DUISTERMAAT HECKMAN MEASURES AND THE EQUIVARIANT INDEX THEOREM

Let N be a symplectic manifold, with a Hamiltonian action of the circle group G and moment map $\mu : N \to \mathbb{R}$. Assume that the level sets of μ are compact manifolds. The Duistermaat-Heckman measure is a locally polynomial function on $\mathbb{R}$ (so called a spline) which measures the symplectic volume of the level sets $\mu^{-1}(t)/G$.

Similarly, we will show that an elliptic (or transversally elliptic) operator D on a G-manifold M produces a locally polynomial function on $\mathbb{R}$ via the "infinitesimal index" map. The index of D can be deduced from the knowledge of the infinitesimal index.

References is De Concini-Procesi-Vergne (arXiv:1012.1049; Box splines and the equivariant index theorem).

1. Duistermaat-Heckman stationary phase

Let M be a symplectic manifold of dimension 2ℓ, with a symplectic form Ω. I am happy to talk in the "Darboux seminar" as of course Darboux theorem is a fundamental theorem for symplectic geometry:

Darboux theorem

There exists local coordinates (p, q) *(why these letters ??) so that* $\Omega = \sum_{i=1}^{\ell} dp_i \wedge dq_i$.

i. e. a symplectic manifold (M, Ω) is locally a symplectic vector space $\mathbb{R}^{2\ell}$. In some sense, Duistermaat-Heckman theorem is a variation on Darboux theorem.

Assume now that the circle group G with generator J,

$$(G := \{\exp(\theta J)\}, \theta \in \mathbb{R}/2\pi\mathbb{Z},)$$

acts on M in a Hamiltonian way: we are in presence of a rotational symmetry on M. We assume that the action of the one parameter group G on M is Hamiltonian: there exists a function μ on M such that the 1-form $d\mu$ is the symplectic gradient of the vector field J_M generated by the action of G

$$\langle d\mu, v\rangle = \Omega(J_M, v).$$

The function μ is thus constant on the orbit of m under $\exp(\theta J)$: Emmy Noether's theorem on the conservation of the Energy .

In other words, $\mu : M \to \mathbb{R}$ satisfies

$$d\mu = \iota(J_M)\Omega$$

where $\iota(J_M)$ is the contraction operator on differential forms.

This equation implies immediately that **the set of critical points of the function $\mu(m)$ is the set of fixed points for the action of the one parameter group $\exp(tJ)$ on M.**

DRAWINGS:

Let us give two simple examples:

The space $\mathbb{R}^2 := (p, q)$ with the rotation action: $J = (p\partial_q - q\partial_q)$, and $\mu = \frac{1}{2}(p^2 + q^2)$.

• Let S_λ be the sphere $x^2+y^2+z^2 = \lambda^2$ and $J = x\partial_y - y\partial_x$ be the rotational symmetry around the z axis. Let Ω be the restriction of $\frac{1}{x^2+y^2+z^2}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dz)$ to S_λ. ($S_\lambda =$ the space $P_1(\mathbb{C})$)

We consider the rotation action around the axe Oz; with generator J; then the moment μ is just $\mu = z$ (the height).

DRAWING:

A symplectic manifold M is endowed with a Liouville measure $d\beta = \frac{1}{(2\pi)^\ell}\frac{\Omega^\ell}{\ell!}$. It is thus natural to compute the integral of a function on M with respect to $d\beta$. For example, if M is compact, the integral of the function 1 is the symplectic volume of M.

Consider f a function on M and the integral

$$\int_M e^{itf} d\beta(m).$$

When t tends to ∞, the asymptotic behavior (in t) of this integral depends only of the knowledge of f near the set of critical points of f (the stationary phase formula).

Duistermaat-Hekmann discovered the extremely beautiful result: if $f(m) = \mu(m)$ then the asymptotic formula is exact.

Let me state Duistermaat-Heckman formula for the case where the set of fixed points for the one parameter group $\exp tJ$ is a finite set F. Then

$$\int_M e^{it\mu(m)} d\beta(m) = \sum_{w\in F} \frac{e^{it\mu(w)}}{t^\ell D_w}$$

where

$$D_w = \det_{T_w M}(\mathcal{L}(J))^1/2.$$

(In Darboux coordinates near w , we have $J = \sum_{i=1}^{\ell} a_i(p_i\partial_{q_i} - q_i\partial_{q_i})$ and $D_w = \prod_{i=1}^{\ell} a_i$.)

When M is a coadjoint orbit $K\Lambda \subset \mathfrak{k}^*$, this is the very familiar Harish-Chandra formula in the theory of compact Lie groups for the Fourier transform of an orbit $K\lambda$. But this "fixed point formula" holds in the general context of a Hamiltonian manifold. and this was discovered by Duistermaat-Heckman.

We will call

$$V_M(\theta) = \int_M e^{i\theta\mu(m)} d\beta(m)$$

the equivariant volume of M: when $\theta = 0$, this is just the volume of M, when M is compact. An important remark is that when M is not compact, it happens that $v_M(\theta)$ is defined in the distribution sense for many important examples (hyperkahler quotients for examples), as an oscillatory integral.

Let me first write the formula for $S_\lambda := \{x^2 + y^2 + z^2 = \lambda^2\}$. It is immediate to see that

$$\int_{S_\lambda} e^{i\theta z} d\beta = \frac{e^{i\lambda\theta} - e^{-i\lambda\theta}}{i\theta}.$$

TWO FIXED POINTS: The symplectic volume of S_Λ is 2λ. (Later λ will belong to $\frac{1}{2}\mathbb{Z}$)

Similarly, when $M = \mathbb{R}^2$, then

$$\int_{\mathbb{R}^2} e^{i\theta} d\beta = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\theta(p^2+q^2)} dp dq = \frac{-1}{i\theta}$$

ONE FIXED POINT.

Duistermaat-Heckman measures

Consider our equivariant volume:

$$V_M(\theta) = \int_M e^{i\theta\mu(m)} d\beta_M.$$

Consider the map $\mu : M \to \mathbb{R}$ and a regular value ξ. It is well known that if we consider the fiber $\mu^{-1}(\xi)$ (a $2\ell - 1$ manifold, and divide by the action of the one parameter group $G := \exp\theta J$, then $\mu^{-1}(\xi)/G$ is a $2\ell - 2$ symplectic manifold (orbifold). This is called the reduced manifold $M_{red}(\xi)$ of M at ξ.

We put $\xi = \mu(m)$ and integrate over the fiber $\mu(m) = \xi$. We then obtain a measure on $\mathbb{R}$, the push-forward of the Liouville measure with support the image of M by μ. We will call $\mu_*(d\beta) = dh(\xi)$ the Duistermaat-Heckman measure on $\mathbb{R}$. Thus by definition:

$$V_M(\theta) = \int_{\mathbb{R}} e^{i\xi\theta} dh(\xi).$$

We see thus that this is well defined, just provided the fibers of μ are compact manifolds.

The measure $dh(\xi)$ is the Fourier transform of the equivariant volume.

Theorem 1.1. *The Duistermaat-Heckman measure is a piecewise polynomial measure on* $\mathbb{R}$ *(a spline).*

$$dh(\xi) = p(\xi) d\xi$$

where p *is piecewise polynomial:*

Moreover, if ξ *is a regular value of the moment map* $\mu : M \to \mathbb{R}$*, then the value* $p(\xi)$ *is equal to the symplectic volume of the reduced fiber* $\mu^{-1}(\xi)/G$*.*

(this theorem is almost equivalent to the stationary phase exact formula)

Thus

$$V_M(\theta) = \int_R e^{i\xi\theta} vol(M_{red}(\xi))d\xi.$$

VERY IMPORTANT THEOREM: the stationary phase formula for $V_M(\theta)$+this theorem can be used to compute volumes of reduced manifolds, such as moduli spaces of flat bundles over a surface of genus g, by Fourier inversion.

Example: $M := S_\lambda$ $\mu = z$ The image is the interval $[-\lambda, \lambda]$.

It is very easy to compute the image measure; the reduced fibers are points, and the image measure is the characteristic function of the interval $[-\lambda, \lambda]$. This is compatible with the formula for the equivariant volume:

$$\int_{-\lambda}^{\lambda} e^{iz\theta} dz$$

is indeed equal to

$$\frac{e^{i\lambda\theta} - e^{-i\lambda\theta}}{i\theta}.$$

It maybe more interesting later on to take products of a certain number of copies $S_\lambda \times \times \cdots \times S_\lambda$ with diagonal action of $\exp \theta J$ and moment map $\mu = \mu_1 + \mu_2 + \cdots + \mu_k$.

Then we see by Fourier transform that we obtain the convolution of the two intervals 2, 3 intervals $[-\lambda, \lambda]$, etc... This is the so called Box splines in numerical analysis.

DRAWINGS:

For example for $M = S_\lambda \times S_\lambda$, we have

$$V_M(\theta) = \frac{e^{-2i\lambda\theta}}{(i\theta)^2} - 2\frac{1}{(i\theta)^2} + \frac{e^{2i\lambda\theta}}{(i\theta)^2}$$
$$= \int_{\mathbb{R}} f(\xi)e^{i\xi\theta}d\xi$$

with

$f(\xi) = 2 * \lambda - \xi$ for $0 \leq \xi \leq 2\lambda$ $f(\xi) = 2 * \lambda + \xi$ for $-2 * \lambda \leq \xi \leq 0$.

For 3 spheres (same radius to simplify), $M = S_\lambda \times S_\lambda \times S_\lambda$,

$$V_M(\theta) = \frac{e^{-3i\lambda\theta}}{(i\theta)^3} - 3\frac{e^{-i\lambda\theta}}{(i\theta)^3} - 3\frac{e^{i\lambda\theta}}{(i\theta)^3} + \frac{e^{3i\lambda\theta}}{(i\theta)^3}.$$

This is the Fourier transform of the locally polynomial density:

$$f(\xi) = 3 * \lambda^2 - \xi^2; \qquad -\lambda \leq \xi \leq \lambda$$

$$f(\xi) = (3\lambda - \xi)^2/2; \qquad \lambda \leq \xi \leq 3\lambda$$

$$f(\xi) = (3\lambda + \xi)^2/2; \qquad -3\lambda \leq \xi \leq -\lambda.$$

for 4 intervals..

Equivariant localization

Duistermaat-Heckmann formula is best understood as a localization formula in equivariant cohomology (Berline-Vergne-Atiyah-Bott);

Let us introduce the equivariant symplectic form (a non homogenous differential form on M: a function+ a 2-form):

$$\Omega(\theta) = \theta\mu(m) + \Omega.$$

Then Duistermaat-Heckman formula can be rewritten as:

$$\frac{1}{(2i\pi)^\ell}\int_M e^{i\Omega(\theta)} = \sum_w \frac{e^{i\theta\mu(w)}}{(\det_{T_wM}\mathcal{L}(\theta J))^{1/2}}.$$

Consider the operator $D = d - \theta\iota(J_M)$. The two equations $d\Omega = 0$, $\iota(J_M)\Omega = d\mu$ are equivalent to the single equation

$$(d - \theta\iota(J_M))\Omega(\theta) = 0$$

we introduced the operator D at the same time than Witten (Supersymmetry and Morse theory, 1982), with the aim of understanding Rossmann formula for Fourier transforms of coadjoint orbits of reductive Lie groups

Let $\alpha(\theta)$ a closed equivariant differential form (invariant by G and satisfying $D\alpha(\theta) = 0$. Then exactly the same localization formula holds (Berline-Vergne, Atiyah-Bott). If M is any manifold and α is any equivariantly closed form:

$$\frac{1}{(2i\pi)^\ell}\int_M \alpha(\theta) = \sum_{w\in F} \frac{i^*\alpha(w)(\theta)}{(\det_{T_wM}\mathcal{L}(\theta J))^{1/2}}.$$

Let us now try to understand in which case the Fourier transform of

$$I(\theta) = \frac{1}{(2i\pi)^\ell}\int_M \alpha(\theta)$$

is a piecewise polynomial measure...

We make a definition.

Definition 1.2. The form $\alpha(\theta)$ will be called of oscillatory type, if at each point $w \in F$, $\alpha(w)(\theta)$ is a sum of products of exponential $e^{i\mu\theta}$ (μ real) times polynomial functions of θ.

Of course, for $\Omega(\theta) = \theta\mu + \Omega$, then $e^{i\Omega(\theta)}$ is of oscillatory type. also $e^{i\Omega(\theta)}b(\theta)$ where $b(\theta)$ is a polynomial, equivariant Chern characters $ch(E)(\theta)$ of vector bundles are of oscillatory type, etc...

Then we have almost the same theorem:

Theorem 1.3. *(Witten non abelian localization theorem)*

If $\alpha(\theta)$ is a closed equivariant differential form of oscillatory type, then

$$\frac{1}{(2i\pi)^\ell}\int_M \alpha(\theta)$$

is the Fourier transform

$$\int_{\mathbb{R}} e^{i\xi\theta} p(\xi) d\xi$$

where $p(\xi)$ is a locally polynomial function + a sum of derivatives of δ-functions of a finite number of points.

Furthermore in case where $\alpha(\theta) = e^{i\Omega(\theta)} b(\theta)$, b polynomial, then $p(\xi)$ computes the integral over the reduced manifold $M_{red}(\xi)$ of the cohomology class $e^{i\Omega_\xi} b(F)$, where F is the curvature of the fibration $\mu^{-1}(\xi \to \mu^{-1}(\xi)/G$.

This theorem has been used to compute intersection rings of reduced manifolds (for example by Jeffrey-Kirwan for the proof of the Verlinde formula).

Equivariant index theorems

The B-V localization formula in equivariant cohomology is very reminiscent of the index formulae of Atiyah-Bott.

Let M be a compact manifold, with an action of the group $G = \{\exp(\theta J)\}$, let $\mathcal{E}^{\pm}$ be two G-equivariant Hermitian vector bundles on M, let $E = E^+ \oplus E^-$; super vector bundle. and let

$$D := \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

an odd elliptic operator on $\Gamma(M, E)$:

$$D^+ : \Gamma(M, E^+) \to \Gamma(M, E^-),$$

$$D^- : \Gamma(M, E^-) \to \Gamma(M, E^+)$$

between the C^∞ sections of $E^{\pm}$ and commuting with G.

The kernel of D^+ as well as the kernel of D^- are finite dimensional vector spaces so that there exists an action of G in the superspace $Ker(D) = Ker(D^+) - Ker(D^-)$. This object depends only of the principal symbol σ of D, up to deformation, thus it depends only of $[\sigma]$ in the equivariant K-theory of $N = T^*M$. We wish to compute the eigenvalues of the generating element J in the space $\operatorname{Ker} D$ and their multiplicities: In other words, we want to compute the Fourier series:

$$Str_{Ker(D)}(\exp \theta J) = \sum_{n \in \mathbb{Z}} c(n) e^{in\theta}.$$

I will assume to simplify that the action of G on M is with connected stabilizers: for $m \in M$, then either $G_m = G$, or $G_m = \{1\}$.

Theorem 1.4. *(De Concini+Procesi+V.) There exists a "natural" piecewise polynomial function $\tilde{c}$ on $\mathbb{R}$, continuous on the lattice $\mathbb{Z}$, such that*

$$Str_{Ker(D)}(\exp \theta J) = \sum_{n} \tilde{c}(n) e^{in\theta}.$$

The theorem is true for general transversally elliptic operators under an action of a compact Lie group K.

In general, we compute a piecewise quasipolynomial function on the dominant weights of K.

We call the function $\tilde{c}$ the infinitesimal index.

I will explain the construction of $\tilde{c}$ in the case of a very popular elliptic operator: the twisted Dirac operator. I I will then explain the general construction.

Assume M compact, D elliptic; Let us first recall the fixed point formula of Atiyah-Bott in the case where the action of G on M has only a finite number of fixed points; **Atiyah-Bott fixed point formula**

$$Str_{Ker(D)}(\exp(\theta J)) = \sum_{w\in F} \frac{STr_{E_w}(\exp\theta J)}{\det_{T_wM}(1 - exp(\theta J))}.$$

Let us analyze this formula when M is symplectic and with spin structure, and L a Kostant line bundle: at Assume that M is Hamiltonian, symplectic (and with spin structure to simplify) with moment map $\mu : M \to \mathbb{R}$ as before. Let L be a Kostant line bundle on M, that is if w is a fixed point by the one parameter group $\exp(\theta J)$, we have

$$\exp(\theta J)|_{L_w} = e^{i\theta\mu(w)}.$$

Consider the twisted Dirac D_L operator on M, operating on $E := S \otimes L$, where $S = S^+ \oplus S^-$ is the spinor bundle. then the general Atiyah-Bott formula is

$$STr_{KerD_L}(\exp(\theta J)) = \sum_{w\in F} \frac{e^{i\theta\mu(w)} Str_{S_w}(\exp\theta J)}{\det_{T_w}(1 - exp(\theta J)}.$$

The supertrace in the spin space S_w is a square root of $\det_{T_w}(1-exp(\theta J)$, thus we obtain

$$STr_{KerD_L}(\exp(\theta J)) = \sum_{w\in F} \frac{e^{i\theta\mu(w)}}{(\det_{T_w}(1 - exp(\theta J)))^{1/2}},$$

a formula which is very similar to

$$\int_M e^{i\theta\mu(m)} d\beta(m) = \sum_{w\in F} \frac{e^{i\theta\mu(w)}}{(\det_{T_w}(\mathcal{L}_w(\theta J)))^{1/2}}.$$

For example, consider $M := S_\lambda$ where $\lambda \in \frac{1}{2} + \mathbb{Z}$ is a half-integer, we consider

$$s = \lambda - 1/2,$$

then the sphere S_λ is prequantizable, with line bundle L_λ, (passing to the double cover of the rotation group G). We can then consider D_λ the twisted Dirac operator by the line bundle L_λ, and the corresponding representation of $\exp(\theta J)$ in $Ker(D_\lambda)$ is the representation of $SO(3)$ corresponding to a particle with spin s. The eigenvalues of J generator of the rotation around the z axis in $SO(3)$ is then the set $\{s, s-1, ..., -s\}$, composed of integers. each with multiplicity 1.

We have then two very similar formulae for $M = S_\lambda$:

$$\int_M e^{i\theta\mu(m)} d\beta(m) = \frac{e^{i\lambda\theta}}{i\theta} - \frac{e^{-i\lambda\theta}}{i\theta}$$

a sum of two terms corresponding to the two fixed points north pole and south pole on S_λ. Similarly

$$STr_{KerD_\lambda}(\exp\theta J) = \frac{e^{i\lambda\theta}}{e^{i\theta/2} - e^{-i\theta/2}} - \frac{e^{-i\lambda\theta}}{e^{i\theta/2} - e^{-i\theta/2}}.$$

In a similar way that we can write

$$\sum_{w \in F} \frac{e^{i\theta\mu(w)}}{(\det_{T_w}(\mathcal{L}_w(\theta J)))^{1/2}}$$

as an integral over the manifold M of an equivariant form ($\int_M e^{i\Omega(\theta)}$), we can rewrite Atiyah-Bott formula as a formula on the symplectic manifold M with symplectic form $\Omega(\theta)$. and we have the "delocalized index formula" :

$$index(D_L)(\exp(\theta J)) = \frac{1}{(2i\pi)^\ell} \int_M e^{i\Omega(\theta)} \hat{A}(M)(\theta)$$

where $\hat{A}(M)(\theta)$ is the equivariant $\hat{A}$ class of the manifold M.

For $\theta = 0$, this is just the Atiyah-Singer index theorem.

$$q(M) := index(D_L) = \int_M ch(L)\hat{A}(M).$$

Now comes the miracle: the equivariant form $\hat{A}(M)(\theta)$ is the characteristic class on M associated to $\prod_{i=1}^{\ell} \frac{c_i}{e^{c_i/2} - e^{-c_i/2}}$ where $c_i(\theta)$ are the equivariant Chern classes of the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$. Let us consider the Bernoulli numbers so that $x/(e^{x/2} - e^{-x/2}) = \sum_{k=0}^{\infty} b_{2k}((1/2)^{2k-1} - 1)x^{2*k}/(2k)!$ and consider the corresponding truncated class

$$Trunc\hat{A}(M) := [\prod_i (\sum_{k=0}^{\infty} b_{2k}((1/2)^{2k-1} - 1)c_i^{2*k}/(2k)!]_{\leq (dimM-2)}$$

Then $Trunc\hat{A}(M)(\theta)$ is polynomial in θ and

$$\alpha(\theta) := e^{i\Omega(\theta)} Trunc\hat{A}(M)(\theta)$$

is a closed equivariant characteristic class of oscillatory type on M.

Then we have the following two formulae:

$$Str_{Ker(D)}(\exp\theta J) = \frac{1}{(2i\pi)^\ell} \int_M e^{i\Omega(\theta)} \hat{A}(M)(\theta) = \sum c(n) e^{in\theta}$$

AND:

$$\frac{1}{(2i\pi)^\ell}\int_M e^{i\Omega(\theta)}Trunc\hat{A}(M)(\theta) = \int_\xi e^{i\xi\theta}C(\xi)d\xi$$

and the distribution $C(\xi)$ is piecewise polynomial on $\mathbb{R}$ and continuous on $\mathbb{Z}$.

Furthermore, C coincides with c on $\mathbb{Z}$, and $C(\xi)$ is the index of the Dirac operator on $M_{red}(\xi)$ twisted by the Kostant line bundle L_ξ on the reduced manifold $M_{red}(\xi)$. Thus this formula is a quantum analogue of the Duistermaat-Heckman formula

$$V_M(\theta) = \int_{\xi\in\mathbb{R}} vol(M_\xi)e^{i\xi\theta}d\xi,$$

$$Tr_{KedD_L}(\exp\theta J) = \sum_{\xi\in\mathbb{Z}} Q(M_\xi)e^{i\xi\theta}$$

Guillemin-Sternberg conjecture: quantization commutes with reduction.

(Proved in the case of general compact groups by Meinrenken, Tian-Zhang, Paradan)

In case where the $\hat{A}$ class of m is a function (that is if all weights at each fixed points on the complexified space T_wM are the same) then we obtain the "Euler-Mac Laurin type of relation".

if

$$V_M(\theta) = \int_{\mathbb{R}} ev(\xi)e^{i\xi\theta}d\xi$$

Then

$$\int_M e^{i\Omega(\theta)}Trunc(A(M))(\theta) = \int_{\mathbb{R}}(\hat{A}(\partial_\xi)v(\xi))e^{i\xi\theta}d\xi$$

That is to compute the multiplicities, we must differential the Duistermaat-Hechman measue by a $\hat{A}$ operator.

Let us verify our formula, for $M = S_\lambda, S_\lambda\times S_\lambda, S_\lambda\times S_\lambda\times S_\lambda$.

For S_λ, $\lambda = s+1/2$, s integers

our duistermaat measure is the characteristic function of the interval $[-\lambda,\lambda]$: it is a constant, so it dose not change when we differentiate by our operator

$$1 - 1/24\partial^2 + ...$$

It indeed coincide with $c(n) = 1$ or $c(n) = 0$ at all integers... DRAWING...

for $S_\lambda\times S_\lambda$, our Duistermaat measure is linear, again, it does not change under differntitation by

$$\hat{A} = 1 - 1/12partial_\xi^2 + ..$$

and indeed $dh(xi)$ coincide with the weights of J in $V_\lambda\otimes V_\lambda$ at all integers...

Start to be amusing for $S_\lambda\times S_\lambda\times S_\lambda$. as our duistermaat-heckman function is given by polynomials of degree two and our operator is $1-1/8*partial_\xi^2+...$

Recall the Duistermaat Heckman function

$$v(\xi) = 3*\lambda^2 - \xi^2$$

for $-\lambda \leq \xi \leq \lambda$

$$v(\xi) = (3\lambda - \xi)^2/2$$

for $\lambda \leq \xi \leq 3\lambda$

$$v(\xi) = (3\lambda + \xi)^2/2$$

for $-3\lambda \leq \xi \leq -\lambda$.

We obtain by differentiation

$$C(\xi) = 3 * \lambda^2 - \xi^2 + 1/4$$

for $-\lambda \leq \xi \leq \lambda$

$$C(\xi) = (3\lambda - \xi)^2/2 - 1/8$$

for $\lambda \leq \xi \leq 3\lambda$

$$C(\xi) = (3\lambda + \xi)^2/2 - 1/8$$

for $-3\lambda \leq \xi \leq -\lambda$.

As λ is a half integer, C has discontinuity only at half integers and coincide with the multiplicities of J in $V_\lambda \otimes V_\lambda \otimes V_\lambda$ at integers...

Some "derivative coincide" however: the values at two near bye integers in both side of the jumps...

General case

We now consider the general case of elliptic operator D. Consider the principal symbol $\sigma(x,\xi)$ of D, an odd morphism of vector bundles

$$\sigma := \begin{pmatrix} 0 & \sigma^-(x,\xi) \\ \sigma^+(x,\xi) & 0 \end{pmatrix}.$$

Using σ, we can construct a chern character $ch(E,\sigma)$ of the bundle E as a compactly supported equivariant class on T^*M. (the difference of $ch(E^+)$ and $ch(E^-)$).

In the same way that we can pass to an integral of an equivariant form to a fixed point formula, we can reverse the process and "delocalize" the fixed pointformula of Atiyah-Bot.

We consider the symplectic manifold $N := T^*M$ with its equivariant symplectic from $\Omega(\theta)$ and we can write

$$\sum_{w \in F} \frac{STr_{E_w}(\exp \theta J)}{\det_{T_w M}(1 - exp(\theta J))}$$
$$= \frac{1}{(2i\pi)^{\dim M}} \int_N e^{i\Omega(\theta)} ch(\sigma(\theta)) \hat{A}(N)(\theta)$$

where $ch(\sigma)$ is the equivariant Chern character

Thus

$$index(D)(\exp(\theta J)) = \frac{1}{(2i\pi)^{\dim M}} \int_N e^{i\Omega(\theta)} \mathrm{ch}(\sigma)(\theta) \hat{A}(N)(\theta)$$

where $\hat{A}(N)(\theta)$ is the equivariant $\hat{A}$ class of the manifold N.

For $\theta = 0$, this is just the Atiyah-Singer index theorem.

Now comes the miracle: the equivariant form $\hat{A}(N)(\theta)$ is the characteristic class on M associated to $\prod_i \frac{c_i}{e^{c_i/2} - e^{-c_i/2}}$ where $c_i(\theta)$ are the equivariant Chern classes of the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$. Let us consider the Bernoulli numbers so that $x/(e^{x/2} - e^{-x/2}) = \sum_{k=0}^{\infty} b_{2k} x^{2*k}/(2k)!$ and consider the corresponding truncated class

$$Trunc\hat{A}(N) := \prod_i (\sum_{k=0}^{\dim N} b_{2k} c_i^{2k}/(2k!)$$

Then $TA(N)(\theta)$ is of polynomial type and

$$\alpha(\theta) := e^{i\Omega(\theta)} \mathrm{ch}(\sigma)(\theta) TA(N)(\theta)$$

is a closed equivariant characteristic class of oscillatory type. Furthermore, it has compact support on $N = T^*M$. We use a further truncation procedure to construct a form $TruncA(N)$ of polynomial type.

The following two formulae (still conjectural, we proved a weaker version with DeConcini+Procesi, using a stabilisation of the tangent bundle):

$$Str_{Ker(D)}(\exp \theta J) = \frac{1}{(2i\pi)^{\dim M}} \int_N e^{i\Omega(\theta)} \mathrm{ch}(\sigma)(\theta) \hat{A}(N)(\theta) = \sum c(n) e^{in\theta}$$

$$\frac{1}{(2i\pi)^{\dim M}} \int_N e^{i\Omega(\theta)} \mathrm{ch}(\sigma)(\theta) TRuncA(N)(\theta) = \int_\xi e^{i\xi\theta} C(\xi) d\xi$$

and the distribution $C(\xi)$ is piecewise polynomial on $\mathbb{R}$ and continuous on $\mathbb{Z}$. Furthermore, it coincides with $\tilde{c}$ on $\mathbb{Z}$.