Algèbre homologique et topologie algébrique
Séance de TD n°1 - exercice 6

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1)

Let R be a ring and $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ a short exact sequence of left R-module. Show that the following propositions are equivalent:

a) f admits a retraction, i.e, there exists $B \xrightarrow{r} A$ such that $rf = \mathrm{id}_A$.

b) g admits a section, i.e, there exists $C \xrightarrow{s} B$ such that $gs = \mathrm{id}_C$.

c) f admits a retraction r and g a section s such that $fr + sg = \mathrm{id}_B$.

d) There exists an isomorphism $B \xrightarrow{h} A \oplus C$ such that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow h & & \| & & \\
0 & \longrightarrow & A & \xrightarrow{i_A} & A \oplus C & \xrightarrow{\pi_C} & C & \longrightarrow & 0
\end{array}$$

When these propositions are satisfied the sequence is said to *split*.

Solution:

$a) \implies d)$: Take $h := i_A r + i_c g$, this makes the diagram commute and then by five lemma (Exercise 3) h is an isomorphism.

$b) \implies d)$: Proceed as above with $h^{-1} := s\pi_C + f\pi_A$.

$d) \implies c)$: As the diagram shows we have $hf = i_A$ and $gh^{-1} = \pi_C$ so composing both sides of each equation by π_A and i_C respectively we get

$$(\pi_A h)f = \pi_A i_A = \mathrm{id}_A \quad \text{and} \quad g(h^{-1} i_C) = \pi_C i_C = \mathrm{id}_C$$

So $r := \pi_A h$ is a retraction for f and $s := h^{-1} i_C$ is a section for g. Even more we have

$$fr + sg = (h^{-1} i_A)(\pi_A h) + (h^{-1} i_C)(\pi_C h) = h^{-1}(i_A \pi_A + i_C \pi_C)h = id_B$$

Now as $c) \implies a)$ and $c) \implies b)$ are obvious we have that all proposition are equivalent.

2)

Show that every short exact sequence of vector spaces splits.

Solution: Consider an exact sequence $0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$ of vector spaces and choose a basis $\{v_i''\}_{i \in I}$ of V''. Since g is surjective for every $v_i'' \in V''$ there exist $v_i \in V$ such that $g(v_i) = v_i''$.

Now define $s : V'' \to V$ by $s(v_i'') = v_i$ over the basis. We have $gs(v_i'') = g(v_i) = v_i''$ for every $i \in I$ so $gs = \mathrm{id}_{V''}$ and therefore s is a section of g. By exercise 1 the sequence splits.

3)

Show that the following short exact sequence of $\mathbb{Z}$-modules doesn't split

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Solution: If it splits we would have an isomorphism $\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. However the left-hand side is a torsion-free module and the right-hand side isn't since $2 \cdot (0,1) = (0,0) \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

4)

Find all the short exact sequence of $\mathbb{Z}$-modules of the form

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} M \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and see which of them splits.

Solution: We must have $M/\ker(g) = M/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ so $|M| = |\mathbb{Z}/2\mathbb{Z}| \cdot |\mathbb{Z}/2\mathbb{Z}| = 4$ then by the structure theorem of abelian groups we have two cases

$$M \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \quad \text{or} \quad M \cong \mathbb{Z}/4\mathbb{Z}$$

In the first case every module in the chain would have $2\mathbb{Z}$ as annihilator and therefore it can be seen as a chain of $\mathbb{Z}/2\mathbb{Z}$-vector spaces, then by exercise 2 it must splits and hence it also splits as a chain of $\mathbb{Z}$-modules. Then we have the diagram of exercise 1

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\quad f \quad} & M & \xrightarrow{\quad g \quad} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle h} & & \| & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\quad i \quad} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\quad \pi \quad} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0
\end{array}$$

And therefore we have $f = hi$ and $g = \pi h$ where $h \in \operatorname{Aut}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$, these are all the exact sequence of this case (note that there are 3 such automorphisms).

Now if $M \cong \mathbb{Z}/4\mathbb{Z}$ we have that M is cyclic, then every morphism is determined by the image of 1, in particular $|\operatorname{Hom}(M, \mathbb{Z}/2\mathbb{Z})| = 2$ and the only surjective morphism is the usual projection $\pi : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ so $g = \pi$. Then we have $\ker(g) = \{\bar{0}, \bar{2}\}$ so $\operatorname{im}(f) = \{0, 2\}$ and hence f is given by $f(1) = 2$. This is the only exact sequence in this case and it dosen't split because $\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.