Vol. 82, No. 3 DUKE MATHEMATICAL JOURNAL © March 1996

EQUIVARIANT INDEX FORMULAS FOR ORBIFOLDS

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1. Introduction. Let P be a smooth manifold. Let H be a compact Lie group acting on P. We assume that the action of H is infinitesimally free, that is, the stabilizer $H(y)$ of any point $y \in P$ is a finite subgroup of H. We write the action of H on the right. The quotient space P/H is an orbifold. (If H acts freely, then P/H is a manifold.) Reciprocally, any orbifold M can be presented this way: for example, one might choose P to be the bundle of orthonormal frames for a choice of a metric on M and $H = O(n)$ if $n = \dim M$. We will assume that there is a compact Lie group G acting on P such that its action commutes with the action of H. We will write the action of G on the left. Then the space P/H is provided with a G-action. Such data (P, H, G) will be our definition of a presented G-orbifold. We will say shortly that P/H is a G-orbifold.

Consider a compact G-orbifold P/H. A tangent vector on P tangent at $y \in P$ to the orbit $H \cdot y$ will be called a vertical tangent vector. Let T_H^*P be the subbundle of T^*P orthogonal to all vertical vectors. We will say that T_H^*P is the horizontal cotangent space. We denote by (y, ξ) a point in T^*P. Consider two $(G \times H)$-equivariant vector bundles $\mathscr{E}^{\pm}$ on P. Let $\Gamma(P, \mathscr{E}^{\pm})$ be the spaces of smooth sections of $\mathscr{E}^{\pm}$. Let

$$\Delta \colon \Gamma(P, \mathscr{E}^+) \to \Gamma(P, \mathscr{E}^-)$$

be a $(G \times H)$-invariant differential operator. Consider the principal symbol $\sigma(\Delta)$ of Δ. The operator Δ is said to be H-transversally elliptic if

$$\sigma(\Delta)(y, \xi_0) \colon \mathscr{E}_y^+ \to \mathscr{E}_y^-$$

is invertible for all $\xi_0 \in (T_H^*P)_y - \{0\}$. When Δ is H-transversally elliptic, the equivariant index of Δ is defined as in [1] and is a trace-class virtual representation of $G \times H$. Introduce $(G \times H)$-invariant metrics on P and on $\mathscr{E}^{\pm}$. Let Δ^* be the formal adjoint of Δ. The virtual space $Q(\Delta)$ of H-invariant "solutions" of Δ

$$Q(\Delta) = [(\mathrm{Ker}(\Delta))^H] - [(\mathrm{Ker}(\Delta^*))^H]$$

is a finite-dimensional virtual representation space for G. More generally, we consider $(G \times H)$-transversally elliptic operators on P. Then the space $Q(\Delta)$ of H-invariant "solutions" of Δ is a trace-class virtual representation of G.

Received 5 December 1994. Revision received 7 July 1995.

Let us first consider the case where Δ is H-transversally elliptic and H acts freely. It is then easy to describe what is the virtual representation $Q(\Delta)$ of G. Since Δ commutes with H, the operator Δ determines a map

$$\Delta^{P/H}\colon \Gamma(P, \mathscr{E}^+)^H \to \Gamma(P, \mathscr{E}^-)^H.$$

We have $\Gamma(P, \mathscr{E}^\pm)^H = \Gamma(P/H, \mathscr{E}^\pm/H)$, and $\Delta^{P/H}$ is a G-invariant elliptic operator on P/H. Thus, we have, for $s \in G$,

$$\operatorname{Tr} Q(\Delta)(s) = \operatorname{index}(\Delta^{P/H})(s).$$

Let $(P/H)(s)$ be the set of fixed points for the action of s on P/H. The equivariant index formula of Atiyah-Segal-Singer [2], [4] allows us to write $\operatorname{index}(\Delta^{P/H})(s)$ as an integral over $T^*(P/H)(s)$. If H acts only infinitesimally freely, we will give an integral formula for $\operatorname{Tr} Q(\Delta)(s)$ generalizing the formula for $\operatorname{index}(\Delta^{P/H})(s)$ in the case of free action.

More generally, if Δ is a $(G \times H)$-transversally elliptic operator on P, we state in Theorem 2 a formula for the character of the trace-class virtual representation $Q(\Delta)$ of G in terms of the equivariant cohomology of $T^*(P/H)$. This theorem generalizes the cohomological index formula given in [7], [9] for the equivariant index of G-transversally elliptic operators on compact manifolds to the case of compact orbifolds.

If $G = \{e\}$, we identify $Q(\Delta)$ with an integer. Several authors gave an integral formula for this integer in various degrees of generality. The notion of an orbifold was introduced by Satake who proved a Gauss-Bonnet formula [16] for orbifolds. For any H-transversally elliptic operator Δ, a formula for the number $Q(\Delta)$ was given by Atiyah [1, Corollary 9.12] in the case where H is a torus. When P/H is a complex algebraic variety, $\mathscr{F}/H$ an holomorphic orbifold bundle on P/H, and Δ the $\bar{\partial}$ operator on the space of sections of $\mathscr{F}/H$, the number $Q(\Delta)$ was computed by Kawasaki [12]. It is the Riemann-Roch number of a sheaf on P/H. For H an arbitrary compact group and any H-transversally elliptic operator Δ, a formula for the number $Q(\Delta)$ was given by Kawasaki [13].

In our case as well as in Kawasaki's proof in [13], Atiyah's algorithm to compute the equivariant index of an H-transversally elliptic operator is a fundamental ingredient. Indeed, our proof of the general formula for index of transversally elliptic operators [9] relies heavily on Atiyah's results in [1]. Once this general formula is established, it is a pleasant exercise on Fourier inversion for compact groups to deduce the formula given here for G-transversally elliptic operators on orbifolds from our index formula for transversally elliptic operators on manifolds. I feel it is useful to do this exercise in order to extend to symplectic orbifolds the universal formula [17] for the character of a quantized representation. In fact, G-orbifolds appear naturally when studying the quantized representation associated to a prequantized symplectic manifold M. Let M be a symplectic manifold with Hamiltonian action of $G \times H$. Let $\mathscr{L}$ be a Kostant-

Souriau line bundle on M, and let $\mu: M \to \mathfrak{h}^*$ be the moment map for the H-action. Consider the space $M_{\text{red}} = \mu^{-1}(0)/H$. When 0 is a regular value of μ, the space M_{red} is a symplectic orbifold with a G-action. The quantized representation $Q(M, \mathscr{L})$ is a virtual representation of $G \times H$ constructed as the $(\mathbb{Z}/2\mathbb{Z})$-graded space of solutions of the $\mathscr{L}$-twisted Dirac operator on M. If M_{red} is an orbifold, the virtual representation $Q(M_{\text{red}}, \mathscr{L}_{\text{red}})$ of G can be constructed in a similar way [19]. We give in Proposition 4 an integral formula for the character of the quantized representation $Q(M_{\text{red}}, \mathscr{L}_{\text{red}})$ of the symplectic orbifold M_{red}.

2. Equivariant index formula on orbifolds

2.1. Differential forms and integration. Let N be a manifold with infinitesimally free action of a compact group H. Let G be a compact Lie group acting on N such that the action of G commutes with the action of H. A differential form $\alpha \in \mathscr{A}(N)$ will be called H-horizontal (or simply horizontal if H is understood) if $\iota(Y_N)\alpha = 0$ for all $Y \in \mathfrak{h}$. A form α on N is called H-basic if α is H-horizontal and H-invariant. If the action of H on N is free, a basic form is the pullback of a form on N/H. Thus, we will also say that an H-basic differential form α on N is a differential form on N/H. The operator $d_{\mathfrak{g}}$ on G-equivariant differential forms on N is defined as in [5, Chapter 7]. For $X \in \mathfrak{g}$, we denote by d_X the operator $d - \iota(X_N)$ on forms on N. A G-equivariant differential form on N is called H-basic if, for all $X \in \mathfrak{g}$, the differential form $\alpha(X)$ is H-basic. We will also say that α is a G-equivariant differential form on N/H. The operator $d_{\mathfrak{g}}$ preserves the space of G-equivariant differential forms on N/H.

We identify the bundle of vertical vectors with $N \times \mathfrak{h}$. Choose a $(G \times H)$-invariant decomposition

$$TN = T_{\text{hor}} N \oplus (N \times \mathfrak{h}). \tag{1}$$

This decomposition allows us to identify $T_H^* N$ with $T_{\text{hor}}^* N$.

The decomposition (1) gives us a connection form

$$\theta \in (\mathscr{A}^1(N) \otimes \mathfrak{h})^{H \times G}. \tag{2}$$

We denote by $\Theta \in \mathscr{A}^2(N) \otimes \mathfrak{h}$ the curvature of θ. Let ϕ be a smooth function on $\mathfrak{h}$. Then we define the horizontal form $\phi(\Theta)$ on N using Taylor's expansion of ϕ at 0. If ϕ is invariant, then $\phi(\Theta)$ is basic.

The stabilisers $H(y)$ of points $y \in N$ are finite subgroups of H. The set B of conjugacy classes of stabilizers of elements of N is a partially ordered set. Let N_a be a connected component of N. Then the set $\{H(y),\ y \in N_a\}$ has a unique minimal element [10]. This element S_a is referred to as the *generic stabilizer* on N_a. We consider the generic stabilizer as a locally constant function from N to conjugacy classes of subgroups of H writing $S(y) = S_a$ if $y \in N_a$. Let $|S(y)|$ be the order of $S(y)$. In particular, $y \to |S(y)|$ is a locally constant function on N. We denote this function by $|S|$ (or $|S^N|$ when we need to specify the manifold N). An element $y \in N$ such that $H(y)$ is conjugated to $S(y)$ is called *regular*. We denote by N_{reg} the set of regular elements. It is an H-invariant open subset of N, and N_{reg}/H is a manifold.

Assume the bundle $T^*_{\text{hor}}N$ has an H-invariant orientation o. We will then say that N/H is oriented. If N is connected, we define $\dim(N/H)$ to be $\dim N - \dim H$. Otherwise, we consider $\dim(N/H)$ as a locally constant function on N.

An H-basic differential form α defines a differential form on N_{reg}/H. If α is compactly supported on N, then the component $\alpha_{[\dim(N/H)]}$ of exterior degree $\dim(N/H)$ of α is integrable on the oriented manifold N_{reg}/H. By definition,

$$\int_{N/H} \alpha = \int_{N_{\text{reg}}/H} \alpha_{[\dim(N/H)]} \,. \tag{3}$$

Let us give a formula for $\int_{N/H} \alpha$ as an integral over N. Let $n = \dim \mathfrak{h}$. Let $E^1, E^2, \ldots, E^n$ be a basis of $\mathfrak{h}$. We write the connection form $\theta \in \mathscr{A}^1(N) \otimes \mathfrak{h}$ as

$$\theta = \sum_1^n \theta_k E^k.$$

Let $E_1, E_2, \ldots, E_n$ be the dual basis of $\mathfrak{h}^*$. It defines a Euclidean volume form dY on $\mathfrak{h}$ and an orientation $o^{\mathfrak{h}}$ on $\mathfrak{h}$. We denote by dh the Haar measure on H tangent to dY at the identity of H. Notice that the form

$$v_{o^{\mathfrak{h}}} = (\mathrm{vol}(H, dh))^{-1} \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n$$

depends only of θ and $o^{\mathfrak{h}}$.

Assume N/H is oriented. Let $o^{N/H}$ be the corresponding orientation. Then N is oriented. We choose as positive volume form $\omega \wedge v_{o^{\mathfrak{h}}}$ if ω is a positive H-invariant section of $\Lambda^{\max} T^*_H N$. We denote this orientation by $o^{N/H} \wedge o^{\mathfrak{h}}$. If α is a basic form on N with compact support, then

$$\int_{N/H} \alpha = \int_N |S| \alpha \wedge v_{o^{\mathfrak{h}}} \,. \tag{4}$$

In this formula, the orientation on N is the orientation $o^{N/H} \wedge o^{\mathfrak{h}}$.

If $\mathscr{V} \to N$ is an H-equivariant vector bundle over N with projection p_0, then the integration over the fiber of an H-basic differential form on $\mathscr{V}$ is an H-basic differential form on N. If α is compactly supported, we have the integration formula

$$\int_{\mathscr{V}/H} \alpha = \int_{N/H} |S^{\mathscr{V}}|/|S^N| (p_0)_* \alpha. \tag{5}$$

Let us define the cotangent bundle to an orbifold N/H. When H acts freely on N, then N/H is a smooth manifold and we have a canonical identification $T^*(N/H) = (T_H^*N)/H$. In our case, the action of H on T_H^*N is infinitesimally free, and we define $T^*(N/H)$ as an orbifold by $T^*(N/H) = (T_H^*N)/H$. It is important to notice that the orbifold $T^*(N/H)$ is orientable. Indeed, the restriction of the canonical 1-form ω^N of T^*N to T_H^*N is a basic 1-form; that is, a form on $T^*(N/H)$. We denote it by $\omega^{N/H}$ and refer to it as the canonical 1-form on $T^*(N/H)$. The 2-form $d\omega^{N/H}$ is nondegenerate on $T_{\mathrm{hor}}(T_H^*N)$. We will choose on $T^*(N/H)$ the symplectic orientation given by $-d\omega^{N/H}$.

2.2. Index formula. Let $M = P/H$ be a compact G-orbifold. Consider two $(G \times H)$-equivariant vector bundles $\mathscr{E}^\pm$ on P. Let

$$\Delta\colon \Gamma(P, \mathscr{E}^+) \to \Gamma(P, \mathscr{E}^-)$$

be a $(G \times H)$-invariant differential operator. We assume that Δ is a $(G \times H)$-transversally elliptic operator on P. We will give an integral formula for $\operatorname{Tr} Q(\Delta)$ in terms of the equivariant cohomology of T^*M. We need some definitions.

Let $\mathscr{E}$ be an H-equivariant bundle over P. If ∇ is an H-invariant connection on $\mathscr{E}$, we define its moment $\mu \in \Gamma(P, \mathrm{End}(\mathscr{E})) \otimes \mathfrak{h}^*$ and the equivariant curvature of ∇ as in [5, Chapter 7]. Our conventions for characteristic classes will be those of [11]. They differ slightly from those of [5]. In particular, if $F(Y)$ $(Y \in \mathfrak{h})$ is the equivariant curvature of ∇, the equivariant Chern character will be $\mathrm{ch}(\mathscr{E}, \nabla)(Y) = \mathrm{Tr}(e^{F(Y)})$.

We will say that ∇ is an H-horizontal connection if $\mu(Y) = 0$ for all $Y \in \mathfrak{h}$. It is always possible to choose a horizontal connection on $\mathscr{E}$. This can be done as follows. Consider a connection form $\theta \in \mathscr{A}^1(P) \otimes \mathfrak{h}$ for the action of H on P. Let ∇ be an H-invariant connection on $\mathscr{E}$ with moment $\mu \in \Gamma(P, \mathrm{End}(\mathscr{E})) \otimes \mathfrak{h}^*$. Then the contraction (μ, θ) is an $\mathrm{End}(\mathscr{E})$-valued 1-form on P. Define $\nabla' = \nabla + (\mu, \theta)$. Then ∇' is horizontal.

If $\mathscr{E}$ is a $(G \times H)$-equivariant vector bundle on P, it is always possible to choose on $\mathscr{E}$ a $(G \times H)$-invariant horizontal connection ∇. Then the equivariant Chern character of $(\mathscr{E}, \nabla)$ is a G-equivariant basic form on P. An important example in the following is the case of a trivial vector bundle $[V_\tau] = P \times V_\tau$, where V_τ is a representation space of H. Let us denote also by τ the infinitesimal representation of $\mathfrak{h}$ in V_τ. It is easy to see that $d + \tau(\theta)$ is a horizontal connection with equivariant Chern character the basic equivariant form $\mathrm{ch}([V_\tau])(X) = \mathrm{Tr}(\tau(\exp \Theta(X)))$ where, for $X \in \mathfrak{g}$, $\Theta(X) = -(\theta, X_P) + \Theta$ is the equivariant curvature.

If $(s, u) \in G \times H$, the manifold

$$P(s, u) = \{p \in P;\ sp = pu\}$$

is a $(G(s) \times H(u))$-manifold, where $G(s)$ is the centralizer of $s \in G$ and $H(u)$ the centralizer of $u \in H$. The group $H(u)$ acts infinitesimally freely on $P(s, u)$. We denote by $M(s,u)$ the orbifold $P(s,u)/H(u)$. If γ is conjugated to u, the orbifold $M(s,\gamma)$ is diffeomorphic to $M(s,u)$.

Consider the horizontal bundle $T_{\mathrm{hor}}P(s,u) \subset T_{\mathrm{hor}}P|_{P(s,u)}$ and the horizontal normal bundle

$$T_{\mathrm{hor},P(s,u)}P = T_{\mathrm{hor}}P|_{P(s,u)}/T_{\mathrm{hor}}P(s,u).$$

The vector bundles $T_{\mathrm{hor}}P(s,u)$ and $T_{\mathrm{hor},P(s,u)}P$ are $(G(s)\times H(u))$-equivariant vector bundles on $P(s,u)$.

Define $T_{M(s,u)}M$ to be the orbifold bundle $(T_{\mathrm{hor},P(s,u)}P)/H(u)$ over $M(s,u)$. If M is a G-manifold, then $T_{M(s,u)}M$ is the normal bundle to $M(s,u)$ in M.

Let ∇ be a $(G\times H)$-invariant horizontal connection on $T_{\mathrm{hor}}P$. Then ∇ induces $H(u)$-horizontal connections ∇_0 on $T_{\mathrm{hor}}P(s,u)$ and ∇_1 on $T_{\mathrm{hor},P(s,u)}P$. Let $R_0(X)$, $R_1(X)$ be the $G(s)$-equivariant curvatures of ∇_0 and ∇_1. On $P(s,u)$ the action of (s,u) induces an endomorphism $g(s,u)$ of the bundle $T_{\mathrm{hor},P(s,u)}P$. Define the $G(s)$-equivariant closed forms on $P(s,u)/H(u)$

$$J(M(s,u))(X) = \det\left(\frac{e^{R_0(X)/2}-e^{-R_0(X)/2}}{R_0(X)}\right) \tag{6}$$

and

$$D_{(s,u)}(T_{M(s,u)}M)(X) = \det(1-g(s,u)e^{R_1(X)}) \tag{7}$$

for $X \in \mathfrak{g}(s)$.

We denote by p_0 the projection $T_H^*P \to P$. We denote by σ_0 the restriction of the principal symbol σ of Δ to T_H^*P. Let $\nabla^{\mathscr{E}^\pm}$ be horizontal connections on $\mathscr{E}^\pm$. Consider the superconnection $\mathbb{A}_0(\sigma_0)$ on $p_0^*\mathscr{E} = p_0^*\mathscr{E}^+ \oplus p_0^*\mathscr{E}^-$ defined by

$$\mathbb{A}_0(\sigma_0) = \begin{pmatrix} p_0^*\nabla^{\mathscr{E}^+} & i\sigma_0^* \\ i\sigma_0 & p_0^*\nabla^{\mathscr{E}^-} \end{pmatrix}.$$

Then the equivariant Chern character $\mathrm{ch}_{s,u}(\mathbb{A}_0(\sigma_0))(X)$ is a $G(s)$-equivariant form on the space $(T_{\mathrm{hor}}^*P(s,u))/H(u) = T^*M(s,u)$. Thus, we can define a $G(s)$-equivariant closed, basic differential form on $T_{\mathrm{hor}}^*P(s,u)$ given for $X \in \mathfrak{g}(s)$ small by

$$I(s,u,\sigma_0)(X) = \frac{\mathrm{ch}_{s,u}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s,u))(X)D_{s,u}(T_{M(s,u)}M)(X)}. \tag{8}$$

For $X=0$, we write

$$I(s,u,\sigma_0) = I(s,u,\sigma_0)(0). \tag{9}$$

Assume first that Δ is H-transversally elliptic. Then the restriction σ_0 of the principal symbol of Δ is homogeneous of positive order on each fiber of the vector bundle $T^*_{\mathrm{hor}}P$. Furthermore, $\sigma_0(y, \xi_0)$ is invertible when ξ_0 is not zero. Thus, for $X \in \mathfrak{g}(s)$, the form $\mathrm{ch}_{s,u}(\mathbb{A}_0(\sigma_0))(X)$ is rapidly decreasing on $T^*_{\mathrm{hor}}P(s, u)$ (this is seen as in [7]), so that $I(s, u, \sigma_0)(X)$ can be integrated over $T^*M(s, u)$.

For $s \in G$, we denote by $C(s)$ the set of elements $\gamma \in H$ such that $P(s, \gamma) \neq \emptyset$. Then $C(s)$ is invariant by conjugacy and the set $(C(s)) = C(s)/\mathrm{Ad}(H)$ is a finite set. Let $M(s, \gamma)$ be the orbifold $P(s, \gamma)/H(\gamma)$. We denote by $S(s, \gamma)$ the generic stabilizer for the action of $H(\gamma)$ on $P(s, \gamma)$. The functions $\dim M(s, \gamma)$ and $|S(s, \gamma)|$ are locally constant functions on $P(s, \gamma)$.

THEOREM 1. *Let $M = P/H$ be an orbifold. Let Δ be a $(G \times H)$-invariant differential operator on P. Assume that Δ is H-transversally elliptic. Then, for each $s \in G$, the trace of the virtual finite-dimensional representation $Q(\Delta)$ of G satisfies the formula*

$$\mathrm{Tr}\, Q(\Delta)(s \exp X) = \sum_{\gamma \in (C(s))} \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} |S(s, \gamma)|^{-1} \times \frac{\mathrm{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s, \gamma))(X) D_{s,\gamma}(T_{M(s,\gamma)}M)(X)}$$

for X small in $\mathfrak{g}(s)$.

Assume now that Δ is only $(G \times H)$-transversally elliptic. Let ω^M be the canonical 1-form of T^*M. Similarly we obtain canonical 1-forms on $\omega^{M(s,\gamma)}$ on $T^*M(s, \gamma)$. Define then

$$I^{\omega}(s, \gamma, \sigma_0)(X) = \frac{e^{-id_X \omega^{M(s,\gamma)}} \mathrm{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s, \gamma))(X) D_{s,\gamma}(T_{M(s,\gamma)}M)(X)}.$$

Then the form $I^{\omega}(s, \gamma, \sigma_0)(X)$ is a $G(s)$-equivariant form on $T^*M(s, \gamma)$, which can be integrated in $\mathfrak{g}(s)$-mean [8].

The formula for $\mathrm{Tr}\, Q(\Delta)$ given in Theorem 1 for Δ an H-transversally elliptic operator has to be modified to obtain a meaningful formula in the case of a $(G \times H)$-transversally elliptic operator Δ where $\mathrm{Tr}\, Q(\Delta)$ is only a generalized function on G. The next theorem extends the cohomological formula for the index of G-transversally elliptic operators on manifolds [8], [9] to the case of G-transversally elliptic operators on orbifolds.

THEOREM 2. *Let $M = P/H$ be an orbifold. Let Δ be a $(G \times H)$-invariant differential operator on P. Assume that Δ is $(G \times H)$-transversally elliptic. Then, for* *each $s \in G$, the trace of the virtual trace-class representation $Q(\Delta)$ of G satisfies the equality*

$$\operatorname{Tr} Q(\Delta)(s \exp X) = \sum_{\gamma \in (C(s))} \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} |S(s,\gamma)|^{-1} \times \frac{e^{-id_X \omega^{M(s,\gamma)}} \operatorname{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s,\gamma))(X) D_{s,\gamma}(T_{M(s,\gamma)}M)(X)}$$

as an equality of generalized functions on a neighborhood of 0 *in* $\mathfrak{g}(s)$.

Remark 2.1. If Δ is only pseudodifferential, the formula above holds, provided we choose a "good" representative σ_0 [8] of the symbol of Δ.

Before proving these theorems, let us write more explicitly the formula of Theorem 1 in the case where $G = \{e\}$. Then we must consider the set $C(e)$ of elements $\gamma \in H$ such that the set $P(\gamma) = \{p \in P,\ p\gamma = p\}$ is not empty. We define $M(\gamma) = P(\gamma)/H(\gamma)$. The formula obtained for the number $Q(\Delta) = \dim(\operatorname{Ker}(\Delta))^H - \dim(\operatorname{Ker} \Delta^*)^H$ is thus Kawasaki's formula:

$$(10) \qquad Q(\Delta) = \sum_{\gamma \in (C(e))} \int_{T^*M(\gamma)} (2i\pi)^{-\dim M(\gamma)} |S(\gamma)|^{-1} \frac{\operatorname{ch}_\gamma(\mathbb{A}_0(\sigma_0))}{J(M(\gamma)) D_\gamma(T_M(\gamma)M)}.$$

Let us give two examples where this formula is easily seen to be true.

(1) Assume H is a finite group. Then the dimension of the space $Q(\Delta)$ is evidently given by the average of the equivariant index

$$Q(\Delta) = |H|^{-1} \sum_{\gamma \in H} \operatorname{index}(\Delta)(\gamma).$$

Using the equivalent expression given in [7] of the Atiyah-Segal-Singer formula [2], [4], we have

$$\operatorname{index}(\Delta)(\gamma) = \int_{T^*P(\gamma)} (2i\pi)^{-\dim P(\gamma)} \frac{\operatorname{ch}_\gamma(\mathbb{A}_0(\sigma_0))}{J(P(\gamma)) D_\gamma(T_{P(\gamma)}P)}.$$

In particular, $\operatorname{index}(\Delta)(\gamma)$ is 0 if γ does not belong to $C(e)$. Let $\gamma \in C(e)$. In this case, $T^*M(\gamma) = T^*P(\gamma)/H(\gamma)$. On each connected component of $P(\gamma)$, the map $T^*P(\gamma) \to T^*P(\gamma)/H(\gamma)$ is a cover of order $|H(\gamma)/S(\gamma)|$ and, by definition, for α a differential form on $P(\gamma)$

$$\int_{T^*M(\gamma)} (2i\pi)^{-\dim M(\gamma)} \alpha = \int_{T^*P(\gamma)} |H(\gamma)|^{-1} |S(\gamma)| (2i\pi)^{-\dim P(\gamma)} \alpha.$$

Rewriting the set $C(e)$ as union of conjugacy classes, we see that the formula for $Q(\Delta)$ is indeed just the average of the Atiyah-Segal-Singer formula.

(2) Assume H acts freely on P. Then $C(e) = \{e\}$. Let $M = P/H$. The restriction σ_0 of σ to T_H^*P determines an elliptic symbol still denoted by σ_0 on $T^*M = T_H^*P/H$ which is the principal symbol of $\Delta^{P/H}$. We have $Q(\Delta) = \operatorname{index}(\Delta^{P/H})$. Formula (10) for $Q(\Delta)$ as an integral over T^*M of an equivariant characteristic class agrees with the Atiyah-Singer formula for the index of $\Delta^{P/H}$ in function of its principal symbol.

Proof. Let us now prove Theorem 1 and Theorem 2. We give only the proof of the first theorem, as both proofs are very similar to the proof of the Frobenius reciprocity for free actions [9, Theorem 26]. We give the main steps. Define

$$v(s,\gamma,\sigma_0)(X) = \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} |S(s,\gamma)|^{-1} \frac{\operatorname{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s,\gamma))(X) D_{s,\gamma}(T_{M(s,\gamma)}M)(X)}.$$

We must prove that

$$\operatorname{Tr} Q(\Delta)(s \exp X) = \sum_{\gamma \in (C(s))} v(s,\gamma,\sigma_0)(X). \tag{11}$$

Consider the virtual character $\operatorname{index}(\Delta)$ of $G \times H$. Let $\hat{H}$ be the set of classes of irreducible finite-dimensional representations of H. For $\tau \in \hat{H}$, consider the operator

$$\Delta \otimes I_{V_\tau}\colon \Gamma(P,\mathscr{E}^+) \otimes V_\tau \to \Gamma(P,\mathscr{E}^-) \otimes V_\tau.$$

For $\tau \in \hat{H}$, let $[V_\tau]$ be the trivial bundle on P with fiber V_τ. We have

$$\Gamma(P,\mathscr{E}^\pm) \otimes V_\tau = \Gamma(P,\mathscr{E}^\pm \otimes [V_\tau]).$$

We denote by Δ^τ the operator $\Delta \otimes I_{V_\tau}$. It has symbol $\sigma_\tau = \sigma \otimes I_{p^*[V_\tau]}$. The map $\Gamma(P,\mathscr{E}^\pm) \otimes V_\tau \otimes V_{\tau^*} \to \Gamma(P,\mathscr{E}^\pm)$ given by $(\phi \otimes f) \mapsto (\phi, f)$ for $f \in V_\tau^*$ and ϕ in $\Gamma(P,\mathscr{E}^\pm) \otimes V_\tau$ induces an isomorphism from $(\Gamma(P,\mathscr{E}^\pm) \otimes V_\tau)^H \otimes V_{\tau^*}$ to the isotypic space of type τ^* in $\Gamma(P,\mathscr{E}^\pm)$. By definition, the trace of the action of G in $[(\operatorname{Ker}(\Delta \otimes I_{V_\tau})^H] - [(\operatorname{Ker}(\Delta^* \otimes I_{V_\tau})^H]$ is $Q(\Delta^\tau)$. Thus, we see that

$$\operatorname{index}(\Delta)(s,h) = \sum_{\tau \in \hat{H}} \operatorname{Tr} Q(\Delta^\tau)(s) \operatorname{Tr} \tau^*(h).$$

To verify equation (11) for $Q(\Delta)$, it is sufficient to verify, for each $s \in G$ and $X \in \mathfrak{g}(s)$ small, that we have the equality of generalized functions of H

$$\operatorname{index}(\Delta)(s \exp X, h) = \sum_{\tau \in \hat{H}} \sum_{\gamma \in (C(s))} v(s,\gamma,\sigma_0^\tau)(X) \operatorname{Tr} \tau^*(h). \tag{12}$$

To simplify formulas, we compute only for $X = 0$. We write $v(s, \gamma, \sigma_0^\tau)$ for $v(s, \gamma, \sigma_0^\tau)(0)$.

Let $u \in H$ and let ϕ be an H-invariant test function on H with support in a small neighborhood of the conjugacy class of u. In particular, we assume that if $\gamma \in (C(s))$ is not conjugated to u, the support of ϕ does not intersect the orbit of γ. Let $\mathfrak{h}(u)$ be the Lie algebra of $H(u)$. Let

$$v_1(\phi) = \int_H \operatorname{index}(\Delta)(s, h)\phi(h)\, dh \tag{13}$$

and

$$v_2(\phi) = \sum_{\tau \in \hat{H}} \sum_{\gamma \in (C(s))} v(s, \gamma, \sigma_0^\tau) \int_H \operatorname{Tr} \tau^*(h)\phi(h)\, dh. \tag{14}$$

We need to verify the equality

$$v_1(\phi) = v_2(\phi). \tag{15}$$

Let us first state the main technical lemma. Let N be the manifold

$$N = P \times \mathfrak{h}^*.$$

We denote by $f\colon P \times \mathfrak{h}^* \to \mathfrak{h}^*$ the second projection. We consider the 1-form

$$\nu = (\theta, f)$$

on N. We choose a basis $E_1, E_2, \ldots, E_n$ of $\mathfrak{h}^*$. This determines the form $v_{o\mathfrak{h}}$ on P. We write $f = \sum f^i E_i$. We denote by $df = df^1 \wedge df^2 \wedge \cdots \wedge df^n$. We denote by p_1 the projection of $N = P \times \mathfrak{h}^*$ on P with fiber $\mathfrak{h}^*$. The integration over the fiber is defined once an orientation is chosen on each fiber. We use the orientation given by df. Furthermore, the integration over the fiber is defined with conventions of signs as in [5]: if $p\colon P \to B$ is an oriented fibration, $p_*(\alpha \wedge p^*\beta) = p_*(\alpha) \wedge \beta$ if α is a form on P and β a form on B.

The following lemma is obtained as Proposition 28 of [9].

LEMMA 3. *If ϕ is a test function on $\mathfrak{h}$, we have*

$$(2i\pi)^{-\dim H}(p_1)_*\left(\int_{\mathfrak{h}} e^{-id_Y \nu}\phi(Y)\, dY\right) = (-1)^{n(n+1)/2}(\operatorname{vol} H, dh) v_{o\mathfrak{h}}\phi(\Theta).$$

Let us return to the proof of the identity (15).

We first compute $v_1(\phi)$. The generalized function index(Δ) can be computed as a special case of the index formula for $(G \times H)$-transversally elliptic operators. Let, for $Y \in \mathfrak{h}(u)$,

$$J_{\mathfrak{h}(u)}(Y) = \det_{\mathfrak{h}(u)} \frac{e^{\operatorname{ad} Y/2} - e^{-\operatorname{ad} Y/2}}{\operatorname{ad} Y}.$$

Using the Weyl integration formula, we have

$$v_1(\phi) = \operatorname{vol}(H/H(u)) \int_{\mathfrak{h}(u)} \operatorname{index}(\Delta)(s, ue^Y)\phi(ue^Y) J_{\mathfrak{h}(u)}(Y) \times \det_{\mathfrak{h}/\mathfrak{h}(u)}(1 - ue^Y)\, dY. \tag{16}$$

Let $p\colon T^*P \to P$ the projection. Define on the superbundle $p^*\mathscr{E} = p^*\mathscr{E}^+ \oplus p^*\mathscr{E}^-$ the superconnection

$$\mathbb{A}(\sigma) = \begin{pmatrix} p^*\nabla^{\mathscr{E}^+} & i\sigma^* \\ i\sigma & p^*\nabla^{\mathscr{E}^-} \end{pmatrix}.$$

Let $T^*P = T^*_{\mathrm{hor}}P \oplus P \times \mathfrak{h}^*$. We can assume by homotopy the symbol σ of Δ of the form $\sigma(y, \xi) = \sigma_0(y, \xi_0)$ where ξ_0 is the projection of ξ on $(T^*_{\mathrm{hor}}P)_y$. We choose on TP the direct sum of a horizontal connection on $T_{\mathrm{hor}}P$ and of the trivial connection on $P \times \mathfrak{h}$.

Let ω^P be the canonical 1-form on T^*P. Its restriction to $N = P \times \mathfrak{h}^*$ is the 1-form $\nu = (\theta, f)$.

Let $(s, u) \in G \times H$. The index formula for Δ gives in particular for $Y \in \mathfrak{h}(u)$ sufficiently small:

$$\operatorname{index}(\Delta)(s, ue^Y) = \int_{T^*P(s,u)} (2i\pi)^{-\dim P(s,u)} \frac{e^{-id_Y\omega^P|T^*P(s,u)} \operatorname{ch}_{s,u}(\mathbb{A}(\sigma))(Y)}{J(P(s,u))(Y) D_{s,u}(T_{P(s,u)}P)(Y)}.$$

The restriction of the connection form θ to $P(s,u)$ is valued in $\mathfrak{h}(u)$ and is a connection form for the $H(u)$-action on $P(s,u)$. We have $T^*P(s,u) = T^*_{\mathrm{hor}}P(s,u) \oplus P(s,u) \times \mathfrak{h}(u)^*$. Thus, the bundle $T^*P(s,u)$ projects on $N(s,u) = P(s,u) \times \mathfrak{h}^*(u)$ as well as on $T^*_{\mathrm{hor}}P(s,u)$. We still denote by α the pullback to $T^*P(s,u)$ of a form α on $N(s,u)$ and by β the pullback to $T^*P(s,u)$ of a form β on $T^*_{\mathrm{hor}}P(s,u)$. For our choices of connections and symbols, we have

$$\operatorname{ch}_{s,u}(\mathbb{A}(\sigma))(Y) = \operatorname{ch}_{s,u}(\mathbb{A}_0(\sigma_0))$$

$$J(P(s,u))(Y) = J(M(s,u)) J_{\mathfrak{h}(u)}(Y)$$

$$D_{s,u}(T_{P(s,u)}P)(Y) = D_{s,u}(T_{M(s,u)}M) \det_{\mathfrak{h}/\mathfrak{h}(u)}(1 - ue^Y).$$

Thus, we obtain

$$\operatorname{index}(\Delta)(s, ue^Y) J_{\mathfrak{h}(u)}(Y) \det_{\mathfrak{h}/\mathfrak{h}(u)}(1 - ue^Y)$$

$$= \int_{T^*P(s,u)} (2i\pi)^{-\dim P(s,u)} \frac{e^{-id_Y \omega^P|_{T^*P(s,u)}} \operatorname{ch}_{s,u}(\mathbb{A}_0(\sigma_0))}{J(M(s,u)) D_{s,u}(T_{M(s,u)}M)}.$$

Let $(y, \xi) \in T^*P(s,u) = T^*_{\mathrm{hor}}P(s,u) \oplus P(s,u) \times \mathfrak{h}(u)^*$. If $\xi = \xi_0 + f$ with $\xi_0 \in (T^*_{\mathrm{hor}}P(s,u))_y$ and $f \in \mathfrak{h}(u)^*$, the Chern character $\operatorname{ch}_{s,u}(\mathbb{A}_0(\sigma_0))$ is rapidly decreasing with respect of the variable ξ_0. The factor $e^{-id_Y\omega^P|_{T^*P(s,u)}}$ integrated against a test function of $Y \in \mathfrak{h}(u)$ is rapidly decreasing in the variable f. A transgression argument similar to those proven in [8] allows us to replace ω^P in $tv + (1-t)\omega_P$ with $t \in [0,1]$. Then we have also

$$\operatorname{index}(\Delta)(s, ue^Y) J_{\mathfrak{h}(u)}(Y) \det_{\mathfrak{h}/\mathfrak{h}(u)}(1 - ue^Y)$$

$$= \int_{T^*P(s,u)} (2i\pi)^{-\dim P(s,u)} e^{-id_Y v|_{T^*P(s,u)}} \frac{\operatorname{ch}_{s,u}(\mathbb{A}_0(\sigma_0))}{J(M(s,u)) D_{s,u}(T_{M(s,u)}M)}.$$

We denote by v_0 the restriction of v to $P(s,u) \times \mathfrak{h}(u)^*$. Consider the fibration $p_1^u \colon T^*P(s,u) \mapsto T^*_{\mathrm{hor}}P(s,u)$ with fiber $\mathfrak{h}(u)^*$. Using notation (9), we thus have

$$\operatorname{index}(\Delta)(s, ue^Y) J_{\mathfrak{h}(u)}(Y) \det_{\mathfrak{h}/\mathfrak{h}(u)}(1 - ue^Y)$$

$$= \int_{T^*_{\mathrm{hor}}P(s,u)} (2i\pi)^{-\dim P(s,u)} (p_1^u)_*(e^{-id_Y v_0}) I(s,u,\sigma_0).$$

Let Θ_0 be the restriction of Θ to $P(s,u)$. The function $Y \mapsto \phi(u \exp Y)$ is an $H(u)$-invariant function on $\mathfrak{h}(u)$ and the form $\phi(u \exp \Theta_0)$ is a basic form on $P(s,u)$. Applying Lemma 3 to the manifold $P(s,u) \times \mathfrak{h}(u)^*$ and integration formula (16), we obtain

$$v_1(\phi) = \varepsilon \operatorname{vol}(H, dh) \int_{T^*_{\mathrm{hor}}P(s,u)} (2i\pi)^{-\dim M(s,u)} v_{o\,\mathfrak{h}(u)} \phi(u \exp \Theta_0) I(s,u,\sigma_0),$$

where ε is a sign.

Finally applying formula (4) to the basic form $\phi(u \exp \Theta_0) I(s,u,\sigma_0)$, we obtain

$$(17) \quad v_1(\phi) = \operatorname{vol}(H, dh) \int_{T^*M(s,u)} |S(s,u)|^{-1} (2i\pi)^{-\dim M(s,u)} \phi(u \exp \Theta_0) I(s,u,\sigma_0).$$

(A check of orientations shows that the sign ε disappears.)

We now compute $v_2(\phi)$. Define

$$v_2(\gamma,\phi) = \sum_{\tau\in\hat{H}} v(s,\gamma,\sigma_0^\tau) \int_H \operatorname{Tr} \tau^*(h)\phi(h)\,dh.$$

Let $\tau \in \hat{H}$. Let us compute

$$v(s,\gamma,\sigma_0^\tau) = \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} |S(s,\gamma)|^{-1} \frac{\operatorname{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0^\tau))}{J(M(s,\gamma))D_{s,\gamma}(T_{M(s,\gamma)}M)}.$$

We have

$$\operatorname{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0^\tau)) = \operatorname{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0))\operatorname{ch}_{s,\gamma}([V_\tau]).$$

For the horizontal connection $d+\tau(\theta)$ on $[V_\tau]$, we have $\operatorname{ch}_{s,\gamma}([V_\tau]) = \operatorname{Tr}(\tau(\gamma\exp\Theta_0))$. Thus,

$$v(s,\gamma,\sigma_0^\tau) = \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} \frac{I(s,\gamma,\sigma_0)}{|S(s,\gamma)|} \operatorname{Tr}(\tau(\gamma\exp\Theta_0)).$$

We obtain

$$\begin{aligned} v_2(\gamma,\phi) &= \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} \frac{I(s,\gamma,\sigma_0)}{|S(s,\gamma)|} \\ &\quad\times\left(\sum_{\tau\in\hat{H}} \operatorname{Tr}\tau(\gamma\exp\Theta_0)\left(\int_H \operatorname{Tr}\tau^*(h)\phi(h)\,dh\right)\right) \\ &= \operatorname{vol}(H,dh)\int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} \frac{I(s,\gamma,\sigma_0)}{|S(s,\gamma)|}\phi(\gamma\exp\Theta_0), \end{aligned}$$

using the Fourier inversion formula.

The basic form $\phi(\gamma\exp\Theta_0)$ depends on the Taylor expansion of ϕ at $\gamma\in H$. Recall that ϕ vanishes on a neighborhood of γ if γ is not conjugated to u. Thus, only the class (u) makes a nonzero contribution to $v_2(\phi) = \sum_{\gamma\in(C(s))} v_2(\gamma,\phi)$, and we obtain

$$(18)\quad v_2(\phi) = \operatorname{vol}(H,dh)\int_{T^*M(s,u)} (2i\pi)^{-\dim M(s,u)} |S(s,u)|^{-1} I(s,u,\sigma_0)\phi(u\exp\Theta_0).$$

Comparing formulas (17) and (18), we obtain formula (15). □

3. Quantization on orbifolds. We here consider the special case of Dirac operators. Consider the case where P has a $(G \times H)$-invariant metric and where T_H^*P is a $(G \times H)$-equivariant oriented even-dimensional bundle with spin structure. Let

$$TP = T_{\text{hor}}P \oplus P \times \mathfrak{h}$$

be the orthogonal decomposition of the tangent bundle. We identify T_H^*P with $T_{\text{hor}}P$ with the help of the metric. Let $\mathscr{S}_{\text{hor}}$ be the spin bundle for $T_{\text{hor}}P$. Choose a $(G \times H)$-invariant orientation o on $T_{\text{hor}}P$. The orientation o determines a $\mathbb{Z}/2\mathbb{Z}$-gradation $\mathscr{S}_{\text{hor}} = \mathscr{S}_{\text{hor}}^+ \oplus \mathscr{S}_{\text{hor}}^-$. If $v \in (T_{\text{hor}}P)_y$, then the Clifford multiplication $c(v)$ is an odd operator on $(\mathscr{S}_{\text{hor}})_y$. Let $\mathscr{F}$ be a $(G \times H)$-equivariant Hermitian vector bundle on P. Let $\mathscr{S}_{\text{hor}} \otimes \mathscr{F}$ be the twisted horizontal spin bundle. With the help of a choice of a $(G \times H)$-invariant unitary connection $\nabla = \nabla^+ \oplus \nabla^-$ on $\mathscr{S}_{\text{hor}} \otimes \mathscr{F} = \mathscr{S}_{\text{hor}}^+ \otimes \mathscr{F} \oplus \mathscr{S}_{\text{hor}}^- \otimes \mathscr{F}$, we may define the formally selfadjoint "horizontal" Dirac operator $D_{\text{hor},\mathscr{F}}$ by

$$D_{\text{hor},\mathscr{F}} = \sum_i c(e_i)\nabla_{e_i},$$

where e_i runs over an orthonormal basis of $T_{\text{hor}}P$. We have $D_{\text{hor},\mathscr{F}} = D_{\text{hor},\mathscr{F}}^+ \oplus D_{\text{hor},\mathscr{F}}^-$ with

$$D_{\text{hor},\mathscr{F}}^+ \colon \Gamma(P, \mathscr{S}_{\text{hor}}^+ \otimes \mathscr{F}) \to \Gamma(P, \mathscr{S}_{\text{hor}}^- \otimes \mathscr{F})$$

and

$$D_{\text{hor},\mathscr{F}}^- \colon \Gamma(P, \mathscr{S}_{\text{hor}}^- \otimes \mathscr{F}) \to \Gamma(P, \mathscr{S}_{\text{hor}}^+ \otimes \mathscr{F}).$$

Clearly, the operators $D_{\text{hor},\mathscr{F}}^{\pm}$ are H-transversally elliptic operators and commute with the natural action of G. The principal symbol of $D_{\text{hor},\mathscr{F}}^+$ is given by

$$\sigma(D_{\text{hor},\mathscr{F}}^+)(y, \xi) = c^+(\xi_0) \otimes I_{\mathscr{F}_y},$$

where ξ_0 is the projection of $\xi \in (T^*P)_y$ on $(T_H^*P)_y$. We define

$$Q^o(P/H, \mathscr{F}) = (-1)^{\dim M/2} Q(D_{\text{hor},\mathscr{F}}^+).$$

When H acts freely, this coincides with the quantization assignment defined in [17]. We generalize to this case the universal formula for the virtual representation $Q^o(P/H, \mathscr{F})$ [6], [17], [18].

Consider the vector bundle $T_H^*P \to P$ with projection p_0. We have chosen a $(G \times H)$-invariant orientation o of T_H^*P.

The horizontal connection ∇_0 of $T^*_{\mathrm{hor}}P$ determines a connection on $\mathscr{S}_{\mathrm{hor}}$. Consider on the equivariant bundle $\mathscr{F}$ a horizontal connection. Then $\mathrm{ch}_{s,u}(\mathscr{F})$ is a $G(s)$-equivariant form on $M(s,u)$.

Consider the pullback of $\mathscr{S}_{\mathrm{hor}} \otimes \mathscr{F}$ to T^*P. Then

$$\mathbb{A}(\sigma) = -\mathbf{c}_0 \otimes I_{p^*\mathscr{F}} + p^*\nabla^{\mathscr{S}_{\mathrm{hor}}\otimes\mathscr{F}},$$

where $\mathbf{c}_0$ is the odd-bundle endomorphism of $p^*\mathscr{S}_{\mathrm{hor}}$ given by $\mathbf{c}_0(y,\xi) = c(\xi_0)$, where c is the Clifford action of $(T^*_H P)_y$ on $(\mathscr{S}_{\mathrm{hor}})_y$ and ξ_0 the projection of ξ on $(T^*_H P)_y$.

Let $\mathbf{B}$ be the superconnection on $p_0^*(\mathscr{S}_{\mathrm{hor}}) \to T^*_{\mathrm{hor}}P$ defined by

$$\mathbf{B} = -\mathbf{c}_0 + p_0^*\nabla^{\mathscr{S}_{\mathrm{hor}}}. \tag{19}$$

Let $(s,u) \in G \times H$. We have for $X \in \mathfrak{g}(s)$

$$\mathrm{ch}_{s,u}(\mathbb{A}(\sigma))(X) = \mathrm{ch}_{s,u}(\mathbf{B})(X)\,\mathrm{ch}_{s,u}(\mathscr{F})(X).$$

Consider the bundle $T^*_{\mathrm{hor}}P(s,u) \to P(s,u)$. It is a $(G(s) \times H(u))$ even-dimensional equivariant orientable vector bundle (see [5, Lemma 6.10]).

Let us choose an orientation o' on the vector bundle $T^*_{\mathrm{hor}}P(s,u) \to P(s,u)$. The rank of this vector bundle is $\dim M(s,u)$. If $U^{s,u}_{o'}$ is the Thom form of the vector bundle $T^*_{\mathrm{hor}}P(s,u) \to P(s,u)$, we have

$$\begin{aligned} &i^{\dim M/2}\mathrm{ch}_{s,u}(\mathbf{B})(X) \\ &\quad = \varepsilon((s,u),o,o')(-2\pi)^{\dim M(s,u)/2} J^{1/2}(T^*M(s,u))(X) D^{1/2}_{s,u}(T^*_{M(s,u)}M)(X) U^{s,u}_{o'}(X), \end{aligned}$$

where $\varepsilon((s,u),o,o')$ is a sign. This follows from [14] (see also [5, Chapter 7]). The equation determines the sign $\varepsilon((s,u),o,o')$. Here the generic stabilizer of the action of $H(u)$ on $T^*_{\mathrm{hor}}P(s,u)$ is equal to the generic stabilizer $S(s,u)$ for the action of $H(u)$ on $M(s,u)$. Thus, integrating over the fibers the formula of Theorem 1 for the index of $D^+_{\mathrm{hor},\mathscr{F}}$ and using Formula 5, we obtain the following proposition, which is the analogue of the equivariant Hirzebruch-Riemann-Roch theorem in the form given in [6], [18].

Proposition 4. *Let $M = P/H$ be an even-dimensional orbifold such that $T_{\mathrm{hor}}P$ is a $(G \times H)$-oriented spin vector bundle with orientation o. Let $\mathscr{F}$ be a $(G \times H)$-equivariant complex vector bundle on P. Then*

$$\begin{aligned} \mathrm{Tr}\, Q^o(P/H,\mathscr{F})(s\exp X) = i^{-\dim M/2} \sum_{\gamma \in (C(s))} \int_{M(s,\gamma),o'} (2\pi)^{-\dim M(s,\gamma)/2} |S(s,\gamma)|^{-1} \\ \times \frac{\varepsilon((s,\gamma),o,o')\,\mathrm{ch}_{s,\gamma}(\mathscr{F})(X)}{J^{1/2}(M(s,\gamma))(X) D^{1/2}_{s,\gamma}(T_{M(s,\gamma)}M)(X)} \end{aligned}$$

for X small in $\mathfrak{g}(s)$.

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