Rigidification of quasi-categories

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Introduction

Adjunction

We have the following (Quillen) adjunction between the categorification functor and the coherent nerve functor:

$$\mathfrak{C}\colon \hat{\Delta}_J \rightleftarrows Cat_{\Delta}\colon N^{\Delta}$$

Introduction

For a simplex Δ^n, Lurie defines $\mathfrak{C}(\Delta^n) := \tilde{P}_n$, where $\tilde{P}_n$ is the simplicial category whose objects are $\{0, 1, \ldots, n\}$ and morphisms $\tilde{P}_n(i,j) = N(P_{i,j})$, where $P_{i,j}$ denotes the poset category of subsets of $\{i, i+1, \ldots, j\}$ containing i and j (ordered by inclusion).
For $\mathcal{D} \in Cat_\Delta$, the coherent nerve $N^\Delta(\mathcal{D})$ of $\mathcal{D}$ is then defined as the simplicial set given by

$$[n] \mapsto Cat_\Delta(\mathfrak{C}(\Delta^n), \mathcal{D}).$$

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Since any simplicial set K may be written as $\operatorname{colim}_{\Delta^n \to K} \Delta^n$ (in $\hat{\Delta}$), we see that a left adjoint of N^Δ is given by

$$\mathfrak{C}(K) \cong \operatorname{colim}_{\Delta^n \to K} \mathfrak{C}(\Delta^n).$$

(The colimit is computed in Cat_Δ)

Introduction

The main issue is that finding the mapping spaces of $\mathfrak{C}(K)$ via the definition above is a mess... We therefore need to introduce new notions so these colimits become computable.

Necklaces

Necklaces

A necklace T is a simplicial set of the form

$$T = \Delta^{n_0} \vee \Delta^{n_1} \vee \cdots \vee \Delta^{n_k},$$

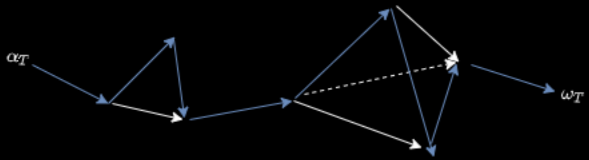
where each $n_i \in \mathbb{Z}_{\geq 0}$ and where each final vertex of Δ^{n_i} has been glued to the initial vertex of Δ^{n_i+1}.
Each Δ^{n_i} is called a bead of T.
A joint is either an initial or final vertex of some bead, the union of which is denoted by J_T. Note that this set is totally ordered. Thus any necklace T comes equipped with a canonical map

$$\partial\Delta^1 \to T$$

sending 0 to the inital vertex α_T of T and 1 to the final vertex ω_T of T.

Necklaces

e.g. $T = \Delta^1 \vee \Delta^2 \vee \Delta^1 \vee \Delta^3 \vee \Delta^1$ (in blue we have $\mathrm{Spi}[T]$, the spine of T)



Necklaces

A spine is a necklace in which every bead is a Δ^1.
Every necklace T has an associated spine denoted $\mathrm{Spi}[T] \hookrightarrow T$.
A simplex is a necklace with one bead.
Every necklace T has an associated simplex $\Delta[T]$ ($T \hookrightarrow \Delta[T]$), namely the simplex with vertex set T_0.
The assignment $T \to \Delta[T]$ is functorial, $T \to \mathrm{Spi}[T]$ is not.

Necklaces

We wish to show that the necklaces form a full subcategory *Nec* of the category $\hat{\Delta}_{*,*} := (\partial\Delta^1 \downarrow \hat{\Delta})$. To do this, we work in greater generality and make use of ordered simplicial sets:
Let $X \in \hat{\Delta}$, for $x, y \in X_0$ we say $x \preceq y$ if there exists a spine T and a map $T \to X$ such that $\alpha_T \mapsto x$ and $\omega_T \mapsto y$. This relation is clearly reflexive and transitive, but not nescesarily antisymmetric.

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Ordered Simplical Set

A simplicial set X is ordered if

- the relation $\preceq$ is antisymmetric;
- an n-simplex in X is determined by its sequence of vertices $x(0) \preceq \cdots \preceq x(n)$, i.e. no two distinct n-simplices have the same vertex sequences.

Necklaces

Simple Inclusion

Let $A, X \in \hat{\Delta}$. A map $A \to X$ is called a simple inclusion if it has the RLP with respect to the canonical inclusion $\partial\Delta^1 \hookrightarrow T$ for all necklaces T.

The idea is that if there is a path in X that starts and ends in A, it must entirely lie in A.

Lemma 1

A simple inclusion $A \hookrightarrow X$ has the RLP with respect to the maps $\partial\Delta^k \hookrightarrow \Delta^k$ for all $k \geq 1$.

Necklaces

Lemma 2

Let X, Y be ordered simplicial sets and $f\colon X \to Y$ a map.

2. Every necklace is an ordered simplicial set.
5. If T is a necklace and $y\colon T \to X$ is map, then the image is a necklace.
6. Suppose that $X \leftarrow A \to Y$ is a diagram of ordered simplicial sets and both $A \to X$ and $A \to Y$ are simples inclusions. Then the pushout $B = X \coprod_A Y$ is an ordered simplicial set and the inclusions $X \hookrightarrow B$ and $Y \hookrightarrow B$ are simple.

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Thanks to Lemma 2 2.+ 5., we see that $Nec \subset \hat{\Delta}_{*,*}$ is a full subcategory.

Let $S, T \in Nec$, then the pushout along the canonical maps $S \leftarrow \partial\Delta^1 \to T$, is denoted by $S \vee T$ and is clearly a necklace again.

Categorification of necklaces

Let T be a necklace, we will now describe $\mathfrak{C}(T)$. The object set is T_0.
For $a, b \in T_0$, let $V_T(a, b)$ denote the set of vertices in T between a and b (included). $J_T(a, b)$ denotes union of $\{a, b\}$ and the set of joints between a and b. There is a unique subnecklace of T (by Lemma 2 2.) with joints $J_T(a, b)$ and vertex set $V_T(a, b)$. Let $\tilde{B}_1, \ldots, \tilde{B}_k$ denote its beads. There are canonical inclusions of the $\tilde{B}_i$'s into T.

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$$\mathfrak{C}(\tilde{B}_k)(j_k, b) \times \mathfrak{C}(\tilde{B}_{k-1})(j_{k-1}, j_k) \times \cdots \times \mathfrak{C}(\tilde{B}_0)(a, j_1) \to \mathfrak{C}(T)(a, b)$$

obtained by first including the $\tilde{B}_i$'s into T and then using the composition in $\mathfrak{C}(T)$.

Categorification of necklaces

Since the description of $\mathfrak{C}(\Delta^n)(-,-)$ is simple, we obtain a description for $\mathfrak{C}(T)(-,-)$:
Let $C_T(a,b)$ denote the poset whose elements are $V_T(a,b)$ which contain $J_T(a,b)$ (ordered by inclusion). The union of sets yields the following pairing:

$$C_T(b,c) \times C_T(a,b) \to C_T(a,c).$$

Applying the (regular) nerve functor, we obtain a simplicial category NC_T with object set T_0. An n-simplex in $NC_T(a,b)$ can be viewed as a flag $\vec{T} := T^0 \subset \cdots \subset T^n$, where $J_T(a,b) \subset T^0$ and $T^n \subset V_T(a,b)$.

Categorification of necklaces

Proposition 3

Let $T \in Nec$. There is a natural isomorphism of simplicial categories $\mathfrak{C}(T)$ and NC_T.

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Idea of the proof:

- $\mathfrak{C}(T) = \mathfrak{C}(B_1) \sqcup \mathfrak{C}(B_2) \sqcup \cdots \sqcup \mathfrak{C}(B_k)$ and $\mathfrak{C}(B_i) = NC_{B_i}$.
- $\vee$ factors through the functors $\mathfrak{C}(B_i) \to NC_{B_i} \to NC_T$ and thus yields a functor $f \colon \mathfrak{C}(T) \to NC_T$.
- This functor is the identity on object, thus one just needs to show that it is full and faithful.
- Build an inverse $g \colon NC_T(a,b) \to \mathfrak{C}(T)(a,b)$, i.e. take an n-simplex, split it into multiple simplices starting and ending in between joints $J_T(a,b)$, push them forward to $\mathfrak{C}(T)(a,b)$ and glue them back together there.

Categorification of necklaces

Corollary 4

Let $T = B_0 \vee \cdots \vee B_k$ be a necklace. Let $a, b \in T_0$ such that $a < b$. Let $j_r, j_{r+1}, \ldots, j_s, j_{s+1}$ be the elements of $J_T(a, b)$ and B_i the bead containing j_i and j_{i+1} for $r \leq i \leq s$. Then the map

$$\mathfrak{C}(B_s)(j_s, j_{s+1}) \times \cdots \times \mathfrak{C}(B_r)(j_r, j_{r+1}) \to \mathfrak{C}(T)(a, b)$$

is an isomorphism. Therefore $\mathfrak{C}(T)(a, b) \cong (\Delta^1)^N$, with $N = |V_T(a, b)| - |J_T(a, b)|$. In particular, it is contractible if $a \leq b$ and empty otherwise.

The categorification functor

Fix $S \in \hat{\Delta}$ and $a, b \in S_0$. For any necklace T and map $T \to S_{a,b}$, there is an induced map $\mathfrak{C}(T)(\alpha_T, \omega_T) \to \mathfrak{C}(S)(a, b)$. We denote by $(Nec \downarrow S)_{a,b}$ the category whose objects are pairs $[T, T \to S_{a,b}]$ and whose morphisms are maps $T \to T'$ yielding a commutative triangle over S. Then we obtain a map

$$E_S(a, b) := \operatorname*{colim}_{T \to S \in (Nec \downarrow S)_{a,b}} [\mathfrak{C}(T)(\alpha_T, \omega_T)] \to \mathfrak{C}(S)(a, b).$$

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We have a composition law on E_s:

$$E_S(b, c) \times E_S(a, b) \to E_S(a, c)$$

induced as follows: let $T \to S_{a,b}$ and $U \to S_{b,c}$, then $T \vee U \to S_{a,c}$. Thus, we have the composite

$$\begin{aligned} \mathfrak{C}(U)(\alpha_U, \omega_U) \times \mathfrak{C}(T)(\alpha_T, \omega_T) &\to \mathfrak{C}(T \vee U)(\omega_T, \omega_U) \times \mathfrak{C}(T \vee U)(\alpha_T, \omega_T) \\ &\to \mathfrak{C}(T \vee U)(\alpha_T, \omega_U) \end{aligned}$$

which induces the pairing.

The categorification functor

Proposition 5

For every simplicial set S, the map $E_S \to \mathfrak{C}(S)$ is an isomorphism of simplicial categories.

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Idea of the proof: playing around with colimits.

The categorification functor

Corollary 6

For any simplicial set S and elements $a, b \in S_0$, the simpicial set $\mathfrak{C}(S)(a,b)$ admits the following description. An n-simplex in $\mathfrak{C}(S)(a,b)$ consists of an equivalence class of triples $[T, T \to S, \vec{T}]$, where

- $T \in Nec$;
- $T \to S$ is a map of simplicial sets, such that $\alpha_T \mapsto a$ and $\omega_T \mapsto b$;
- $\vec{T}$ is a flag $T^0 \subset \cdots \subset T^n$ such that $J_T \subset T^0$ and $T^n \subset V_T$.

The equivalence is generated by maps $f\colon T \to U$ over S such that $\vec{U} = f_*(\vec{T})$.

The i^{th}-boundary means deleting the i^{th}-set in the flag, the degeneracy means copying it.

The categorification functor

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Proof: proven by considering colimits as coequaliser.

Flagged necklaces and totally nondegenerate maps

Flagged necklaces

A flagged necklace is a pair $[T, \overrightarrow{T}]$ where $T \in Nec$ and $\overrightarrow{T}$ is a flag of subsets of V_T containing J_T. Its length is the cardinality of the set of subset symbols.

A flagged necklace is called flanked if $T^0 = J_T$ and $T^n = V_T$.

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Lemma 7

Under the equivalence relation of corollary 6, each triple $[T, T \to S, \vec{T}]$ is equivalent to one in which the flag is flanked. Moreover, two flanked triples are equivalent if they can by connected by a zig-zag of morphisms of flagged necklaces in which every triple of the zig-zag is flanked.

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The proof of this lemma uses a functorial construction called flankification: let $[T, \vec{T}]$ be a flagged necklace, then there is a unique necklace $T' \hookrightarrow T$ whose set of joints is T^0 and vertex set is T^n.

Flagged necklaces and totally nondegenerate maps

Totally nondegenerate map

Let $T \in Nec$ and $S \in \hat{\Delta}$, then a map $T \to S$ is called totally nondegenerate if the image of each bead in T is a nondegenerate simplex of S.

For any necklace T and map $T \to S$, there exists a necklace $\bar{T}$, a totally nondegenerate map $\bar{T} \to S$ and a surjection $T \to \bar{T}$ making the obvious triangle commute.

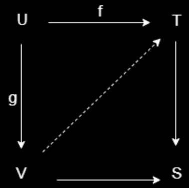
Flagged necklaces and totally nondegenerate maps

Proposition 8

Let $S \in \hat{\Delta}$ and $a, b \in S_0$.

(a) Suppose that T and U are necklaces, $U \to S$ and $T \to S$ two maps and the second one totally nondegenerate. Then there is at most one surjection $U \to T$ making the obvious triangle commute.

(b) Suppose one has



where T, U and V are flagged necklaces, $T \to S$ tot.nondegen. and f, g surjections. Then there is a unique map of flagged necklaces $V \to T$ making the diagram commute.

Flagged necklaces and totally nondegenerate maps

Corollary 9

Let $S \in \hat{\Delta}$ and $a, b \in S_0$. Under the equivalence relation of corollary 6, each triple $[T, T \to S_{a,b}, \vec{T}]$ is equivalent to a unique triple $[U, U \to S_{a,b}, \vec{U}]$ which is both flanked and totally nondegenerate.

Flagged necklaces and totally nondegenerate maps

Corollary 9

Let $S \in \hat{\Delta}$ and $a, b \in S_0$. Under the equivalence relation of corollary 6, each triple $[T, T \to S_{a,b}, \vec{T}]$ is equivalent to a unique triple $[U, U \to S_{a,b}, \vec{U}]$ which is both flanked and totally nondegenerate.

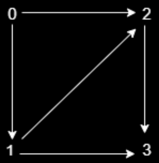
Idea of the proof: via the flankification and tot.nondegen replacement, we know at least one such triple exists. If $[U, U \to S_{a,b}, \vec{U}]$ and $[V, V \to S_{a,b}, \vec{V}]$ are both flanked and tot.nondegen. Then by lemma 7 we have a zig-zag of morphisms of flanked necklaces over S connecting U and V, using proposition 8 (b) on the zig-zag, we obtain two maps $U \to V$ and $V \to U$. By proposition 8 (a), these two maps must be inverses of each other.

An example

If $D \in \hat{\Delta}$ is ordered, then Dugger-Spivak show that to describe $\mathfrak{C}(D)(a, b)$, one can restrict oneself to flanked triples $[T, T \to S, \vec{T}]$ where $T \to S$ in an injection, i.e. T "lives" in D.

An example

Let us take $S = \Delta^2 \sqcup_{\Delta^1} \Delta^2$:



We will compute $\mathfrak{C}(S)(0,3)$. There are five necklaces injecting into $S_{0,3}$, namely $T_{1,2} = \Delta^1 \vee \Delta^1$, $U = \Delta^1 \vee \Delta^1 \vee \Delta^1$, $V = \Delta^1 \vee \Delta^2$ and $W = \Delta^2 \vee \Delta^1$.

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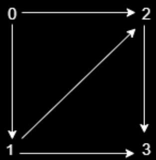

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$\mathfrak{C}(S)(0,3)_0 = \{[T_1, \{0,1,3\}], [T_2, \{0,2,3\}], [U, \{0,1,2,3\}]\}$, others are degeneracies.

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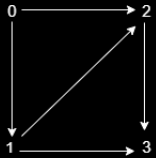

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$\mathfrak{C}(S)(0,3)_1 = \{[V, \{0,1,3\} \subset \{0,1,2,3\}], [W, \{0,2,3\} \subset \{0,1,2,3\}]\}$.

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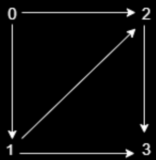

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$\mathfrak{C}(S)(0,3)_1 = \{[V, \{0,1,3\} \subset \{0,1,2,3\}], [W, \{0,2,3\} \subset \{0,1,2,3\}]\}$.
Thus $\mathfrak{C}(S)(0,3)$ looks like $\bullet \leftarrow \bullet \rightarrow \bullet$.

Homotopy theory

We now have a nice description of the mapping spaces, but these are not particularly helpful to do homotopy theory, we will now introduce one type of categorification functor $\mathfrak{C}^{nec}$ which has better homotopic properties and another one $\mathfrak{C}^{hoc}$ which will enable us to create a zig-zag of weak equivalences between $\mathfrak{C}$ and $\mathfrak{C}^{nec}$.

Homotopy theory

For $S \in \hat{\Delta}$, we treat a choice of $a, b \in S_0$ as a map $\partial\Delta^1 \to S$. We denote by $(Nec \downarrow S)_{a,b}$ the overcategory for the inclusion $Nec \hookrightarrow (\partial\Delta^1 \downarrow S)$ and define

$$\mathfrak{C}^{nec}(S)(a, b) := N(Nec \downarrow S)_{a,b}.$$

The other functor is given by the formula

$$\mathfrak{C}^{hoc}(S)(a, b) := \operatorname*{hocolim}_{T \in (Nec \downarrow S)_{a,b}} \mathfrak{C}(T)(\alpha_T, \omega_T).$$

Homotopy theory

Theorem 10

For every simplicial set S, the maps $\mathfrak{C}(S) \leftarrow \mathfrak{C}^{hoc}(S) \rightarrow \mathfrak{C}^{nec}(S)$ are weak equivalences of simplicial categories.

Meta-theory via gadgets

Let $\mathcal{P} \subset \hat{\Delta}_{*,*}$ containing the terminal object. For any $S \in \hat{\Delta}$ and $a, b \in S_0$, let $(\mathcal{P} \downarrow S)_{a,b}$ denote the overcategory whose objects are pairs $[P, P \to S]$, where $P \in \mathcal{P}$ and $P \to S$ such that $\alpha_P \mapsto a$ and $\omega_P \mapsto b$. We define an assignment:

$$\mathfrak{C}^{\mathcal{P}}(S)(a, b) := N(\mathcal{P} \downarrow S)_{a,b}.$$

It is functorial only if $\mathcal{P}$ is "closed under wedges".

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Category of gadgets

We call $\mathcal{G} \subset \hat{\Delta}_{*,*}$ a category of gadgets if it satisfies the following properties:

1. $Nec \subset \mathcal{G}$;
2. $\forall X \in \mathcal{G}\ \forall T \in Nec$, all maps $T \to X$ are in $\mathcal{G}$;
3. $\forall X \in \mathcal{G}$, the simplicial set $\mathfrak{C}(X)(\alpha, \omega)$ is contractible.

The category $\mathcal{G}$ is said to be closed under wedges if it is also true that: for any $X, Y \in \mathcal{G}$, the wedge $X \vee Y$ also belongs to $\mathcal{G}$.

Meta-theory via gadgets

Proposition 11

Let $\mathcal{G}$ be a category of gadgets. Then for any simplicial set S and any $a, b \in S_0$, the natural map

$$\mathfrak{C}^{nec}(S)(a, b) \to \mathfrak{C}^{\mathcal{G}}(S)(a, b)$$

induced by the inclusion $Nec \hookrightarrow \mathcal{G}$, is a Kan equivalence. If $\mathcal{G}$ is closed under wedges, then the map of simplicial categories $\mathfrak{C}^{nec}(S) \to \mathfrak{C}^{\mathcal{G}}(S)$ is a weak equivalence.

Meta-theory via gadgets

Proposition 11

Let $\mathcal{G}$ be a category of gadgets. Then for any simplicial set S and any $a, b \in S_0$, the natural map

$$\mathfrak{C}^{nec}(S)(a, b) \to \mathfrak{C}^{\mathcal{G}}(S)(a, b)$$

induced by the inclusion $Nec \hookrightarrow \mathcal{G}$, is a Kan equivalence. If $\mathcal{G}$ is closed under wedges, then the map of simplicial categories $\mathfrak{C}^{nec}(S) \to \mathfrak{C}^{\mathcal{G}}(S)$ is a weak equivalence.

The proof relies on Qillen's theorem A, saying that if the overcategories of $Nec \hookrightarrow \mathcal{G}$ are contractible, the induced map is a Kan-equivalence. This is the case by the assumptions on $\mathcal{G}$ and theorem 10.

Properties of categorification

Proposition 12

For any simplicial sets X and Y, both $\mathfrak{C}(X \times Y)$ and $\mathfrak{C}(X) \times \mathfrak{C}(Y)$ are simplicial categories with object set $X_0 \times Y_0$. For any $a_0, b_0 \in X_0$ and $a_1, b_1 \in Y_0$, the natural map

$$\mathfrak{C}(X \times Y)((a_0, a_1), (b_0, b_1)) \to \mathfrak{C}(X)(a_0, b_0) \times \mathfrak{C}(Y)(a_1, b_1),$$

induced by the projections, is a Kan equivalence. Therefore, the map of simplicial categories

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Idea of the proof: take $\mathcal{G}$ the category of objects containing finite product of necklaces, show that it is a category of gadgets. Then use theorem 10 and proposition 11 to work with $\mathfrak{C}^{\mathcal{G}}$ instead of $\mathfrak{C}$.

It has been a long battle...

It has been a long battle...
By now, most of you are probably as lost as the guy in the middle:

La Liberté guidant le peuple • Eugène Delacroix • 1830

But anyway...

Thank you for sticking with me until the end !

Any questions ?

Extras

Lemma 2

Let X, Y be ordered simplicial sets and $f\colon X \to Y$ a map.

1. The category of ordered simplicial sets is closed under finite limits.
2. Every necklace is an ordered simplicial set.
3. If $X' \subset X$, then X' is ordered.
4. The map f is entirely determined by $f_0\colon X_0 \to Y_0$ on vertices.
5. If f_0 is injective, so is f.
6. The image of an n-simplex $x\colon \Delta^n \to X$ is of the form $\Delta^k \hookrightarrow X$ for some $k \leq n$.
7. If T is a necklace and $y\colon T \to X$ is map, then the image is a necklace.
8. Suppose that $X \leftarrow A \to Y$ is a diagram of ordered simplicial sets and both $A \to X$ and $A \to Y$ are simples inclusions. Then the pushout $B = X \coprod_A Y$ is an ordered simplicial set and the inclusions $X \hookrightarrow B$ and $Y \hookrightarrow B$ are simple.