Multicomplexes and Spectral Sequences
a Homotopy point of view

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Section 1

Spectral Sequences

Spectral sequences: quick and simple

R is the underlying ring.

Definition

A *spectral sequence* (A, ψ) is

- a family of complexes of R-modules $\{(A_r, d_r)\}_{r \geq 0}$,
- a family of isomorphisms of R-modules $\{\psi_r : H_*(A_r) \leftarrow A_{r+1}\}_{r \geq 0}$ called *characteristic maps*

A *morphism of spectral sequences* is a family of morphisms of complexes $\{f_r : A_r \to B_r\}_{r \geq 0}$ which is *compatible with characteristic maps*.

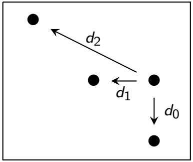
SpSe is the category of spectral sequences.

Spectral sequences: convention and properties

$\{(A_r, d_r)\}_{r\geq 0}$ a spectral sequence.

In practice: A_r is bigraded (over $\mathbb{Z}\times\mathbb{Z}$) and d_r has bidegree $(\pm r, \pm r \pm 1)$.

Today:
d_r is of bidegree $(-r, r-1)$
(A_r, d_r) is called
an *r-bigraded complex*



Proposition

The category **SpSe** *is*

- *an additive category*
- *not complete or cocomplete*

Homotopy Theory of Spectral Sequences

Definition

Let $f : A \to B$ in **SpSe** and $r \geq 0$. f is an *E_r-quasi-isomorphism* if the morphism $f_r : A_r \to B_r$ is a quasi-isomorphism of r-bigraded complexes. Equivalently f_k are isomorphisms for $k > r$.

This gives a category with weak equivalences $(\mathbf{SpSe}, \mathcal{E}_r)$:

- → contains all isomorphisms
- → satisfies the two-out-of-three property

Section 2

The picture

The picture

$$\begin{tikzcd}
 & 2\text{-}\mathbf{mC}_R \arrow[d] & & & & \\
N\text{-}\mathbf{mC}_R \arrow[r] & \mathbf{mC}_R \arrow[r] \arrow[rrr, bend left, "E"] & \mathbf{tC}_R \arrow[r, "\mathrm{Tot}"] & \mathbf{fC}_R \arrow[r, "E"] & \mathbf{SpSe} \arrow[r] & \mathbf{ESpSe} \\
 & 4\text{-}\mathbf{mC}_R \arrow[u] & \mathbf{DiGraph} \arrow[u, dashed] \arrow[ur] & & &
\end{tikzcd}$$

Notation

- $\mathbf{fC}_R$ filtered complexes
- $\mathbf{mC}_R$ multicomplexes and $\mathbf{tC}_R$ twisted complexes
- N-$\mathbf{mC}_R$ truncated versions of multicomplexes
- **ESpSe** extended spectral sequences
- **DiGraph** directed graphs

The plan, part 1

$$\begin{tikzcd}
N\text{-}\mathbf{mC}_R \arrow[r] & \mathbf{mC}_R \arrow[r] \arrow[rrr, bend left, "E"] & \mathbf{tC}_R \arrow[r, "\mathrm{Tot}"] & \mathbf{fC}_R \arrow[r, "E"] & \mathbf{SpSe}
\end{tikzcd}$$

Plan

- $E : \mathcal{C} \to$ **SpSe** functor
- $\mathcal{E}_r = E^{-1}(\mathcal{E}_r)$ as weak equivalences
- Model category structures with $\mathcal{E}_r$? Cofibrantly generated ?
- Comparisons with fixed r?
- Comparisons when r varies?

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N\text{-}\mathbf{mC}_R \arrow[r] & \mathbf{mC}_R \arrow[r] \arrow[rrr, bend left, "E"] & \mathbf{tC}_R \arrow[r, "\mathrm{Tot}"] & \mathbf{fC}_R \arrow[r, "E"] & \mathbf{SpSe}
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Plan

- $E : \mathcal{C} \to$ **SpSe** functor
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Section 3

Chain complexes and filtered complexes: $\mathbf{fC}_R \to \mathbf{SpSe}$

Model category structure on chain complexes

Theorem

The category of (unbounded) chain complexes admits a right proper cofibrantly generated model structure, where:

1. *weak equivalences are quasi-isomorphisms*
2. *fibrations are surjective morphisms*

Model category structure on chain complexes: representing objects

Representing object	Explicit complex	$\mathrm{Hom}(-, C)$
Discs $D_n(R)$	$R^{(n)} \to R^{(n-1)}$	C_n
Spheres $S_n(R)$	$R^{(n)}$	$Z_n(C)$
Boundary: $\iota : S_{n-1}(R) \to D_n(R)$	$\begin{tikzcd} 0 \arrow[r] \arrow[d] & R^{(n)} \arrow[d] \\ R^{(n-1)} \arrow[r] & R^{(n-1)} \end{tikzcd}$	$C_n \xrightarrow{d=\iota_n^*} Z_{n-1}(C)$ $\mathrm{Coker}\, \iota_n^* = H_n(C)$

Proposition

A morphism $f : C \to D$ of chain complexes has the right lifting property with respect to

- $0 \to D_n(R)$ *if and only if f_n is surjective*
- $\{S_{n-1}(R) \to D_n(R)\}_{n\in\mathbb{Z}}$ *if and only if f is a quasi-isomorphism.*

Model category structure on chain complexes: generating cofibrations and generating acyclic cofibrations

Representing object	Explicit complex	$\mathrm{Hom}(-, C)$
Discs $D_n(R)$	$R^{(n)} \to R^{(n-1)}$	C_n
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Proposition

The category of chain complexes is a cofibrantly generating model category where

- $\{0 \to D_n(R)\}_{n\in\mathbb{Z}}$ *is the set of generating acyclic cofibrations*
- $\{S_{n-1}(R) \to D_n(R)\}_{n\in\mathbb{Z}}$ *is the set of generating cofibrations*

The spectral sequence of a filtered complex

(C, F, d) filtered chain complex

$$\ldots \subset F_pC \subset F_{p+1}C \subset \ldots \subset C$$

r-cycles: $Z_r^{p,n}(C) = F_pC^n \cap d^{-1}(F_{p-r}C^{n-1})$
r-page: $E_r^{p,n}(C) := Z_r^{p,n}(C)/Z^p_{r-1,n}C + dZ^{p+r-1,n+1}_{r-1}C$

Representing object	Explicit complex	$\mathrm{Hom}(-, C)$
Discs $\mathcal{Z}_r^{p,n}(R)$	$R^{(n)}_{(p)} \to R^{(n-1)}_{(p-r)}$	$Z_r^{p,n}(C)$
Boundary: $\varphi_r : \mathcal{Z}_r^{p,n} \longrightarrow \mathcal{Z}_{r-1}^{p-1,n} \oplus \mathcal{Z}_{r-1}^{p+r-1,n+1}$	$\begin{tikzcd} & R^{(n)}_{(p)} \arrow[r] \arrow[d, "\begin{pmatrix}1\\1\end{pmatrix}"'] & R^{(n-1)}_{(p-r)} \arrow[d] \\ R^{(n+1)}_{(p+r-1)} \arrow[r] & R^{(n)}_{(p-1)} \oplus R^{(n)}_{(p)} \arrow[r] & R^{(n-1)}_{(p-r)} \end{tikzcd}$	$\mathrm{Coker}\,\varphi_r^*$ $= E_r^{p,n}(C)$

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Section 4

Multicomplexes:

$\mathbf{mC}_R \to \mathbf{tC}_R \to \mathbf{fC}_R \to \mathbf{SpSe}$

The categories $\mathbf{mC}_R$ and $\mathbf{tC}_R$

A *multicomplex or twisted complex* (A, d) is

- a bigraded R-module A
- for every $i \geq 0$ a map $d_i : A \to A$ of bidegree $(-i, i-1)$ satisfying for all $l \geq 0$,

$$\sum_{i+j=l} \pm d_i d_j = 0.$$

A *morphism of twisted complexes* $f = (A, d) \to (A', d')$ is a family of maps $f_i : A \to A'$ of bidegree $(-i, i)$ satisfying for all $l \geq 0$,

$$\sum_{i+j=l} d'_i f_j = \sum_{i+j=l} \pm f_j d_i$$

It gives rise to the category $\mathbf{tC}_R$.
The category $\mathbf{mC}_R$ has the same objects but *strict morphisms*

$$f_i = 0, i > 0.$$

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$$f_i = 0, i > 0.$$

The categories $\mathbf{mC}_R$ and N-$\mathbf{mC}_R$

An *N-multicomplex* is a multicomplex (A, d) such that $d_i = 0, i \geq N$. The category N-$\mathbf{mC}_R$ is the full subcategory of $\mathbf{mC}_R$ of N-multicomplexes.

Example

- $N = 1$: vertical bicomplexes (bigraded)
- $N = 2$: bicomplexes.
- $N = 4$:

$$\begin{aligned}
d_0^2 = d_3^2 &= 0\\
d_0d_1 - d_1d_0 = d_3d_2 - d_2d_3 &= 0\\
d_0d_2 - d_1^2 + d_2d_0 = d_3d_1 - d_2^2 + d_1d_3 &= 0\\
d_0d_3 - d_1d_2 + d_2d_1 - d_3d_0 &= 0
\end{aligned}$$

Symmetry $d_i \leftrightarrow d_{N-1-i}$

The functor Tot

$$\begin{tikzcd}
N\text{-}\mathbf{mC}_R \arrow[r] & \mathbf{mC}_R \arrow[r] & \mathbf{tC}_R \arrow[r, "\mathrm{Tot}"] \arrow[rr, bend right, "E"'] & \mathbf{fC}_R \arrow[r, "E"] & \mathbf{SpSe}
\end{tikzcd}$$

Theorem (Cirici, Egas Santander, Livernet, Whitehouse, 2018)

$$\mathrm{Tot} : \mathbf{tC}_R \to \mathbf{fC}_R$$

is an isomorphism of categories when restricted to its essential image.

$$\mathrm{Tot}(A)^n := \prod_{i\leq 0} A^{i,n-i} \oplus \bigoplus_{i>0} A^{i,n-i} \quad F_p\mathrm{Tot}(A)^n := \prod_{i\leq p} A^{i,n-i}$$

$$d(a)_i := \sum_{m\geq 0} \pm d_m(a_{i+m}), \quad (\mathrm{Tot}(f)(a))_i := \sum_{m\geq 0} \pm f_m(a_{i+m})$$

The model category structures

Theorem

Let $\mathcal{C}$ be filtered complexes or (truncated) multicomplexes. For each $r \geq 0$, the category $\mathcal{C}$ admits a right proper cofibrantly generated model structure, denoted $\mathcal{C}_r$, where:

- *weak equivalences are E_r-quasi-isomorphisms,*
- *fibrations are morphisms satisfying surjectivity conditions ($E_i(f)$ is bidegree-wise surjective for $0 \leq i \leq r$, or closely related conditions).*

- filtered complexes, bicomplexes [Cirici–Egas-Santander–L–Whitehouse 2020]
- (truncated) multicomplexes [Fu–Guan–L–Whitehouse 2022]

The model category structure of $\mathbf{mC}_R$ via witnesses

$$\begin{tikzcd}
N\text{-}\mathbf{mC}_R \arrow[r] & \mathbf{mC}_R \arrow[r] & \mathbf{tC}_R \arrow[r, "\mathrm{Tot}"] \arrow[rr, bend right, "E"'] & \mathbf{fC}_R \arrow[r, "E"] & \mathbf{SpSe}
\end{tikzcd}$$

Let (A, d) be a multicomplex

$$E_r^{p,q} \circ Tot(A) = Z_r^{p,q}(A) / B_r^{p,q}(A)$$

$$Z_r^{p,q}(A) := \Big\{ a_0 \in A^{p,q} \mid \text{for all } 0 \leq l \leq r-1 \\ \textstyle\sum_{i+j=l} (-1)^i d_i a_j = 0 \text{ for some } a_j \in A^{p-j,q+j},\ 1 \leq j \leq r-1 \Big\},$$

$\Rightarrow$ Explicit formula for the differential [L., Whitehouse, Ziegenhagen, 2020]

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Let (A, d) be a multicomplex

$$E_r^{p,q} \circ Tot(A) = ZW_r^{p,q}(A) / BW_r^{p,q}(A)$$

$$\begin{aligned} ZW_r^{p,q}(A) := \Big\{ (a_0, \dots, a_{r-1}) \mid & \text{ for all } 0 \le l \le r-1 \\ & \textstyle\sum_{i+j=l} (-1)^i d_i a_j = 0 \Big\}, \end{aligned}$$

$\Rightarrow$ Has the advantage to be representable in N-$\mathbf{mC}_R$

Comparisons

Proposition

$\mathbf{mC}_R$ and N-$\mathbf{mC}_R$ are abelian categories.

Proof.

The category $\mathbf{mC}_R$ (resp. N-$\mathbf{mC}_R$) is equivalent to the category of

- modules over the bigraded ring: $\mathcal{A} := R < d_i, i \geq 0 > /\langle \sum d_i d_j \rangle$ (resp. $\mathcal{A}_N$)
- modules over a ring in 1-$\mathbf{mC}_R$: $\mathcal{C} := R < d_i, i \geq 1 >$ with $\delta_0(d_l) = \sum_{i+j=l} d_i d_j$ (resp. $\mathcal{C}_N$)

☐

In particular: for $2 \leq n \leq l \leq \infty$, the inclusion functor

$$i_{l,n} : n\text{-}\mathbf{mC}_R = \mathcal{C}_n - \mathrm{Mod} \rightarrow l\text{-}\mathbf{mC}_R = \mathcal{C}_l - \mathrm{Mod}$$

has a left adjoint.

Comparisons

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$\mathbf{mC}_R$ *and* N-$\mathbf{mC}_R$ *are abelian categories.*

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Comparisons

Theorem (X. Fu, A. Guan, L., Whitehouse, 2022)

For $2 \leq n \leq l \leq \infty$ *and* $r \geq 0$ *the adjunction*

$$(n\text{-}\mathbf{mC}_R)_r \underset{i_{l,n}}{\overset{p_{l,n}}{\rightleftarrows}} (l\text{-}\mathbf{mC}_R)_r$$

is a Quillen equivalence.

Section 5

Truncated multicomplexes and involution

Model structures on N-$\mathbf{mC}_R$ with involution

The symmetry $d_i \leftrightarrow d_{N-1-i}$ induces

- An equivalence of categories $(-)^{op} : N\text{-}\mathbf{mC}_R \to N\text{-}\mathbf{mC}_R$, *the involution*
- Two spectral sequences associated to an N-multicomplex $E^{I}(A) := E(A)$ and $E^{(II)}(A) := E(A^{op})$.

Example

For bicomplexes: the two spectral sequences correspond to the spectral sequences associated to the column and to the row filtrations.

Model structures on N-$\mathbf{mC}_R$ with involution

Definition

A morphism of N-multicomplexes $f\colon A \to B$ is said to be an *$E_{r,s}$-quasi-isomorphism* if $E_r^{(I)}(f)$ and $E_s^{(II)}(f)$ are quasi-isomorphism. The class of those maps is denoted $\mathcal{E}_{r,s}$.

Theorem (J. Cirici, L., Whitehouse, arXiv: 25.01)

For every $r, s \geq 0$, the category N-$\mathbf{mC}_R$ admits a right proper cofibrantly generated model structure, where:

1. *$\mathcal{E}_{r,s}$ is the class of weak equivalences.*
2. *A fibration f is a morphism such that $E_i^{(I)}(f)$ and $E_j^{(II)}(f)$ are bidegree-wise surjective for every $0 \leq i \leq r$ and $0 \leq j \leq s$.*

Comparing the model structures $(N\text{-}\mathbf{mC}_R)_{r,s}$

- Unfortunatly the inclusion functor $i_{l,n} : n\text{-}\mathbf{mC}_R \to l\text{-}\mathbf{mC}_R$ is not stable under the involution for $l \geq n$.
- We would like to have functors stable under the involution.
- $N\text{-}\mathbf{mC}_R \cong \mathcal{A}_N - \mathrm{Mod}$ is better than $N\text{-}\mathbf{mC}_R \cong \mathcal{C}_N - \mathrm{Mod}$
- We illustrate this by considering the case

$$j : 2\text{-}\mathbf{mC}_R \to 4\text{-}\mathbf{mC}_R$$

  which sends (A, d_0, d_1) to $(A, 0, d_0, d_1, 0)$ with an appropriate shift.
- It comes from a map of bigraded ring $\mathcal{A}_4 \to \mathcal{A}_2$, hence has a left adjoint q

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Comparing the model structures $(N\text{-}\mathbf{mC}_R)_{r,s}$

Proposition (J. Cirici, L., Whitehouse, arXiv: 25.01)

The pair of functors (q, j) *is a Quillen adjunction.*

$$\begin{tikzcd}(2\text{-}\mathbf{mC}_R)_{0,0} \arrow[r, shift right, "j"'] & (4\text{-}\mathbf{mC}_R)_{1,1} \arrow[l, shift right, "q"']\end{tikzcd}$$

Moreover the functors q *and* j *preserve the involution* $(-)^{op}$

Question

Is it a Quillen equivalence?

Section 6

Extended spectral sequences:
SpSe → **ESpSe**

(“Spectral sequences via linear presheaves”, L., Whitehouse: arXiv 24.06)

Extended spectral sequences: quick and simple

Definition

An *extended spectral sequence* (A, ψ) is

- a family of complexes of R-modules $\{(A_r, d_r)\}_{r\geq 0}$,
- a family of morphisms of R-modules $\{\psi_r : A_{r+1} \to H_*(A_r)\}_{r\geq 0}$ called *characteristic maps*

A *morphism of extended spectral sequences* is a family of morphisms of complexes $\{f_r : A_r \to B_r\}_{r\geq 0}$ which is *compatible with characteristic maps*.

ESpSe is the category of spectral sequences.

Categorical properties of **ESpSe**

Proposition

The category **ESpSe** *has all small limits and colimits, with limits computed pagewise and colimits via a reflector* $\mathcal{N}\mathcal{Q}$.

$$\begin{tikzcd} \mathrm{LWB} \arrow[r, shift left, "\mathcal{Q}"] \arrow[r, phantom, "\perp"] & \mathbf{ESpSe} \arrow[l, shift left, "\mathcal{N}"] \end{tikzcd}$$

- LWB the category of linear witness books, a presheave category with values in R-modules, (thus abelian)
- **ESpSe** is isomorphic to $(\mathrm{LWB})^{\mathrm{e}}$
- **SpSe** is isomorphic to $(\mathrm{LWB})^{\mathrm{s}}$

Shift and Décalage

Theorem

- *On the category* **ESpSe** *we have adjunctions*

$$\mathrm{LDec} \dashv \mathrm{Shift} \dashv \mathrm{Dec}\,.$$

- *It restricts to the category* **SpSe**
- *It is compatible with Deligne's shift and decalage in* $\mathbf{fC}_R$

Model category structure in **ESpSe**

Choice of weak equivalences in **ESpSe**

- $\mathcal{E}_r = \{f : X \to Y \mid f_r \text{ quasi-isomorphism}\}$
- $\mathcal{E}'_r = \{f : X \to Y \mid f_r \text{ quasi-isomorphism and } f_s \text{ isomorphism } s > r\}$

Both restricts to $\mathcal{E}_r$ in **SpSe**.

Theorem

There are model category structures on **ESpSe** *with fibrations the class of maps $f : X \to Y$ such that f_i is bidegreewise surjective for $0 \leq i \leq r$:*

- *The class of weak equivalences in* $\mathbf{ESpSe}_r$ *is* $\mathcal{E}_r$
- *The class of weak equivalences in* $\mathbf{ESpSe}'_r$ *is* $\mathcal{E}'_r$

Model category structures in **ESpSe**

$$\mathcal{E}'_r = \{f : X \to Y \in \mathcal{E}_r \mid f_s \text{ isomorphism } s > r\}$$

Theorem

- **ESpSe**$_r$ *is Quillen equivalent to the projective model category structure on chain complexes,* $\forall r \geq 0$
- **ESpSe**$'_r \to$ **ESpSe**$_r$ *is a right Bousfield localization which is not a Quillen equivalence.*
- *Via Shift and Décalage, the model category structures* **ESpSe**$'_r$ *are all Quillen equivalent when* r *varies*

The relative category $(\mathbf{SpSe}, \mathcal{E}_r)$

Theorem

- $(\mathbf{SpSe}, \mathcal{E}_0)$ *is a relative category which is homotopically full in* $(\mathbf{ESpSe}, \mathcal{E}'_0)$
- *Hence the relative category* $(\mathbf{SpSe}, \mathcal{E}_0)$ *is an* $(\infty, 1)$ *category in the model of Barwick-Kan.*
- $(\mathbf{SpSe}, \mathcal{E}_r)$ *has* $(\mathbf{SpSe}, \mathcal{E}_0)$ *as fibrant replacement in the model of Barwick-Kan.*