

Box splines

Let C_N be the N dimensional hypercube:

$$C_N := \{t_1, t_2, \dots, t_N; 0 \leq t_i \leq 1\}$$

We slice it with the hyperplane:

$$H_t = \left\{ \sum_{i=1}^N t_i = t \right\}$$

$$\text{Box}_N(t) = \text{volume}(C_N \cap H_t).$$

Then $\text{Box}_N(t)$ is supported on $0 \leq t \leq N$. On each interval $[k, k + 1]$ given by a polynomial of degree $N - 1$. But different polynomials on each interval. Their $N - 2$ first derivatives agree at the extreme of intervals.

Remark the symmetry:

$$Box_N(t) = Box_N(N - t)$$

We also see that

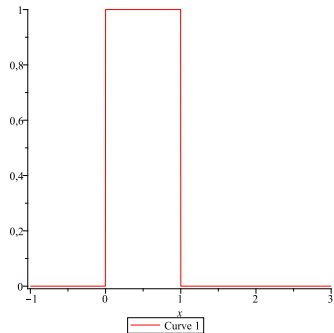
$$\frac{d}{dt} Box_N(t) = Box_{N-1}(t) - Box_{N-1}(t - 1)$$

That is

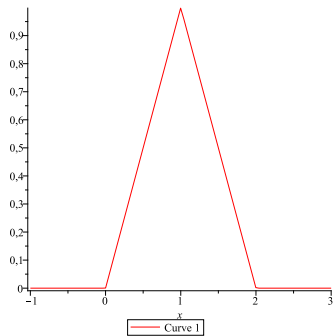
$$\frac{d}{dt} Box_N(t) = (\nabla Box_{N-1})(t)$$

where ∇ is the difference operator. (We will not use this equation in this elementary talk: it holds in the distribution sense, for $N > 1$, because of C^{N-2} -differentiability properties)

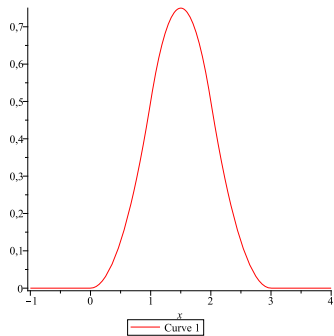
Example: $Box_1(t)$



$Box_2(t)$



$Bot_3(t)$

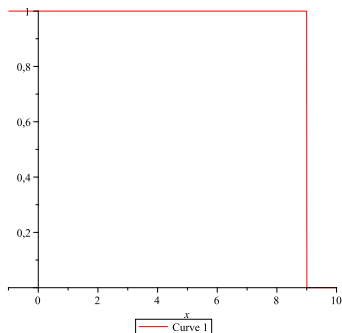


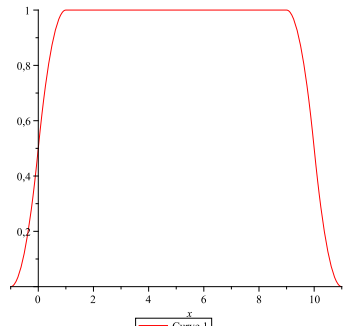
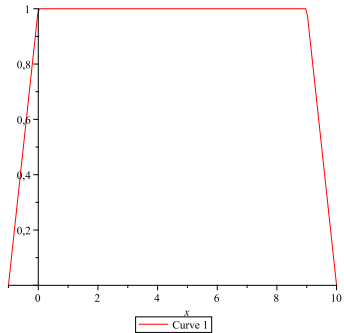
Wonderful properties of Box splines

For example

$$\sum_{n \in \mathbb{Z}} \text{Box}_N(t - n) = 1$$

The following pictures (see Procesi document) shows the sum of $\text{Box}_1(t - n)$, $\text{Box}_2(t - n)$, $\text{Box}_3(t - n)$ over the integers $n = -1, 0, \dots, 8$.





This property follows right away from the geometric definition:

Example: Box_2 : By drawing... Compute $\sum_n \text{Box}_2(t - n)$: We have to sum all the volumes of $x_1 + x_2 = t - n$ for any n .

Now x_1 ranges between 0 and 1. Put $x_2 = t - x_1 - n$ where n is the integer $\text{floor}(t - x_1)$ so that x_2 is between 0 and 1. So it is just parametrized by $0 \leq x_1 \leq 1$.

More generally, we will see that:

Theorem

For any polynomial P of degree strictly less than N ,

$$t \rightarrow \sum_n P(n) \text{Box}_N(t - n)$$

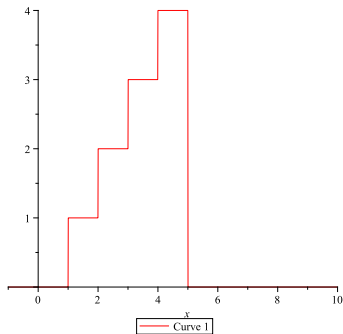
is a polynomial function of t .

$$N = 1$$

- Constant function: already seen.
- $P(t) = t$ of degree too big:

Drawing: I think I sum over $n = 0, 1, 2, 3, 4$

Compute $\sum_n n \text{Box}_1(t - n)$ not a polynomial !!



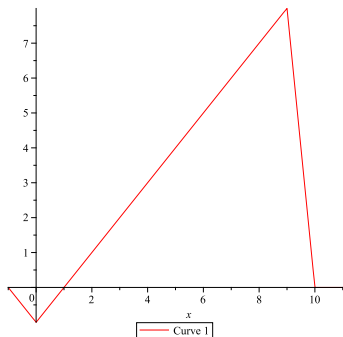
$$N = 2$$

- Constant function: already seen.

-

Function $P(t) = t$ of degree 1: true

Drawing: I think I summed over $n = -1, 0, 1, 2, 3, 4, \dots, 8$



Series of differential operators

If $\mathbf{T}\left(\frac{d}{dt}\right) = \sum_{n=0}^{\infty} t_n \left(\frac{d}{dt}\right)^n$ is a series of differential operators, we can act on polynomials.

$(\mathbf{T}\left(\frac{d}{dt}\right)P)(t)$ can be computed as for n large $\left(\frac{d}{dt}\right)^n P = 0$.

We will use the series

$$\left(\frac{1 - \exp\left(-\frac{d}{dt}\right)}{\frac{d}{dt}}\right) = 1 - \frac{1}{2} \frac{d}{dt} + \frac{1}{3!} \left(\frac{d}{dt}\right)^2$$

and the inverse

Definition

We define the Todd operator

$$\begin{aligned} \mathbf{Todd}\left(\frac{d}{dt}\right) &= \frac{\frac{d}{dt}}{\left(1 - \exp\left(-\frac{d}{dt}\right)\right)} \\ &= 1 + \frac{1}{2} \frac{d}{dt} + \frac{1}{12} \left(\frac{d}{dt}\right)^2 - \frac{1}{720} \left(\frac{d}{dt}\right)^4 + \dots \end{aligned}$$

Integration against the Box Spline

Let $test$ be a smooth function of t . Then we can integrate $test(t_1 + t_2 + \cdots + t_N)$ over the hypercube. By Fubini theorem, we obtain

$$\int_0^1 \int_0^1 \cdots \int_0^1 test(t_1 + t_2 + \cdots + t_N) dt_1 \cdots dt_N = \int_{\mathbb{R}} test(t) Box_N(t) dt$$

Definition

If $test$ is a smooth function of t , define

- The usual convolution

$$(Box_N *_c test)(t) = \int_{u \in \mathbb{R}} test(u) Box_N(t - u) du$$

- The semi-discrete convolution

$$(Box_N *_d test)(t) = \sum_{u \in \mathbb{Z}} test(u) Box_N(t - u)$$

If $test$ is a **polynomial**, we have (use Taylor expansion)

$$\int_0^1 test(t - u) du = \left(\frac{1 - e^{-d/dt}}{d/dt} \right) test(t).$$

Thus

$$\begin{aligned} (Box_N *_c test)(t) &= \int_{u \in \mathbb{R}} test(u) Box_N(t - u) du \\ &= \int_{u_1=0}^1 \cdots \int_{u_N=0}^1 test(t - (u_1 + u_2 + \cdots + u_N)) du_1 \cdots du_N \\ &= \left(\left(\frac{1 - e^{-\frac{d}{dt}}}{\frac{d}{dt}} \right)^N test \right)(t). \end{aligned}$$

Theorem



If P is a polynomial function, then

$$\text{Box}_N *_c P = \left(\left(\frac{1 - e^{-\frac{d}{dt}}}{\frac{d}{dt}} \right)^N P \right)(t).$$



If P is a polynomial function of degree $< N$, then

$$\text{Box}_N *_c P = \text{Box}_N *_d P = \left(\left(\frac{1 - e^{-\frac{d}{dt}}}{\frac{d}{dt}} \right)^N P \right)(t).$$

Proof

Let Box_N : it is enough to prove this for $P(t) = t^{N-1}$

$$t \rightarrow \sum_n n^{N-1} Box_N(t-n)$$

is a polynomial; For smaller degree, we derivate, and we can use the recurrence formula.

I will compute

$$\sum_n P(n) Box_N(t-n)$$

not exactly for $P(t) = t^{N-1}$ but for the polynomial of degree $N-1$ given by

$$t \rightarrow \frac{(t+1)(t+2)\cdots(t+N-1)}{(N-1)!}$$

Consider the standard $(N - 1)$ -dimensional simplex (dilated)

$$S_{N-1}(t) = \{t_i \geq 0; t_1 + t_2 + \cdots + t_N = t.\}$$

$S_{N-1}(t)$ has volume $\frac{t^{N-1}}{(N-1)!}$.

If $t = n$ is an integer the number of integral points in $S_{N-1}(t)$ is

$$\mathcal{P}_N(n) = \frac{(n+1) \cdots (n+N-1)}{(N-1)!}.$$

Now let us integrate over the first quadrant $t_i > 0$, the function $e^{-(t_1+t_2+\dots+t_N)y}$.

$$I = \int_{t_1>0} \dots \int_{t_N>0} e^{-(t_1+t_2+\dots+t_N)y} dt_1 dt_2 \dots dt_N.$$

Using Fubini, we compute I by integrating first over the simplices $S_{N-1}(t)$ then over t , thus

$$I := \int_{t>0} \frac{t^{N-1}}{(N-1)!} e^{-ty} dt$$

But we can also decompose the quadrant in cubes
 $[n_1, n_2, n_3, \dots, n_N] + \text{Hypercube } n_i = 0, 1, \dots, 0 \leq t_i \leq 1$ and
 obtain that I is equal to

$$\begin{aligned}
 \sum_{\mathbf{n}} \int_{t_1=0}^1 \cdots \int_{t_N=0}^1 e^{-((n_1+t_1)-(n_2+t_2)-(n_3+t_3)-\cdots-(n_N+t_N))y} dt_1 dt_2 \cdots dt_N. \\
 &= \int_{t \in \mathbb{R}} \sum_{n_i} e^{-(\sum n_i)y} \text{Box}_N(t) e^{-ty} dt \\
 &= \int_{t \in \mathbb{R}} \sum_{n \geq 0} \mathcal{P}_N(n) e^{-ny} \text{Box}_N(t) e^{-ty} dt \\
 &= \int_{t \in \mathbb{R}} \sum_{n \geq 0} \mathcal{P}_N(n) \text{Box}_N(t - n) e^{-ty} dt.
 \end{aligned}$$

We obtain thus that for every $y > 0$

$$I := \int_{t>0} \frac{t^{N-1}}{(N-1)!} e^{-ty} dt$$

and also

$$= \int_{t \in \mathbb{R}} \sum_{n \geq 0} \mathcal{P}_N(n) \text{Box}_N(t-n) e^{-ty} dt.$$

CONCLUSION: For $t > 0$, we have almost everywhere

$$\sum_{n \geq 0} \mathcal{P}_N(n) \text{Box}_N(t-n) = \frac{t^{N-1}}{(N-1)!}.$$

So we have it on each interval where Box_N is continuous.

Recall that we want to compute

$$\sum_{n \in \mathbb{Z}} P(n) \text{Box}_N(t - n)$$

for the polynomial

$$t \rightarrow \frac{(t+1)(t+2)\cdots(t+N-1)}{(N-1)!}$$

The sum is over the integers n such that $t - n \leq N$, as Box_N is supported on $[0, N]$. thus over the integers $-(N-1), -(N-2), -(N-3), \dots, -1, 0$. But my polynomial vanishes there, and for $n \geq 0$ coincide with $\mathcal{P}_N(n)$. Thus I obtain
For $t \geq 0$

$$\sum_{n \in \mathbb{Z}} P(n) \text{Box}_N(t - n) = \frac{t^{N-1}}{(N-1)!}$$

Same calculation for $t < 0$, using $\text{Box}_N(t) = \text{Box}_N(N - t)$

$$\begin{aligned}\sum_{n \in \mathbb{Z}} P(n) \text{Box}_N(t - n) &= \sum_{n \in \mathbb{Z}} P(n) \text{Box}_N(N + n - t) \\ &= \sum_{n \in \mathbb{Z}} P(-m - N) \text{Box}_N(-t - m)\end{aligned}$$

Remark that $P(-m - N) = (-1)^{(N-1)} P(m)$

and we obtain the same formula

For $t < 0$

$$\sum_{n \in \mathbb{Z}} P(n) \text{Box}_N(t - n) = \frac{t^{N-1}}{(N-1)!}.$$

It remains to see that

$$\left(\frac{1 - e^{-\partial_t}}{\partial_t}\right)^N \text{binomial}(t + N - 1, N - 1) = \frac{t^{N-1}}{(N-1)!}.$$

For example, by induction.

Using distributions, we could have seen directly that

$$\left(\frac{d}{dt}\right)^N (\text{Box}_N *_d P) = 0$$

so that the result is a polynomial of degree $< N$.

as

$$\frac{d}{dt} * (\text{Box}_N *_d P) = (\nabla \text{Box}_{N-1}) *_d P = \text{Box}_{N-1} * \nabla P.$$

Consequence of this theorem

Let f be any function on \mathbb{Z} . Consider the function on \mathbb{R}

$$F(t) := (\text{Box}_N *_{\mathcal{D}} f)(t) := \sum_n f(n) \text{Box}_N(t - n).$$

Then F is a locally polynomial function of t (on each interval it is given by a polynomial function of t). We can derivate F over any open interval by any series of differential operator $P(\frac{d}{dt})$.

The Todd operator

Theorem

Let f be any function on \mathbb{Z} :

Let

$$F_N(t) = (\text{Box}_N *_d f)(t) := \sum_n f(n) \text{Box}_N(t - n).$$

Then $F_N(t)$ is a function on \mathbb{R} , polynomial on each interval.

Then $f(n)$ (n an integer) is obtained by the limit from the right of

$$f(n) = \lim_{\epsilon > 0, \epsilon \rightarrow 0} ((\text{Todd}(d/dt))^N (\text{Box}_N *_d f))(n + \epsilon)$$

Why we want to do it

For some very interesting cases, the function $(Box_N *_d f)$ is known and related to the "classical" geometry.

If we know regularity properties of $(Box_N *_d f)$, then we deduce regularities properties for f .

Going from F_N to f is going from the classical mechanics to quantum mechanics.

We will see examples later.

Proof

We want to prove this equation, for $n = 0$ (enough). Then $F_N(t)$ for $t > 0$ near 0 (HERE I USE THE LIMIT on the RIGHT) involves only the values $f(0), f(-1), f(-2), \dots, f(-(N-1))$ of f , indeed $Box_N(t+N) = 0$ for $t > 0$, as Box_N is supported on $[0, N]$. We can choose a unique polynomial P of degree $N-1$ which coincide with f at $0, -1, \dots, -(N-1)$. Near $t > 0$, small,

$$(Box_N *_d f)(t) = (Box_N *_d P)(t)$$

Differentiate with the reverse operator, we obtain our identity $f(0) = (Todd(\frac{d}{dt}))^N (Box_N *_d f)(0)$.

A wonderful property of the Box spline

Apply this to $f(n) = 0$ except for $n = 0$ where $f(0) = 1$.
THAT IS

$$f = \delta_0.$$

Then

Theorem

Consider the locally polynomial function

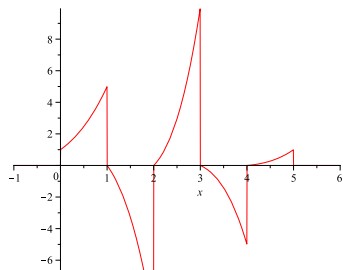
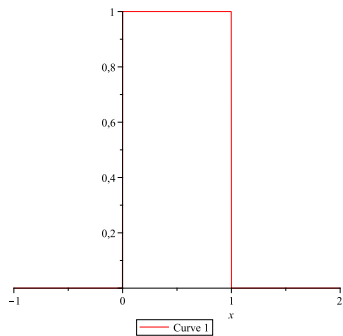
$$((\text{Todd}(d/dt))^N \text{Box}_N)(t)$$

Then

$$((\text{Todd}(d/dt))^N \text{Box}_N)|_{\mathbb{Z}} = \delta_0$$

(limits from the right)

Examples



We have used essentially the relation that $\mathcal{P}_N(n)$, the number of integral points in the standard simplex can be obtained from the volume $\text{vol}(S_N(t))$ by applying the Todd operator.

Consider $M := P_{N-1}(\mathbb{C})$ realized by

$$\{(z_1, z_2, \dots, z_N); \sum_i |z_i|^2 = t\} / e^{i\theta}$$

with symplectic form $\Omega_t = tc$. Here c is the Fubini-Study canonical 2-form on M , with $\int c^{N-1} = 1$. We compute

$$\text{vol}(M_t) := \int_M e^{tc} = \frac{t^{N-1}}{(N-1)!}$$

Let $t = n$ an integral value,

Let \mathcal{L}_n be the line bundle on M with holomorphic sections polynomials on degree n : Thus $H^0(M, \mathcal{L}_n)$ has basis $z_1^{n_1} \dots z_N^{n_N}$ with $n_i \geq 0$; $\sum n_i = n$. That is the number of integral points in $S_{N-1}(n)$.

If we apply the Todd operator

$$\text{Todd}\left(\frac{d}{dt}\right) \int_M e^{tc}$$

we obtain

$$\int_M e^{tc} \left(\frac{c}{1 - e^{-c}}\right)^N$$

For $t = n$, we then obtain

$$\text{Todd}\left(\frac{d}{dt}\right) \int_M e^{tc} \Big|_{t=n} = \int_M \text{chern}(L_n) \text{Todd}(M).$$

The Riemann Roch theorem asserts that this is the dimension of $H^0(M, L_n)$ (no higher cohomology).

The Mother Formula

CONCLUSION:

The relation

$$\left(\left(\text{Todd} \left(\frac{d}{dt} \right) \right)^N \text{Box}_N \right) |_{\mathbb{Z}} = \delta_0$$

is the mother formula:

Children

- Inversion formula for semi-discrete convolution

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Riemann-Roch theorem for $P_{N-1}(\mathbb{C})$.

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Multiplicities formulae: last talk.