 \geq 2$, set
$$\widehat{\zeta}(y_{\underline{s}}) = \zeta(\underline{s}).$$
This defines $\widehat{\zeta}(w)$ for $w \in x_0X^*x_1$.

By linearity, extend the definition of $\widehat{\mathrm{Li}}_w(z)$ to $w \in \mathfrak{H}^1$ and the definition of $\widehat{\zeta}(w)$ to $w \in \mathfrak{H}^0$. Notice that, for $w \in \mathfrak{H}^0$,

$$\widehat{\zeta}(w) = \widehat{\mathrm{Li}}_w(1).$$

Proposition. *For any w and w' in $\mathfrak{H}^1$,*

$$\widehat{\mathrm{Li}}_w(z)\widehat{\mathrm{Li}}_{w'}(z) = \widehat{\mathrm{Li}}_{w ш w'}(z).$$

In particular, for $z = 1$, we find

$$\widehat{\zeta}(w)\widehat{\zeta}(w') = \widehat{\zeta}(w ш w')$$

for any w and w' in $\mathfrak{H}^0$.

Hoang Ngoc Minh, M. Petitot, van der Hoeven:
The map $w \mapsto \mathrm{Li}_w(z)$ defines an **injective** homomorphism of algebras from $\mathfrak{H}^1_ш$ into the algebra of analytic functions in the unit disc.

The proof rests on the study of the monodromy of the Knizhnik-Zamolodchikov differential equation

$$\frac{d}{dz}\widehat{\mathrm{Li}}(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)\widehat{\mathrm{Li}}(z).$$

A solution satisfying the initial condition

$$\lim_{z\to 0} e^{-x_0 \log z}\widehat{\mathrm{Li}}(z) = 1$$

is the generating series

$$\widehat{\mathrm{Li}}(z) = \sum_{w\in X^*} \mathrm{Li}_w(z) w,$$

with a suitable definition of $\mathrm{Li}_w(z)$ for $w \in X^*$, $w \notin X^* x_1$.

V The Harmonic Algebra

For $s \geq 2$ and $s' \geq 2$:

$$\sum_{n\geq 1} n^{-s} \sum_{m\geq 1} m^{-s'} = \sum_{n>m\geq 1} n^{-s} m^{-s'} + \sum_{m>n\geq 1} m^{-s'} n^{-s} + \sum_{n\geq 1} n^{-s-s'},$$

$$\zeta(s)\zeta(s') = \zeta(s,s') + \zeta(s',s) + \zeta(s+s')$$

For instance

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4).$$

First on X^*, the map $\star : X^* \times X^* \to \mathfrak{H}$ is defined by induction, starting with

$$x_0^n \star w = w \star x_0^n = w x_0^n$$

for any $w \in X^*$ and any $n \geq 0$ (for $n = 0$ it means $e \star w = w \star e = w$ for all $w \in X^*$), and then

$$y_s u \star y_t v = y_s(u \star y_t v) + y_t(y_s u \star v) + y_{s+t}(u \star v)$$

for u and v in X^*, s and t positive integers.

Example.

$$y_2^{\star 3} = y_2 \star y_2 \star y_2 = 6y_2^3 + 3y_2y_4 + 3y_4y_2 + y_6.$$

Lemma. *The following syntaxic identity holds:*

$$y_2^* \star (-y_2)^* = (-y_4)^*.$$

Proof. Denote by $\underline{t} = (t_1, t_2, \ldots)$ a sequence of commutative variables. Consider the quasisymmetric series

$$\phi(y_{\underline{s}}) = \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^{s_1} \cdots t_{n_k}^{s_k}$$

and extend by linearity. Then

$$\phi(u \star v) = \phi(u)\phi(v).$$

On the other hand

$$\phi(y_2^*) = \sum_{k=0}^{\infty} \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^2 \cdots t_{n_k}^2,$$

$$\phi((-y_2)^*) = \sum_{k=0}^{\infty} (-1)^k \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^2 \cdots t_{n_k}^2$$

$$\phi((-y_4)^*) = (-1)^k \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^4 \cdots t_{n_k}^4.$$

Hence from the identity

$$\prod_{n=1}^{\infty} (1 + t_n t) = \sum_{k=0}^{\infty} t^k \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1} \cdots t_{n_k}$$

one deduces

$$\phi(y_2^*) = \prod_{n=1}^{\infty} (1 + t_n^2), \quad \phi((-y_2)^*) = \prod_{n=1}^{\infty} (1 - t_n^2)$$

$$\phi((-y_4)^*) = \prod_{n=1}^{\infty} (1 - t_n^4),$$

which implies the Lemma.

Here is the Zagier-Broadhurst formula.

Theorem. *For any $n \geq 1$,*

$$\zeta(\{3,1\}_n) = 4^{-n} \zeta(\{4\}_n).$$

This formula was originally conjectured by D. Zagier and first proved by D. Broadhurst.

Remark.

$$\zeta(\{2\}_n) = \frac{\pi^{2n}}{(2n+1)!}$$

and

$$\frac{1}{2n+1} \zeta(\{2\}_{2n}) = \frac{1}{2^{2n}} \zeta(\{4\}_n).$$

hence

$$\zeta(\{3,1\}_n) = 2 \cdot \frac{\pi^{4n}}{(4n+2)!}.$$

Proof.

Goal: to prove

$$y_4^n - (4 y_3 y_1)^n \in \ker \widehat{\zeta}.$$

From

$$y_2^* \star (-y_2)^* = (-y_4)^* \quad \text{and} \quad y_2^* ш (-y_2)^* = (-4 y_3 y_1)^*$$

one deduces, for any $n \geq 1$,

$$\sum_{i+j=2n} (-1)^j y_2^{2i} \star y_2^{2j} = (-y_4)^n$$

and

$$\sum_{i+j=2n} (-1)^j y_2^{2i} ш y_2^{2j} = (-4 y_3 y_1)^n,$$

hence

$$y_4^n - (4 y_3 y_1)^n = \sum_{i+j=2n} (-1)^{n-j} (y_2^{2i} \star y_2^{2j} - y_2^{2i} ш y_2^{2j}) \in \ker \widehat{\zeta}.$$

VI Hoffman Third Standard Relations

$$\widehat{\zeta}(x_1 ш w - x_1 \star w) = 0 \quad \text{for any} \quad w \in \mathfrak{H}^0.$$

Example. For $w = x_0x_1$,

$$x_1 ш x_0x_1 = x_1x_0x_1 + 2x_0x_1^2 = y_1y_2 + 2y_2y_1,$$

$$x_1 \star x_0x_1 = y_1 \star y_2 = y_1y_2 + y_2y_1 + y_3,$$

hence

$$y_2y_1 - y_3 \in \ker \widehat{\zeta}$$

and

(Euler) $$\zeta(2,1) = \zeta(3)$$

Example. From

$$a ш (ba)^* = (2(ba)^* - 1)a(ba)^*.$$

one deduces

$$\sum_{n\ge 0} y_1 ш y_2^n = \left(2\sum_{h\ge 0} y_2^h - e\right) y_1 \sum_{k\ge 0} y_2^k$$

The homogeneous part of weight $2n+1$ is

$$y_1 ш y_2^n = 2\sum_{h+k=n} y_2^h y_1 y_2^k - y_1y_2^n = 2\sum_{i=1}^{n} y_2^i y_1 y_2^{n-i} + y_1y_2^n.$$

On the other hand for the harmonic product

$$y_1 \star y_2^n = \sum_{i=0}^{n} y_2^i y_1 y_2^{n-i} + \sum_{h=0}^{n-1} y_2^h y_3 y_2^{n-h-1}.$$

Hence

$$y_1 ш y_2^n - y_1 \star y_2^n = \sum_{i=1}^{n} y_2^i y_1 y_2^{n-i} - \sum_{h=0}^{n-1} y_2^h y_3 y_2^{n-h-1}.$$

and

$$\sum_{i=1}^{n} \zeta(\{2\}_i, 1, \{2\}_{n-i}) = \sum_{h=0}^{n-1} \zeta(\{2\}_h, 3, \{2\}_{n-1-h}).$$

For instance

$$\zeta(2,1) = \zeta(3),$$

$$\zeta(2,1,2) + \zeta(2,2,1) = \zeta(3,2) + \zeta(2,3),$$

$$\zeta(2,1,2,2) + \zeta(2,2,1,2) + \zeta(2,2,2,1) = \zeta(3,2,2) + \zeta(2,3,2) + \zeta(2,2,3).$$

VII Regularized Double Shuffle Relations

The map $\widehat{\zeta} : \mathfrak{H}^0 \to \mathbb{R}$ is a morphism of algebra of $\mathfrak{H}^0_ш$ into $\mathbb{R}$ and also a morphism of algebras of $\mathfrak{H}^0_\star$ into $\mathbb{R}$:

$$\widehat{\zeta}(u ш v) = \widehat{\zeta}(u)\widehat{\zeta}(v) \quad \text{and} \quad \widehat{\zeta}(u \star v) = \widehat{\zeta}(u)\widehat{\zeta}(v).$$

Question : is-it possible to extend $\widehat{\zeta}$ into a morphism of algebras $\mathfrak{H}^1 \to \mathbb{R}$ for both laws ш and $\star$?

Answer : NO! One can do it for each product, but not simultaneously for both.

$$x_1 ш x_1 = 2x_1^2, \qquad x_1 \star x_1 = y_1 \star y_1 = 2y_1^2 + y_2$$

and

$$\widehat{\zeta}(y_2) = \zeta(2) \neq 0.$$

Radford's structure Theorem for the shuffle algebra:

$$\mathfrak{H}_{\text{ш}} = \mathfrak{H}^1_{\text{ш}}[x_0]_{\text{ш}} = \mathfrak{H}^0_{\text{ш}}[x_0,x_1]_{\text{ш}} \quad \text{and} \quad \mathfrak{H}^1_{\text{ш}} = \mathfrak{H}^0_{\text{ш}}[x_1]_{\text{ш}}.$$

Hoffman's structure Theorem for the harmonic algebra:

$$\mathfrak{H}_\star = \mathfrak{H}^1_\star[x_0]_\star = \mathfrak{H}^0_\star[x_0,x_1]_\star \quad \text{and} \quad \mathfrak{H}^1_\star = \mathfrak{H}^0_\star[x_1]_\star.$$

From $\mathfrak{H}^1_{\text{ш}} = \mathfrak{H}^0_{\text{ш}}[x_1]_{\text{ш}}$ and $\mathfrak{H}^1_\star = \mathfrak{H}^0_\star[x_1]_\star$ we deduce that there are two uniquely determined algebra morphisms

$$\widehat{Z}_{\text{ш}} : \mathfrak{H}^1_{\text{ш}} \longrightarrow \mathbb{R}[T] \quad \text{and} \quad \widehat{Z}_\star : \mathfrak{H}^1_\star \longrightarrow \mathbb{R}[T]$$

which extend $\widehat{\zeta}$ and map x_1 to T.

Theorem (Boutet de Monvel, Zagier). *There is a $\mathbb{R}$-linear isomorphism $\varrho : \mathbb{R}[T] \to \mathbb{R}[X]$ which makes commutative the following diagram:*

$$\begin{tikzcd} & \mathbb{R}[X] \\ \mathfrak{H}^1 \arrow[ur, "\widehat{Z}_{\text{ш}}"] \arrow[dr, "\widehat{Z}_\star"'] & \\ & \mathbb{R}[T] \arrow[uu, "\varrho"'] \end{tikzcd}$$

An explicit formula for ϱ is given by means of the generating series

$$\sum_{\ell\geq 0} \varrho(T^\ell)\frac{t^\ell}{\ell!} = \exp\left(Xt + \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n} t^n\right).$$

Denote by $\text{reg}_{\text{ш}}$ the $\mathbb{Q}$-linear map $\mathfrak{H} \to \mathfrak{H}^0$ which maps $w \in \mathfrak{H}$ onto its constant term when w is written as a polynomial in x_0, x_1 in the shuffle algebra $\mathfrak{H}^0[x_0,x_1]_{\text{ш}}$. Then $\text{reg}_{\text{ш}}$ is a morphism of algebras $\mathfrak{H}_{\text{ш}} \to \mathfrak{H}^0_{\text{ш}}$.

Theorem (Boutet de Monvel, Ihara-Kaneko).
For $w \in \mathfrak{H}^1$ and $w_0 \in \mathfrak{H}^0$,

$$\boxed{\text{reg}_{\text{ш}}(w \,\text{ш}\, w_0 - w \star w_0) \in \ker \widehat{\zeta}.}$$

For $w = x_1$ since $x_1 \text{ш} w_0 - x_1 \star w_0 \in \mathfrak{H}^0$ for any $w_0 \in \mathfrak{H}^0$, one recovers the third standard relations of Hoffman.

VIII Diophantine Conjectures

Conjecture (Petitot-Hoang Ngoc Minh, Zagier, Cartier, Ihara-Kaneko,...). *All existing algebraic relations between the real numbers $\zeta(\underline{s})$ lie in the ideal generated by the ones described above.*

Petitot and Hoang Ngoc Minh: up to weight $s_1 + \cdots s_k \leq 16$, the three standard relations

$$\widehat{\zeta}(u)\widehat{\zeta}(v) = \widehat{\zeta}(u \text{ш} v),$$
$$\widehat{\zeta}(u)\widehat{\zeta}(v) = \widehat{\zeta}(u \star v)$$

and

$$\widehat{\zeta}(x_1 \text{ш} w - x_1 \star w) = 0$$

for u, v and w in $x_0X^*x_1$ suffice.

References

(*) See on MathSciNet or MathDataBase references to the works of

Apéry, R.; Arakawa, T.; Ball, K.; Beukers, F.; Bigotte, M.; Bowman, D.; Borwein, J.M.; Bradley, D.M.; Broadhurst, D.J.; Cartier, P.; Connes, A.; Cresson, J.; Écalle, J.; Euler, L.; Flageolet, Ph.; Girgensohn, R.; Goncharov, A.B.; Granville, A.; Hoffman, M.E.; Ihara, K.; Jacob, G.; Kaneko, M.; Kontsevich, M.; Kreimer, D.; Lee, T.T.Q.; Lisoněk, P.; Minh Hoang Ngoc; Müller, U.; Murukami, J.; Oussous, N.E.; Ohno, Y.; Petitot, M.; Racinet, G.; Rivoal, T.; Salvy, B.; Schubert, C.; Sorokin, V.N.; Van der Hoeven, J.; Vasiliev, D.V.; Zagier, D.; Zudilin, W.

...and others

(*) Added after the lecture

`http://www.math.jussieu.fr/~miw/articles/pdf/graz.pdf`