Complete Segal spaces (Part III)

1) Homotopical point of view on Segal spaces
2) Why are Segal spaces not enough?
3) Complete Segal spaces and classifying diagram
4) Comparaison between ~~the~~ CSS and QCat model structure

Recall:

category ⇝ Segal space

$\mathcal{C}$ category (horizontal functor)

$[n] \longmapsto c((N\mathcal{C})_n)$ $\quad s\mathcal{S}$

$$sSet \longrightarrow Fun(\Delta^{op}, sSet)$$
$$N\mathcal{C} \longmapsto [n] \mapsto c(N\mathcal{C})_n$$

$$\begin{array}{ccccc} (N\mathcal{C})_0 & \leftrightarrows & (N\mathcal{C})_1 & \equiv & (N\mathcal{C})_2 \\ \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow \\ (N\mathcal{C})_0 & \leftleftarrows & (N\mathcal{C})_1 & \equiv & (N\mathcal{C})_2 \; \cdots \end{array}$$

- objects of $c(N\mathcal{C})$ are objects of $\mathcal{C}$
- composition is well defined not just up to homotopy ($c(N\mathcal{C})_n$ is a set)

space ⇝ Segal space

K Kan complex

$$sSet \longrightarrow s\mathcal{S} := Fun(\Delta^{op}, sSet) \quad \text{where } Map(\Delta^n, K)_m = Hom_{sSet}(\Delta^n \times \Delta^m, K)$$
$$K \longmapsto [n] \to Map(\Delta^n, K)$$

$$\begin{array}{ccccc} Hom_{sSet}(\Delta^0 \times \Delta^0, K) & \leftrightarrows & Hom_{sSet}(\Delta^1 \times \Delta^0, K) & \equiv & Hom_{sSet}(\Delta^2 \times \Delta^0, K) \\ \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\ Hom_{sSet}(\Delta^0 \times \Delta^1, K) & \leftrightarrows & Hom_{sSet}(\Delta^1 \times \Delta^1, K) & \equiv & Hom_{sSet}(\Delta^2 \times \Delta^1, K) \end{array}$$

- Objects of $K^{\Delta[\cdot]}$ are just object of K
- A morphism is a point in $K^{\Delta[\cdot]}$ and so is a point in the path space → path in the space
- Composition of morphism corresponds to concatenation of path in the space

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1) Homotopical point of view on Segal Spaces

In the previous part ~~we~~ → categorical aspect of Segal spaces $\left\{ \begin{array}{l} \cdot \text{ objects} \\ \cdot \text{ morphism} \\ \cdot \text{ composition} \end{array} \right.$

Now we will study → homotopical analogue of $\left\{ \begin{array}{l} \cdot \text{ composition} \\ \cdot \text{ identity rule} \\ \cdot \text{ associativity} \end{array} \right.$

Recall:

Definition

Two maps $f, g : L \to K$ between Kan complexes are called homotopic if there exists a map $H : L \times \Delta^1 \to K$ such that $H|_0 = f$ and $H|_1 = g$

(NB: For Kan complexes it is a equivalence relation).

Definition (K Kan complex)
Two points $x, y : \Delta^0 \to K$ are said homotopic or equivalent if there exist a map $\gamma : \Delta^1 \to K$ such that $\gamma(0) = x$ and $\gamma(1) = y$

Definition
Let x, y two objects of a segal space T (ie $x, y \in T_{0}$). ~~Fo~~ Two morphism f, g in $map_T(x,y)$ are homotopic if $f, g : \Delta^0 \to map_T(x,y)$ are homotopic in $\underbrace{map_T(x,y)}_{\text{Kan complex}}$

Example:
- In $({}^tN\mathcal{C})$ two maps are homotopic iff they are equal to each other.
- In $K^{\Delta[\cdot]}$ two maps are homotopic iff they are homotopic in the usual sense.

Notation:
For every object $x \rightsquigarrow id_x$ the image of x under degeneracy map $s_0 T_0 \to T_1$

→ The composition is associative and has unit up to homotopy

Proposition
Let $f \in map_T(x,y)$, $g \in map_T(y,z)$ and $h \in map_T(z,w)$. Then $h \circ (g \circ f) \sim (h \circ g) \circ f$ and $f \circ id_x \sim id_y \circ f \sim f$.

↳ ~~The compos~~
For T a Segal space we define the homotopy category as follows
→ The objects of $H_0(T)$ are the object of T
→ For x, y in $H_0(T)$, $Hom_{H_0(T)}(x,y) = \pi_0(map_T(x,y))$
in other words it is the set of connected component of $map_T(x,y)$.
(comp. induced by the one of map_T).

Example:
- The homotopy category $H_0({}^tN\mathcal{C})$ is equivalent to $\mathcal{C}$ → All the ~~homotopy~~ cat. property but no homotopical info.
- The homotopy category of $K^{\Delta[\cdot]}$ has objects the points of K and has morphisms homotopy classes of path in K → Fundamental groupoid of K

Definition

Let T be a Segal space. A morphism f in $\mathrm{map}_T(x,y)$ is a homotopy equivalence if there exist g in $\mathrm{map}_T(y,x)$ such that

$$g \circ f \sim id_x \text{ and } f \circ g \sim id_y$$

Example:

- In $t(N\mathcal{C})$ a map is a homotopy equivalence iff it is an isomorphism.
- In $K^{\Delta[\cdot]}$ as every path in a space is reversible, every morphism is an equivalence.

Definition

We define $i : T_{hoequiv} \hookrightarrow T_1$ as the subspace generated by the set of weak equivalences.

Definition

We say that a Segal space is a Segal space groupoid if every map is a homotopy equivalence.

Example:

If K is a Kan complex, $K^{\Delta[\cdot]}$ is a Segal space groupoid.

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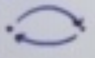
2) Why are Segal spaces not enough

Construction

$I(1)$

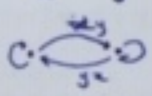


$E_{(1)} = t(N(I(1)))$

$E_{(1)_n}$ (2^n elements)

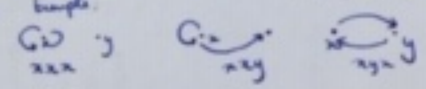
$E_{(1)_0}$ $\cdot x$ $\cdot y$ $c(\{x,y\})$

$E_{(1)_1}$



$c(\{xx, xy, yx, yy\})$

$E_{(1)_2}$ Example



$c(\{xxx, xxy, \ldots\})$

A Segal space has a category theory and an homotopy theory... but they are not compatible with each other.

→ There are object that are equivalent in a Segal space T but not in the space T_0.

In fact in $E(1)$ the two objects x, y are equivalent (in the sense that there is a homotopy equivalence between them) → In discrete simplicial space we saw that equivalence means iso and xy, yx are iso

But they are not equivalent in the space $E_{(1)_0}$ as there is no path between them (constant simp set $\cdot\ \cdot$)

→ The notion of equivalence ~~of~~ for Segal spaces don't coincide with fully faithfull & and essentially surjective (see later on)

→ It does not satisfy the "homotopy hypothesis".

In the second example, K Kan complex ⤳ $\mathcal{S}K^{\Delta[\cdot]}$ Segal groupoid
But the opposite is not true: $E(1)$ Segal groupoid but not equivalent to a space.
*every morphism is an ~~∞~~ equiv.

Remark:

In higher category theory, a higher category $\begin{cases} \nearrow \text{homotopical data} \\ \searrow \text{categorical} \end{cases}$ *discrete
A groupoid every morphism is invertible (ie: non trivial categorical data)
→ should correspond to a space.

3) Complete Segal spaces and classifying diagram

For T a Segal space, the map $s_0 : T_0 \to T_1$ sends objects → identity maps which are homotopy equivalence. So s_0 factors through $T_{hoequiv}$.

Definition

A complete Segal space (CSS) is a Segal space T such that
$$s_0 : T_0 \longrightarrow T_{hoeq},$$
is an equivalence.

Remark:

A morphism between constant simplicial ~~s~~ set is an equivalence of Kan complex iff the underlying map is bijection.

Example (of non ~~S~~CSS)

In the Segal space $E(1)$ we have two objects, $E(1) = c(\{x,y\})$ but four equivalences $\{xx, xy, yx, yy\}$ ⤳ ~~⇒~~ the map from objects to equivalences is ~~clearly~~ not surjective.

Theorem

Let $f : W \to V$ be a map of CSS. The following are equivalent:

1) f is a levelwise equivalence (ie: $f_n : W_n \to V_n$ is an equivalence)

2) f is fully faithfull and essentially surjective

i) <u>Fully faithfull</u>. For any two object x, y in W the induced map of ~~spaces~~ Kan complexes
$$\mathrm{map}_W(x,y) \longrightarrow \mathrm{map}_V(fx, fy)$$ is an equivalence of Kan complex

ii) <u>Essentially surjective</u>: For any object y in V there is an object in W such that fx is equivalent to y in V.

Remark:

It does not always holds in Segal spaces. We have that $^t N(\Delta^\circ) \longrightarrow E(1)$ is fully faithfull and essentially surjective but it is not a levelwise equivalence.
In fact $E(1)_1$ is not levelwise contractible ($* \longrightarrow E(1)_1$ is not induced by a bijection).

Proposition

A CSS W is a CSS groupoid iff W is homotopically constant (ie $W_0 \simeq W_n$)

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Remember it was not true for Segal spaces (with $E(1)$) and goes in the direction of the homotopy hypothesis.

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Build a CSS out of category

$\mathcal{C}$ category $\rightsquigarrow$ $^t(N\mathcal{C})$ is a Segal space but is not always complete ex: $E(1)$

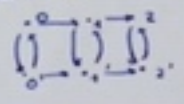
[horizontal embedding functor only consider categorical aspect but ignore the homotopy theory $\to$ no way to get a CSS

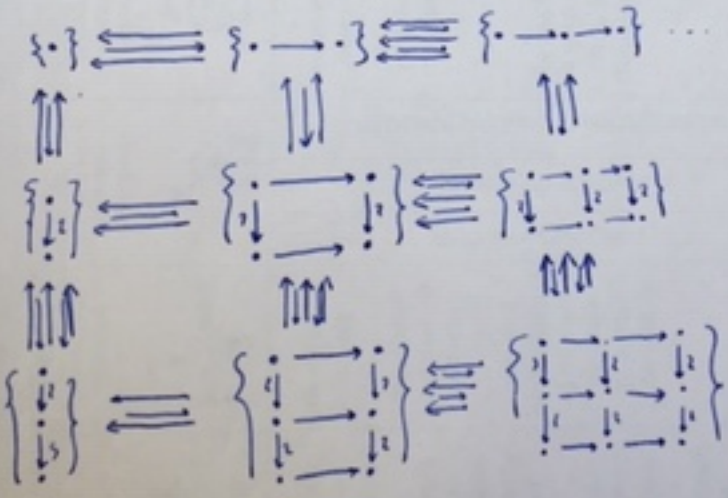
Classifying diagramm $\quad \mathcal{N}(\mathcal{C}) : \Delta^{op} \longrightarrow sSet$ ✓ simplicial space

$[n] \longmapsto [\ell] \mapsto \mathrm{Hom}_{cat}(I(\ell)\times[n], \mathcal{C})$

Notation: $I(n)$ ... n

ex. $[2] \times I(1)$





Theorem

Let $\mathcal{C}$ be a category then $\mathcal{N}(\mathcal{C})$ is a CSS.

4) Comparaison between CSS and QCat model structure

Theorem

There is a simplicial model structure on $s\mathcal{S}$ called CSS model structure, such that

1) Cofibrations are the ~~the~~ monomorphism (so every object is cofibrant)
2) The weak equivalences are the maps f such that ~~Map~~ $\mathrm{Map}(f, W)$ ~~Map~~ is a Kan equivalence for every CSS W.
3) The fibrant object are the complete segal spaces.

Ideas: Localization of the Reedy model structure (with respect to the set of $j^* G(n) \to \Delta^n$ [illegible])

Theorem

A map $f : V \to W$ of Segal space is a CSS equivalence iff it is fully faithfull and essentially surjective.

Properties of CSS model structure

- Cofibrantly generated
- Left Proper (but not right proper)
- Cartesian closed

Conclusion:

* A complete Segal space is a bisimplicial set where
  1) The vertical axis has a homotopical behavior (Reedy fibrancy condition)
  2) The horizontal axis has a categorical behavior (Segal condition)
  3) The two interact well with each other (Completeness condition)

References:

. Bergner's book
. Introduction to complete Segal spaces Nima Rasekh.