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Simultaneous Approximation of Logarithms of Algebraic Numbers

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1 Introduction.

Recently, close connections have been established between simultaneous diophantine approximation and algebraic independence. A survey of this topic is given by M. Laurent in these proceedings [7]. These connections are one of the main motivations to investigate systematically the question of algebraic approximation to transcendental numbers. In view of the applications to algebraic independence, a special attention is paid to the dependence on the degree.

To each qualitative transcendence result telling:

one at least of the numbers $\theta_1, \ldots, \theta_m$ is transcendental

one can associate a quantitative refinement, which is a lower bound for

$$\max_{1 \le i \le m} |\theta_i - \gamma_i|$$

when $\gamma_1, \ldots, \gamma_m$ are algebraic numbers. Such estimates will depend on two parameters: the degree $[\boldsymbol{Q}(\gamma_1, \ldots, \gamma_m) : \boldsymbol{Q}]$ of the number field generated by the algebraic approximations, and the *height*

$$\max_{1 \le i \le m} \mathrm{h}(\gamma_i).$$

Here it will be convenient to use the *absolute logarithmic height* $\mathrm{h}(\gamma)$ of an algebraic number γ, which has several equivalent definitions (see for instance [12], Chap. 3). One of these is

$$\mathrm{h}(\gamma) = \frac{1}{d} \log \mathrm{M}(\gamma),$$

where $d = [\boldsymbol{Q}(\gamma) : \boldsymbol{Q}]$ is the degree of γ over $\boldsymbol{Q}$ and $\mathrm{M}(\gamma)$ is Mahler's measure of γ: if $f \in \boldsymbol{Z}[X]$ is the minimal polynomial of γ over $\boldsymbol{Z}$, with leading coefficient $a_0 > 0$ and roots $\gamma^{(1)}, \ldots, \gamma^{(d)}$, so that

$$f(X) = a_0(X - \gamma^{(1)}) \cdots (X - \gamma^{(d)}),$$

then

$$\mathrm{M}(\gamma) = a_0 \prod_{i=1}^{d} \max\{1, |\gamma^{(i)}|\} = \exp\left(\int_0^1 \log|f(e^{2i\pi t})|dt\right).$$

Another equivalent definition for $\mathrm{h}(\gamma)$ is

$$\mathrm{h}(\gamma) = \frac{1}{[K : \boldsymbol{Q}]} \sum_{v \in M_K} D_v \log\max\{1, |\gamma|_v\},$$

when K is any number field containing γ, M_K denotes the set of (normalized) places of K and D_v denotes the local degree at $v \in M_K$. The normalization is done in such a way that the product formula reads

$$\prod_{v \in M_K} |\gamma|_v^{D_v} = 1$$

for any non zero $\gamma \in K$.

In the classical theory of simultaneous *rational* approximation, given a tuple $(\vartheta_1, \ldots, \vartheta_m)$ of real numbers, Khinchine's transference theorem ([2] Chap. V § 3 Th. IV) exhibits a duality between lower bounds for

$$q \longmapsto \min_{(p_1,\ldots,p_m) \in \boldsymbol{Z}^m} \max_{1 \le i \le m} \left|\vartheta_i - \frac{p_i}{q}\right|$$

and for

$$(p_1, \ldots, p_m) \longmapsto \min_{q \in \boldsymbol{Z}} |p_1\vartheta_1 + \cdots + p_m\vartheta_m + q|.$$

It is not known whether there is a similar transference theorem in the context of algebraic diophantine approximation.

Here, we shall consider both questions: measures of simultaneous algebraic approximation and measures of linear independence.

Such a study is worth of consideration in a general context; just to give an example, the situation concerning almost all tuples (either in $\boldsymbol{R}^m$ or in $\boldsymbol{C}^m$, for Lebesgue's measure) is not yet described in a satisfactory way (see [1] for recent results on this context).

For simplicity, we shall restrict here our attention to a special case, where we assume that the numbers e^{θ_i} are algebraic. We denote by

$$\mathcal{L} = \exp^{-1}(\overline{\boldsymbol{Q}}) = \{z \in \boldsymbol{C} \; ; \; e^z \in \overline{\boldsymbol{Q}}^{\times}\}$$

the set of complex logarithms of algebraic numbers. It is a $\boldsymbol{Q}$-vector space, which contains numbers like $i\pi$, $\log 2$, $\log 3 \ldots$.

It will be convenient to introduce the following definition.

Definition. Let $\underline{\theta} = (\theta_1, \ldots, \theta_m)$ be a tuple of complex numbers. A function $\varphi : \boldsymbol{N} \times \boldsymbol{R}_{>0} \to \boldsymbol{R}_{>0} \cup \{\infty\}$ is a *simultaneous approximation measure for* $\underline{\theta}$ if there exist a positive integer D_0 together with a real number $h_0 \geq 1$ such that, for any integer $D \geq D_0$, any real number $h \geq h_0$ and any m-tuple $\underline{\gamma} = (\gamma_1, \ldots, \gamma_m)$ of algebraic numbers satisfying

$$[\boldsymbol{Q}(\underline{\gamma}) : \boldsymbol{Q}] \leq D \quad \text{and} \quad \max_{1 \leq i \leq m} \mathrm{h}(\gamma_i) \leq h,$$

we have

$$\max_{1 \leq i \leq m} |\theta_i - \gamma_i| \geq \exp\{-\varphi(D, h)\}.$$

2 Main Conjectures.

Let $\lambda_1, \ldots, \lambda_m$ be logarithms of algebraic numbers with $\alpha_i = e^{\lambda_i} \quad (1 \leq i \leq m)$. Let $\beta_0, \ldots, \beta_m$ be algebraic numbers. Denote by D the degree of the number field $\boldsymbol{Q}(\alpha_1, \ldots, \alpha_m, \beta_0, \ldots, \beta_m)$. Finally let $h \geq 1/D$ satisfy

$$h \geq \max_{1 \leq i \leq m} h(\alpha_i), \quad h \geq \frac{1}{D} \max_{1 \leq i \leq m} |\lambda_i| \quad \text{and} \quad h \geq \max_{0 \leq j \leq m} h(\beta_j).$$

Conjecture 1. *Assume $\lambda_1, \ldots, \lambda_m$ are linearly independent over $\boldsymbol{Q}$. Then*

$$\sum_{i=1}^{m} |\lambda_i - \beta_i| \geq \exp\{-c_1 m D^{1+(1/m)} h\},$$

where c_1 is a positive absolute constant.

Conjecture 2. *Assume that the number*

$$\Lambda = \beta_0 + \beta_1 \lambda_1 + \cdots + \beta_m \lambda_m$$

is non zero. Then

$$|\Lambda| \geq \exp\{-c_2 m D^2 h\},$$

where c_2 is a positive absolute constant.

These conjectures are very simple and describe the situation in a clear way. On the opposite, as we shall see, known results are more complicated to state, so far.

In case $m = 1$, both conjectures 1 and 2 coincide:

$$|\lambda - \beta| \geq \exp\{-cD^2h\} \qquad (?)$$

For $D = 1$ (and $m = 1$), this is an open problem of Mahler [8]:

Does there exist an absolute constant $c > 0$ such that, for any positive rational integers a and b,

$$|e^b - a| \geq a^{-c}?$$

If $|e^b - a|$ is small, then b and $\log a$ are of the same order of magnitude, hence one can replace $a^{-c} = e^{-c\log a}$ in the right hand side by e^{-cb}. For the same reason, since $|e^b - a|/a = |e^{b-\log a} - 1|$ is close to $|b - \log a|$, one can replace $|e^b - a|$ in the left hand side by $|b - \log a|$ (replacing at the same time c by $c + 1$ in the right hand side).

The best known estimates on this question are due to K. Mahler [8]:

$$|e^b - a| \geq b^{-cb}$$

and

$$|b - \log a| \geq a^{-c\log\log a} \quad \text{for} \quad a \geq 3.$$

Mahler found a sharp explicit numerical value for c, namely $c = 33$ (for both estimates), provided that a (hence also b) is sufficiently large. A refinement is due to Franck Wielonsky [13]: for sufficiently large a, these last two estimates hold with $c = 20$.

Stronger estimates than Conjecture 2 are suggested in [6] in the special case $D = 1$ and $\beta_0 = 0$. When $a_1, \ldots, a_m$ are positive rational numbers and $b_1, \ldots, b_m$ are rational numbers, one can remove the logarithms from the statement, replacing

$$b_1 \log a_1 + \cdots + b_m \log a_m$$

by the number

$$|a_1^{b_1} \cdots a_m^{b_m} - 1|$$

which is a close aproximation:

Conjecture 3. *For any $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that, for any non-zero rational integers $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ with $a_1^{b_1} \cdots a_m^{b_m} \neq 1$,*

$$\left| a_1^{b_1} \cdots a_m^{b_m} - 1 \right| \geq \frac{C(\epsilon)^m}{B^{m-1+\epsilon} A^{m+\epsilon}},$$

where $A = \max_{1 \leq i \leq m} |a_i|$ and $B = \max_{1 \leq i \leq m} |b_i|$.

Links between measures of linear independence of logarithms and the *abc*-conjecture are discussed in [9].

3 Results: Simultaneous Approximation.

Here is the state of the art concerning Conjecture 1. Until recently, only N.I. Fel'dman considered such a question [3] and [4]; see also [5] Th. 3.34:

Theorem 1 (Fel'dman) **.** *Let $\lambda_1, \ldots, \lambda_m$ be $\boldsymbol{Q}$-linearly independent logarithms of algebraic numbers. There exists a positive constant $c = c(\lambda_1, \ldots, \lambda_m)$ such that*

$$cD^{2+1/m}(h + \log D)(\log D)^{-1}$$

is a simultaneous approximation measure for the numbers $\lambda_1, \ldots, \lambda_m$.

Further estimates have been produced more recently [10], [11], [12]. We select a few examples.

A rather general statement is the following (cf. Chap. 16 of [12]).

Theorem 2. *Let m and n be two positive rational integers. Define*

$$c = 2^{23} m^3 n^2 (2m)^{m/n}.$$

Let λ_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$ be elements of $\mathcal{L}$, K a number field of degree $D = [K : \boldsymbol{Q}]$ such that the algebraic numbers $\alpha_{ij} = e^{\lambda_{ij}}$ belong to $K^\times$, $\beta_1, \ldots, \beta_n$, $\beta'_1, \ldots, \beta'_m$ elements of K, A_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$, B, B' and E positive real numbers satisfying, for $1 \leq i \leq m$ and $1 \leq j \leq n$, the following conditions:

$$\mathrm{h}(\alpha_{ij}) \leq \log A_{ij}, \quad |\lambda_{ij}| \leq \frac{D}{E} \log A_{ij},$$

$$\mathrm{h}(1 : \beta_1 : \cdots : \beta_n) \leq \log B, \qquad \mathrm{h}(1 : \beta'_1 : \cdots : \beta'_m) \leq \log B',$$

$$B \geq e, \quad B' \geq e, \quad B \geq D \log B', \quad B' \geq D \log B$$

and

$$1 \leq \log E \leq D \log A_{ij} \leq \min\{B, B'\}.$$

Assume that the $m \times n$ matrix $(\log A_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ has rank 1:

$$\log A_{ij} \log A_{11} = \log A_{i1} \log A_{1j}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Define

$$U_2^{mn} = D^{mn+m+n} (\log B)^n (\log B')^m \left(\prod_{i=1}^{m} \prod_{j=1}^{n} \log A_{ij} \right) (\log E)^{-m-n}.$$

Assume further that for any $\underline{t} \in \boldsymbol{Z}^m \setminus \{0\}$ satisfying $|t_i| \leq (cU_2)^2$ for $1 \leq i \leq m$, we have

$$t_1 \beta'_1 + \cdots + t_m \beta'_m \neq 0,$$

and that for any $\underline{s} \in \boldsymbol{Z}^n \setminus \{0\}$ satisfying $|s_j| \leq (cU_2)^2$ for $1 \leq j \leq n$, we have

$$s_1 \beta_1 + \cdots + s_n \beta_n \neq 0.$$

Assume furthermore

$$D \log B \leq U_2, \quad D \log B' \leq U_2,$$

$$D \log B' \log A_{11} \cdots \log A_{1n} \geq (\log A_{1j})^n \log E$$

for $1 \leq j \leq n$ and

$$D \log B \log A_{11} \cdots \log A_{m1} \geq (\log A_{i1})^m \log E$$

for $1 \leq i \leq m$. Then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\lambda_{ij} - \beta_j \beta'_i| \geq e^{-cU_2}.$$

In the special case $m = 1$ the statement is slightly simpler:

Corollary 1. *Let n be a positive integer. Define*

$$c = 2^{24} n^2.$$

Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be algebraic numbers, let D be the degree of the number field they generate, and let $A_1, \ldots, A_n$, A, B, B', E be real numbers which satisfy

$$B \geq e, \quad B' \geq e, \quad A = \max_{1 \leq j \leq n} A_j,$$

$$\mathrm{h}(\alpha_j) \leq \log A_j \quad (1 \leq j \leq n) \quad \textit{and} \quad \mathrm{h}(1 : \beta_1 : \cdots : \beta_n) \leq \log B.$$

For $1 \leq j \leq n$, assume that the number α_j is non zero, choose $\lambda_j \in \mathcal{L}$ such that $e^{\lambda_j} = \alpha_j$ and assume

$$|\lambda_j| \leq \frac{D}{E} \log A_j.$$

Let U be a positive real number satisfying

$$U \geq D^{2+(1/n)}(\log B)(\log B' \log A_1 \cdots \log A_n)^{1/n}(\log E)^{-1-(1/n)};$$

$$U \geq D^2(\log B)(\log A)(\log E)^{-1-(1/n)}.$$

Further, assume

$$1 \leq \log E \leq D \log A_j \leq B, \quad \log B' \leq D \log A,$$

$$B' \geq D \log A, \quad U \geq D \log B,$$

$$\log E \leq D \log B \leq B' \quad \textit{and} \quad \log E \leq D \log B' \leq B.$$

Furthermore, assume

$$s_1\beta_1 + \cdots + s_n\beta_n \neq 0$$

for any $\underline{s} \in \boldsymbol{Z}^n \setminus \{0\}$ with

$$0 < \max_{1 \leq j \leq n} |s_j| \leq (cU)^2.$$

Then, we have

$$\sum_{j=1}^{n} |\lambda_j - \beta_j| \geq e^{-cU}.$$

Before giving a few examples, we introduce the following definition.

Definition. A tuple $\underline{\theta} = (\theta_1, \ldots, \theta_n) \in \boldsymbol{C}^n$ of complex numbers satisfies a *linear independence measure condition* if, for any $\epsilon > 0$, there exists $S_0 > 0$ such that, for any $S \geq S_0$ and any $\underline{s} \in \boldsymbol{Z}^n$ satisfying $0 < \max_{1 \leq j \leq n} |s_j| \leq S$, we have

$$|s_1\theta_1 + \cdots + s_n\theta_n| \geq e^{-S^\epsilon}.$$

The following three examples are easily deduced from Corollary 1.

EXAMPLE 1. *Let* $(x_1,\dots,x_n)$ *be a tuple of complex numbers which satisfies a linear independence measure condition. There exists a positive constant* $c = c(n,x_1,\dots,x_n)$ *such that the function*

$$cD^{2+(1/n)}h(h+\log D)(\log h+\log D)^{-1}$$

is a simultaneous approximation measure for the $2n$ *numbers*

$$x_1,\dots,x_n,e^{x_1},\dots,e^{x_n}.$$

EXAMPLE 2. *Let* $\beta_1,\dots,\beta_m$ *be* $\boldsymbol{Q}$*-linearly independent algebraic numbers. There exists a positive constant* $c = c(\beta_1,\dots,\beta_m)$ *such that the function*

$$cD^{1+(1/m)}h(\log h+D\log D)(\log h+\log D)^{-1}$$

is a simultaneous approximation measure for the numbers $e^{\beta_1},\dots,e^{\beta_m}$.

EXAMPLE 3. *Let* $\alpha_1,\dots,\alpha_m$ *be non zero algebraic numbers. For* $1 \le i \le m$, *let* λ_i *be a determination of the logarithm of* α_i. *Assume the numbers* $\lambda_1,\dots,\lambda_m$ *are* $\boldsymbol{Q}$*-linearly independent. Then there exists a positive constant* $c = c(\lambda_1,\dots,\lambda_m)$ *such that*

$$cD^{2+1/m}(h+\log D)(\log h+\log D)^{1/m}(\log D)^{-1-1/m}$$

is a simultaneous approximation measure for the numbers $\lambda_1,\dots,\lambda_m$.

The next three examples are consequences of Theorem 2.

EXAMPLE 4. *Let* $m \ge 1$ *and* $n \ge 1$ *be positive integers,* $(x_1,\dots,x_m)$ *be a* m*-tuple of complex numbers satisfying a linear independence measure condition, and* $(y_1,\dots,y_n)$ *be also a* n*-tuple of complex numbers satisfying a linear independence measure condition. There exists a constant* $c > 0$ *such that a simultaneous approximation measure for the* $m+n+mn$ *numbers*

$$x_i,\quad y_j,\quad e^{x_iy_j}\qquad (1\le i\le m,\ 1\le j\le n)$$

is

$$cD^{1+\frac{m+n}{mn}}h(h+\log D)^{\frac{m+n}{mn}}(\log h+\log D)^{-\frac{m+n}{mn}}.$$

Example 5. *Let K be a number field of degree D, β, β_1', β_2' be elements of K, λ_1, λ_2 λ_1', λ_2' elements in $\mathcal{L}$ such that the algebraic numbers*

$$\alpha_1 = e^{\lambda_1},\ \alpha_2 = e^{\lambda_2},\ \alpha_1' = e^{\lambda_1'},\ \alpha_2' = e^{\lambda_2'}$$

are in K. Assume λ_1, λ_2 are linearly independent over $\boldsymbol{Q}$ and β is irrational. Let $B \geq e$ and $B' \geq e$ be real numbers with

$$\mathrm{h}(\beta) \leq \log B, \quad \mathrm{h}(1 : \beta_1' : \beta_2') \leq \log B'.$$

Let A_1, A_2, A_1', A_2' be positive numbers, all $\geq e^2$, and E a real number $\geq e$, which satisfy

$$\log A_1 \log A_2' = \log A_2 \log A_1'$$

and, for $i = 1, 2$,

$$\mathrm{h}(\alpha_i) \leq \log A_i, \quad \mathrm{h}(\alpha_i') \leq \log A_i',$$

and

$$|\lambda_i| \leq \frac{D}{E} \log A_i, \quad |\lambda_i'| \leq \frac{D}{E} \log A_i'.$$

Assume

$$\log E \leq D \log A_i \leq \min\{B, B'\}, \quad \log E \leq D \log A_i' \leq \min\{B, B'\},$$

$$\log E \leq D \log B', \quad \log B' \leq B, \quad \log B \leq B'$$

and

$$\log E \leq D \log B \frac{\log A_1}{\log A_2}, \quad \log E \leq D \log B \frac{\log A_2}{\log A_1}.$$

Define

$$U = D^2 (\log B)^{1/2} (\log B')^{1/2} (\log A_1 \log A_2 \log A_1' \log A_2')^{1/4} (\log E)^{-1}.$$

Then

$$|\lambda_1 - \beta_1'| + |\lambda_2 - \beta_2'| + |\beta\lambda_1 - \lambda_1'| + |\beta\lambda_2 - \lambda_2'| > \exp\{-2^{30} U\}.$$

Example 6. *Let λ_1, λ_2 be two elements of $\mathcal{L}$ which are linearly independent over $\boldsymbol{Q}$ and let θ be a complex irrational number which satisfies a linear independence measure condition. Then there exists a constant $c > 0$ such that the function*

$$cD^2 (h + \log D) h^{1/2} (\log D)^{-1}$$

is a simultaneous approximation measure for the five numbers $\lambda_1, \lambda_2,\ \theta,$ $e^{\theta\lambda_1}, e^{\theta\lambda_2}$.

Theorem 2 can be extended: in place of $\beta_j\beta'_i$, on may consider more generally algebraic numbers β_{ij}. Here is an example dealing with simultaneous approximation of logarithms of algebraic numbers (compare with [11] § 10, Th. 10.1 and remark p. 423–424).

Theorem 3. *Let* d_1 *and* ℓ_1 *be positive integers and let* $M = (\lambda_{ij})$ *be a* $d_1 \times \ell_1$ *matrix with coefficients in* $\boldsymbol{\mathcal{L}}$. *Let* r *be the rank of* M. *Assume the the* $d_1\ell_1$ *numbers* λ_{ij} *are linearly independent. Set* $\kappa = (1/d_1) + (1/\ell_1)$. *Then, there exists a positive constant* c *such that the function*

$$cD^{r\kappa+1}(h + \log D)^{r\kappa}(\log D)^{-r\kappa}$$

is a simultaneous approximation measure for the $d_1\ell_1$ *numbers* λ_{ij} $(1 \leq i \leq d_1,$ $1 \leq j \leq \ell_1)$.

4 Results: Measures of Linear Independence.

The story concerning Conjecture 2 is quite rich. We refer to [5] and [12] for extensive references, including works of A.O. Gel'fond, N.I. Fel'dman and A. Baker, just to name a few.

Here is the state of the art on this topic.

Theorem 4. *For each* $m \geq 1$ *there exists a positive number* $C(m)$ *with the following property. Let* $\lambda_1, \ldots, \lambda_m$ *be logarithms of algebraic numbers, define* $\alpha_j = \exp(\lambda_j)$ $(1 \leq j \leq m)$, *and let* $\beta_0, \ldots, \beta_m$ *be algebraic numbers. Denote by* D *the degree of the number field* $\boldsymbol{Q}(\alpha_1, \ldots, \alpha_m, \beta_0, \ldots, \beta_m)$ *over* $\boldsymbol{Q}$. *Further, let* B, E, E^* *be positive real numbers, each* $\geq e$ *and let* $A_1, \ldots, A_m$ *be positive real numbers. Assume*

$$\log A_j \geq \max\left\{\mathrm{h}(\alpha_j),\ \frac{E|\lambda_j|}{D},\ \frac{\log E}{D}\right\} \quad (1 \leq j \leq m)$$

$$\log E^* \geq \max\left\{\ \frac{1}{D}\log E,\ \log\left(\frac{D}{\log E}\right)\right\}$$

and $B \geq E^*$. *Further, assume either*

(i) (general case)

$$B \geq \max_{1\leq i\leq m} \frac{D\log A_i}{\log E} \qquad \textit{and} \qquad \log B \geq \max_{0\leq i\leq m} \mathrm{h}(\beta_i)$$

or

(ii) (homogeneous rational case)

$$b_0 = 0, \quad \beta_i = b_i \in \boldsymbol{Z} \quad (1 \leq i \leq m), \quad b_m \neq 0$$

and

$$B \geq \max_{1\leq j\leq m-1} \left\{ \frac{|b_m|}{\log A_j} + \frac{|b_j|}{\log A_m} \right\}.$$

If the number

$$\Lambda = \beta_0 + \beta_1\lambda_1 + \cdots + \beta_m\lambda_m$$

is non zero, then

$$|\Lambda| > \exp\{-C(m)D^{m+2}(\log B)(\log A_1)\cdots(\log A_m)(\log E^*)(\log E)^{-m-1}\}.$$

A discussion of the explicit value for $C(m)$ is given in Chapter 12 of [12].

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