HIGHER CATEGORIES

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These few notes are intended to present the progression of the course, some references and exercises.

Lecture 1 (9th march 2022). Introduction and presentation of the course; mostly two models of $(\infty, 1)$-categories (quasicategories and simplicial categories) and their comparison.

There are other models, and we refer to the book of J. Bergner, *The homotopy Theory of* $(\infty, 1)$*-categories*, for the models presented by Segal categories and complete Segal spaces. There are other models developped by Barwick and Kan in *Relative categories: another model for the homotopy theory of homotopy theories*. If one would like to understand theory of (∞, n)-categories, the paper by D. Ara, *Higher quasi-categories vs Higher Rezk spaces*, explains generalization of quasicategories and complete Segal spaces and proves the Quillen equivalence between them. Some of them will be explored during the course.

Note on the bibliography:

On model categories– Hirschorn, *Localization of model categories*

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Chapter 1: Quasicategories

This chapter follows indifferently the following references: D.C. Cisinski, *Higher categories and homotopical algebra*, A. Joyal, *The Theory of quasicategories and its applications*, J. Lurie, *Higher topos theory.*

1. Reminder

1.1. **Presheaves.** We refer to section 1.1 of Cisinski.

1.2. **Localisation.** We recall here the meaning of localization and take benefit of it to talk about equivalence of categories.

Exercise 1.2.1. Let $F : \mathbf{C} \to \mathcal{D} : G$ be an adjunction, F being the left adjoint.

(1) Show that G is fully faithful if and only if the counit of the adjunction is an isomorphism.
(2) Show F is fully faithful if and only if the unit of the adjunction is an isomorphism.
(3) Deduce that the adjunction is an equivalence of categories if and only if F is fully faithful and essentially surjective.

1.3. **Model categories.** Presentation using weak factorization systems (see E. Riehl, *A concise definition of a model category*).
Recollection from the course by Idrissi *Homotopy 2*: homotopy category of a model category $\mathcal{C}$ denoted $Ho(\mathcal{C})$, Whitehead Theorem and its corollary: a map f in $\mathcal{C}$ is a weak equivalence if and only if $Ho(f)$ is an isomorphism.

A useful Proposition (due to Dugger) concerning Quillen adjunction

Proposition 1.3.1 (Hirschorn 8.6.3)**.** *An adjunction (F, G) is a Quillen adjunction if and only if one of the following holds:*

(1) *F preserves cofibrations between cofibrant objects and all acyclic cofibrations;*
(2) *G preserves fibrations between fibrant objects and all acyclic fibrations;*
(3) *F preserves cofibrations between cofibrant objects and G preserves fibrations;*
(4) *G preserves fibrations between fibrant objects and F preserves cofibrations.*

Example 1.3.2. Let $\hat{\mathbf{\Delta}}$ be the category of simplicial sets. These are examples of model category structures:

- Kan-Quillen on $\hat{\mathbf{\Delta}}$: mono, we, Kan fibration
- Quillen on **Top**: retract of cell attachments, we, Serre fibrations.
- Joyal-Tierney on **Cat**: mono, equivalences of categories, isofibrations.

Exercise 1.3.3. (1) Show that the notion of isofibration is self-dual: a functor $F : \mathbf{X} \to \mathbf{Y}$ is an isofibration iff the opposite functor $F^o : \mathbf{X}^o \to \mathbf{Y}^o$ is an isofibration.
(2) Show that a functor $F : \mathbf{X} \to \mathbf{Y}$ is an isofibration iff it has the right lifting property with respect to the inclusion $\{0\} \subset J$, where J is the groupoid generated by one isomorphism $0 \cong 1$

Lecture 2 (11th march 2022).

1.4. **Simplicial sets.** Notation: $\Delta^n, \partial\Delta^n, \Lambda^n_k$; recollection on Kan fibrations, Kan complexes.

The constant functor $\mathbf{Set} \to \hat{\mathbf{\Delta}}$ admits a left adjoint denoted by τ_0:

$$\pi_0(X) = \text{ coeq}(X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0)$$

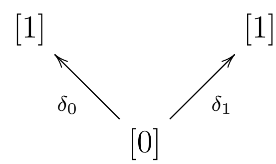
When X is a Kan complex, $\pi_0(X) = X_0/\sim$ where $x \sim y$ if and only if there exists $f \in X_1$ such that $d_1 f = x; d_0 f = y$, is an equivalence relation. The constant functor has a right adjoint given by $X \mapsto X_0$.

1.5. **Nerve of a category.** (see Section 1.4 of Cisinski)

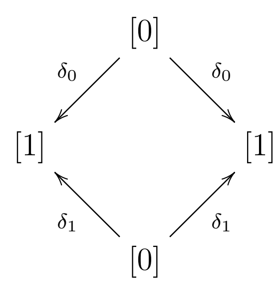
Definition of the nerve functor: $\hat{\mathbf{\Delta}} \to \mathbf{Cat}$. It admits a left adjoint τ_1 defined by left Kan extension along the Yoneda functor of the functor $\mathbf{\Delta} \to \mathbf{Cat}$ which associated to $[n]$ the Poset category $0 < 1 \ldots < n$. τ_1 is not easy to describe since it is not easy to describe colimits of categories (see exercise 1.5.1). Statement of the nerve theorem (see Proposition 1.4.11 in Cisinski's book).

Exercise 1.5.1. We denote by $[n]$ the category (poset) $0 < 1 < \ldots < n$. We denote by $\delta_i : [n-1] \to [n]$ the functor that "misses" i.

(1) Show that the colimit of the following diagramme is [2]:



(2) What is the colimit of the diagramme:



(3) Show that $\tau_1(\Lambda^3_1) = \tau_1(\partial\Delta^3) = \tau_1(\Lambda^3_2) = [3]$.

Exercise 1.5.2. Let X be a topological space. Compute $\tau_1(Sing(X))$.

Exercise 1.5.3. Let $\mathcal{C}$ be a category. Show that the nerve of $\mathcal{C}$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

2. Quasicategories

2.1. **First definitions.** Where we define quasicategories (see definition 1.5.1 of Cisinski's). Definition of the opposite of a simplicial set, denoted X^{op}.

Exercise 2.1.1. Show that $N(\mathcal{C})^{op}$ is isomorphic to $N(\mathcal{C}^{op})$. Show that X is a quasicategory if and only if X^{op} is.

2.2. **Weak composition and homotopy category associated to a quasicategory.** Notation: let X be a simplicial set, $a, b \in X_0$. We define $X_1(a,b)$ as the pullback (in sets) of the diagramme

$$\begin{array}{ccc} & & X_1 \\ & & \downarrow{\scriptstyle (d_1,d_0)} \\ * & \xrightarrow[(a,b)]{} & X_0 \times X_0 \end{array}$$

We define a relation on $X_1(a,b)$ (defined by Boardman-Vogt) and prove that it is an equivalence relation and that it endows hX with a structure of category. hX is called the homotopy category associated to X or its fundamental category. (see Section 1.6 of Cisinski's book).

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We prove that that category hX is isomorphic to the category $\tau_1(X)$. For $f \in X_1(a,b)$ we denote by $[f]$ its image in in $hX(a,b)$.

Remark 2.2.1. Let X be a quasicategory and f, g, k in X_1. The equality $[k] = [g] \circ [f]$ in hX amounts to the existence of a 2-cell σ in X_2 such that $d_0\sigma = g, d_1\sigma = k, d_2\sigma = f$.

Definition 2.2.2. Let X be a quasicategory.

(1) An element f in X_1 is *invertible* (or is an equivalence) if $[f]$ is an isomorphism in hX.
(2) A quasicategory is an ∞*-groupoid if every element $f \in X_1$ is invertible, or equivalently if hX is a groupoid.*

Example: a Kan complex is an ∞-groupoid. If X is a topological space, then $Sing(X)$ is a Kan complex and $hSing(X)$ is the fundamental groupoid of X.

Exercise 2.2.3. Let X be a simplicial set having the right lifting property with respect to inner horns inclusion $\Lambda^n_k \to \Delta^n$ for $n = 2$ and $n = 3$. Show that the relation defined by Boardmann Vogt is an equivalence relation, that it induces a structure of category on hX and that hX is isomorphic to $\tau_1(X)$.

2.3. **The category of quasicategories.**

2.3.1. *Natural transformations.* Let X, Y be two quasicategories. A morphism of simplicial sets $f : X \to Y$ is called a *functor*. We denote by **Qcat** the category whose objects are quasicategories and morphisms are functors, that is the full subcategory of quasicategories in $\hat{\boldsymbol{\Delta}}$.

Definition 2.3.1. Let X, Y be two quasicategories. A morphism of simplicial sets $H : X \times \Delta^1 \to Y$ is called a natural transformation from $f = H \circ (1_X \times \delta_1)$ to $g = H \circ (1_X \times \delta_0)$. We say that the natural transformation H is *invertible* if for every $x \in X_0$ the induced 1-simplex $f(x) \to g(x) \in Y_1$ is invertible (that is its image in hY is an isomorphism). We say that two quasicategories X, Y are *equivalent* if there exist functors $F : X \to Y$ and $G : Y \to X$ and invertible natural transformations from GF to the identity and from the identity to FG.

Exercise 2.3.2. Let $\mathcal{C}, \mathcal{D}$ be categories. Let $[1]$ be the category $0 < 1$. Let δ_0(resp. δ_1) : $* \to [1]$ denote the functors sending $*$ to 1 (resp. 0). Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors.

(1) Show that a natural transformation α from F to G is an isomorphism (that is there is a natural transformation β from G to F such that $\alpha\circ\beta$ and $\beta\circ\alpha$ are the identity natural transformations) if and only if for every object c of $\mathcal{C}$ the morphism $\alpha_c : Fc \to Gc$ in $\mathcal{D}$ is an isomorphism.
(2) Show that given a natural transformation τ from F to G is equivalent to given $H : \mathcal{C} \times [1] \to \mathcal{D}$ such that $H \circ (1_X \times \delta_0) = G$ and $H \circ (1_X \times \delta_1) = F$.
(3) Deduce that τ is a natural transformation (in the classical sense of categories) if and only if $N\tau$ is a natural transformation (in the sense of quasicategories).
(4) Prove that τ is an isomorphism if and only if $N\tau$ is invertible.
(5) Deduce that $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if $N\mathcal{C}$ and $N\mathcal{D}$ are.

2.3.2. *The category* **Qcat** *is cartesian closed.* For X, Y simplicial sets we denote by Y^X the simplicial set defined by $(Y^X)_n = \mathrm{Hom}_{\hat{\mathbf{\Delta}}}(X \times \Delta^n, Y)$. We wil show later that if Y is a quasicategory then Y^X is.

Exercise 2.3.3. Show that for $\mathcal{C}, \mathcal{D}$ categories, if we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the functor category, then $N\mathcal{D}^{N\mathcal{C}} = N(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$. Show that if X is a simplicial set we have $(N\mathcal{D})^X$ is isomorphic to $N(\mathrm{Fun}(\tau_1(X), \mathcal{D}))$ (providing that we know that τ_1 commutes with finite products).

Exercise 2.3.4. Let X, Y be quasicategories; we assume that Y^X is a quasicategory. Show that if $H \in (Y^X)_1$ is invertible, then H is an invertible natural transformation as defined in Definition 2.3.1. Can you prove the converse?

Let X be a simplicial set. Define $X(a, b)$ to be the pullback of the diagram

$$\begin{array}{ccc} & & X^{\Delta^1} \\ & & \downarrow \\ * & \xrightarrow[(a,b)]{} & X \times X \end{array}$$

We have that $X(a, b)_0 = X_1(a, b)$ and if X is a quasicategory the relation defined on $X(a, b)$ by $f \simeq g$ if there exists $H \in X(a, b)_1$ such that $d_0 H = g$ and $d_1 H = f$ is equivalent to the relation defined on $X_1(a, b)$ in Section 2.2. Hence $\pi_0(X(a, b)) = hX(a, b)$.

For $f, g \in X(a, b)_0$ we define $Comp(f, g)$ as the pullback of the diagram

$$\begin{array}{ccc} & & X^{\Delta^2} \\ & & \downarrow \\ * & \xrightarrow[(f,g)]{} & X^{\Lambda^2_1} \end{array}$$

We will see later that $Comp(f, g)$ is a contractible Kan complex.

2.3.3. *Limits and colimits in* **QCat**. We have proved that **QCat** is stable under products, coproducts and filtered colimits.

Exercise 2.3.5. Show that Λ^2_1 is not a quasicategory. Explain it provides a counter example to the stability of **QCat** under finite colimits.

Lecture 4 (Friday 18th of march)

2.4. **Inner anodyne extensions and inner fibrations.** Notation: for S a set of monomorphisms in $\hat{\mathbf{\Delta}}$ we denote by $\bar{S}$ the smallest saturated class of morphisms containing S. We recall that $(\bar{S}, (S)^{\boxslash})$ is a weak factorization system. Let $\mathcal{M}_{in} = \{\Lambda^n_k \to \Delta^n\}_{n\geq 2, 0<k<n}$. An inner anodyne extension is a map in $\overline{\mathcal{M}_{in}}$ and an inner fibration is a map in $(\mathcal{M}_{in})^{\boxslash})$.

Proposition 2.4.1. *Every functor $X \to NC$ with X a quasicategory and C a category is an inner fibration.*

For $i : X \to Y$, $j : A \to B$ and $f : U \to V$ we recall the notation:

$$i\Box j : X \times B \cup_{X\times A} Y \times A \to Y \times B$$
$$f^{\Box j} : U^Y \to V^Y \times_{V^X} U^X.$$

Proposition 2.4.2. *We have* $\overline{\mathcal{M}_{in}} = \overline{\{(\Lambda^2_1 \to \Delta^2)\Box(\partial\Delta^n \to \Delta^n)\}_{n\geq 0}}$

Theorem 2.4.3. *If i is an inner anodyne extension and j is a monomorphism, then $i\Box j$ is an inner anodyne extension.*
If j is a mono and f is an inner fibration then $f^{\Box j}$ is an inner fibration.
If i is an inner anodyne extension and f is an inner fibration, then $f^{\Box i}$ is a trivial (Kan) fibration.

Corollary 2.4.4. *For K a simplicial set and X a quasicategory (resp. Kan complex) une quasicategory (resp complexe de Kan) alors X^K is a quasicategory (resp. Kan complex) .*

Corollary 2.4.5. *Let X be a simplicial set. The following propositions are equivalent.*

(1) *$X$is a quasicategory*
(2) *$X^{\Delta^2} \to X^{\Lambda^2_1}$ is a trivial (Kan) fibration.*
(3) *$\forall n \geq 2$, $X^{\Delta^n} \to X^{I_n}$ is a trivial (Kan) fibration.*

In particular comp(f, g) is contractible.

This corollary has been proved, except for the fact that the inclucion $I_n \to \Delta^n$ is an inner anodyne extension. The following exercise will help to finish the proof.

Exercise 2.4.6. We prove by induction on n, that the spine inclusion $I_n \to \Delta^n$ is an inner anodyne extension. It is clearly true for $n = 2$. Assume that $I_n \to \Delta^n$ is an inner anodyne extension.

(1) Show that $I_{n+1} \to I_{n+1} \cup \partial_0\Delta^{n+1} \cup \partial_{n+1}\Delta^{n+1}$ is an inner anodyne extension.
(2) Show that for all $1 \leq i \leq n-1$, $I_{n+1} \to I_{n+1} \cup (\cup^i_{j=0}\partial_j\Delta^{n+1}) \cup \partial_{n+1}\Delta^{n+1}$ is an inner anodyne extension.
(3) Conclude.

Exercise 2.4.7. Find a simplicial set that has the right lifting property with respect to all spine inclusions, but which is not a quasicategory. Hint: show that the pushout S of the diagram

$$\begin{array}{ccc} I_3 & \longrightarrow & \partial\Delta^3 \\ \downarrow & & \\ \Delta^3 & & \end{array}$$

is not a quasicategory but has the right lifting property with respects to all spine inclusions.

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Exercise 2.4.8. Let $p : X \to Y$ be a map between simplicial sets. Show that the following assertions are equivalent

1. p is an inner fibration.
2. For every inner anodyne extension i, $p^{\Box i}$ is a trivial fibration.
3. For every monomomorphism j, $p^{\Box j}$ is an inner fibration
4. For $i = \Lambda^2_1 \to \Delta^2$, $p^{\Box i}$ is a trivial fibration.

Proposition 2.4.9. *If $j : X \to Y$ is an inner anodyne extension then $\tau_1(j)$ is an equivalence of categories.*

2.5. **Equivalences.**

Proposition 2.5.1. *The functor τ_1 preserves finite products.*

We have defined: the core of a category (right adjoint to the inclusion functor from groupoids to categories);

Exercise 2.5.2. For a given quasicategory X and a given subcategory S of hX we have defined S' as the pullback of the inclusion $S \to hX$ along the unit of the adjunction $X \to NhX$: the subquasicategory of X generated by S'. Show that hS' is isomorphic to S.

This permits us to define $\text{Core}(X)$ has the subquasicategory of X generated by $\text{Core}(hX)$.

Exercise 2.5.3. Show that Core defines a functor $\mathbf{QCat} \to \infty - \mathbf{gpd}$, right adjoint to the inclusion functor.

We define τ_0 as the composite $\pi_0 \circ \text{Core} \circ \tau_1$, namely on objects $\tau_0(X)$ is the isomorphism classes of objects in $\tau_1(X)$. We proved that τ_0 preserves finite products.

Following Joyal notation, let S^{τ_0} be the category whose objects are simplicial sets and given two simplicial sets X, Y

$$S^{\tau_0}(X, Y) = \tau_0(Y^X).$$

We have proved that this forms a category and that there is a well defined functor $\tau_0 : \hat{\mathbf{\Delta}} \to S^{\tau_0}$

Definition 2.5.4. A morphism $f : X \to Y$ of simplicial sets is a categorical equivalence if $\tau_0(f)$ is an isomorphism in S^{τ_0}.

We explained that if X, Y are quasicategories then this amounts to having a map $g : Y \to X$ and $H \in (X^X)_1$ and $K \in (Y^Y)_1$ invertible so that H is a homotopy from fg to id_X and K a homotopy from fg to id_Y.

Exercise 2.5.5. We recall that two maps $f, g : X \to Y$ are homotopic if there exists $H : X \times \Delta^1 \to Y$ such that $H|_{X\times\{0\}} = f, H|_{X\times\{1\}} = g$. Recall that this relation is not necessarily an equivalence relation.

1. Recall why it is an equivalence relation when Y^X is a Kan complex.
2. Deduce that if X and Y are Kan complexes, then $f : X \to Y$ is a categorical equivalence if and only if it is a homotopy equivalence.

Example 2.5.6. Any trivial fibration between quasicategories is a categorical equivalence. Any functor $f : C \to D$ between two categories is an equivalence of category if and only if Nf is a categorical equivalence.

Proposition 2.5.7. *Let $f : X \to Y$ be a categorical equivalence between two quasicategories. We have:*

(1) *$hf : hX \to hY$ is an equivalence of categories.*
(2) *$\tau_0 f : \tau_0 X \to \tau_0 Y$ is a bijection.*

Exercise 2.5.8. Let $f : X \to Y$ be a morphism of simplicial sets. Show that

(1) If f is a trivial fibration, then f is a categorical equivalence. (check that the proof we presented for X, Y quasicategories, work the same.
(2) If f is a categorical equivalence, then $\tau_1(f)$ is an equivalence of categories.

Definition 2.5.9. Let $f : X \to Y$ be a map of simplicial sets. We say that f is a weak categorical equivalence if for every quasicategory Z, $\tau_0(f, Z) : S^{\tau_0}(Y, Z) \to S^{\tau_0}(X, Z)$ is a bijection.

Theorem 2.5.10. *Let $f : X \to Y$ be a map of simplicial sets. TFAE*

(1) *for every quasicategory Z, $Z^{\square f} = f^* : Z^Y \to Z^X$ is a categorical equivalence*
(2) *for every quasicategory Z, $h(f^*) : h(Z^Y) \to h(Z^X)$ is an equivalence of categories.*
(3) *for every quasicategory Z, $\mathrm{Core}(f^*) : \mathrm{Core}(Z^Y) \to \mathrm{Core}(Z^X)$ is a homotopy equivalence*
(4) *f is a weak categorical equivalence.*

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Exercise 2.5.11. Prove (1) $\Rightarrow$ (3).

Example 2.5.12. Any inner anodyne extension is a weak categorical equivalence.
Any categorical equivalence between quasicategories is a weak categorical equivalence (and conversely).

2.6. **Isofibrations.**

Definition 2.6.1. A functor $F : C \to D$ between quasicategories is an isofibration if

- F is an inner fibration
- $hF : hC \to hD$ is an isofibration of categories

Example 2.6.2. For X a quasicategory, the unique map $p : X \to *$ is an isofibration.

Definition 2.6.3. A functor $F : C \to D$ between quasicategories is conservative if for every $f \in C_1$, if Ff is invertible so is f.

Exercise 2.6.4. Show that $F : C \to D$ is an isofibration if and only if F^{op} is an isofibration.

Example 2.6.5. Left and right fibrations are conservative isofibrations.

Theorem 2.6.6 (Joyal lifting lemma)**.** *Let $p : X \to Y$ be an inner fibration between quasicategories and let $f : x \mapsto x'$ in X_1 such that pf is invertible in Y_1. TFAE:*

(1) *f is invertible*
(2) *every map from $\Lambda^n_0 \to \Delta^n$ to p where the initial arrow is invertible admits a lift.*
(3) *every map from $\Lambda^n_n \to \Delta^n$ to p where the terminal arrow is invertible admits a lift.*

Corollary 2.6.7. *Let X be a quasicategory. X is a Kan complex if and only if hX is a groupoid.*

Corollary 2.6.8. *Let X be a quasicategory and let $f \in X_1$, that is, $f : \Delta^1 \to X$. f is invertible if and only if f extends to a morphism $J \to X$, where J is the groupoid generated by $[1]$.*

Corollary 2.6.9. *Let $p : X \to Y$ be an inner fibration between quasicategories. TFAE*

(1) *p is an isofibration*
(2) *p has the RLP with respect to $\{0\} \to J$.*
(3) *$Core(p)$ is a Kan fibration between Kan complexes.*

Proposition 2.6.10. *Let $p : X \to Y$ be an isofibration between quasicategories. For every monomorphism $i : K \to L$, $p^{\square i}$ is an isofibration. Same result holds if p is an inner fibration and $i_0 : K_0 \to L_0$ is a bijection. For such a monomorphism, if X is a quasicategory then $i^* : X^L \to X^K$ is a conservative isofibration.*

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Exercise 2.6.11. Show that conservative isofibrations are preserved by pullbacks

Corollary 2.6.12. *Let $F, G : X \to Y$ be two functors between quasicategories and $\eta : X \times \Delta^1 \to Y$ be a natural transformation between them. η is invertible in $(Y^X)_1$ if and only if $\eta_x : F(x) \to G(x)$ is invertible in Y_1.*
If X is a quasicategory, then for every $x, y \in X_0$ the simplicial set $Map_X(x, y)$ is a Kan complex.

Corollary 2.6.13. *Let $p : X \to Y$ be a conservative isofibration between quasicategories. We have*

(1) *The fibers of p are Kan complexes.*
(2) *If Y is a Kan complex then p is a Kan fibration.*

Exercise 2.6.14. Let $p : X \to Y$ is be a functor between quasicategories. Show that p is conservative if and only if we have a cartesian square in simplicial sets

$$\begin{array}{ccc} \mathrm{Core}(X) & \longrightarrow & X \\ {\scriptstyle \mathrm{Core}(p)}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Core}(Y) & \longrightarrow & Y \end{array}$$

Exercise 2.6.15. Let $p : X \to Y$ is be a functor between quasicategories. Let $f : B \to Y$ be a functor of quasicategories. Let $A = B \times_Y X$ be the pullback of f along p in simplicial sets, namely we have the cartesian square $(\mathcal{D})$ in $\hat{\mathbf{\Delta}}$:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[f]{} & Y \end{array}$$

(1) Explain why $(\mathcal{D})$ is not necessarily a cartesian square in **QCat**.
(2) Show that if p is an inner fibration then $(\mathcal{D})$ is a cartesian square in **QCat**.

(3) In case p is an inner fibration, is $\mathrm{Core}(\mathcal{D})$

$$\begin{array}{ccc} \mathrm{Core}(A) & \longrightarrow & \mathrm{Core}(X) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Core}(p)} \\ \mathrm{Core}(B) & \xrightarrow[\mathrm{Core}(f)]{} & \mathrm{Core}(Y) \end{array}$$

a cartesian square in the category of Kan complexes **Kan** (the full subcategory of $\hat{\mathbf{\Delta}}$ whose objects are Kan complexes)?

(4) Show that $\mathrm{Core}(\mathcal{D})$ is a cartesian square in **Kan** if p is an isofibration. Deduce that the functor Core preserves pullback along isofibrations.

(5) Why couldn't we use the fact that Core is right adjoint to the inclusion $\mathbf{Kan} \to \mathbf{QCat}$ to prove the previous proposition?

Theorem 2.6.16. *Let $p : X \to Y$ be a functor between quasicategories.*

(1) *p is a trivial fibration if and only if p is an isofibration and a (weak) categorical equivalence.*

(2) *p is an isofibration if and only if p has the RLP with respect to monomorphisms that are weak categorical equivalences.*

2.7. **Joyal model structure.**

Theorem 2.7.1. *There exists a Quillen model structure on $\hat{\Delta}$ where cofibrations are monomorphisms, weak equivlalences are weak categorical equivalences. Furthermore fibrant objects are quasicategories, and fibrations between quasicategories are isofibrations.*

Exercise 2.7.2. Show the end of the Theorem: in the Joyal model structure, fibrant objects are quasicategories, and fibrations between quasicategories are isofibrations.

Exercise 2.7.3. Show that the Joyal model category is not right proper, by exploring the counterexample of Lurie, that is the pullback of the diagram

$$\begin{array}{ccc} & & \Delta^{\{0,2\}} \\ & & \downarrow \\ \Lambda^2_1 & \longrightarrow & \Delta^2 \end{array}$$

Lecture 8 (Friday 1st of april) Reference: Rune Haugsgeng, course on ∞-categories.

3. Limits and colimits in quasicategories

3.1. **Joins in categories.** Definition and adjunction properties.

3.2. **Joins for simplicial set.** Definition of the join of two simplicial sets.

Exercise 3.2.1. Prove that $\Delta^n * \Delta^m \simeq \Delta^{n+m+1}$, $\Delta^0 * \partial\Delta^n \simeq \Lambda^{n+1}_0$, $\partial\Delta^n * \Delta^0 \simeq \Lambda^{n+1}_{n+1}$.

Proposition 3.2.2. *Let X be a simplicial set. The functor $- * X : \hat{\mathbf{\Delta}} \to X/\hat{\mathbf{\Delta}}$ admits a right adjoint, which associates to $p : X \to Y$ the simplcial set $Y_{/p}$. The functor $X * - : \hat{\mathbf{\Delta}} \to X/\hat{\mathbf{\Delta}}$ admits a right adjoint, which associates to $p : X \to Y$ the simplcial set $Y_{p/}$.*

Exercise 3.2.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We have $N\mathcal{D}_{/NF} \simeq N(\mathcal{D}_{/F})$.

Lemma 3.2.4. *If X and Y are quasicategories, so is $X * Y$.*

3.3. Properties of the slices.

Special case: When $y : \Delta^0 \to X$, the slice $Y_{y/}$ is the simplicial sets whose n-simplices are thoses $n+1$-simpices σ of X satsifying $\sigma(0) = y$.

Functoriality: Any $j : T \to S$ and $p : S \to X$ and $f : X \to Y$ morphisms in $\hat{\mathbf{\Delta}}$ yield commutative diagrams:

$$\begin{array}{ccc} X_{/p} & \longrightarrow & Y_{/fp} \\ \downarrow & & \downarrow \\ X_{/pj} & \longrightarrow & Y_{/fpj} \end{array} \qquad \begin{array}{ccc} X_{p/} & \longrightarrow & Y_{fp/} \\ \downarrow & & \downarrow \\ X_{pj/} & \longrightarrow & Y_{fpj/} \end{array}$$

3.4. Pushout/join, Pullback/slices.

Definition 3.4.1. Let $i : A \to B$, $j = K \to L$, define $i \boxtimes j : A * L \cup_{A*K} B * K \to B * L$

Exercise 3.4.2. Show that

- $(\Lambda^n_j \to \Delta^n) \boxtimes (\partial\Delta^m \to \Delta^m) \simeq \Lambda^{n+m+1}_j \to \Delta^{n+m+1}$
- $(\partial\Delta^m \to \Delta^m) \boxtimes (\Lambda^n_j \to \Delta^n) \simeq \Lambda^{n+m+1}_{j+m+1} \to \Delta^{n+m+1}$
- $(\partial\Delta^m \to \Delta^m) \boxtimes (\partial\Delta^n \to \Delta^n) \simeq \partial\Delta^{n+m+1} \to \Delta^{n+m+1}$

Definition 3.4.3. Let $i : A \to B$, $p : B \to X$ and $h = X \to Y$, morphisms of simplicial sets. Define $< i, p, h >^{/-}: X_{/p} \to X_{/pi} \times_{Y_{/hpi}} Y_{/hp}$ and $< i, p, h >^{-/}: X_{p/} \to X_{pi/} \times_{Y_{hpi/}} Y_{hp/}$

Proposition 3.4.4. *Let $i : A \to B, j : K \to L, h : X \to Y$. The following proposition are equivalent*

1. *$i \boxtimes j$ has the LLP against h*
2. *For every $q : L \to X$, i has the left lifting property against $< j, q, h >^{/-}$*
3. *For any $p : B \to X$ j has the left lifting property against $< i, p, h >^{-/}$*

Corollary 3.4.5. *Let $\mathcal{C}$ be a quasicategory. $p : S \to \mathcal{C}$ and $j : T \to S$.*

1. *If j is a monomorphism then $\mathcal{C}_{p/} \to \mathcal{C}_{pj/}$ is a left fibration.*
2. *If j is a monomorphism then $\mathcal{C}_{/p} \to \mathcal{C}_{/pj}$ is a right fibration.*
3. *If j is right anodyne then $\mathcal{C}_{p/} \to \mathcal{C}_{pj/}$ is a trivial fibration.*
4. *If j is left anodyne then $\mathcal{C}_{/p} \to \mathcal{C}_{/pj}$ is a trivial fibration.*

In addition $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ are quasicategories.

3.5. Initial and terminal object.

Definition 3.5.1. An object x in a quasicategory $\mathcal{C}$ is initial (resp. terminal) if $\mathcal{C}_{x/} \to \mathcal{C}$ (resp. $\mathcal{C}_{/x} \to \mathcal{C}$) is a trivial fibration.

Proposition 3.5.2. *x is initial in $\mathcal{C}$ if and only if every map $\sigma : \partial\Delta^n \to \mathcal{C}$ such that $\sigma(0) = x$ extends to a map from Δ^n.*

Remark 3.5.3. Let $\mathcal{C}$ be a quasicategory. The full subquasicategory $\mathcal{C}_{\text{init}}$ of $\mathcal{C}$ spanned by initial objects is a contractible Kan complex.

3.6. **Limits and colimits.**

Definition 3.6.1. Let $p : K \to \mathcal{C}$ a morphism of simplicial sets with $\mathcal{C}$ a quasicategory. A colimit of p is an initial object of $\mathcal{C}_{p/}$. A limit of p is a terminal object in $\mathcal{C}_{/p}$.

Proposition 3.6.2. *Let $\bar{p} : K * \Delta^0 \to \mathcal{C}$ be such that $\bar{p}|_K = p$. We have $\bar{p} \in (\mathcal{C}_{p/})_0$ is a colimit of p if and only if $\mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/}$ is a trivial fibration.*

Chapter 2: Simplicial categories

Lecture 9 (Wednesday 6th of april)

For references on enriched categories, see Emily Riehl *Categorical homotopy theory*, Part I-Chapter 3.

For references on the model category structure for simplicial categories, see the book by Julie Bergner, *The homotopy Theory of* $(\infty, 1)$*-categories*

1. Enriched categories

We assume that the notions of symmetric monoidal categories and (lax) monoidal functors between symmetric monoidal categories is known. Given a symmetric monoidal category $\mathcal{V}$, we define the notion of $\mathcal{V}$-enriched categories (or shortly $\mathcal{V}$-categories).

1.1. First definitions.

Definition 1.1.1. Let $\mathcal{C}, \mathcal{D}$ be two $\mathcal{V}$-enriched categories. A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ is the following data: to each object x of $\mathcal{C}$ is associated an object $F(x)$ of $\mathcal{D}$; to each pair x, y of objects of $\mathcal{C}$ is associated an arrow $F_{xy} : \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ in $\mathcal{V}$, satisfying the usual axioms.

Proposition 1.1.2. *Let $F : \mathcal{V} \to \mathcal{U}$ be a (lax) monoidal functor between symmetric monoidal categories. Let $\mathcal{C}$ be a $\mathcal{V}$-category. The functor F induces a structure of $\mathcal{U}$-category on $\mathcal{C}$ denoted $F_*\mathcal{C}$, whose objets are those of $\mathcal{C}$ and morphisms are defined by $F(\mathcal{C}(x, y))$.*

Example 1.1.3. Let $\mathcal{V}$ be a small symmetric monoidal category and $I_{\mathcal{V}}$ be the unit for this monidal structure. The functor $\mathcal{V}(I_{\mathcal{V}}, -) : \mathcal{V} \to \mathbf{Set}$ is lax monoidal; hence every $\mathcal{V}$-category $\mathcal{C}$ induces a (normal) category denoted $\mathcal{C}_0$: its objects are those of $\mathcal{C}$ and morphisms are defined by $\mathcal{C}_0(x, y) = \mathrm{Hom}_{\mathcal{V}}(I_{\mathcal{V}}, \mathcal{C}(x, y))$. Note that every $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to a functor $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$.
Every $f : I_{\mathcal{V}} \to \mathcal{C}(x, y)$ in $\mathcal{V}$ is denoted $f : x \to y$ seen as an arrow in $\mathcal{C}_0(x, y)$.

Exercise 1.1.4. Prove Proposition 1.1.2. Show that if $F : \mathcal{V} \to \mathcal{U}$ is a (lax) monoidal functor between symmetric monoidal categories and $\mathcal{C}$ is a $\mathcal{V}$-category, then F induces a functor $\mathcal{C}_0 \to (F_*\mathcal{C})_0$.

Example 1.1.5. A closed symmetric monoidal category $\mathcal{V}$ is a category for which $X \otimes -$ admits a right adjoint denoted $\underline{\mathrm{Hom}}(X, -)$. It thus induces a $\mathcal{V}$-enriched category $\underline{\mathcal{V}}$. Furthermore $\underline{\mathcal{V}}_0 = \mathcal{V}$.

Definition 1.1.6. A simplicial category $\mathcal{C}$ is a $\hat{\Delta}$-enriched category (where the symmetric monoidal structure is given by the product, which is closed). Given a simplicial category amounts to given the following data: a class of objects, and for every pair of objects x, y a simplicial set $\mathcal{C}(x, y)$. We have: $\mathcal{C}_0(x, y) = \mathcal{C}(x, y)_0$. We denote by $\pi_0\mathcal{C}$ the category having the same objects as those of $\mathcal{C}$ and for morphisms $\pi_0(\mathcal{C}(x, y))$. A simplicial functor (or functor for short) between two simplicial categories is a $\hat{\Delta}$-functor. Denote by $\mathbf{Cat}_{\Delta}$ the category of simplicial categories and simplicial functors.

1.2. **$\mathcal{V}$-enriched natural transformations.** Let $\mathcal{D}$ be a $\mathcal{V}$-category and $g \in \mathcal{D}_0(y,z)$. We define an arrow in $\mathcal{V}$: $g_* : \mathcal{D}(x,y) \cong I_{\mathcal{V}} \otimes \mathcal{D}(x,y) \to \mathcal{D}(y,z) \otimes \mathcal{D}(x,y) \to \mathcal{D}(x,z)$. One can deduce 3 (representable) functors from g_*:

- $\mathcal{D}(x,-) : \mathcal{D}_0 \to \mathcal{V}$ a functor;
- $\mathcal{D}_0(x,-) : \mathcal{D}_0 \to$ **Set** a functor;
- IF $\mathcal{V}$ is closed monoidal: $\underline{\mathcal{D}}(x,-) : \mathcal{D} \to \underline{\mathcal{V}}$ a $\mathcal{V}$-functor.

Note that $(\underline{\mathcal{D}}(x,-))_0 : \mathcal{D}_0 \to \mathcal{V}$ corresponds to the functor $\mathcal{D}(x,-)$.

Definition 1.2.1. A natural $\mathcal{V}$-transformation τ from a $\mathcal{V}$-functor F to a $\mathcal{V}$-functor G is the following data: to each object x of $\mathcal{C}$ is associated $\tau_x : Fx \to Gx$ (an arrow in $\mathcal{C}_0(Fx,Gx)$) such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}(x,y) & \xrightarrow{F_{xy}} & \mathcal{D}(Fx,Fy) \\ {\scriptstyle G_{xy}}\downarrow & & \downarrow{\scriptstyle (\tau_y)_*} \\ \mathcal{D}(Gx,Fy) & \xrightarrow[(\tau_x)^*]{} & \mathcal{D}(Fx,Gy) \end{array}$$

Definition 1.2.2. Two objects x,y of a $\mathcal{V}$-category $\mathcal{C}$ are said to be isomorphic if they are isomorphic in $\mathcal{C}_0$.

Theorem 1.2.3. *Assume $\mathcal{V}$ is closed monoidal. The objects x and y are isomorphic if and only if their representable functors are naturally isomorphic (in the third case, naturally $\mathcal{V}$-isomorphic).*

Definition 1.2.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor. F is a $\mathcal{V}$-equivalence of $\mathcal{V}$-categories if

(1) For each pair x,y of objects of $\mathcal{C}$, F_{xy} is an isomorphism in $\mathcal{V}$.
(2) $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ is essentially surjective.

Exercise 1.2.5. Under condition (1) in the previous definition, show that the second condition is equivalent to the condition: F_0 is an equivalence of categories.

2. $\mathcal{V}$-enriched categories when $\mathcal{V}$ is a Quillen model category

Let $\mathcal{V}$ be a Quillen model category, and denote $\gamma : \mathcal{V} \to Ho(\mathcal{V})$ its localization with respect to weak equivalences.

2.1. Monoidal model categories.

Definition 2.1.1. $\mathcal{V}$ is a monoidal model category if it is endowed with both a Quillen model category structure and a symmetric monoidal structure satisfying:

(1) If $f : A \to B$ and $g : K \to L$ are cofibrations, then $f \boxtimes g : A \otimes L \cup_{A\otimes K} B \otimes K \to K \otimes L$ is a cofibration, acyclic if f or g is.
(2) For every cofibrant X and for every cofibrant replacement QI of $I_{\mathcal{V}}$, the map $X \otimes QI \to X \otimes I \cong X$ is a weak equivalence.

2.2. Exercises.

Exercise 2.2.1. Let $\mathcal{V}$ be a closed monoidal model category, we denote by e its initial object.

1. Show that for every X object in $\mathcal{V}$, $e \otimes X \cong e$.
2. Show that if K, L are cofibrant so is $K \otimes L$.
3. Deduce that if f is a cofibration and K is cofibrant then $f \otimes \mathrm{Id}_K$ is a cofibration (acyclic if f is).
4. Deduce that if f, g are cofibrations between cofibrant objects then so is $f \otimes g$, acyclic if f and g are.
5. Show that $Ho(\mathcal{V})$ is endowed with a monoidal structure and γ is lax monoidal.

Exercise 2.2.2. Show that $\hat{\mathbf{\Delta}}$ with the Kan-Quillen model structure and the cartesian product is a closed monoidal model category. We denote by $\mathcal{H} = Ho(\hat{\mathbf{\Delta}})$ and by $\gamma : \hat{\mathbf{\Delta}} \to \mathcal{H}$ the localization functor. Let $\mathcal{C}$ be a simplicial category. Denote by $Ho(\mathcal{C}) := \gamma_* \mathcal{C}$ the $\mathcal{H}$-enriched category induced by γ. Prove that its underlying category is $\pi_0(\mathcal{C})$, which is called *The homotopy category of the simplicial category* $\mathcal{C}$.

3. Model structure on $\mathbf{Cat}_\Delta$.

3.1. Dwyer-Kan equivalences, Dwyer-Kan fibrations.

Definition 3.1.1. A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ is a Dwyer-Kan equivalence if $HoF : Ho(\mathcal{C}) \to Ho(\mathcal{D})$ is an $\mathcal{H}$-equivalence of $\mathcal{H}$-enriched categories. Equivalently, F is a DK-equivalence if and only if

(W1) for every objects c, c' of $\mathcal{C}$, $F_{cc'}$ is a weak homotopy equivalence of simplicial sets.
(W2') $\pi_0 F : \pi_0 \mathcal{C} \to \pi_0 \mathcal{D}$ is an equivalence of categories.

Note that under hypothesis $(W1)$, hypothesis $(W2')$ is equivalent to hypothesis $(W2)$: $\pi_0 F$ is essentially surjective.

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Remark 3.1.2. Recall that the constant embedding $\iota : \mathbf{Cat} \to \mathbf{Cat}_\Delta$ admits for right adjoint $(-)_0$ and for left adjoint π_0 and that $\pi_0 \iota(\mathcal{C}) \cong C$. If F is a functor between two categories, then F is an equivalence of categories if and only if ιF is a DK-equivalence between simplicial categories.

Definition 3.1.3. A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ is a Dwyer-Kan fibration if

(F1) for every pair of objects c, c' of $\mathcal{C}$, $F_{cc'}$ is a Kan fibration.
(F2) $\pi_0 F : \pi_0 \mathcal{C} \to \pi_0 \mathcal{D}$ is an isofibration.

3.2. Bergner's Theorem.

Theorem 3.2.1 (Bergner). *There exists a cofibrantly generated model category structure on* $\mathbf{Cat}_\Delta$ *where weak equivalences are Dwyer-Kan equivalences and fibrations are Dwyer-Kan fibrations. Fibrant objects are categories enriched in Kan complexes (we say that they are locally Kan categories). This model structure is left and right proper. But not a cartesian closed model category.*

Proposition 3.2.2. $F : \mathcal{C} \to \mathcal{D}$ *is a DK-fibration and a DK-equivalence, if and only if for every* x, y *objects of* $\mathcal{C}$, F_{xy} *is a trivial fibration and* $\pi_0 F$ *is surjective on objects (equivalently* F *is surjective on objects).*

Remark 3.2.3. If you are interested in conditions for which the category of $\mathcal{V}$-enriched categories with $\mathcal{V}$ a monoidal model category admits a model category structure with weak equivalences DK-equivalences and fibrations DK-fibrations, a good reference is F. Muro, *Dwyer-Kan homotopy theory of enriched categories*, (J. Topol, 2015).

4. Quillen's equivalence

4.1. **The adjunction.** We define an adjunction:

$$\mathfrak{C} : \hat{\Delta} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{Cat}_\Delta \ : N^\Delta$$

using a cosimplicial object in $\mathbf{Cat}_\Delta$. Let **PosCat** be the category of categories enriched in Posets, with enriched functors as morphisms. For all $n \geq 0$, let P_n be the enriched category in Posets defined as:

(1) Objects of P_n are $0, 1, \ldots, n$.
(2) For $0 \leq i \leq j \leq n$, $P_n(i,j) = \{A | \{i,j\} \subset A \subset [[i,j]]\}$, ordered by inclusion.
(3) For $i > j, P_n(i,j) = \emptyset$.
(4) The composition $P_n(j,k) \times P_n(i,j) \to P_n(i,k)$ is defined by $(A, B) \mapsto A \cup B$.

Note that for every $i < j$, we have $P_n(i,j) = [1]^{\times j-i-1}$.

Since the nerve functor $\mathbf{Cat} \to \hat{\mathbf{\Delta}}$ is strong monoidal, we have an enriched category in simplicial sets: $\tilde{P}_n = N_*(P_n)$. Note that every $\varphi : [n] \to [m] \in \mathbf{\Delta}$ induces an enriched functor $\varphi_* : P_n \to P_m$ hence a functor $\hat{\Delta} \to \mathbf{Cat}_\Delta$.

Exercise 4.1.1. Show that for every $n, 0 \leq i \leq j \leq n$, $\tilde{P}_n(i,j)$ is a quasicategory and that for every $\varphi : [n] \to [m] \in \mathbf{\Delta}$, $\varphi_* : \tilde{P}_n(i,j) \to \tilde{P}_m(\varphi(i), \varphi(j))$ is an isofibration between quasicategories. *Remark: it is not in general a trivial fibration unless φ is surjective*

Definition 4.1.2. Let $\mathcal{C}$ be a simplicial category. The homotopy coherent nerve of $\mathcal{C}$ (or simplicial nerve) is defined by: $N^\Delta(\mathcal{C})_n = \mathrm{Hom}_{\mathbf{Cat}_\Delta}(\tilde{P}_n, \mathcal{C})$. It admits a left adjoint denoted $\mathfrak{C}$ and $\mathfrak{C}(\Delta^n) \cong \tilde{P}_n$.

We have

- $N^\Delta(\mathcal{C})_0 = Ob(\mathcal{C}) = N(\mathcal{C}_0)_0$
- $N^\Delta(\mathcal{C})_1 = N(\mathcal{C}_0)_1$
- An element in $N^\Delta(\mathcal{C})_2$ consists in the following data: a triangle $(f, g, h) : \partial\Delta_2 \to N(\mathcal{C}_0)$ of arrows in $\mathcal{C}_0$, namely $f : a \to b, g : b \to c, h : a \to c$, together with an element $H \in \mathcal{C}(a,c)_1$ such that $d_0 H = g \circ f$, $d_1 H = h$.

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Note that $N^\Delta(\mathcal{C}) \neq N(\mathcal{C}_0)$. However we have the following diagram of categories

$$\begin{array}{ccccc} \hat{\Delta} & \underset{N^\Delta}{\overset{\mathfrak{C}}{\rightleftarrows}} & \mathbf{Cat}_\Delta & \underset{\iota}{\overset{\pi_0}{\rightleftarrows}} & \mathbf{Cat} \\ & & & \searrow^{\gamma_*} \quad \nearrow_{(-)_0} & \\ & & & \mathbf{Cat}_{\mathcal{H}} & \end{array}$$

where the adjunction on the right is induced by the adjunction $\pi_0 : \hat{\mathbf{\Delta}} \to \mathbf{Set} : \iota$ and γ_* is the functor between enriched categories induced by the localization functor $\gamma : \hat{\mathbf{\Delta}} \to \mathcal{H}$ with respect to the Kan-Quillen model structure on $\hat{\mathbf{\Delta}}$.

In particular we have

$$\tau_1(X) \cong \pi_0(\mathfrak{C}(X)).$$

Exercise 4.1.3. Assuming that for any fibrant simplicial category $\mathcal{C}$, $N^\Delta(\mathcal{C})$ is a quasicategory, show that $h(N^\Delta(\mathcal{C})) \cong \pi_0(\mathcal{C})$.

Theorem 4.1.4. *The adjunction* $(\mathfrak{C}(-), N^\Delta)$ *is a Quillen equivalence.*

Proof. Here is an idea of the proof: We first prove that it is a Quillen adjunction using (4) of 1.3.1:

(1) $\mathfrak{C}$ preserves cofibrations.
(2) N^Δ preserves fibrations between fibrant objects.

We use two strong results in Lurie:

- $\mathfrak{C}[f]$ is a DK-equivalence if and only if f is a weak categorical equivalence. (Proposition 2258)
- For every $\mathcal{C}$ fibrant in $\mathbf{Cat}_\Delta$ we have $\mathfrak{C}(N^\Delta(\mathcal{C})) \to \mathcal{C}$ is a DK-equivalence.

These two results imply that the derived counit is a DK equivalence and the derived unit is a weak categorical equivalence. □

Chapter 3: From model categories to $(\infty, 1)$-categories

1. A little bit of "size" and "set"

We refer to ncat lab Grothendieck Universe (https://ncatlab.org/nlab/show/Grothendieck+universe) and Mike Schulman *Set theory for category theory*, for more details and Murfet's notes "http://therisingsea.org/notes/FoundationsForCategoryTheory.pdf" We have defined the notion of Grothendieck universe $\mathcal{U}$, with the axioms

(1) Transitivity: if $t \in \mathcal{U}$ and $x \in t$ then $x \in \mathcal{U}$.
(2) If $x, y \in \mathcal{U}$ then $\{x, y\} \in \mathcal{U}$.
(3) If x in $\mathcal{U}$ then $\mathcal{P}(x) \in \mathcal{U}$.
(4) If $I \in \mathcal{U}$ and $(x_i)_{i \in I}$ is a family of elements of $\mathcal{U}$, then $\cup_{i \in I} x_i \in \mathcal{U}$.

Exercise 1.1. Show that (3) implies that $\emptyset \in \mathcal{U}$. Show that for $x, y \in U$, $x \times y \subset \mathcal{P}(\mathcal{P}(x \cup y))$. Show that the set of all functions $f : x \to y$ is a subset of $\mathcal{P}(x \times y)$.

Elements (or members) of $\mathcal{U}$ are called $\mathcal{U}$-small set. Subsets of $\mathcal{U}$ are called $\mathcal{U}$-moderate sets $\mathcal{U}\mathbf{Set}$ is the category consisting of $\mathcal{U}$-small sets and functions between them.

Definition 1.2. A $\mathcal{U}$-small category is a category $\mathcal{C}$ such that $Ob(\mathcal{C})$ is a $\mathcal{U}$-small set and $Mor(\mathcal{C})$ is a $\mathcal{U}$-small set, or equivalently because of axiom (4), for every objects x, y of $\mathcal{C}$, $\mathcal{C}(x, y)$ is a $\mathcal{U}$-small set. A category satisfying the latter condition is called a locally $\mathcal{U}$-small category.

From Quillen Theory one gets:

Proposition 1.3. If $\mathcal{C}$ is a $\mathcal{U}$-small model category (resp. locally $\mathcal{U}$-small model category), with class of weak equivalences denoted by $\mathcal{W}$ then $\mathcal{C}[\mathcal{W}^{-1}]$ is $\mathcal{U}$-small (resp. locally $\mathcal{U}$-small).

Remark 1.4. If $\mathcal{C}$ is a $\mathcal{U}$-small category then the category $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{U}\mathbf{Set})$ is locally $\mathcal{U}$-small. The objects of this category are $\mathcal{U}$-moderate.