Complete Segal Spaces as $(\infty, 1)$-categories

Victor Saunier

M2 Paris

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Preliminaries — Reedy model structure

Recall that Δ is a Reedy category, thus every functor category $\mathrm{Fun}(\Delta^{op}, \mathcal{C})$ where $\mathcal{C}$ itself is a model category, has a model structure, the *Reedy model structure*, which might coincide with either the injective or the projective structure.

Let $Set_{\Delta}^{(2)} = \mathrm{Fun}(\Delta^{op} \times \Delta^{op} \to Set)$. It is equivalent to $\mathrm{Fun}(\Delta^{op}, \mathrm{Fun}(\Delta^{op}, Set))$ in **two different "canonical" ways**, linked by the transposition $\tau : ([n], [m]) \mapsto ([m], [n])$.

Hence, we can consider two model structures, the *vertical Reedy model structure* (abbr. v-fibrant), when seeing bisimplicial sets as vertical simplicial objects in Set_{Δ}, and the *horizontal Reedy model structure* (abbr. h-fibrant), where the picture is transposed.

Preliminaries — Local objects, local maps

Let (C, W, F) be an enriched model structure on $\mathcal{C}$ and S is a set of morphisms, then

Definition

A S-**local object** X of $\mathcal{C}$ is such that every map $s : A \to B$ of S induces

$$s_* : \mathrm{Map}_{\mathcal{C}}(B, X) \to \mathrm{Map}_{\mathcal{C}}(A, X)$$

a weak equivalence[a]

A S-**local map** $u : A \to B$ is such that for every S-local object X, u induces

$$u_* : \mathrm{Map}_{\mathcal{C}}(B, X) \to \mathrm{Map}_{\mathcal{C}}(A, X)$$

a weak equivalence (same footnote)

[a] In fact, an acyclic fibration if the model category is simplicial

Preliminaries — Bousfield localisation

If (C, W, F) is a model structure on $\mathcal{C}$ that is left proper, simplicial and combinatorial[1], and S is a set of morphisms, then

Theorem

There is a model structure on $\mathcal{C}$ that is also left proper, simplicial, combinatorial such that

- C is still the class of cofibrations
- Fibrants objects are S-local objects that were already fibrant
- Weak Equivalences are S-local maps of W, and between (newly) fibrant objects, those are exactly the maps that are in W.

The Reedy model structures we will use are simplicial, proper and combinatorial !

[1] locally presented and cofibrantly generated

Preliminaries — The Model Structure on $Set_{\Delta}^{(2)}$

From the vertical Reedy structure, we deduce that $Set_{\Delta}^{(2)}$ of bisimplicial sets has a model structure, the *Rezk model structure*, obtained by left Bousfield localisation for

$$S = \{I_n \to \Delta^n\} \cup \{E(1) \to *\}$$

It can be characterized as follows:

The Rezk model structure

There is a model structure on $Set_{\Delta}^{(2)}$ whose cofibrations are monomorphisms, fibrant objects complete Segal spaces and weak equivalences maps $A \to B$ that induce weak homotopy equivalences $X^B \to X^A$ for every fibrant X.

Preliminaries — The Quillen equivalence

We want to explain the following, proved by Joyal and Tierney.

Theorem [Joyal, Tierney]

There is a Quillen equivalence between the Rezk model structure on $Set_{\Delta}^{(2)}$ and the Joyal model structure on Set_{Δ} given by

$$p_1^* : Set_{\Delta} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Set_{\Delta}^{(2)} : i_1^*$$

where $p_1 : \Delta \times \Delta \to \Delta$ is the first projection and $i_1 : \Delta \to \Delta \times \Delta$ the injection in the first summand (being constant in $[0]$ in the other).

It's a Quillen adjunction (p_1^* clearly preserves cofibrations, and the other bit is annoying to prove).

Preliminaries — The Tools

We will use the following criterion for Quillen equivalences, which has been proved in the course *Homotopy II* of Najib Idrissi:

Theorem

A Quillen adjunction

$$F : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : G$$

is a Quillen equivalence, if and only if the derived unit $\epsilon_A : A \to GRF(A)$ and $\eta_X : FQG(X) \to X$ are weak equivalences for every A cofibrant and X fibrant, with R the fibrant replacement of $\mathcal{D}$ and Q the cofibrant replacement of $\mathcal{C}$.

We will call the first condition having the homotopy localisation property, and dually, the second is the homotopy colocalisation.

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Homotopy Localisation — The Horizontal Reedy Structure

Let X be a bisimplicial set, we say X is *categorically constant* if the simplicial maps $X_{\bullet,0} \to X_{\bullet,n}$ between the rows (induced by $[n] \to [0]$) are weak categorical equivalences. We have the following theorem (4.5 in [JT]):

Theorem

The Rezk model structure is also a left Bousfield localization of the horizontal Reedy structure. A h-fibrant space is a complete Segal space iff it is a categorically constant. The Rezk weak equivalences are exactly weak categorical equivalences on each row.

Homotopy Localisation

From this new characterization, we can deduce the homotopy localisation property.

Corollary

For any complete Segal space X, the derived counit $\eta_X : p_1^* i_1^*(X) \to X$ is a Rezk weak equivalence.

Row-wise, our map is $X_{\bullet,0} \to X_{\bullet,n}$ (i_1^* takes only the first row and p_1 repeats it on every row). What we are asking is thus for X to be categorically constant. But X is complete Segal space, so it is h-fibrant and categorically constant, thus we have proven our corollary.

In fact, combining some of the result of *supra*, we have almost proven the following:

Corollary

Complete Segal spaces are exactly spaces that are both horizontally and vertically Reedy fibrant.

The $\Longrightarrow$ direction is clear because of the two left Bousfield localisations. The $\Longleftarrow$ sense is a consequence of v-fibrant spaces being categorically constant (which is not too hard to prove), and that latter property being equivalent to being a complete Segal space under the h-fibrancy hypothesis.

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Homotopy Colocalisation - The Property

We turn to the other property, and we aim to show something somewhat stronger:

Proposition

The derived unit $\epsilon_A : A \to i_1^* R p_1^*(A)$ is a weak equivalence for every fibrant-cofibrant A. In fact, $i_1^* R p_1^*(A)$ is canonically isomorphic to A, and the counit map is the identity under this canonical isomorphism.

This time, we have to compute a fibrant replacement, so it will be a bit more technical.

Homotopy Colocalisation - The Γ Functor

In order to compute the fibrant replacement of $p_1^*(A)$, we introduce the following functor:

Definition

Let $\Gamma : Cat_\infty \to Set_\Delta^{(2)}$ be the functor such that $\Gamma(X)_m := \iota(X^{\Delta^m})$, with ι the largest sub-Kan complex functor (aka the *core* of the simplicial set).

where Cat_∞ is the subcategory of Set_Δ of quasicategories.

Homotopy Colocalisation - The Fibrant Replacement

By the definition, we have $i_1^*\Gamma(X) = \Gamma(X)_{\bullet,0} = (\iota(X^{\Delta^\bullet}))_0$, thus $i_1^*\Gamma(X) = (X^{\Delta^\bullet})_0 = X^{\Delta^0}$ is no other than X by Yoneda. Thus, we have a map $X \to i_1^*\Gamma(X)$ (the identity), and we can take its adjoint, and then we have:

Proposition

Let X be an ∞-category, then $\Gamma(X)$ is a complete Segal space, and the natural map $p_1^*(X) \to \Gamma(X)$, obtained by adjunction, is a Rezk weak equivalence. Thus, $\Gamma(X)$ is a fibrant replacement of $p_1^*(X)$.

We won't prove it, since the first part is rather long and technical (but the second is rather accessible).

Homotopy Colocalisation - The "Proof" of the Property

We return to our proposition:

Proposition

The derived unit $\epsilon_A : A \to i_1^* R p_1^*(A)$ is a weak equivalence for every fibrant-cofibrant A. In fact, $i_1^* R p_1^*(A)$ is canonically isomorphic to A, and the counit map is the identity under this canonical isomorphism.

We have shown that $i_1^* \Gamma(X) \to X$ is the identity for X a quasicategory (i.e. a fibrant-cofibrant object), and that $R p_1^*(A)$ could be taken to be $\Gamma(A)$, so the combinaison of those two facts concludes.

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Addendum

We presented here one such Quillen equivalence, but Joyal and Tierney also proved:

Theorem

There is an Quillen equivalence:

$$t_! : Set_\Delta^{(2)} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Set_\Delta : t^!$$

where $t_!$ is the *total space functor*, which is left Kan extension along the Yoneda embedding of the bicosimplicial object in Set_Δ :

$$t : ([m],[n]) \mapsto \Delta^m \times (\Delta')^n$$

with $(\Delta')^n$ is the (nerve of) the groupoid freely generated by $[n]$, and $t^!$ its right adjoint, obtained by Kan adjunction theorem.

where the adjunction is in the other direction.

Bibliography

The exposition given here is solely taken from chosen excerpts of the following paper.

André Joyal, Myles Tierney, *Quasi-categories vs Segal Spaces*, 2006

which can be found on the arxiv:
https://arxiv.org/abs/math/0607820.