DUISTERMAAT HECKMAN MEASURES AND THE EQUIVARIANT INDEX THEOREM

Let $N$ be a symplectic manifold, with a Hamiltonian action of the circle group $G$ and moment map $\mu : N \to \mathbb{R}$. Assume that the level sets of $\mu$ are compact manifolds. The Duistermaat-Heckman measure is a locally polynomial function on $\mathbb{R}$ (so called a spline) which measures the symplectic volume of the level sets $\mu^{-1}(t)/G$.

Similarly, we will show that an elliptic (or transversally elliptic) operator $D$ on a $G$-manifold $M$ produces a locally polynomial function on $\mathbb{R}$ via the “infinitesimal index” map. The index of $D$ can be deduced from the knowledge of the infinitesimal index.

References is De Concini-Procesi-Vergne (arXiv:1012.1049; Box splines and the equivariant index theorem).

1. DUISTERMAAT-HECKMAN STATIONARY PHASE

Let $M$ be a symplectic manifold of dimension $2\ell$, with a symplectic form $\Omega$. I am happy to talk in the "Darboux seminar" as of course Darboux theorem is a fundamental theorem for symplectic geometry:

**Darboux theorem**

There exists local coordinates $(p, q)$ (why these letters ??) so that $\Omega = \sum_{i=1}^{\ell} dp_i \wedge dq_i$.

i. e. a symplectic manifold $(M, \Omega)$ is locally a symplectic vector space $\mathbb{R}^{2\ell}$.

In some sense, Duistermaat-Heckman theorem is a variation on Darboux theorem.

Assume now that the circle group $G$ with generator $J$,

$$(G := \{\exp(\theta J)\}, \theta \in \mathbb{R}/2\pi\mathbb{Z})$$

acts on $M$ in a Hamiltonian way: we are in presence of a rotational symmetry on $M$. We assume that the action of the one parameter group $G$ on $M$ is Hamiltonian: there exists a function $\mu$ on $M$ such that the 1-form $d\mu$ is the symplectic gradient of the vector field $J_M$ generated by the action of $G$

$$\langle d\mu, v \rangle = \Omega(J_M, v).$$

The function $\mu$ is thus constant on the orbit of $m$ under $\exp(\theta J)$: Emmy Noether’s theorem on the conservation of the Energy .

In other words, $\mu : M \to \mathbb{R}$ satisfies

$$d\mu = \iota(J_M)\Omega$$

where $\iota(J_M)$ is the contraction operator on differential forms.
This equation implies immediately that the set of critical points of the function $\mu(m)$ is the set of fixed points for the action of the one parameter group $\exp(tJ)$ on $M$.

**DRAWINGS:**

Let us give two simple examples:

The space $\mathbb{R}^2 := (p, q)$ with the rotation action: $J = (p\partial_q - q\partial_p)$, and $\mu = \frac{1}{2}(p^2 + q^2)$.

- Let $S_\lambda$ be the sphere $x^2 + y^2 + z^2 = \lambda^2$ and $J = x\partial_y - y\partial_x$ be the rotational symmetry around the z axis. Let $\Omega$ be the restriction of $\frac{1}{\sqrt{x^2+y^2+z^2}}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dz)$ to $S_\lambda$. ($S_\lambda$ = the space $P_1(\mathbb{C})$)

  We consider the rotation action around the axe $Oz$; with generator $J$; then the moment $\mu$ is just $\mu = z$ (the height).

  **DRAWING:**

  A symplectic manifold $M$ is endowed with a Liouville measure $d\beta = \frac{1}{(2\pi)^\frac{\ell}{2}} \Omega^\ell$. It is thus natural to compute the integral of a function on $M$ with respect to $d\beta$. For example, if $M$ is compact, the integral of the function 1 is the symplectic volume of $M$.

Consider $f$ a function on $M$ and the integral

$$\int_M e^{itf}d\beta(m).$$

When $t$ tends to $\infty$, the asymptotic behavior (in $t$) of this integral depends only of the knowledge of $f$ near the set of critical points of $f$ (the stationary phase formula).

Duistermaat-Heckman discovered the extremely beautiful result: if $f(m) = \mu(m)$ then the asymptotic formula is exact.

Let me state Duistermaat-Heckman formula for the case where the set of fixed points for the one parameter group $\exp tJ$ is a finite set $F$. Then

$$\int_M e^{it\mu(m)}d\beta(m) = \sum_{w \in F} e^{it\mu(w)} \frac{1}{t^\frac{\ell}{2}D_w}$$

where

$$D_w = \det (L(J))^{1/2}.$$

(In Darboux coordinates near $w$, we have $J = \sum_{i=1}^\ell a_i(p_i\partial_{q_i} - q_i\partial_{p_i})$ and $D_w = \prod_{i=1}^\ell a_i$.)

When $M$ is a coadjoint orbit $K\Lambda \subset \mathfrak{k}^*$, this is the very familiar Harish-Chandra formula in the theory of compact Lie groups for the Fourier transform of an orbit $K\Lambda$. But this "fixed point formula" holds in the general context of a Hamiltonian manifold, and this was discovered by Duistermaat-Heckman.

We will call

$$V_M(\theta) = \int_M e^{it\mu(m)}d\beta(m)$$
the equivariant volume of $M$: when $\theta = 0$, this is just the volume of $M$, when $M$ is compact. An important remark is that when $M$ is not compact, it happens that $v_M(\theta)$ is defined in the distribution sense for many important examples (hyperkahler quotients for examples), as an oscillatory integral.

Let me first write the formula for $S_\lambda := \{x^2 + y^2 + z^2 = \lambda^2\}$. It is immediate to see that

$$\int_{S_\lambda} e^{i\theta z} d\beta = \frac{e^{i\lambda \theta} - e^{-i\lambda \theta}}{i \theta}. $$

TWO FIXED POINTS: The symplectic volume of $S_\Lambda$ is $2\lambda$. (Later $\lambda$ will belong to $\frac{1}{2} \mathbb{Z}$)

Similarly, when $M = \mathbb{R}^2$, then

$$\int_{\mathbb{R}^2} e^{i\theta} d\beta = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\theta(p^2 + q^2)} dp dq = \frac{1}{i \theta}.$$

ONE FIXED POINT.

Duistermaat-Heckman measures

Consider our equivariant volume:

$$V_M(\theta) = \int_M e^{i\theta \mu(m)} d\beta_M. $$

Consider the map $\mu : M \to \mathbb{R}$ and a regular value $\xi$. It is well known that if we consider the fiber $\mu^{-1}(\xi)$ (a $2\ell - 1$ manifold, and divide by the action of the one parameter group $G := \exp \theta J$, then $\mu^{-1}(\xi)/G$ is a $2\ell - 2$ symplectic manifold (orbifold). This is called the reduced manifold $M_{\text{red}}(\xi)$ of $M$ at $\xi$.

We put $\xi = \mu(m)$ and integrate over the fiber $\mu(m) = \xi$. We then obtain a measure on $\mathbb{R}$, the push-forward of the Liouville measure with support the image of $M$ by $\mu$. We will call $\mu_*(d\beta) = dh(\xi)$ the Duistermaat-Heckman measure on $\mathbb{R}$. Thus by definition:

$$V_M(\theta) = \int_{\mathbb{R}} e^{i\xi \theta} dh(\xi). $$

We see thus that this is well defined, just provided the fibers of $\mu$ are compact manifolds.

The measure $dh(\xi)$ is the Fourier transform of the equivariant volume.

**Theorem 1.1.** The Duistermaat-Heckman measure is a piecewise polynomial measure on $\mathbb{R}$ (a spline).

$$dh(\xi) = p(\xi) d\xi $$

where $p$ is piecewise polynomial:

Moreover, if $\xi$ is a regular value of the moment map $\mu : M \to \mathbb{R}$, then the value $p(\xi)$ is equal to the symplectic volume of the reduced fiber $\mu^{-1}(\xi)/G$. 

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(this theorem is almost equivalent to the stationary phase exact formula)

Thus

$$V_M(\theta) = \int_{\mathbb{R}} e^{i\xi \theta} \text{vol}(M_{\text{red}}(\xi)) d\xi.$$

VERY IMPORTANT THEOREM: the stationary phase formula for $V_M(\theta)$+this theorem can be used to compute volumes of reduced manifolds, such as moduli spaces of flat bundles over a surface of genus $g$, by Fourier inversion.

Example: $M := S_\lambda \mu = z$ The image is the interval $[-\lambda, \lambda]$.

It is very easy to compute the image measure; the reduced fibers are points, and the image measure is the characteristic function of the interval $[-\lambda, \lambda]$. This is compatible with the formula for the equivariant volume:

$$\int_{-\lambda}^{\lambda} e^{iz\theta} dz$$

is indeed equal to

$$\frac{e^{i\lambda\theta} - e^{-i\lambda\theta}}{i\theta}.$$

It maybe more interesting later on to take products of a certain number of copies $S_\lambda \times \cdots \times S_\lambda$ with diagonal action of $\exp \theta J$ and moment map $\mu = \mu_1 + \mu_2 + \cdots + \mu_k$.

Then we see by Fourier transform that we obtain the convolution of the two intervals $2$, $3$ intervals $[-\lambda, \lambda]$, etc... This is the so called Box splines in numerical analysis.

DRAWINGS:

For example for $M = S_\lambda \times S_\lambda$, we have

$$V_M(\theta) = \frac{e^{-2i\lambda\theta}}{(i\theta)^2} - \frac{1}{(i\theta)^2} + \frac{e^{2i\lambda\theta}}{(i\theta)^2}$$

$$= \int_{\mathbb{R}} f(\xi)e^{i\xi \theta} d\xi$$

with

$$f(\xi) = 2* \lambda - \xi \quad \text{for} \quad 0 \leq \xi \leq 2\lambda \quad f(\xi) = 2* \lambda + \xi \quad \text{for} \quad -2* \lambda \leq \xi \leq 0.$$ 

For 3 spheres (same radius to simplify), $M = S_\lambda \times S_\lambda \times S_\lambda$,

$$V_M(\theta) = \frac{e^{-3i\lambda\theta}}{(i\theta)^3} - \frac{3e^{-i\lambda\theta}}{(i\theta)^3} - \frac{3e^{i\lambda\theta}}{(i\theta)^3} + \frac{e^{3i\lambda\theta}}{(i\theta)^3}.$$ 

This is the Fourier transform of the locally polynomial density:

$$f(\xi) = 3* \lambda^2 - \xi^2; \quad -\lambda \leq \xi \leq \lambda$$

$$f(\xi) = (3\lambda - \xi)^2/2; \quad \lambda \leq \xi \leq 3\lambda$$

$$f(\xi) = (3\lambda + \xi)^2/2; \quad -3\lambda \leq \xi \leq -\lambda.$$ 

for 4 intervals.
Equivariant localization

Duistermaat-Heckman formula is best understood as a localization formula in equivariant cohomology (Berline-Vergne-Atiyah-Bott);

Let us introduce the equivariant symplectic form (a non homogenous differential form on $M$: a function + a 2-form):

$$
\Omega(\theta) = \theta \mu(m) + \Omega.
$$

Then Duistermaat-Heckman formula can be rewritten as:

$$
\frac{1}{(2i\pi)^{\ell}} \int_M e^{i\Omega(\theta)} = \sum_w \frac{e^{i\theta \mu(w)}}{(\det_{T_w M} \mathcal{L}(\theta J))^{1/2}}.
$$

Consider the operator $D = d - \iota(J_M)$. The two equations $d\Omega = 0$, $\iota(J_M)\Omega = d\mu$ are equivalent to the single equation

$$(d - \iota(J_M))\Omega(\theta) = 0
$$

we introduced the operator $D$ at the same time than Witten (Supersymmetry and Morse theory, 1982), with the aim of understanding Rossmann formula for Fourier transforms of coadjoint orbits of reductive Lie groups.

Let $\alpha(\theta)$ a closed equivariant differential form (invariant by $G$ and satisfying $D\alpha(\theta) = 0$. Then exactly the same localization formula holds (Berline-Vergne, Atiyah-Bott). If $M$ is any manifold and $\alpha$ is any equivariantly closed form:

$$
\frac{1}{(2i\pi)^{\ell}} \int_M \alpha(\theta) = \sum_{w \in F} \frac{i^* \alpha(w)(\theta)}{(\det_{T_w M} \mathcal{L}(\theta J))^{1/2}}.
$$

Let us now try to understand in which case the Fourier transform of

$$
I(\theta) = \frac{1}{(2i\pi)^{\ell}} \int_M \alpha(\theta)
$$

is a piecewise polynomial measure...

We make a definition.

**Definition 1.2.** The form $\alpha(\theta)$ will be called of oscillatory type, if at each point $w \in F$, $\alpha(w)(\theta)$ is a sum of products of exponential $e^{i\mu \theta}$ ($\mu$ real) times polynomial functions of $\theta$.

Of course, for $\Omega(\theta) = \theta \mu + \Omega$, then $e^{i\Omega(\theta)}$ is of oscillatory type. also $e^{i\Omega(\theta)}b(\theta)$ where $b(\theta)$ is a polynomial, equivariant Chern characters $ch(E)(\theta)$ of vector bundles are of oscillatory type, etc...

Then we have almost the same theorem:

**Theorem 1.3.** (Witten non abelian localization theorem)

If $\alpha(\theta)$ is a closed equivariant differential form of oscillatory type, then

$$
\frac{1}{(2i\pi)^{\ell}} \int_M \alpha(\theta)
$$
is the Fourier transform
\[ \int_{\mathbb{R}} e^{i\xi \theta} p(\xi) d\xi \]
where \( p(\xi) \) is a locally polynomial function + a sum of derivatives of \( \delta \)-functions of a finite number of points.

Furthermore in case where \( \alpha(\theta) = e^{i\Omega(\theta)} b(\theta) \), \( b(\theta) \) polynomial, then \( p(\xi) \) computes the integral over the reduced manifold \( M_{\text{red}}(\xi) \) of the cohomology class \( e^{i\Omega}\delta b(F) \), where \( F \) is the curvature of the fibration \( \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G \).

This theorem has been used to compute intersection rings of reduced manifolds (for example by Jeffrey-Kirwan for the proof of the Verlinde formula).

### Equivariant index theorems

The B-V localization formula in equivariant cohomology is very reminiscent of the index formulae of Atiyah-Bott.

Let \( M \) be a compact manifold, with an action of the group \( G = \{ \exp(\theta J) \} \), let \( \mathcal{E}^\pm \) be two \( G \)-equivariant Hermitian vector bundles on \( M \), let \( E = E^+ \oplus E^- \); super vector bundle, and let
\[
D := \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}
\]
an odd elliptic operator on \( \Gamma(M, E) \):
\[
D^+ : \Gamma(M, E^+) \to \Gamma(M, E^-),
\]
\[
D^- : \Gamma(M, E^-) \to \Gamma(M, E^+)
\]
between the \( C^\infty \) sections of \( E^\pm \) and commuting with \( G \).

The kernel of \( D^+ \) as well as the kernel of \( D^- \) are finite dimensional vector spaces so that there exists an action of \( G \) in the superspace \( \text{Ker}(D) = \text{Ker}(D^+) - \text{Ker}(D^-) \). This object depends only of the principal symbol \( \sigma \) of \( D \), up to deformation, thus it depends only of \( [\sigma] \) in the equivariant \( K \)-theory of \( N = T^*M \). We wish to compute the eigenvalues of the generating element \( J \) in the space \( \text{Ker} D \) and their multiplicities: In other words, we want to compute the Fourier series:
\[
\text{Str}_{\text{Ker}(D)}(\exp \theta J) = \sum_{n \in \mathbb{Z}} c(n) e^{in\theta}.
\]

I will assume to simplify that the action of \( G \) on \( M \) is with connected stabilizers: for \( m \in M \), then either \( G_m = G \), or \( G_m = \{1\} \).

**Theorem 1.4.** (De Concini+Procesi+V.) There exists a "natural" piecewise polynomial function \( \tilde{c} \) on \( \mathbb{R} \), continuous on the lattice \( \mathbb{Z} \), such that
\[
\text{Str}_{\text{Ker}(D)}(\exp \theta J) = \sum_{n} \tilde{c}(n) e^{in\theta}.
\]
The theorem is true for general transversally elliptic operators under an action of a compact Lie group \( K \).

In general, we compute a piecewise quasipolynomial function on the dominant weights of \( K \). We call the function \( \tilde{c} \) the infinitesimal index.

I will explain the construction of \( \tilde{c} \) in the case of a very popular elliptic operator: the twisted Dirac operator. I will then explain the general construction.

Assume \( M \) compact, \( D \) elliptic; Let us first recall the fixed point formula of Atiyah-Bott in the case where the action of \( G \) on \( M \) has only a finite number of fixed points;

\[
\text{Atiyah-Bott fixed point formula}
\]

\[
\text{Str}_{\text{Ker}(D)}(\exp(\theta J)) = \sum_{w \in F} \frac{\text{Str}_{E_w}(\exp \theta J)}{\det_{T_w M}(1 - \exp(\theta J))}.
\]

Let us analyze this formula when \( M \) is symplectic and with spin structure, and \( L \) a Kostant line bundle: at Assume that \( M \) is Hamiltonian, symplectic (and with spin structure to simplify) with moment map \( \mu : M \to \mathbb{R} \) as before. Let \( L \) be a Kostant line bundle on \( M \), that is if \( w \) is a fixed point by the one parameter group \( \exp(\theta J) \), we have

\[
\exp(\theta J)|_{L_w} = e^{i\theta \mu(w)}.
\]

Consider the twisted Dirac \( D_L \) operator on \( M \), operating on \( E := S \otimes L \), where \( S = S^+ \oplus S^- \) is the spinor bundle. then the general Atiyah-Bott formula is

\[
\text{Str}_{\text{Ker}(D)}(\exp(\theta J)) = \sum_{w \in F} e^{i\theta \mu(w)} \text{Str}_{S_w}(\exp \theta J) \frac{1}{\det_{T_w M}(1 - \exp(\theta J))^{1/2}}.
\]

The supertrace in the spin space \( S_w \) is a square root of \( \det_{T_w M}(1 - \exp(\theta J)) \), thus we obtain

\[
\text{Str}_{\text{Ker}(D)}(\exp(\theta J)) = \sum_{w \in F} e^{i\theta \mu(w)} \frac{1}{(\det_{T_w M}(1 - \exp(\theta J)))^{1/2}}.
\]

a formula which is very similar to

\[
\int_M e^{i\theta \mu(m)} d\beta(m) = \sum_{w \in F} e^{i\theta \mu(w)} (\det_{T_w M}(\mathcal{L}_w(\theta J)))^{1/2}.
\]

For example, consider \( M := S_\lambda \) where \( \lambda \in \frac{1}{2} + \mathbb{Z} \) is a half-integer, we consider

\[
s = \lambda - 1/2,
\]

then the sphere \( S_\lambda \) is prequantizable, with line bundle \( L_\lambda \), (passing to the double cover of the rotation group \( G \)). We can then consider \( D_\lambda \) the twisted Dirac operator by the line bundle \( L_\lambda \), and the corresponding representation of \( \exp(\theta J) \) in \( \text{Ker}(D_\lambda) \) is the representation of \( SO(3) \) corresponding to a particle with spin \( s \). The eigenvalues of \( J \) generator of the rotation around
the z axis in $SO(3)$ is then the set $\{s, s-1, ..., -s\}$, composed of integers, each with multiplicity 1.

We have then two very similar formulae for $M = S_{\lambda}$:

$$\int_M e^{i\theta \mu(m)}d\beta(m) = \frac{e^{i\lambda\theta}}{i\theta} - \frac{e^{-i\lambda\theta}}{i\theta}$$

a sum of two terms corresponding to the two fixed points north pole and south pole on $S_{\lambda}$. Similarly

$$\text{STr}_{KerD_{\lambda}}(\exp \theta J) = \frac{e^{i\lambda\theta}}{e^{i\theta/2} - e^{-i\theta/2}} - \frac{e^{-i\lambda\theta}}{e^{i\theta/2} - e^{-i\theta/2}}.$$

In a similar way that we can write

$$\sum_{w \in F} \frac{e^{i\theta \mu(w)}}{(\det_{T_w}(C_w(\theta J)))^{1/2}}$$

as an integral over the manifold $M$ of an equivariant form ($\int_M e^{i\Omega(\theta)}$), we can rewrite Atiyah-Bott formula as a formula on the symplectic manifold $M$ with symplectic form $\Omega(\theta)$. and we have the "delocalized index formula" :

$$\text{index}(D_L)(\exp(\theta J)) = \frac{1}{(2i\pi)^7} \int_M e^{i\Omega(\theta)} \hat{A}(M)(\theta)$$

where $\hat{A}(M)(\theta)$ is the equivariant $\hat{A}$ class of the manifold $M$.

For $\theta = 0$, this is just the Atiyah-Singer index theorem.

$$q(M) := \text{index}(D_L) = \int_M ch(L)\hat{A}(M).$$

Now comes the miracle: the equivariant form $\hat{A}(M)(\theta)$ is the characteristic class on $M$ associated to $\prod_{i=1}^\ell e^{ci/2}e^{-ci/2}$ where $c_i(\theta)$ are the equivariant Chern classes of the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$. Let us consider the Bernoulli numbers so that $x/(e^{x/2} - e^{-x/2}) = \sum_{k=0}^\infty b_k((1/2)^{2k-1} - 1)x^{2k}/(2k)!$ and consider the corresponding truncated class

$$\text{Trunc}\hat{A}(M) := (\prod_i (\sum_{k=0}^\infty b_{2k}((1/2)^{2k-1} - 1)c_i^{2k}/(2k)!))_{\leq(dimM-2)}.$$

Then $\text{Trunc}\hat{A}(M)(\theta)$ is polynomial in $\theta$ and

$$\alpha(\theta) := e^{i\Omega(\theta)}\text{Trunc}\hat{A}(M)(\theta)$$

is a closed equivariant characteristic class of oscillatory type on $M$.

Then we have the following two formulae:

$$\text{STr}_{Ker(D)}(\exp \theta J) = \frac{1}{(2i\pi)^7} \int_M e^{i\Omega(\theta)} \hat{A}(M)(\theta) = \sum c(n)e^{i\alpha(n)}$$

AND:
\[
\frac{1}{(2i\pi)^l} \int_M e^{i\Omega(\theta)} \text{Trunc} \hat{A}(M)(\theta) = \int_\xi e^{i\xi \theta} C(\xi) d\xi
\]
and the distribution \( C(\xi) \) is piecewise polynomial on \( \mathbb{R} \) and continuous on \( \mathbb{Z} \).

Furthermore, \( C \) coincides with \( c \) on \( \mathbb{Z} \), and \( C(\xi) \) is the index of the Dirac operator on \( M_{\text{red}}(\xi) \) twisted by the Kostant line bundle \( L_\xi \) on the reduced manifold \( M_{\text{red}}(\xi) \). Thus this formula is a quantum analogue of the Duistermaat-Heckman formula
\[
V_M(\theta) = \int_{\xi \in \mathbb{R}} \text{vol}(M_\xi) e^{i\xi \theta} d\xi,
\]
\[
\text{Tr}K_{\text{KedD}}(\exp \theta J) = \sum_{\xi \in \mathbb{Z}} Q(M_\xi) e^{i\xi \theta}
\]
Guillemin-Sternberg conjecture: quantization commutes with reduction. (Proved in the case of general compact groups by Meinrenken, Tian-Zhang, Paradan)

In case where the \( \hat{A} \) class of \( m \) is a function (that is if all weights at each fixed points on the complexified space \( T_w M \) are the same) then we obtain the "Euler-Mac Laurin type of relation".

If
\[
V_M(\theta) = \int_{\xi \in \mathbb{R}} e\nu(\xi) e^{i\xi \theta} d\xi
\]
Then
\[
\int_M e^{i\Omega(\theta)} \text{Trunc} (A(M))(\theta) = \int_{\mathbb{R}} (\hat{A}(\partial_\xi) v(\xi)) e^{i\xi \theta} d\xi
\]
That is to compute the multiplicities, we must differential the Duistermaat-Heckman measure by a \( \hat{A} \) operator.

Let us verify our formula, for \( M = S_\lambda, S_\lambda \times S_\lambda, S_\lambda \times S_\lambda \times S_\lambda \).

For \( S_\lambda, \lambda = s + 1/2, s \) integers
our duistermaat measure is the characteristic function of the interval \([-\lambda, \lambda] \): it is a constant, so it dose not change when we differentiate by our operator
\[
1 - 1/24 \delta^2 + ...
\]
It indeed coincide with \( c(n) = 1 \) or \( c(n) = 0 \) at all integers... DRAWING...
for \( S_\lambda \times S_\lambda \), our Duistermaat measure is linear, again, it does not change under differntiation by
\[
\hat{A} = 1 - 1/12 \partial^2 + ...
\]
and indeed \( dh(x_i) \) coincide with the weights of \( J \) in \( V_\lambda \otimes V_\lambda \) at all integers...
Start to be amusing for \( S_\lambda \times S_\lambda \times S_\lambda \). as our duistermaat-heckman function is given by polynomials of degree two and our operator is \( 1 - 1/8 \partial^2 + ... \)
Recall the Duistermaat Heckman function
\[
v(\xi) = 3 * \lambda^2 - \xi^2
\]
for $-\lambda \leq \xi \leq \lambda$

$$v(\xi) = (3\lambda - \xi)^2/2$$

for $\lambda \leq \xi \leq 3\lambda$

$$v(\xi) = (3\lambda + \xi)^2/2$$

for $-3\lambda \leq \xi \leq -\lambda$.

We obtain by differentiation

$$C(\xi) = 3\lambda^2 - \xi^2 + 1/4$$

for $-\lambda \leq \xi \leq \lambda$

$$C(\xi) = (3\lambda - \xi)^2/2 - 1/8$$

for $\lambda \leq \xi \leq 3\lambda$

$$C(\xi) = (3\lambda + \xi)^2/2 - 1/8$$

for $-3\lambda \leq \xi \leq -\lambda$.

As $\lambda$ is a half integer, $C$ has discontinuity only at half integers and coincide with the multiplicities of $J$ in $V_\lambda \otimes V_\lambda \otimes V_\lambda$ at integers...

Some "derivative coincide" however: the values at two near bye integers in both side of the jumps...

**General case**

We now consider the general case of elliptic operator $D$. Consider the principal symbol $\sigma(x, \xi)$ of $D$, an odd morphism of vector bundles

$$\sigma := \begin{pmatrix} 0 & \sigma^-(x, \xi) \\ \sigma^+(x, \xi) & 0 \end{pmatrix}.$$

Using $\sigma$, we can construct a Chern character $ch(E, \sigma)$ of the bundle $E$ as a compactly supported equivariant class on $T^*M$. (the difference of $ch(E^+)$ and $ch(E^-)$).

In the same way that we can pass to an integral of an equivariant form to a fixed point formula, we can reverse the process and "delocalize" the fixed point formula of Atiyah-Bott.

We consider the symplectic manifold $N := T^*M$ with its equivariant symplectic from $\Omega(\theta)$ and we can write

$$\sum_{w\in F} \frac{ST_{T^*M}(\exp \theta J)}{\det_{T^*M}(1 - \exp(\theta J))}$$

$$= \frac{1}{(2i\pi)^{\dim M}} \int_N e^{i\Omega(\theta)} ch(\sigma(\theta)) \hat{A}(N)(\theta)$$

where $ch(\sigma)$ is the equivariant Chern character.

Thus
index(D)(\exp(\theta J)) = \frac{1}{(2i\pi)^{\dim M}} \int_N e^{i\Omega(\theta)} \operatorname{ch}(\sigma)(\theta) \hat{A}(N)(\theta)

where \( \hat{A}(N)(\theta) \) is the equivariant \( \hat{A} \) class of the manifold \( N \).

For \( \theta = 0 \), this is just the Atiyah-Singer index theorem.

Now comes the miracle: the equivariant form \( \hat{A}(N)(\theta) \) is the characteristic class on \( M \) associated to \( \prod_i \frac{c_i}{e^{c_i/2} - e^{-c_i/2}} \) where \( c_i(\theta) \) are the equivariant Chern classes of the complexified tangent bundle \( TM \otimes_{\mathbb{R}} \mathbb{C} \). Let us consider the Bernoulli numbers so that \( x/(e^{x/2} - e^{-x/2}) = \sum_{k=0}^{\infty} b_{2k} x^{2k}/(2k)! \) and consider the corresponding truncated class

\[
\text{Trunc}\hat{A}(N) := \prod_i \left( \sum_{k=0}^{\dim N} b_{2k} c_i^{2k}/(2k!) \right)
\]

Then \( TA(N)(\theta) \) is of polynomial type and

\[
\alpha(\theta) := e^{i\Omega(\theta)} \operatorname{ch}(\sigma)(\theta) TA(N)(\theta)
\]

is a closed equivariant characteristic class of oscillatory type. Furthermore, it has compact support on \( N = T^* M \). We use a further truncation procedure to construct a form \( \text{Trunc} A(N) \) of polynomial type.

The following two formulae (still conjectural, we proved a weaker version with DeConcini+Procesi, using a stabilisation of the tangent bundle):

\[
\text{Str}_{\text{Ker}(D)}(\exp \theta J) = \frac{1}{(2i\pi)^{\dim M}} \int_N e^{i\Omega(\theta)} \operatorname{ch}(\sigma)(\theta) \hat{A}(N)(\theta) = \sum c(n) e^{in\theta}
\]

\[
\frac{1}{(2i\pi)^{\dim M}} \int_N e^{i\Omega(\theta)} \operatorname{ch}(\sigma)(\theta) \text{Trunc} A(N)(\theta) = \int_{\xi} e^{i\xi \theta} C(\xi) d\xi
\]

and the distribution \( C(\xi) \) is piecewise polynomial on \( \mathbb{R} \) and continuous on \( \mathbb{Z} \). Furthermore, it coincides with \( \tilde{c} \) on \( \mathbb{Z} \).