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KOSTANT PARTITIONS FUNCTIONS AND FLOW POLYTOPES

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Dedicated to Bertram Kostant on the occasion of his 80th birthday

Abstract. This paper discusses volumes and Ehrhart polynomials in the context of flow polytopes. The general approach that studies these functions via rational functions with poles on arrangement of hyperplanes and the total residue of such functions allows us, via a unified approach, to reobtain many interesting calculations existing in the literature. In particular we generalize Lidskii formula relating the Ehrhart polynomial to the volume function.

Introduction

The function computing the number of ways one can decompose a vector as a linear combination with nonnegative integral coefficients of a fixed finite set of integral vectors is called a *partition function.* This problem can be expressed in terms of polytopes as follows. Let A be an r by N integral matrix with column vectors $\alpha_1, \ldots, \alpha_N$. Let a be an r-dimensional integral column vector and let $P(a) = \{y \in \mathbb{R}^N_{\geqslant 0} \mid Ay = a\}$ be the convex polytope associated to A and a. The number of ways one can decompose a as a linear combination with nonnegative integral coefficients of the vectors α_i is the number of integral points in $P(a)$. The function $a \to |P(a) \cap \mathbb{Z}^N|$ will be called the partition function $k(a)$. It is intuitively clear that $k(a)$ is related to the volume function $v(a) = \text{volume}(P(a))$, and that this last function varies polynomially in the function of a, provided the polytope $P(a)$ does not change "shape". In fact, $k(a)$ is obtained from $v(a)$ by applying the Khovanskii–Pukhlikov differential operator [11], [15].

Partition functions play a fundamental role in understanding characters of representations of compact Lie groups K. For example, the multiplicity of a weight in a finite-dimensional irreducible representation of K is given by the Kostant formula in terms of the partition function associated to the system of positive

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roots. More generally, the decomposition of a representation of K restricted to a compact subgroup H is expressed in terms of partition functions. Also, when G is a noncompact reductive Lie group, with maximal compact subgroup K, partition functions occur in the formulae for the K-multiplicities of an irreducible Harish-Chandra module for G. One instance is Blattner's formula, which gives the K-decomposition of a discrete series representation of G.

In general, it is difficult to give "concrete" formulae for the partition functions. We refer to [1] for a complete survey on the subject, and efficient implementation of the computation of Kostant partition functions in the case of classical root systems and to [8] and [9] for applications to tensor product multiplicities.

Let us describe briefly the content of this paper. We start by recalling the formulae for $v(a)$ and $k(a)$ (Proposition 12 and Theorem 13) in terms of residues of rational functions with poles on hyperplanes. Then we consider a particular instance of polytopes, the flow polytopes, that is, polytopes for which the columns of the determining matrix are roots for the linear group. It is in this context that we came across some very interesting calculations of Chan, Robbins, and Yuen and various specific examples ([7], [16], [23], [25]) related to the root system A_n. We show that indeed these results can be obtained using our formulae for $k(a)$ and $v(a)$ by residues of rational functions with poles on the hyperplanes $x_i = x_j$.

Furthermore, for flow polytopes, we generalize a formula due to Lidskii [18] relating the partition function $k(a)$ to the volume function $v(a)$. The relation (Theorem 38) consists in replacing multinomials $a^I/I!$ by appropriate multibinomials, and is not (to our knowledge) easy to deduce from the Khovanskii–Pukhlikov formula [15]. We use here a change of variables in residues specific to the root system A_n. We give an application of this generalized Lidskii formula to the parking polytope considered by Pitman and Stanley [21].

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1. Volume and Ehrhart function

The ideas of this section are valid for general convex polytopes [4], [24], thus we describe first the general setting. Later we will use particular properties of the root system A_n to deduce some results on flow polytopes.

1.1. Polytopes

Let U be an r-dimensional real vector space and let V be its dual vector space. We will usually denote by x the variable in U and by a the variable in V. Let $\Phi = [\alpha_1, \alpha_2, \ldots, \alpha_N]$ be a sequence of nonzero, not necessarily distinct, linear forms on U all lying in an open half-space. We denote by $\Delta^+ \subset V$ the set $\{\Phi\}$ (we mean Δ^+ and Φ are the same sets, but Φ may have multiplicities), and by Δ the set $\Delta^+ \cup -\Delta^+$. We assume that Δ^+ spans V.

We consider $\mathbb{R}^N$ with basis $(w_1, \ldots, w_N)$ and we let A be the surjective linear map from $\mathbb{R}^N$ to the vector space V defined by $A(w_k) = \alpha_k$, $1 \leqslant k \leqslant N$. The elements α_k are the column vectors of the matrix A. For $a \in V$, we consider the convex polytope

$$P(\Phi, a) := \{y = (y_1, y_2, \dots, y_N) \in \mathbb{R}^N_{\geqslant 0} \mid Ay = a\}.$$

Let $C(\Phi) = C(\Delta^+)$ be the cone generated by $\{\alpha_1, \dots, \alpha_N\}$. The cone $C(\Delta^+)$ is a pointed polyhedral cone. The dual cone $C(\Delta^+)^*$ of $C(\Delta^+)$ is defined by $C(\Delta^+)^* = \{x \in U \mid \langle \alpha, x \rangle \geqslant 0 \text{ for all } \alpha \in \Delta^+\}$ and its interior is nonempty.

The polytope $P(\Phi, a)$ is empty if a is not in $C(\Delta^+)$. If a is in the interior of the cone $C(\Delta^+)$, then the polytope $P(\Phi, a)$ has dimension d where $d = N - r = \dim(\mathrm{Ker}(A))$.

1.2. Total residue

Each $\alpha \in \Delta$ determines a linear form on U and a complex hyperplane $\{x \in U_{\mathbb{C}} \mid \alpha(x) = 0\}$ in $U_{\mathbb{C}}$. The ring R_Δ of rational functions with poles on the hyperplane arrangement $\mathcal{H}_{\mathbb{C}}(\Delta) = \bigcup_{\alpha\in\Delta}\{x \in U_{\mathbb{C}} \mid \alpha(x) = 0\}$ is the ring $\Delta^{-1}S(V)$ generated by the ring $S(V)$ of polynomial functions on U, together with inverses of the linear functions $\alpha \in \Delta$. Thus a function in R_Δ can be written $R(x) = P(x)/\prod_{\alpha\in\Delta}\alpha(x)^{n_\alpha}$ where P is a polynomial function on r complex variables and n_α are nonnegative integers. We may write $R(x) = P(x)/\prod_{\alpha\in\Delta}\alpha(x)^{n_\alpha}$ indicating the variable $x \in U$ or, more abstractly, $R = P/\prod_{\alpha\in\Delta}\alpha^{n_\alpha}$, where $P \in S(V)$.

A subset σ of Δ is called a *basis* of Δ if the elements $\alpha \in \sigma$ form a basis of V. In this case, we set

$$f_\sigma(x) := \frac{1}{\prod_{\alpha\in\sigma}\alpha(x)}$$

and call such an element a *simple fraction* or just *simple*. Denote by S_Δ the linear subspace of R_Δ spanned by simple elements.

The space U acts on R_Δ by differentiation: $(\partial(u)f)(x) = (d/d\epsilon)f(x + \epsilon u)|_{\epsilon=0}$. We denote by $\partial(U)R_\Delta$ the span of derivatives of functions in R_Δ.

Theorem 1 ([6], Prop. 7**).**

$$R_\Delta = \partial(U)R_\Delta \oplus S_\Delta.$$

The projection map $\mathrm{Tres}_\Delta : R_\Delta \to S_\Delta$ according to this decomposition is called the total residue map and it is easy to see that the function $\mathrm{Tres}_\Delta(f)$, $f \in R_\Delta$, depends only on the smallest hyperplane arrangement containing the poles of f, therefore we will just write $\mathrm{Tres}(f)$ for $\mathrm{Tres}_\Delta(f)$. The definition of total residue formalizes notions introduced in [14]. The vector space S_Δ is contained in the homogeneous component of degree $-r$ of R_Δ, so that the total residue vanishes outside the homogeneous component of degree $-r$ of R_Δ. In particular, if P is analytic and f is homogeneous of degree $-r$, $\mathrm{Tres}(Pf) = P(0)\,\mathrm{Tres}(f)$.

Example 2. If V is of dimension 1 and $\Delta = \{\pm e_1\}$, then R_Δ is the ring of Laurent series

$$L = \Big\{f(x) = \sum_{k\geqslant -q} a_k x^k\Big\}.$$

The total residue of a function $f(x) \in L$ is the function a_{-1}/x. The usual residue denoted $\mathrm{Res}_{x=0} f$ is the constant a_{-1}. We will also use the constant term a_0 of f. We write $\mathrm{Ct}_{x=0}(f) = a_0$.

Example 3. Let U be a two-dimensional vector space with basis e^1, e^2 and let x_1, x_2 be the dual basis of V. Let $\Delta = \{\pm x_1, \pm x_2, \pm(x_1 - x_2)\}$. Consider the element $x_1/(x_1 - x_2)x_2^2$ in R_Δ of degree -2. We have

$$x_1/(x_1 - x_2)x_2^2 = \frac{1}{x_2^2} + \frac{1}{(x_1 - x_2)x_2}.$$

The function $1/x_2^2 = -\partial_{x_2}(1/x_2)$ is a derivative. Thus $\mathrm{Tres}\big(x_1/(x_1 - x_2)x_2^2\big) = 1/(x_1 - x_2)x_2$.

We extend the definition of total residue to the space $\widehat{R}_\Delta$ which is the space consisting of functions P/Q where Q is a finite product of powers of the linear forms α_i and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series. Indeed, suppose that $P/Q \in \widehat{R}_\Delta$ where we may assume that Q is of degree q, and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series with P_k of degree k. As the total residue vanishes outside the homogeneous component of degree $-r$ of Δ, we just define

$$\mathrm{Tres}(P/Q) = \mathrm{Tres}(P_{q-r}/Q).$$

Let $F : U_\mathbb{C} \to U_\mathbb{C}$ be an analytic map preserving each hyperplane $\alpha = 0$, and invertible at the origin. If $f \in \hat{R}_\Delta$, the function $(F^* f)(x) = f(F(x))$ is again in $\hat{R}_\Delta$. Let $\mathrm{Jac}(F)$ be the Jacobian of the map F.

Proposition 4. *For any f in $\hat{R}_\Delta$, we have the equality in S_Δ:*

$$\mathrm{Tres}(f) = \mathrm{Tres}(\mathrm{Jac}(F)(F^* f)).$$

Proof. We have $F^*(\alpha) = g_\alpha \alpha$ where g_α is analytic and $g_\alpha(0) \neq 0$. If $f_\sigma := 1/\prod_{\alpha \in \sigma} \alpha$ is a simple fraction, then $F^*(f_\sigma) = f_\sigma r_\sigma$, where r_σ is analytic. Furthermore, we see that $r_\sigma(0)$ is the inverse of $\mathrm{Jac}(F)(0)$. Thus $\mathrm{Tres}(\mathrm{Jac}(F) F^* f_\sigma) = \mathrm{Jac}(F)(0) r_\sigma(0) f_\sigma = f_\sigma$. We obtain $\mathrm{Tres}(\mathrm{Jac}(F)(F^* s)) = s$ if $s \in S_\Delta$.

Let $f \in \hat{R}_\Delta$. Choose a system of coordinates x_i on U and write $dx = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_r$. Then $(f - \mathrm{Tres}(f))\, dx$ is the differential of some $(r-1)$-form $\sum_{k=1}^{r} f_k \, dx_1 \wedge dx_2 \wedge \widehat{dx_k} \wedge \cdots \wedge dx_r$, with $f_k \in \hat{R}_\Delta$. The vector space $\sum_{k=1}^{r} f_k \, dx_1 \wedge dx_2 \wedge \widehat{dx_k} \wedge \cdots \wedge dx_r$ with $f_k \in \hat{R}_\Delta$ is stable by the action of F^* on differential forms. As F^* commutes with d, $(F^* f) F^* dx - F^*(\mathrm{Tres}(f)) F^* dx$ is the differential of some $(r-1)$-form. It follows that $(F^* f)\,\mathrm{Jac}(F) - F^*(\mathrm{Tres}(f))\,\mathrm{Jac}(F) \in \partial(U) R_\Delta$. Using the remark above for $s = \mathrm{Tres}(f)$, we obtain the formula of the proposition by projection on S_Δ. □

The following definitions are important.

Let $\Phi = [\alpha_1, \alpha_2, \ldots, \alpha_N]$ be our sequence of elements of Δ. For any $a \in V$, the function $e^{\langle a, x \rangle}/\prod_{k=1}^{N} \alpha_k(x)$ is in $\widehat{R}_\Delta$.

Definition 5. We define $J_\Phi(a) \in S_\Delta$ by

$$J_\Phi(a) = \mathrm{Tres}\left(\frac{e^a}{\prod_{k=1}^{N} \alpha_k}\right) = \frac{1}{(N-r)!}\mathrm{Tres}\left(\frac{a^{N-r}}{\prod_{k=1}^{N} \alpha_k}\right).$$

For $\alpha \in \Phi$, the function $\alpha/(1 - e^{-\alpha}) = 1 + \frac{1}{2}\alpha + \frac{1}{12}\alpha^2 + \cdots$ is analytic at the origin. Thus $1/(1 - e^{-\alpha}) = (1/\alpha)(\alpha/(1 - e^{-\alpha}))$ is an element of $\widehat{R}_\Delta$.

Definition 6. We define $K_\Phi(a) \in S_\Delta$ by

$$K_\Phi(a) = \operatorname{Tres}\left(\frac{e^a}{\prod_{k=1}^N (1 - e^{-\alpha_k})}\right).$$

The following lemma is immediate.

Lemma 7.

- *The function* $a \to J_\Phi(a)$ *is a* polynomial *function from* V *to* S_Δ *which is homogeneous of degree* $(N - r)$.
- *The function* $a \to K_\Phi(a)$ *is a* polynomial *function from* V *to* S_Δ. *Its top degree term is the function* $J_\Phi(a)$.

1.3. Counting functions

Suppose now that our sequence Φ generates a lattice $V_\mathbb{Z}$ in V: $V_\mathbb{Z} = \sum_{i=1}^N \mathbb{Z}\alpha_i$. In this case, the lattice $V_\mathbb{Z}$ determines a measure da on V so that the fundamental domain of the lattice $V_\mathbb{Z}$ is of measure 1 for da. If σ is a basis of Δ, we write $\operatorname{vol}(\sigma)$ for the volume of the parallelepiped $\bigoplus_{\alpha\in\sigma}[0,1]\alpha$, relative to our Lebesgue measure da. Observe that $\operatorname{vol}(\sigma) = |\det(\sigma)|$, where σ is the matrix whose columns are the $\alpha_i \in \sigma$. We say that Φ is unimodular, if $\operatorname{vol}(\sigma) = 1$ for all basis σ of Δ.

Let du be the Lebesgue measure on $\mathbb{R}^N$. The vector space $\operatorname{Ker}(A) = A^{-1}(0)$ is of dimension $d = N-r$ and it is equipped with the quotient Lebesgue measure du/da. For $a \in V$, $A^{-1}(a)$ is an affine space parallel to $\operatorname{Ker}(A)$, and thus also equipped with the Lebesgue measure du/da. Volumes of subsets of $A^{-1}(a)$ are computed for this measure, in particular, we can speak of the volume of the polytope $P(\Phi, a)$.

Definition 8.

- If $a \in V$, define $v(\Phi, a) = \operatorname{volume}(P(\Phi, a))$.
- If $a \in V_\mathbb{Z}$, define $k(\Phi, a) = |P(\Phi, a) \cap \mathbb{Z}^N|$.

Thus $k(\Phi, a)$ is the number of solutions $(y_1, y_2, \ldots, y_N)$, in nonnegative integers y_j, of the equation $\sum_{j=1}^N y_j\alpha_j = a$.

Let F be a face of the cone $C(\Delta^+)$ and let $\Phi_F = \Phi \cap F$. It is clear that for $a \in F$, we have $k(\Phi, a) = k(\Phi_F, a)$.

The function $k(\Phi, a)$ is called the *Ehrhart function* associated to Φ. If Φ is a root system of a semi-simple Lie algebra, this is the *Kostant partition function.*

We will also use the relative volume of $P(\Phi, a)$, which compares the volume of $P(\Phi, a)$ to the volume of the standard simplex, so it is defined by

$$\operatorname{vol}_{\mathrm{rel}} P(\Phi, a) = (N - r)!\, v(\Phi, a).$$

This number may have a combinatorial meaning: the number of elements of a simplicial decomposition of $P(\Phi, a)$.

1.4. Chambers

We briefly recall the notion of a chamber. We refer to [2] for further reference and for the description of an effective algorithm to compute chambers. Here is a word of *caution*: The notion of chambers we will use in this paper is unrelated to the notion of Weyl chamber associated to root systems.

For any subset ν^+ of Δ^+, we denote by $C(\nu^+)$ the closed cone generated by ν^+. We denote by $C(\Delta^+)_{\rm sing}$ the union of the cones $C(\nu^+)$ where ν^+ is any subset of Δ^+ of cardinal strictly less than $r = \dim(V)$. By definition, the set $C(\Delta^+)_{\rm reg}$ of Δ^+-regular elements is the complement of $C(\Delta^+)_{\rm sing}$. A connected component of $C(\Delta^+)_{\rm reg}$ is called a chamber. If $\mathfrak{c}$ is a chamber, and σ a basis of Δ contained in Δ^+, then either $\mathfrak{c} \subset C(\sigma)$ or $\mathfrak{c} \cap C(\sigma) = \varnothing$, as the boundary of $C(\sigma)$ does not intersect $\mathfrak{c}$. Thus the closure of the chamber $\mathfrak{c}$ is the intersection of the simplicial cones $C(\sigma)$, σ a basis of Δ^+, containing $\mathfrak{c}$. A wall of Δ is a (real) hyperplane generated by $r-1$ linearly independent elements of Δ. It is clear that a chamber $\mathfrak{c}$ is limited by walls.

Remark 9. When a varies in a chamber $\mathfrak{c}$, the combinatorial nature of $P(\Phi, a)$ remains the same, and the family of polytopes $P(\Phi, a)$ has parallel facets. Recall that the Minkowski sum $A+B$ of two convex subsets A, B is given by $A+B = \{a+b, a \in A, b \in B\}$. The notion of Minkowski sums and of chambers are intimately related: if $a_1, \dots, a_m$ are vectors in the closure of a chamber $\mathfrak{c}$, then the polytope $P(\Phi, a_1 + \cdots + a_m)$ is isomorphic to the Minkowski sum of the polytopes $P(\Phi, a_k)$. This follows, for example, from [5, Sect. 3.1].

It is very difficult to determine the number of chambers $\mathfrak{c}$. For example, consider A_n^+ the positive root system for a root system of type A. For $n = 1, 2, 3, 4, 5, 6$, the number of chambers is 1, 2, 7, 48, 820, 44 288. A Maple program to compute chambers for any set Δ^+ is available at [10].

1.5. Jeffrey–Kirwan residue and the Laplace transform

We now go back to the problem of writing formulae for the volume and the number of integral points of a polytope. The following easy result (see, e.g. [2]) computes the Laplace (or discrete Laplace) transform of these functions, thus showing that what we need is efficient ways to compute the inverse Laplace transform. The Jeffrey–Kirwan residue is the tool.

Lemma 10. *For x in the interior of the dual cone of $C(\Delta^+)$,*

$$\sum_{a \in C(\Delta^+) \cap V_{\mathbb{Z}}} k(\Phi, a) e^{-\langle a, x\rangle} = \frac{1}{\prod_{\alpha \in \Phi} 1 - e^{-\langle \alpha, x\rangle}},$$

$$\int_{C(\Delta^+)} v(\Phi, a) e^{-\langle a, x\rangle}\, da = \frac{1}{\prod_{\alpha \in \Phi} \langle \alpha, x\rangle}.$$

The Jeffrey–Kirwan residue [14] associated to a chamber $\mathfrak{c}$ of $C(\Delta^+)$ is a linear form $f \mapsto \langle\langle \mathfrak{c}, f\rangle\rangle$ on the vector space S_Δ of simple fractions. Any function f in S_Δ can be written as a linear combination of functions f_σ, with σ a basis of Δ contained in Δ^+. To determine the linear map $f \mapsto \langle\langle \mathfrak{c}, f\rangle\rangle$, it is enough to determine it on this set of functions f_σ. So assume that σ is a basis of Δ contained in Δ^+ and define

Definition 11.

- If $\mathfrak{c} \subset C(\sigma)$, then $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle = 1/\mathrm{vol}(\sigma)$.
- If $\mathfrak{c} \cap C(\sigma) = \varnothing$, then $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle = 0$.

The following proposition is easy to prove using elementary properties of Laplace transforms and it formalizes the Jeffrey–Kirwan formula ([6], [14], see also Appendix of [4]) for the volume of polytopes.

Proposition 12. *Let $\mathfrak{c}$ be a chamber of $C(\Delta^+)$ and denote by a an element of V. Then, for $a \in \overline{\mathfrak{c}}$, the volume of $P(\Phi, a)$ is given by*

$$v(\Phi, a) = \langle\langle \mathfrak{c}, J_\Phi(a) \rangle\rangle.$$

In particular, the function $a \to v(\Phi, a)$ is given by a polynomial formula on the closure $\overline{\mathfrak{c}}$ of the chamber $\mathfrak{c}$.

Thus the calculation of the volume of a polytope $P(\Phi, a), a \in \overline{\mathfrak{c}}$, is divided into two problems:

(A) Compute the linear form $f \to \langle\langle \mathfrak{c}, f \rangle\rangle$.
(B) Compute the function $J_\Phi(a)$, that is, compute the function

$$\frac{e^{\langle a, x \rangle}}{\prod_{\alpha \in \Phi} \langle \alpha, x \rangle}$$

up to derivatives.

The linear form $f \to \langle\langle \mathfrak{c}, f \rangle\rangle$ is determined in a combinatorial way using the notion of maximal nested subsets of Δ introduced in [12]. We refer to [1] for the description of the corresponding algorithm.

We now state the formula for the Ehrhart function $k(\Phi, a)$ which counts the number of integral points in the polytope $P(\Phi, a)$. We restrict ourselves to unimodular systems. A remarkable result of Dahmen–Micchelli [11] states that the partition function $k(\Phi, a)$ is obtained by applying a specific differential operator to the volume function $v(\Phi, a)$. The same result also follows from the Khovanskii–Pukhlikov relation [15] between the number of integral points and the volumes of deformed polytopes. Thus, from Proposition 12, we obtain the following theorem.

Theorem 13 (Dahmen and Micchelli)**.** *Assume that Φ is unimodular. Let $Z(\Phi)= \sum_{k=0}^{N} [0,1]\alpha_k$ be the zonotope generated by Φ. Let $\mathfrak{c}$ be a chamber of $C(\Delta^+)$ and denote by a an element of $V_{\mathbb{Z}}$. Then, for $a \in \mathfrak{c} - Z(\Phi)$, the number of integral points in $P(\Phi, a)$ is given by*

$$k(\Phi, a) = \langle\langle \mathfrak{c}, K_\Phi(a) \rangle\rangle.$$

In particular, the function $a \to k(\Phi, a)$ is given by a polynomial formula on the cone $\overline{\mathfrak{c}} \cap V_{\mathbb{Z}}$.

For a formula in the general case, see [24]. See also [13].

We denote by $v(\Phi, \mathfrak{c})(a)$ the polynomial function of a coinciding with $v(\Phi, a)$, when $a \in \overline{\mathfrak{c}}$. It is a homogeneous polynomial of degree $(N - r)$. We denote by

$k(\Phi, \mathfrak{c})(a)$ the polynomial function of a coinciding with $k(\Phi, a)$, when $a \in (\mathfrak{c} - Z(\Phi)) \cap V_{\mathbb{Z}}$. The highest degree term of $k(\Phi, \mathfrak{c})$ is $v(\Phi, \mathfrak{c})$.

If $\mathfrak{c}_1, \mathfrak{c}_2$ are adjacent chambers, the polynomial function $k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2)$ is determined in [20]. It vanishes on a certain number of affine hyperplanes parallel to the wall separating $\mathfrak{c}_1$ and $\mathfrak{c}_2$. We state the corresponding property only for a chamber on the boundary of the polytope. This particular case would also follow from reciprocity relations for the partition function.

Let $f \in U$ be such that the half-space $\langle f, a\rangle \geqslant 0$ contains the cone $C(\Delta^+)$ and such that $\{\langle f, a\rangle = 0\} \cap C(\Delta^+)$ is a facet F of $C(\Delta^+)$. We choose f to be a primitive vector, that is, such that $\langle f, V_{\mathbb{Z}}\rangle = \mathbb{Z}$. We simply say that $f = 0$ is a face of $C(\Delta^+)$. Let $\mathfrak{c}$ be a chamber, with a facet on F. Let $\kappa_F = \langle f, \sum_{k=1}^{N} \alpha_k\rangle \in \mathbb{N}$.

Corollary 14. *Assume that $\overline{\mathfrak{c}}$ has a facet included in the face F of $C(\Delta^+)$. Then $k(\Phi, \mathfrak{c})$ is divisible by $\prod_{b=1}^{\kappa_F - 1}(f + b)$.*

2. Flow polytopes and the special linear group

We now study in more detail what can be said when Φ is a sequence of elements in the root system A_n.

2.1. Definitions and examples

Let E be an $(r+1)$-dimensional vector space with basis e_i $(i = 1, \ldots, r+1)$, and consider the set

$$\mathsf{A}_r^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant r+1\}.$$

These are the positive roots for a system of type A_r. The number of elements in A_r^+ is $(r+1)r/2$. We let V_r be the vector space generated by the elements in A_r^+. When r is fixed, we write $V_r = V$:

$$V = \left\{ a = \sum_{i=1}^{r+1} a_i e_i \in E \;\middle|\; \sum_{i=1}^{r+1} a_i = 0 \right\}.$$

It is easy to see that the walls of A_r are the kernels of the linear forms $\sum_{i \in J} a_i$ where J is a subset of $\{1, 2, \ldots, r\}$.

The vector space V is of dimension r and the map $p: \mathbb{R}^r \to V$ defined by $(a_1, a_2, \ldots, a_r) \mapsto a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1}$, explicitly provides an isomorphism of V with the Euclidean space $\mathbb{R}^r$. In the rest of this paper, the element $(a_1, a_2, \ldots, a_r) \in \mathbb{R}^r$ refers to the element $a \in V$.

Define $U = V^*$ as in the general setting. We identify U with $\mathbb{R}^r$ by p^*. Then the root $e_i - e_j$ $(1 \leqslant i < j < r+1)$ produces the linear function $x_i - x_j$ on U, while the root $e_i - e_{r+1}$ produces the linear function x_i. In this identification, a function in R_{A_r} is thus a rational function $f(x_1, x_2, \ldots, x_r)$ on $U_{\mathbb{C}}$, with poles on the hyperplanes $x_i = x_j$ or $x_i = 0$.

The vector space V is provided with a lattice, the lattice spanned by A_r^+, which is given by

$$V_{\mathbb{Z}} = \left\{ a = \sum_{i=1}^{r+1} a_i e_i, a_i \in \mathbb{Z} \;\middle|\; \sum_{i=1}^{r+1} a_i = 0 \right\}.$$

In our identification of V with $\mathbb{R}^r$, $V_{\mathbb{Z}}$ is identified with the standard lattice $\mathbb{Z}^r$. We choose on V the measure da determined by $V_{\mathbb{Z}}$. We have $da = da_1 \cdots da_r$. It is well known and easy to prove that A_r is unimodular.

The cone $C(\mathsf{A}_r^+)$ is simplicial with generators the $r-1$ simple roots $\{e_1-e_2, e_2-e_3, \ldots, e_r - e_{r+1}\}$ and it is described as

$$C(\mathsf{A}_r^+) = \{a \in V \mid a_1 + a_2 + \cdots + a_i \geqslant 0 \text{ for all } i, 1 \leqslant i \leqslant r\}. \tag{1}$$

Let $a = a_1e_1 + a_2e_2 + \cdots + a_re_r + a_{r+1}e_{r+1} \in V_{\mathbb{Z}}$. The Kostant partition function for the system A_r is the function $k(\mathsf{A}_r^+, a)$ on $V_{\mathbb{Z}} \cap C(\mathsf{A}_r^+)$. Observe that $k(\mathsf{A}_r^+, a_1e_1 + a_2e_2 + \cdots + a_re_r + a_{r+1}e_{r+1}) = k(\mathsf{A}_r^+, -a_{r+1}e_1 - a_re_2 - \cdots - a_2e_r - a_1e_{r+1})$, as follows from the automorphism of the Dynkin diagram of A_r induced by the permutation $(1, 2, \ldots, r, r+1) \mapsto (r+1, r, \ldots, 2, 1)$.

More generally, we will consider partition functions $k(\Phi, a)$ where Φ is a subset of A_r^+, eventually with multiplicities. Thus, let $\Phi = [\alpha_1, \ldots, \alpha_N]$ be a sequence of N elements of A_r^+. Given the root $e_i - e_j$ in Φ, we write m_{ij} $(i<j)$ for its multiplicity.

Definition 15. A flow polytope is a polytope isomorphic to a polytope $P(\Phi, a)$ where Φ is a sequence of elements of A_r^+ and $a \in V$.

By definition, the graph associated to Φ is the graph with $r+1$ vertices $1, 2, \ldots \ldots, r+1$, and m_{ij} edges from $i \to j$ if $e_i - e_j$ is in Φ. It looks like a cascade if the vertices $1, 2, \ldots, r+1$ are located on a slope, with 1 on top, and $r+1$ at the bottom (see Figure 1). The graph associated to A_r^+ is the complete flow graph with one edge from i to j if $i<j$. Let us give a description of the polytope $P(\Phi, a)$ in visual terms which will explain the name “flow polytope”. Let $a = a_1e_1 + a_2e_2 + \cdots + a_re_r - (a_1 + a_2 + \cdots + a_r)e_{r+1}$. By definition

$$P(\Phi, a) = \left\{ y_{ij} \in \mathbb{R}_{\geqslant 0},\ i<j \;\middle|\; \sum_{i<j} y_{ij}(e_i - e_j) = a \right\}.$$

We can imagine that the positive quantity y_{ij} is the quantity of water at time t in the branch $i \mapsto j$ of this cascade. We can also visualize a_1, $a_2, \ldots, a_r$ as additional sources of water. Then the equations of the polytope $P(\Phi, a)$ express that the quantity of water arriving at the place j is equal to the quantity of water flowing down.

Example 16 (Pitman–Stanley polytope)**.** Let $a \in V,\ a_i \geqslant 0$ and

$$\Pi_r(a) = \{y \in \mathbb{R}^r_{\geqslant 0} \mid y_1 + \cdots + y_i \leqslant a_1 + a_2 + \cdots + a_i\}.$$

These are the inequalities of the parking polytope, where y_i is the number of cars arriving at the gate and a_i the capacities of each slot located on a row. Let us give a description of this polytope in the setting of flow polytopes. Consider the graph with $r+1$ vertices, and edges from i to $r+1$ and from i to $i+1$, for $1 < i \leqslant r$. We can imagine that the positive quantity $y_i := y_{i(r+1)}$ is the quantity of water at time t in the branch $i \mapsto r+1$ of this cascade. There is also a leak $z_{i(i+1)} \geqslant 0$ running from the vertex i to $i+1$. We can also visualize a_1, a_2, $\ldots$,

a_r as additional sources of water, so that the quantities of water y_i in the streams $i \to r+1$ satisfy the inequalities stated above (see Figure 1). In other words, let

$$\Phi_{PS} = [(e_i - e_{r+1}), (e_k - e_{k+1}), 1 \leqslant i \leqslant r, 1 \leqslant k \leqslant r]. \tag{2}$$

The multiplicity of a root in Φ_{PS} is always 1, except for the root $(e_r - e_{r+1})$ for which it is 2. Thus $\Pi_r(a)$ is isomorphic to $P(\Phi_{PS}, a)$.

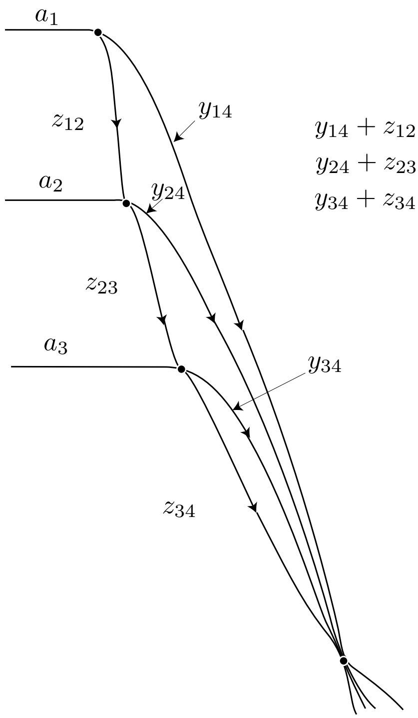

FIGURE 1. Graph for $\Pi_3(a)$.

Example 17 (Chan–Robbins–Yuen polytope**).** Let Y_n be the vector space of $(n \times n)$-matrices $\{\sum x_{\ell k} E_{k\ell} \mid 1 \leqslant k \leqslant n, 1 \leqslant \ell \leqslant n,\ \ell \leqslant (k+1)\}$, where $E_{k\ell}$ is the matrix with unique nonzero entry at the kth row and ℓth column. Let B_n be the Birkhoff polytope. This is the polytope of double stochastic matrices b: the $(n \times n)$-matrix b has nonnegative entries, and entries on each line of b sum up to 1, as well as the entries on each row. The Chan–Robbins–Yuen polytope is the polytope $CRY_n = B_n \cap Y_n$. Thus any doubly stochastic matrix in CRY_n has nonzero entries only on the diagonal lines below the diagonal, on the diagonal and the line just above the diagonal, see, for example, equation (3) describing CRY_3.

Lemma 18. *The polytope CRY_n is isomorphic to $P(\mathsf{A}_n^+, (e_1 - e_{n+1}))$.*

Proof. Let $\mathfrak{n}$ be the vector space of strictly lower triangular $(n+1) \times (n+1)$-matrices, with basis the matrices $E_{k\ell}, 1 < \ell < k < (n+1)$. Then the polytope

$P(\mathsf{A}_n^+, (e_1 - e_{n+1}))$ is, by definition, the set of strictly lower triangular $(n+1) \times (n+1)$-matrices $y = \sum_{i<j} y_{ij} E_{ji}$ with nonnegatives entries y_{ij} and such that $\sum_{i<j} y_{ij}(e_i - e_j) = e_1 - e_{n+1}$. We send the matrix $b = \sum_{i \leqslant j+1} b_{ij} E_{ji}$ in CRY_n to the strictly lower $(n+1) \times (n+1)$-matrix $y = \sum_{1 \leqslant i < j \leqslant (n+1)} b_{i(j-1)} E_{ji}$. This map produces an isomorphism between CRY_n and $P(\mathsf{A}_n^+, (e_1 - e_{n+1}))$. This fact is most transparent on an example. Take for example $n = 3$. By definition, the polytope $P(\mathsf{A}_3^+, (e_1 - e_4))$ consists of all (4×4) lower triangular matrices

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{12} & 0 & 0 & 0 \\ y_{13} & y_{23} & 0 & 0 \\ y_{14} & y_{24} & y_{34} & 0 \end{pmatrix}$$

with nonnegative coefficients y_{ij} and such that $\sum y_{ij}(e_i - e_j) = e_1 - e_4$. This gives the four equations:

$$\begin{aligned} y_{12} + y_{13} + y_{14} = 1, \quad -y_{12} + y_{23} + y_{24} = 0, \\ -y_{13} - y_{23} + y_{34} = 0, \quad -y_{14} - y_{24} - y_{34} = -1. \end{aligned}$$

The polytope CRY_3 consists of the matrices

$$b = \begin{pmatrix} b_{11} & b_{21} & 0 \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix} \tag{3}$$

such that $b_{21} = 1 - b_{11}$, $b_{32} = 1 - (b_{12} + b_{22})$, $b_{13} + b_{23} + b_{33} = 1$ and so that the sum of each column is equal to 1. It is immediate to verify that the matrix

$$y(b) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_{11} & 0 & 0 & 0 \\ b_{12} & b_{22} & 0 & 0 \\ b_{13} & b_{23} & b_{33} & 0 \end{pmatrix}$$

is in $P(\mathsf{A}_3^+, (e_1 - e_4))$. $\square$

2.2. The space R_{A_r}

We collect here some results of the ring R_Δ when $\Delta = \mathsf{A}_r$.

Let Σ_r be the permutations on the set $\{1, \dots, r\}$. We recall the following result [17] (see [3] for a proof).

Lemma 19. *Let $w \in \Sigma_r$ and let f_π, $f_w, w \in \Sigma_r$, be defined by*

$$f_\pi := \frac{1}{(e_1 - e_2)(e_2 - e_3) \cdots (e_{r-1} - e_r)(e_r - e_{r+1})},$$

and

$$f_w = w \cdot f_\pi = \frac{1}{\prod_{i=1}^{r-1} (e_{w(i)} - e_{w(i+1)})(e_{w(r)} - e_{r+1})},$$

then $\{f_w,\, w \in \Sigma_r\}$ *is a basis for* S_{A_r} *and* $\dim S_{\mathsf{A}_r} = r!$.

Recall that a function in R_{A_r} is identified to a rational function $f(x_1, x_2, \ldots, x_r)$ on $U_{\mathbb{C}}$, with poles on the hyperplanes $x_i = x_j$ or $x_i = 0$. In this identification, the basis $f_w, w \in \Sigma_r$, of S_{A_r} is given by the elements

$$f_w(x_1, \ldots, x_r) = w \cdot f_\pi(x_1, \ldots, x_r) = \frac{1}{\prod_{i=1}^{r-1}(x_{w(i)} - x_{w(i+1)})x_{w(r)}}.$$

For a permutation $\sigma \in \Sigma_r$, define the linear form on R_{A_r} by

$$\operatorname{Ires}^{\sigma}_{x=0} f := \operatorname{Res}_{x_{\sigma(1)}=0} \operatorname{Res}_{x_{\sigma(2)}=0} \cdots \operatorname{Res}_{x_{\sigma(r)}=0} f(x_1, x_2, \ldots, x_r).$$

Such linear forms are called *iterated residues*, their value depends on the order of the variables: in computing $\operatorname{Res}_{x_1=0} \operatorname{Res}_{x_2=0} f(x_1, x_2)$ we first consider x_1 as fixed, so that $\operatorname{Res}_{x_2=0} f(x_1, x_2)$ is a function of x_1. Then we take the residue in x_1. For $\sigma = 1$, the identity permutation, we write simply

$$\operatorname{Ires}^{1}_{x=0} f = \operatorname{Ires}_{x=0} f = \operatorname{Res}_{x_1=0} \operatorname{Res}_{x_2=0} \cdots \operatorname{Res}_{x_r=0} f(x_1, x_2, \ldots, x_r).$$

The following lemma is easy to prove.

Lemma 20.

- *The linear form* $f \mapsto \operatorname{Ires}^{\sigma}_{x=0} f$ *on* R_{A_r} *vanishes on the vector space of derivatives* $\sum_{i=1}^{r} \partial_i R_{\mathsf{A}_r}$.
- *For* $\sigma, w \in \Sigma_r$, $\operatorname{Ires}^{\sigma}_{x=0} f_w = \delta^{\sigma}_{w}$, *where* δ^{σ}_{w} *is the Kronecker* δ*-function.*

Thus the $r!$ linear forms $\operatorname{Ires}^{\sigma}_{x=0} f, \sigma \in \Sigma_r$, on S_{A_r} are dual to the basis f_w.

For $w \in \Sigma_r$, we denote by $C^+_w \subset C({\mathsf{A}_r}^+)$ the simplicial cone generated by the basic subset σ^+_w of A^+_r defined by

$$\sigma^+_w = \{\epsilon(i)(e_{w(i)} - e_{w(i+1)}) \mid i = 1, \ldots, r-1\} \cup \{(e_{w(r)} - e_{r+1})\},$$

where $\epsilon(i)$ is 1 or -1 depending on whether $w(i) < w(i+1)$ or not. The function f_w is proportional to the basic fraction $f_{\sigma^+_w}$. When $w = 1$, then $C^+_1 = C(\mathsf{A}^+_r)$.

We denote by $\mathfrak{c}_{\text{nice}}$ the open subset of $C(\mathsf{A}^+_r)$ defined by

$$\mathfrak{c}_{\text{nice}} = \{a \in C({\mathsf{A}_r}^+) \mid a_i > 0,\, i = 1, \ldots, r\}.$$

Lemma 21.

- *The set* $\mathfrak{c}_{\text{nice}}$ *is a chamber for the system* A^+_r *and will be called the* nice chamber.
- *We have* $\langle\langle \mathfrak{c}_{\text{nice}}, f\rangle\rangle = \operatorname{Ires}_{x=0} f$.

Proof. The set $\mathfrak{c}_{\text{nice}}$ does not intersect any wall of A_r, so that $\mathfrak{c}_{\text{nice}}$ is contained in a chamber. But $\mathfrak{c}_{\text{nice}}$ is the interior of the simplicial cone generated by the basis $(e_i - e_{r+1})$, $1 \leqslant i \leqslant r$. Thus $\mathfrak{c}_{\text{nice}}$ is a chamber.

Let us prove the second point. Consider now $f \in S_{\mathsf{A}_r}$. To compute $\langle\langle \mathfrak{c}_{\text{nice}}, f\rangle\rangle$, we need only to compute it on the basis $f_{\sigma^+_w}$ of S_{A_r}. According to the definition, to compute $\langle\langle \mathfrak{c}, f_{\sigma^+_w}\rangle\rangle$, we need to check if $\mathfrak{c} \subset C^+_w$ or not. It is easy to see that this happens only if $w = 1$ thus proving the last point. □

Let Φ be a sequence of N elements of A_r^+. Let $a \in V$. We explicitly write the functions $J_\Phi(a)$ and $K_\Phi(a)$. We have

$$J_\Phi(a)(x_1,\dots,x_r) = \frac{1}{(N-r)!}\mathrm{Tres}\left(\frac{(a_1x_1+\cdots+a_rx_r)^{N-r}}{\prod_{i=1}^r x_i^{m_{i,r+1}}\prod_{1\leqslant i<j\leqslant r}(x_i-x_j)^{m_{i,j}}}\right). \quad (4)$$

If $i = (i_1, i_2, \dots, i_r)$ is a sequence of nonnegative integers, we write $|i| := i_1 + i_2 + \cdots + i_r$. Then, more explicitly,

$$J_\Phi(a) = \sum_{|i|=N-r} \frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!}\cdots\frac{a_r^{i_r}}{i_r!} f_\Phi(i),$$

where

$$f_\Phi(i) = \mathrm{Tres}\left(\frac{x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r}}{\prod_{i=1}^r x_i^{m_{i,r+1}}\prod_{1\leqslant i<j\leqslant r}(x_i-x_j)^{m_{i,j}}}\right). \quad (5)$$

We have

$$K_\Phi(a)(x_1,\dots,x_r)=\mathrm{Tres}\left(\frac{e^{a_1x_1}e^{a_2x_2}\cdots e^{a_rx_r}}{\prod_{i=1}^r(1-e^{-x_i})^{m_{i,r+1}}\prod_{1\leqslant i<j\leqslant r}(1-e^{-(x_i-x_j)})^{m_{i,j}}}\right). \quad (6)$$

Recall that $a \in V$ is written as $a = a_1e_1 + \cdots + a_re_r - (\sum_{i=1}^r a_i)e_{r+1}$ or $(a_1, a_2, \dots, a_r)$. The chamber $\mathfrak{c}_{\mathrm{nice}}$ has a face $a_1 = 0$ included in a face of $C(\mathsf{A}_r^+)$.

Lemma 22.

- *The polynomial* $v(\mathsf{A}_r^+, \mathfrak{c}_{\mathrm{nice}})(a_1, a_2, \dots, a_r)$ *is divisible by* a_1^{r-1}.
- *The polynomial* $k(\mathsf{A}_r^+, \mathfrak{c}_{\mathrm{nice}})(a_1, a_2, \dots, a_r)$ *is divisible by* $(a_1+1)(a_1+2)\cdots(a_1+r-1)$. *Furthermore,*

$$k(\mathsf{A}_r^+, \mathfrak{c}_{\mathrm{nice}})(0, a_2, \dots, a_r) = k(\mathsf{A}_{r-1}^+, \mathfrak{c}_{\mathrm{nice}})(a_2, \dots, a_r).$$

Proof. The second assertion follows from the general division property (Corollary 14) and from the fact that $a_1 = 0$ is a face F of $C(\mathsf{A}_r^+)$ such that $F \cap \mathsf{A}_r^+ = \mathsf{A}_{r-1}^+$. The first follows from the second. □

Similarly, we have

Lemma 23. *Let* Φ *be a sequence of elements of* A_r^+. *Let* $q^\Phi = (\sum_{k=2}^{r+1} m_{1k})$.

- *The polynomial* $v(\Phi, \mathfrak{c}_{\mathrm{nice}})(a_1, a_2, \dots, a_r)$ *is divisible by* $a_1^{q^\Phi-1}$.
- *The polynomial* $k(\Phi, \mathfrak{c}_{\mathrm{nice}})$ *is divisible by* $(a_1+1)(a_1+2)\cdots(a_1+q^\Phi-1)$.

As explained before, to compute the volume of the polytope $P(\mathsf{A}_r^+, a)$, we need to compute the total residue of the rational function $a^{r(r+1)/2}/\prod_{\alpha\in\mathsf{A}_r^+}\alpha$. In particular, to compute the volume of the Chan–Robbins–Yuen polytope, we need to compute the total residue of this function for $a = e_1 - e_{r+1}$, the highest root of A_r^+. Morris identities (see [3] for a proof) imply the following result.

Theorem 24. *Let $d \geqslant 0$. Then*

$$\operatorname{Tres}\left(\frac{(e_1-e_{r+1})^{rd+r(r-1)/2-r}}{\prod_{i=1}^{r}(e_i-e_{r+1})^d\prod_{1\leqslant i<j\leqslant r}(e_i-e_j)}\right) = \prod_{i=d-1}^{r+d-4}\frac{1}{2i+1}\binom{r+d+i-2}{2i}\Big[\sum_{w\in\Sigma[2,r]}\epsilon(w)w.f_\pi\Big].$$

Here the function f_π is the function

$$f_\pi := \frac{1}{(e_1-e_2)(e_2-e_3)\cdots(e_{r-1}-e_r)(e_r-e_{r+1})}$$

and $w \in \Sigma[2,r] \sim \Sigma_{r-1}$ is the symmetric group operating on the set $[2,3,\ldots,r]$.

For $d=1$, we will also use that

$$\prod_{i=d-1}^{r+d-4}\frac{1}{2i+1}\binom{r+d+i-2}{2i} = \prod_{i=1}^{r-2}\frac{(2i)!}{i!\,(i+1)!}.$$

3. Flow polytopes and applications

In this section, we apply residue formulae to the computation of volumes or number of integral points of specific flow polytopes.

3.1. Some values of the Kostant partition function

Almost by definition, the function $k(\mathsf{A}_r^+,\mathbf{a})$ is given by an iterated constant term.

Lemma 25. *Let $a = a_1e_1 + a_2e_2 + \cdots + a_re_r + a_{r+1}e_{r+1}$, with $\sum_{i=1}^{r+1} a_i = 0$. The value $k(\mathsf{A}_r^+, a)$ is given by the iterated constant term formula*

$$k(\mathsf{A}_r^+, a) = \operatorname{Ct}_{x_1=0}\cdots\operatorname{Ct}_{x_{r+1}=0}\left(\frac{x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r}x_{r+1}^{a_{r+1}}}{\prod_{1\leqslant i<j\leqslant r+1}(1-x_j/x_i)}\right).$$

In this iterated constant term, we start by expanding $1/\prod_{i=1}^{r}(1-x_{r+1}/x_i)$ as a power series in x_{r+1}. The constant term at $x_{r+1}=0$ of $x_{r+1}^{a_{r+1}}/\prod_{1<i\leqslant r}(1-x_{r+1}/x_i)$ is in the ring $\mathbb{Z}[x_1^{-1},x_2,\ldots,x_r^{-1},x_r]$ and we reiterate.

Thus we can obtain some particular values of the Kostant partition function from Morris identities. These remarkable values were observed in [16].

Proposition 26. *Let*

$$a = (1+d)e_1 + (2+d)e_2 + \cdots + (r+d)e_r - (rd + r(r+1)/2)e_{r+1}.$$

Then

$$k(\mathsf{A}_r^+, a) = \prod_{i=d+1}^{r+d-1}\frac{1}{2i+1}\binom{r+d+i+1}{2i}.$$

For $d=0$, we have also

$$k(\mathsf{A}_r^+, a) = \prod_{i=1}^{r}\frac{2i!}{i!\,(i+1)!}.$$

Proof. Recall that $k(\mathsf{A}_r^+, a_1e_1 + a_2e_2 + \cdots + a_re_r + a_{r+1}e_{r+1}) = k(\mathsf{A}_r^+, -a_{r+1}e_1 - a_re_2 - \cdots - a_2e_r - a_1e_{r+1})$. Thus $k(\mathsf{A}_r^+, a)$ is given as the iterated constant term of the function

$$\frac{x_1^{rd+r(r+1)/2}x_2^{-(r+d)}\cdots x_{r+1}^{-(1+d)}}{\prod_{1\leqslant i<j\leqslant r+1}(1-x_j/x_i)} = \frac{x_1^{(r+1)(d+1)+r(r+1)/2}}{x_1^{(1+d)}x_2^{(1+d)}\cdots x_{r+1}^{(1+d)}\prod_{1\leqslant i<j\leqslant r+1}(x_i-x_j)}.$$

The iterated constant term $\mathrm{Ct}_{x_1=0}\cdots\mathrm{Ct}_{x_{r+1}=0}\, f$ of a function f coincide with the iterated residue $\mathrm{Res}_{x_1=0}\cdots\mathrm{Res}_{x_{r+1}=0}(f/x_1x_2\cdots x_{r+1})$. Then we apply Theorem 24 (for the values "r" $= r+1$ and "d" $= d+2$) combined with the fact that $\mathrm{Ires}(f_w) = 0$ if $w \neq 1$. □

3.2. Divisibility property of the volume and of the Kostant partition function

In this subsection, we list some divisibility properties of the polynomial functions $k(\mathsf{A}_r^+, \mathfrak{c}_{\mathrm{nice}})$ and $v(\mathsf{A}_r^+, \mathfrak{c}_{\mathrm{nice}})$. Recall that $a \in V$ is written as $a = a_1e_1 + \cdots + a_re_r - (\sum_{i=1}^r a_i)e_{r+1}$. We also write $a = (a_1, a_2, \ldots, a_r)$. We have already seen that $k(\mathsf{A}_r^+, \mathfrak{c}_{\mathrm{nice}})$ is divisible by $(a_1+1)\cdots(a_1+r-1)$. The following divisibility property is more mysterious.

Proposition 27. *Let Φ be a sequence of N vectors in A_r^+ generating V. Assume $m_{r,r+1} = 1$ and $m_{r-1,r+1} + m_{r-1,r} = 2$. Furthermore, assume that*

$$\frac{m_{j,r+1} + m_{j,r} + m_{j,r-1}}{m_{j,r-1}} = c$$

is independent of j for $1 \leqslant j \leqslant r-2$. Then

- $v(\Phi, \mathfrak{c}_{\mathrm{nice}})(a_1, \ldots, a_{r-1}, a_r)$
$$= \Big(\frac{a_1 + \cdots + a_{r-2}}{c} + a_{r-1}\Big) v(\Phi\backslash(e_{r-1}-e_r), \mathfrak{c}_{\mathrm{nice}})(a_1, a_2, \ldots, a_{r-2}, 0, 0).$$
- $k(\Phi, \mathfrak{c}_{\mathrm{nice}})(a_1, \ldots, a_{r-1}, a_r)$
$$= \Big(\frac{a_1 + \cdots + a_{r-2}}{c} + a_{r-1} + 1\Big) k(\Phi\backslash(e_{r-1}-e_r), \mathfrak{c}_{\mathrm{nice}})(a_1, a_2, \ldots, a_{r-2}, 0, 0).$$

By $\Phi\backslash(e_{r-1} - e_r)$ we mean the set Φ without the element $e_{r-1} - e_r$. As a corollary, we obtain the following result.

Corollary 28 (Schmidt–Bincer [22])**.**

$$\begin{aligned} 3v(\mathsf{A}_r^+, &\mathfrak{c}_{\mathrm{nice}})(a_1, a_2, \ldots, a_r) \\ &= (a_1 + a_2 + \cdots + a_{r-2} + 3a_{r-1}) v(\mathsf{A}_r^+\backslash(e_{r-1}-e_r), \mathfrak{c}_{\mathrm{nice}}) \\ &\quad \times (a_1, a_2, \ldots, a_{r-2}, 0, 0). \\ 3k(\mathsf{A}_r^+, &\mathfrak{c}_{\mathrm{nice}})(a_1, a_2, \ldots, a_{r-1}, a_r) \\ &= (a_1 + a_2 + \cdots + a_{r-2} + 3a_{r-1} + 3) k(\mathsf{A}_r^+\backslash(e_{r-1}-e_r), \mathfrak{c}_{\mathrm{nice}}) \\ &\quad \times (a_1, a_2, \ldots, a_{r-2}, 0, 0). \end{aligned}$$

In particular, the polynomial $v(\mathsf{A}_r^+, \mathfrak{c}_{\text{nice}})(a_1, a_2, \ldots, a_r)$ is of homogeneous degree $r(r-1)/2$, independent of a_r, of degree less than 1 in the variable a_{r-1} and divisible by $a_1^{r-1}(a_1 + a_2 + a_3 + \cdots + a_{r-2} + 3a_{r-1})$.

The polynomial $k(\mathsf{A}_r^+, \mathfrak{c}_{\text{nice}})(a_1, a_2, \ldots, a_r)$ is independent of a_r, of degree less than 1 in the variable a_{r-1} and divisible by $(a_1+1)(a_1+2)\cdots(a_1+r-1)(a_1 + a_2 + a_3 + \cdots + a_{r-2} + 3a_{r-1} + 3)$. Its highest degree component is $v(\mathsf{A}_r^+, \mathfrak{c}_{\text{nice}})$.

We give the proof of the corollary in the case of the Kostant partition function, the proof of Proposition 27 being almost identical. The statement for the Kostant partition function implies the statement for the volume.

Proof. We apply the general formula: $k(\mathsf{A}_r^+, \mathfrak{c}_{\text{nice}})(a) = \langle\langle \mathfrak{c}_{\text{nice}}, K_{\mathsf{A}_r^+}(a)\rangle\rangle$. By Lemma 21, we thus have

$$k(\mathsf{A}_r^+, \mathfrak{c}_{\text{nice}})(a_1, a_2, \ldots, a_r) = \operatorname{Ires}_{x=0} \frac{e^{a_1x_1+\cdots+a_{r-1}x_{r-1}+a_rx_r}}{\prod_{1\leqslant i<j\leqslant r}(1-e^{-(x_i-x_j)})\prod_{1\leqslant i\leqslant r}(1-e^{-x_i})}.$$

We first take the residue in $x_r = 0$ which is a simple pole and obtain

$$\frac{e^{a_1x_1+\cdots+a_{r-1}x_{r-1}}}{\prod_{1\leqslant i<j\leqslant r-1}(1-e^{-(x_i-x_j)})\prod_{1\leqslant i\leqslant r-1}(1-e^{-x_i})^2}.$$

We proceed now to take the residue in $x_{r-1} = 0$. There is a double pole in x_{r-1}, so that the dependence in a_{r-1} is of degree at most 1. More precisely, a simple calculation shows that

$$\begin{aligned}
&\operatorname{Res}_{x_{r-1}=0}\left(\frac{e^{a_1x_1+\cdots+a_{r-1}x_{r-1}}}{\prod_{1\leqslant i<j\leqslant r-1}(1-e^{-(x_i-x_j)})\prod_{1\leqslant i\leqslant r-1}(1-e^{-x_i})^2}\right)\\
&\quad= \frac{e^{a_1x_1+\cdots+a_{r-2}x_{r-2}}}{\prod_{1\leqslant i<j\leqslant r-2}(1-e^{-(x_i-x_j)})\prod_{1\leqslant i\leqslant r-2}(1-e^{-x_i})^3}\left(a_{r-1}+1+\sum_{i=1}^{r-2}\frac{e^{-x_i}}{1-e^{-x_i}}\right)\\
&\quad= \left(a_{r-1}+1+\tfrac{1}{3}(a_1+a_2+\cdots+a_{r-2})\right)\\
&\qquad\times \frac{e^{a_1x_1+\cdots+a_{r-2}x_{r-2}}}{\prod_{1\leqslant i<j\leqslant r-2}(1-e^{-(x_i-x_j)})\prod_{1\leqslant i\leqslant r-2}(1-e^{-x_i})^3}\\
&\qquad- \tfrac{1}{3}(\partial_1+\partial_2+\cdots+\partial_{r-2})\frac{e^{a_1x_1+\cdots+a_{r-2}x_{r-2}}}{\prod_{1\leqslant i\leqslant r-2}(1-e^{-x_i})^3\prod_{1\leqslant i<j\leqslant r-2}(1-e^{-(x_i-x_j)})}.
\end{aligned}$$

As iterated residues vanish on derivatives, we obtain

$$\begin{aligned}
k(\mathsf{A}_r^+,\mathfrak{c}_{\text{nice}})(a_1, a_2, \ldots, a_{r-1}, a_r) &= \left(a_{r-1}+1+\tfrac{1}{3}(a_1+a_2+\cdots+a_{r-2})\right)\\
&\times \operatorname{Res}_{x_1=0}\cdots\operatorname{Res}_{x_{r-2}=0}\frac{e^{a_1x_1+\cdots+a_{r-2}x_{r-2}}}{\prod_{1\leqslant i<j\leqslant r-2}(1-e^{-(x_i-x_j)})\prod_{1\leqslant i\leqslant r-2}(1-e^{-x_i})^3}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&k(\mathsf{A}_r^+ \setminus (e_{r-1}-e_r), \mathfrak{c}_{\text{nice}})(a_1, a_2, \ldots, a_{r-2}, 0, 0)\\
&\quad= \operatorname{Res}_{x_1=0}\cdots\operatorname{Res}_{x_{r-2}=0}\frac{e^{a_1x_1+\cdots+a_{r-2}x_{r-2}}}{\prod_{1\leqslant i<j\leqslant r-2}(1-e^{-(x_i-x_j)})\prod_{1\leqslant i\leqslant r-2}(1-e^{-x_i})^3}
\end{aligned}$$

as the step $x_r = 0$ as well as the step $x_{r-1} = 0$ involves only simple poles, and we obtain the divisibility property announced. □

Example 29. We have

$$k(\mathsf{A}_2^+, \mathfrak{c}_{\text{nice}})(a_1, a_2) = a_1 + 1,$$
$$k(\mathsf{A}_3^+, \mathfrak{c}_{\text{nice}})(a_1, a_2, a_3) = \tfrac{1}{6}(a_1+1)(a_1+2)(a_1+3a_2+3),$$
$$k(\mathsf{A}_4^+, \mathfrak{c}_{\text{nice}})(a_1, a_2, a_3, a_4) = \tfrac{1}{360}(a_1+1)(a_1+2)(a_1+3)(a_1+a_2+3a_3+3) \times (a_1^2+5a_1a_2+9a_1+10(a_2+1)(a_2+2)).$$

3.3. Explicit volumes of some flow polytopes

We now show that the conjectures of Chan, Robbins, and Yuen on their polytope CRY_n and a related polytope are consequences of Theorem 24 and Corollary 28. To state the result in the same way they were originally formulated, we use the relative volume.

Theorem 30 (Zeilberger)**.** *The relative volume of CRY_n is equal to*

$$\prod_{i=1}^{n-2} \frac{1}{i+1}\binom{2i}{i}.$$

Remark 31. The volume of CRY_n is computed by Zeilberger [25], using a formula of Postnikov and Stanley, [23] or of Chan, Robbins, and Yuen [7] for the volume of CRY_n, as a particular value of the Kostant partition function (see Corollary 35).

Proof. The polytope CRY_n is isomorphic to the polytope $P(\mathsf{A}_n^+, (e_1 - e_{n+1}))$ and the linear isomorphism described in Lemma 18 preserves the volume. Since the element $e_1 - e_{n+1}$ is in the closure of $\mathfrak{c}_{\text{nice}}$, we can compute the relative volume of $P(\mathsf{A}_n^+, (e_1 - e_{n+1}))$ as $\langle\langle \mathfrak{c}_{\text{nice}}, \operatorname{Tres} J_n\rangle\rangle$ where

$$J_n = \frac{(e_1 - e_{n+1})^{n(n-1)/2}}{\prod_{1\leqslant i<j\leqslant (n+1)}(e_i - e_j)}.$$

Then

$$\langle\langle \mathfrak{c}_{\text{nice}}, J_n\rangle\rangle = \operatorname{Ires}_{x=0} \operatorname{Tres} J_n = \prod_{i=1}^{n-2} \frac{1}{i+1}\binom{2i}{i},$$

where the last equality follows from Theorem 24 (for $d = 1, r = n$) and the fact that $\operatorname{Ires}_{x=0}$ vanishes on f_w for $w \neq 1$. □

Consider the polytope $CRY_n(\widehat{(2,2)})$, consisting of elements of CRY_n with the entry $(2,2)$ equal to 0. This is a face of codimension 1 of CRY_n, isomorphic to the polytope $P(\mathsf{A}_n^+ \backslash (e_2 - e_3), (e_1 - e_{n+1}))$. Using the automorphism of the Dynkin diagram, and the fact that the dimension of CRY_n is $\binom{n}{2}$, we obtain from Corollary 28 the following corollary. This identity is also observed in [16].

Proposition 32 (Conjecture 4 of [7])**.** *We have*

$$\binom{n}{2} \operatorname{vol_{rel}} CRY_n(\widehat{(2,2)}) = 3 \operatorname{vol_{rel}} CRY_n.$$

3.4. Coefficients of the volume function

We consider the nice chamber $\mathfrak{c}_{\text{nice}}$. We have $a = \sum_{i=1}^r a_i(e_i - e_{r+1})$. As all the elements $e_i - e_{r+1}$ are in the closure of the same chamber, the polytope $P(\Phi, a)$ is the Minkowski sum $\sum_{i=1}^r a_i P(\Phi, e_i - e_{r+1})$. Thus the polynomial $v(\Phi, \mathfrak{c}_{\text{nice}})(a_1, a_2, \ldots, a_r)$ is the mixed volume of the polytopes $P(\Phi, e_i - e_{r+1})$. Using Proposition 12 and formula (5), we write

$$v(\Phi, \mathfrak{c}_{\text{nice}})(a_1, a_2, \ldots, a_r) = \sum_i \frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!} \cdots \frac{a_r^{i_r}}{i_r!} f_{\mathfrak{c}_{\text{nice}}}(\Phi, i), \tag{7}$$

where $f_{\mathfrak{c}_{\text{nice}}}(\Phi, i) = \langle\langle \mathfrak{c}_{\text{nice}}, f_\Phi(i)\rangle\rangle$.

Let Φ' be the subsequence of Φ where we have deleted all roots $e_i - e_{r+1}$. The system Φ' is a sequence of vectors in the positive root system of A_{r-1}^+ included in $V_{r-1} = \{a = \sum_{i=1}^r a_i e_i \mid \sum_{i=1}^r a_i = 0\}$.

Definition 33. Define $t_j^\Phi = m_{j,j+1} + \cdots + m_{j,r+1} - 1,\ 1 \leqslant j \leqslant r$, where $m_{i,j}$ is the multiplicity of the root $e_i - e_j$ in Φ.

Let $|\Phi| = N = \sum m_{ij}$. Remark that if $|i| = i_1 + i_2 + \cdots + i_r = N - r$, then the vector $(i_1 - t_1^\Phi)e_1 + (i_2 - t_2^\Phi)e_2 + \cdots + (i_r - t_r^\Phi)e_r$ is in V_{r-1}.

Proposition 34. *Let Φ be a sequence of elements of A_r^+, then*

$$f_{\mathfrak{c}_{\text{nice}}}(\Phi, i) = k(\Phi', (i_1 - t_1^\Phi)e_1 + (i_2 - t_2^\Phi)e_2 + \cdots + (i_r - t_r^\Phi)e_r).$$

For $\Phi = \mathsf{A}_r^+$, we have

$$f_{\mathfrak{c}_{\text{nice}}}(\mathsf{A}_r^+, (i_1, i_2, \ldots, i_{r-1}, i_r)) = 0 \quad \text{if } i_r > 0$$

and

$$\begin{aligned} &f_{\mathfrak{c}_{\text{nice}}}(\mathsf{A}_r^+, (i_1, i_2, \ldots, i_{r-1}, 0)) \\ &\qquad = k(\mathsf{A}_{r-2}^+, (i_1 - (r-1))e_1 + (i_2 - (r-2))e_2 + \cdots + (i_{r-1} - 1)e_{r-1}). \end{aligned}$$

This formula is stated by Postnikov and Stanley (private communication). For A_r^+, it is due to Lidskii [18].

Proof. We use the iterated residue formula for the volume function $v(\Phi, \mathfrak{c}_{\text{nice}})$. We write $t_i = t_i^\Phi$. Following the proof of Proposition 7, we have

$$\begin{aligned} &f_{\mathfrak{c}_{\text{nice}}}(\Phi, i) \\ &\quad = \operatorname{Res}_{x_1=0} \operatorname{Res}_{x_2=0} \cdots \operatorname{Res}_{x_r=0} \left(\frac{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}}{x_1^{m_{1,r+1}} \cdots x_r^{m_{r,r+1}} \prod_{1 \leqslant i < j \leqslant r} (x_i - x_j)^{m_{ij}}} \right) \\ &\quad = \operatorname{Ct}_{x_1=0} \operatorname{Ct}_{x_2=0} \cdots \operatorname{Ct}_{x_r=0} \left(\frac{x_1^{i_1+1-m_{1,r+1}} x_2^{i_2+1-m_{2,r+1}} \cdots x_r^{i_r+1-m_{r,r+1}}}{\prod_{1 \leqslant i < j \leqslant r} (x_i - x_j)^{m_{ij}}} \right) \\ &\quad = \operatorname{Ct}_{x_1=0} \operatorname{Ct}_{x_2=0} \cdots \operatorname{Ct}_{x_r=0} \left(\frac{x_1^{i_1-t_1} x_2^{i_2-t_2} \cdots x_r^{i_r-t_r}}{\prod_{1 \leqslant i < j \leqslant r} (1 - x_j/x_i)^{m_{ij}}} \right) \\ &\quad = k(\Phi', (i_1 - t_1)e_1 + (i_2 - t_2)e_2 + \cdots + (i_r - t_r)e_r). \end{aligned}$$

Consider now the system $\Phi = \mathsf{A}_r^+$, thus Φ' is A_{r-1}^+. Since $t_j^\Phi = r - i$, we have

$$f_{\mathfrak{c}_{\text{nice}}}(\mathsf{A}_r^+, i) = k(\mathsf{A}_{r-1}^+, (i_1 - (r-1))e_1 + (i_2 + (r-2))e_2 + \cdots + i_{r-1}e_{r-1} + i_r e_r).$$

If $i_r > 0$, the element $(i_1 - (r-1))e_1 + \cdots + i_{r-1}e_{r-1} + i_r e_r \in V_{r-1}$ is not in the cone generated by $(e_i - e_j)$ with $1 \leqslant i < j \leqslant r$, as seen by looking at the component on e_r which would be negative. Furthermore, if $i_r = 0$, we are on a facet isomorphic to $C(\mathsf{A}_{r-2}^+)$ of the cone $C(\mathsf{A}_{r-1}^+)$ so that

$$\begin{aligned} k(\mathsf{A}_{r-1}^+, &(i_1 - (r-1))e_1 + (i_2 + (r-2))e_2 + \cdots + i_{r-1}e_{r-1}) \\ &= k(\mathsf{A}_{r-2}^+, (i_1 - (r-1))e_1 + (i_2 + (r-2))e_2 + \cdots + i_{r-1}e_{r-1}). \end{aligned}$$

Thus we obtain the proposition. □

Consider the polytope $P(\mathsf{A}_r^+, (e_1 - e_{r+1})) = P(\mathsf{A}_r^+, (1, 0, 0, \ldots, 0))$. In the formula for $v(\mathsf{A}_r^+, \mathfrak{c}_{\text{nice}})(1, 0, \ldots, 0)$ in equation (7), the only i to consider is $i_1 = r(r-1)/2, i_2 = 0, \ldots, i_r = 0$. From Proposition 34, and using the symmetry of the Dynkin diagram, we then obtain

Corollary 35 **(**Chan–Robbins–Yuen [7]**).**

$$\operatorname{vol}_{\text{rel}} P(\mathsf{A}_r^+, (e_1 - e_{r+1})) = k(\mathsf{A}_{r-2}^+, (1, 2, 3, 4, \ldots, (r-2))).$$

Similarly, for any sequence Φ, the relative volume of the flow polytope $P(\Phi, (e_1 - e_{r+1}))$ is an integer, given by the partition function $k(\Phi', a)$ at a particular point a. Combinatorists are happy with this result, only if they can explain this by giving an explicit simplicial decomposition of the corresponding flow polytope. This is indeed how Chan, Robbins, and Yuen obtained the equality above.

3.5. The Lidskii formula for the Kostant restricted partition function

In the same spirit as the Lidskii formula for the Kostant partition function [18], we will give a closed formula for $k(\Phi, a)$ in a function of $v(\Phi, a)$. In fact, we will express the S_Δ-valued polynomial $K_\Phi(a)$ which governs the computation of $k(\Phi, a)$, in a function of the polynomial $J_\Phi(a)$ which governs the computation of $v(\Phi, a)$

Recall Definition 33 of the integers t_j^Φ. We first start by giving another expression for the element $K_\Phi(a)$ described in equation (6).

Lemma 36. *For $a \in V_{\mathbb{Z}}$,*

$$K_\Phi(a)(x_1, \ldots, x_r) = \operatorname{Tres}\left(\frac{\prod_{i=1}^r (1 + x_i)^{a_i + t_i^\Phi}}{\prod_{i=1}^r x_i^{m_{i,r+1}} \prod_{1 \leqslant i < j \leqslant r} (x_i - x_j)^{m_{i,j}}} \right).$$

Proof. This formula follows from Proposition 4, by using the change of variables $e^{x_i} \to 1 + x_i$. □

Define

$$s_j^\Phi = 1 - \sum_{k=1}^{j-1} m_{kj} \quad \text{and} \quad \left(\!\!\binom{u}{k}\!\!\right) = \frac{u(u+1)(u+2)\cdots(u+(k-1))}{k!}.$$

Theorem 37. *Let $a \in V$. Then*

$$K_\Phi(a) = \sum_{|\mathbf{i}|=N-r} \binom{a_1+t_1^\Phi}{i_1}\binom{a_2+t_2^\Phi}{i_2}\cdots\binom{a_{r-1}+t_{r-1}^\Phi}{i_{r-1}}\binom{a_r+t_r^\Phi}{i_r} f_\Phi(i)$$
$$= \sum_{|\mathbf{i}|=N-r} \left(\!\binom{a_1+s_1^\Phi}{i_1}\!\right)\cdots\left(\!\binom{a_{r-1}+s_{r-1}^\Phi}{i_{r-1}}\!\right)\left(\!\binom{a_r+s_r^\Phi}{i_r}\!\right) f_\Phi(i).$$

Proof. We use Lemma 36, and write $t_i = t_i^\Phi$. We have

$$K_\Phi(a) = \mathrm{Tres}\left(\frac{(1+x_1)^{a_1+t_1}\cdots(1+x_r)^{a_r+t_r}}{\prod_{\alpha\in\Phi}\alpha(x)}\right).$$

To compute the total residue, as the denominator $\prod_{\alpha\in\Phi}\alpha(x)$ is homogeneous of degree N, we seek the homogeneous term of degree $(N-r)$ in the numerator, thus we seek the coefficient of each term of the form $x_1^{i_1}\cdots x_r^{i_r}$ with $i_1+i_2+\cdots+i_r = N-r$. We obtain the first equality.

The second is proved in the same way, by using in $K_\Phi(a)$ the change of variable $1-e^{-x_i} \to x_i$ and hence computing the total residue of the function

$$\frac{(1+x_1)^{-(a_1+s_1)}\cdots(1+x_r)^{-(a_r+s_r)}}{x_1^{m_{1,r+1}}x_2^{m_{2,r+1}}\cdots x_r^{m_{r,r+1}}\prod_{1\leqslant i<j\leqslant r}(x_i-x_j)^{m_{ij}}}. \qquad \square$$

Applying the linear form $\langle\langle \mathfrak{c}, \bullet\rangle\rangle$ to the equations of Theorem 37, we obtain the fact that the polynomial $k(\Phi,\mathfrak{c})$ is immediately deduced from the polynomial function $v(\Phi,\mathfrak{c})$ (i.e., its highest degree component, see equation (7)) by replacing the monomial $a_k^{i_k}/i_k!$ by the function $\binom{a_k+t_k^\Phi}{i_k}$ (with same leading term).

Theorem 38. *Let $\mathfrak{c}$ be a chamber. Then*

$$v(\Phi,\mathfrak{c})(a_1,a_2,\ldots,a_r) = \sum_{|\mathbf{i}|=N-r} f_\mathfrak{c}(\Phi,i)\frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!}\cdots\frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!}$$

with $f_\mathfrak{c}(\Phi,i) = \langle \mathfrak{c}, f_\Phi(i)\rangle$ and

$$k(\Phi,\mathfrak{c})(a_1,a_2,\ldots,a_r)$$
$$= \sum_{|\mathbf{i}|=N-r} f_\mathfrak{c}(\Phi,i)\binom{a_1+t_1^\Phi}{i_1}\binom{a_2+t_2^\Phi}{i_2}\cdots\binom{a_{r-1}+t_{r-1}^\Phi}{i_{r-1}}\binom{a_r+t_r^\Phi}{i_r}$$
$$= \sum_{|\mathbf{i}|=N-r} f_\mathfrak{c}(\Phi,i)\left(\!\binom{a_1+s_1^\Phi}{i_1}\!\right)\cdots\left(\!\binom{a_{r-1}+s_{r-1}^\Phi}{i_{r-1}}\!\right)\left(\!\binom{a_r+s_r^\Phi}{i_r}\!\right).$$

In particular, we obtain

Proposition 39 (Lidskii)**.** *Write*

$$v(\mathsf{A}_r^+,\mathfrak{c})(a_1,a_2,\ldots,a_r)=\sum_{|i|=\binom{r}{2}} f_{\mathfrak{c}}(i)\frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!}\cdots\frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!}$$

with $f_{\mathfrak{c}}(i)=f_{\mathfrak{c}}(\mathsf{A}_r^+,i)$. *Then*

$$\begin{aligned}k(\mathsf{A}_r^+,\mathfrak{c})&(a_1,a_2,\ldots,a_r)\\ &=\sum_{|i|=\binom{r}{2}} f_{\mathfrak{c}}(i)\binom{a_1+r-1}{i_1}\binom{a_2+r-2}{i_2}\cdots\binom{a_{r-1}+1}{i_{r-1}}\binom{a_r}{i_r}\\ &=\sum_{|i|=\binom{r}{2}} f_{\mathfrak{c}}(i)\left(\!\!\binom{a_1+1}{i_1}\!\!\right)\left(\!\!\binom{a_2}{i_2}\!\!\right)\cdots\left(\!\!\binom{a_{r-1}+3-r}{i_{r-1}}\!\!\right)\left(\!\!\binom{a_r+2-r}{i_r}\!\!\right).\end{aligned}$$

3.6. Volumes and Ehrhart polynomials of the Pitman–Stanley polytope

Consider the Pitman-Stanley polytope $\Pi_r(a)$, for $a_i \geqslant 0$. The polytope $\Pi_r(a)$ is identified to the flow polytope $P(\Phi_{PS},a)$ described in Example 2.

We write as before

$$\operatorname{vol}\Pi_r(a)=\sum_{i_1+i_2+\cdots+i_r=r} f_{\mathfrak{c}_{\text{nice}}}(\Phi_{PS},i)\frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!}\cdots\frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!}.$$

We will see that the coefficients $f_{\mathfrak{c}_{\text{nice}}}(\Phi_{PS},i)$ of this polynomial are 0 or 1.

Proposition 40 (Pitman–Stanley)**.** *Let*

$$K_r=\{(i_1,i_2,i_3,\ldots,i_r)\mid i_1\geqslant 1, i_1+i_2\geqslant 2,\ldots,i_1+i_2+\cdots+i_r=r\}.$$

The volume of the polytope $\Pi_r(a)$ *is given by*

$$\sum_{i\in K_r}\frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!}\cdots\frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!}.$$

The number of integral points in $\Pi_r(a)$ *is given by*

$$\sum_{i\in K_r}\left(\!\!\binom{a_1+1}{i_1}\!\!\right)\left(\!\!\binom{a_2}{i_2}\!\!\right)\cdots\left(\!\!\binom{a_{r-1}}{i_{r-1}}\!\!\right)\left(\!\!\binom{a_r}{i_r}\!\!\right).$$

Proof. We write $\Phi=\Phi_{PS}$. As $a\in\mathfrak{c}_{\text{nice}}$, by Proposition 34,

$$f_{\mathfrak{c}_{\text{nice}}}(\Phi,i)=k(\Phi',(i_1-1)e_1+(i_2-1)e_2+\cdots+(i_r-1)e_r).$$

The system Φ' is the set of simple roots $(e_1-e_2),\ldots,(e_{r-1}-e_r)$. They are linearly independent and generate a simplicial cone $C(\Phi')=C(\mathsf{A}_{r-1}^+)$. Thus the function $k(\Phi',a)$ is identically 1 on the cone $C(\Phi')\cap V_{\mathbb{Z}}$. We obtain $f_{\mathfrak{c}_{\text{nice}}}(\Phi,i)=0$ or 1. More precisely, it is 1 if and only $(i_1-1)e_1+(i_2-1)e_2+\cdots+(i_r-1)e_r$ is in the cone $C(\mathsf{A}_{r-1}^+)$ which is described by the corresponding equation (1). We thus need $(i_1,i_2,i_3,\ldots,i_r)\in K_r$. The formula for the number of points follows from Corollary 38. $\square$

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