# Algebraic Structures on prime $m$-Dyck paths

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Joint work with Maria Ronco

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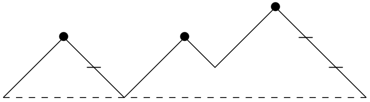
*38 degrés à l'ombre...*

# Section 1– $m$-Dyck paths

- Definitions, notations
- Grafting $m$-Dyck paths
- Enumerating prime $m$-Dyck paths
- The $m$-Tamari lattice

# $m$-Dyck paths of size $n$

- It is a path in $(\mathbb{R}^+)^{\times 2}$, starting at $(0,0)$ and ending at $(2nm, 0)$, consisting of up steps $(m, m)$ and down steps $(1, -1)$
- It is **prime** if it only meets the $x$-axis at these two points.



$(2,2,3)$, not prime

$(2,3,3)$, prime

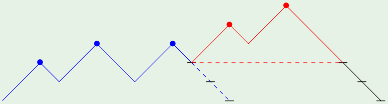
- Notation:
  - $x = (x_1, \ldots, x_n)$: the list of heights of the summits of $x$. Note that $x_1 = m$
  - $\mathrm{Dy}_n^m$: the set of $m$-Dyck paths of size $n$; $d_{m,n}$ its cardinal.
  - $\mathrm{PDy}_n^m$: the set of prime $m$-Dyck paths of size $n$; $p_{m,n}$ its cardinal.

# Grafting $m$-Dyck paths

$$(x_1, \ldots, x_p) \times_i (y_1, \ldots, y_q) = (x_1, \ldots, x_p, y_1 + i, \ldots, y_q + i), \; 0 \leq i \leq x_p$$

## Example

$$(2,3,3) \times_2 (2,3) = (2,3,3,4,5)$$



The operation $\times_0$ is associative. It corresponds to the **concatenation**.
Any $y \in \mathrm{Dy}_n^m$ writes uniquely as

$$y = y^{(1)} \times_0 \ldots \times_0 y^{(r)}$$

with $y^{(j)} \in \mathrm{PDy}_{n_j}^m$ and $\sum_j n_j = n$.

# Generalization: grafting $m$-Dyck paths along a partition

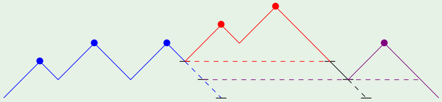
## The notation $x \times_\mu y$

- $x = (x_1, \ldots, x_p) \in \mathrm{Dy}_p^m$
- $y = y^{(1)} \times_0 \ldots \times_0 y^{(r)}$ has $r$ prime components, in $\mathrm{Dy}_q^m$
- $x_p \geq \mu_1 \geq \ldots \geq \mu_r \geq 0$ an $r$ partition **compatible with** $x$, denoted $\mu$

$$x \times_\mu y = (\ldots((x \times_{\mu_1} y^{(1)}) \times_{\mu_2} y^{(2)}) \times \ldots) \times_{\mu_r} y^{(r)}$$

## Example

$$(2,3,3) \times_{2 \geq 1} (2,3;2) = (2,3,3,4,5,3)$$



# Enumerating prime $m$-Dyck paths

- Fuss-Catalan numbers: $A_s(p,r) = \frac{r}{sp+r}\binom{sp+r}{s}$
- $d_{m,n} = \frac{1}{mn+1} \cdot \binom{(m+1)n}{n} = A_{n-1}(m+1, m+1) = A_n(m+1, 1)$

## Proposition (?; L-Ronco)

*For* $n, m \geq 1$

$$p_{m,n} = \frac{1}{n}\binom{(m+1)n-2}{n-1}$$

Idea of the proof:

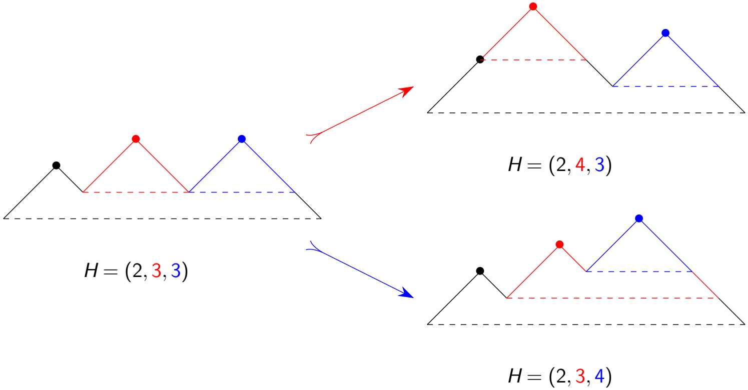
- Concatenation of prime $m$-Dyck paths gives $D_m(x) = \frac{P_m(x)}{1-P_m(x)}$
- use Riordan convolution formula
  $A_s(p, r+t) = \sum_{k=0}^{s} A_k(p,r)A_{s-k}(p,t)$

to get

$$p_{m,n} = A_{n-1}(m+1, m)$$

# The $m$-Tamari lattice [Bergeron, Préville-Ratelle; Bousquet-Mélou, Fusy, Préville-Ratelle]

covering relations:



# Section 2– Results by Lopez, Préville-Ratelle, Ronco, arxiv 1508.01252+ JPAA 2020

- Split of associativity, $\mathcal{D}yck_m$-algebras
- The structure of $\mathcal{D}yck_m$-algebra on the vector space spanned by $m$-Dyck paths
  - Labeling the down steps
  - The valuation map
  - The formula

# Split of associativity

see also Novelli and the notion of $m$-Dendriform algebras.

## Definition

A $\mathcal{D}yck_m$ algebra $D$ is a vector space endowed with bilinear products $*_i$, $0 \leq i \leq m$ satisfying the relations

- $x *_i (y *_j z) = (x *_i y) *_j z$, for $0 \leq i < j \leq m$,
- $\sum_{j=0}^{i} x *_i (y *_j z) = \sum_{k=i}^{m} (x *_k y) *_i z$,

for any elements $x, y$ and $z$ in $D$.

## Facts

- $* = \sum_{i=0}^{m} *_i$ is associative
- $m = 0$, $*_0$ is associative
- $m = 1$, $*_0$ and $*_1$ satisfy the **dendriform relations**

# Free $\mathcal{D}yck_m$-algebras

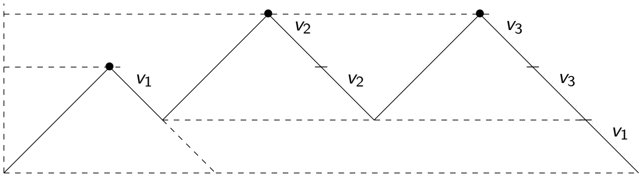
**Theorem (Lopez;Préville-Ratelle;Ronco)**

*The free $\mathcal{D}yck_m$-algebra generated by one element has for underlying vector space the span of the set of m-Dyck paths.*
*Moreover, for $x \in \mathrm{Dy}_p^m, y \in \mathrm{Dy}_q^m$, there exists an interval $J_i(x,y)$ in the m-Tamari lattice $\mathrm{Dy}_{p+q}^m$ such that:*

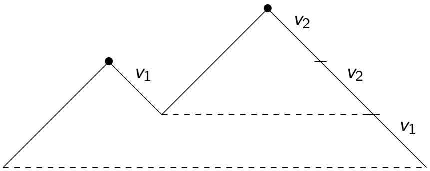
$$x *_i y = \sum_{z \in J_i(x,y)} z$$

# Labeling the down steps of an $m$-Dyck path

$(2, 3, 3)$



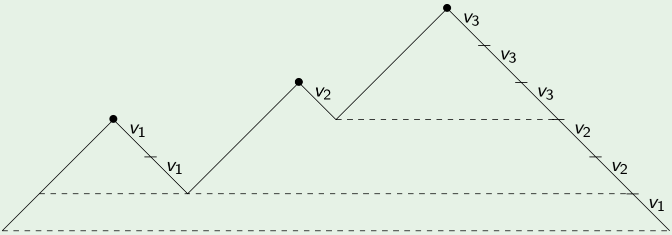

$(2, 3)$



# The valuation map of an $m$-Dyck path $x = (x_1, \dots, x_n)$

$$val(x; -) : [x_n] \to [m] = \{0, \dots, m\}$$

**Example** ($x = (3, 4, 6) \in \mathrm{Dy}_3^3$)




$6 \mapsto 3$
$5 \mapsto 2$
$4 \mapsto 2$
$3 \mapsto 2$
$2 \mapsto 1$
$1 \mapsto 1$
$0 \mapsto 0$

**Idea**

$x *_i y$ will involve $x \times_j y$ with $val(x; j) = i$.
More complicated if $y$ is not prime.

# The valuation map of an $m$-Dyck path $x = (x_1, \ldots, x_n)$

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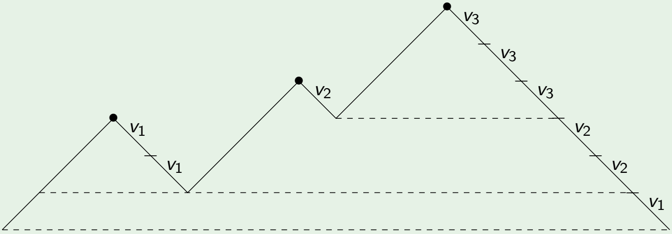



| | | |
|---|---|---|
| 6 | $\mapsto$ | 3 |
| 5 | $\mapsto$ | 2 |
| 4 | $\mapsto$ | 2 |
| 3 | $\mapsto$ | 2 |
| 2 | $\mapsto$ | 1 |
| 1 | $\mapsto$ | 1 |
| 0 | $\mapsto$ | 0 |

## Idea

$x *_i y$ will involve $x \times_j y$ with $val(x; j) = i$.
More complicated if $y$ is not prime.

# The Formula

## Theorem

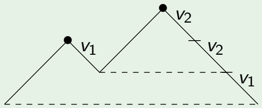
*$x = (x_1, \ldots, x_p) \in \mathrm{Dy}_p^m$, $y \in \mathrm{Dy}_q^m$ with $r$ prime componentes.*

$$x *_i y = \sum_{\mu} x \times_\mu y$$

*where $x_p \geq \mu_1 \ldots \geq \mu_r$ and $val(x; \mu_r) = i$,*
*is the sum of every elements in an m-Tamari interval.*

## Example

$x = (2, 3)$; $val : (0, 1, 2, 3) \mapsto (0, 1, 1, 2)$

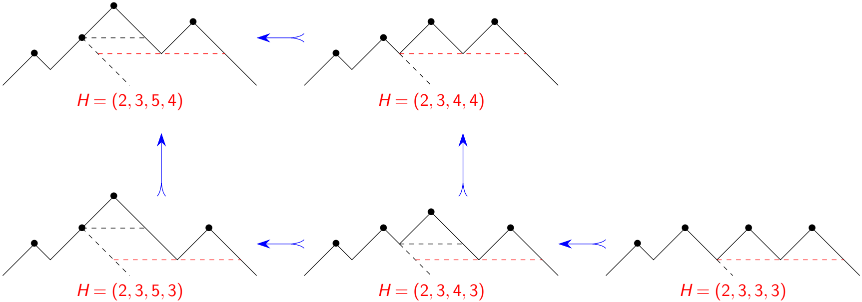



To compute $(2, 3) *_1 (2, 2)$, we need the partitions of the form $3 \geq \mu_1 \geq \mu_2$ avec $\mu_2 \in \{1, 2\}$.

# Example

$$(2,3) *_1 (2,2) = (2,3) \times_{3\geq 2} (2,2) + (2,3) \times_{2\geq 2} (2,2) + (2,3) \times_{3\geq 1} (2,2) + (2,3) \times_{2\geq 1} (2,2) + (2,3) \times_{1\geq 1} (2,2)$$

$$= (2,3,5,4) + (2,3,4,4) + (2,3,5,3) + (2,3,4,3) + (2,3,3,3)$$

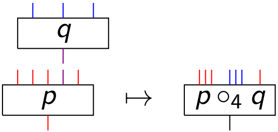


# Section 3– Operad, cooperad structures on *m*-Dyck paths

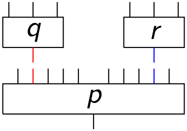
- Quick introduction to operads and co-operads
- The structure of co-operad on $\mathrm{Dy}^m$.
- Reinterpreting the results by L-PR-R

# (non symmetric) operad $\mathcal{P}$

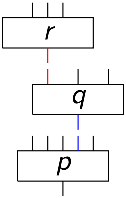
- A collection of sets (or vector spaces) $\mathcal{P}(n)$, $n \geq 1$
- Operations $\circ_i : \mathcal{P}(n) \times \mathcal{P}(n') \to \mathcal{P}(n + n' - 1)$, $1 \leq i \leq n$




- A unit in $\mathcal{P}(1)$
- Relations:




parallel




sequential

**Keep in mind, a non-symmetric operad consists of**

graded objects, that we can either

- graft, as leaves;
- or insert, as boxes;

- The endomorphism operad $\mathcal{P}(n) = \mathrm{Map}(X^{\times n}, X)$
- The terminal operad $\mathcal{P}(n) = \star_n$, $\star_n \circ_i \star_{n'} = \star_{n+n'-1}$
- The operad on the symmetric group, $\mathcal{P}(n) = \Sigma_n$

$$\sigma = (2, \boxed{3}, 1) \quad \circ_3 \quad \tau = (2, 3, 1) \quad = \quad (2, 4, 5, 3, 1)$$

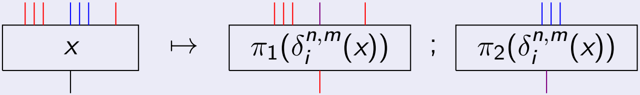
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \end{bmatrix} \circ_3 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

# (non symmetric) co-operad $\mathcal{C}$

## Definition

co-operad

- A collection of sets (or vector spaces) $\mathcal{C}(n)$,
- Co-operations $\delta_i^{n,n'} : \mathcal{C}(n+n'-1) \to \mathcal{C}(n) \times \mathcal{C}(n')$, $1 \le i \le n$



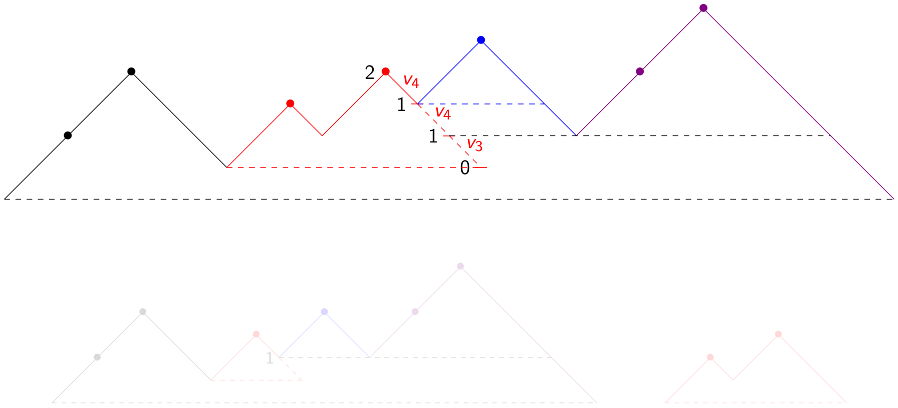

- A co-unit in $\mathcal{C}(1)$ and Co-relations

## Remark

- Co-operad structure "excision of a sub-object"
- If $\mathcal{C}$ is a co-operad in $\mathrm{Vect}$, then $\mathcal{P}(n) = \mathcal{C}(n)^*$ forms an operad.
- If $\mathcal{P}$ is an operad in $\mathrm{Vect}$, and $\mathcal{P}(n)$ is finite dimensional $\forall n$, then $\mathcal{C}(n) = \mathcal{P}(n)^*$ forms a co-operad.

# Example: $\delta_3^{6,2} : \mathrm{Dy}_7^2 \to \mathrm{Dy}_6^2 \times \mathrm{Dy}_2^2$

$x = (2, 4, 3, 4, 5, 4, 6)$




$\pi_1(\delta_3^{6,2}(x)) = (2, 4, 3, 4, 4, 6)$

$\pi_2(\delta_3^{6,2}(x)) = (2, 3)$

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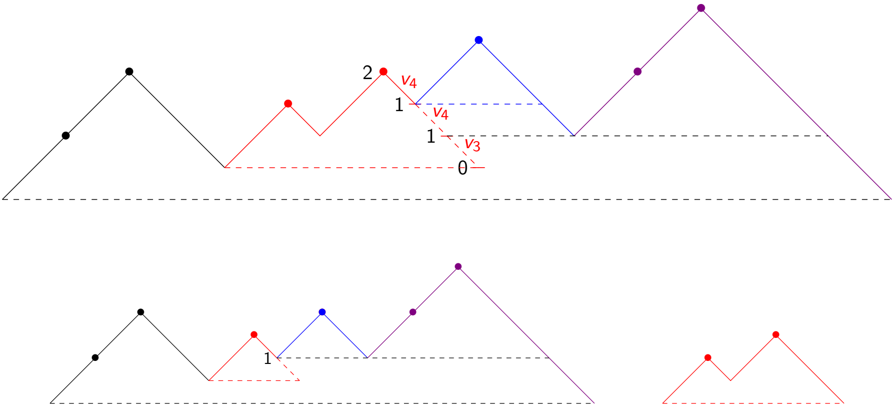



$\pi_1(\delta_3^{6,2}(x)) = (2, 4, 3, 4, 4, 6)$

$\pi_2(\delta_3^{6,2}(x)) = (2, 3)$

# Operadic interpretation of the results by L., P.-R. and R.

## Theorem

- *For* $1 \leq i \leq p$*, the collection of maps*

$$\delta_i^{n,n'} : \mathrm{Dy}^m_{n+n'-1} \to \mathrm{Dy}^m_n \times \mathrm{Dy}^m_{n'}$$

  *endows* $\{\mathcal{C}(n) := \mathrm{Dy}^m_n\}_{n\geq 1}$ *with a co-operadic structure, in* $\mathrm{Sets}$

- *For* $(x, y) \in \mathrm{Dy}^m_n \times \mathrm{Dy}^m_{n'}$ *we have* $(\delta_i^{n,n'})^{-1}(x, y)$ *is an interval in* $\mathrm{Dy}_{n+n'-1}$ *for the m-Tamari order.*
- *The operad* $\mathcal{C}^*$ *in vector spaces is the operad* $\mathcal{D}yck_m$*.*

## Idea of proof

There is a morphism of operads $\mathcal{D}yck_m \to \mathcal{C}^*$ sending $*_i$ to $(m, m+i)^*$.

# Operadic interpretation of the results by L., P.-R. and R.

**Theorem**

- *For $1 \leq i \leq p$, the collection of maps*

$$\delta_i^{n,n'} : \mathrm{Dy}^m_{n+n'-1} \to \mathrm{Dy}^m_n \times \mathrm{Dy}^m_{n'}$$

*endows $\{\mathcal{C}(n) := \mathrm{Dy}^m_n\}_{n\geq 1}$ with a co-operadic structure, in* $\mathrm{Sets}$

- *For $(x, y) \in \mathrm{Dy}^m_n \times \mathrm{Dy}^m_{n'}$ we have $(\delta_i^{n,n'})^{-1}(x, y)$ is an interval in $\mathrm{Dy}_{n+n'-1}$ for the $m$-Tamari order.*
- *The operad $\mathcal{C}^*$ in vector spaces is the operad $\mathcal{D}yck_m$.*

**Corollary**

$$\sum_{x\in \mathrm{Dy}^n_p} x * \sum_{y\in \mathrm{Dy}^m_q} y = \sum_{z\in \mathrm{Dy}^m_{p+q}} z$$

# Section 4– Two algebraic structures on prime $m$-Dyck paths

- Poincaré-Birkoff-Witt Theorem
- Brace algebras
- $m$-brace algebras
- Prime $m$-Dyck paths and binary planar rooted trees
- A formula

# Motivation: Poincaré Birkoff-Witt Theorem

## cas Lie

$$\mathrm{Lie}_{\mathrm{alg}} \underset{[-,]}{\overset{U}{\rightleftarrows}} \mathrm{As}_{\mathrm{alg}} \quad (\perp)$$

$$\mathrm{Lie}_{\mathrm{alg}} \underset{\mathrm{Prim}}{\overset{U}{\rightleftarrows}} \mathrm{As}_{\mathrm{big}} \quad (\perp)$$

$$S^c(\mathbb{L}(V)) = T(V), \;\; e^{-\ln(1-x)} - 1 = \frac{x}{1-x}$$

## Definition

A **brace** algebra is a vector space $B$ equipped with operations $M_{1n} : B^{\otimes n+1} \longrightarrow B$ for $n \geq 0$, satisfying $M_{10} = \mathrm{Id}$ and

$$M_{1n}(M_{1r}(x; y_1, \ldots, y_r); z_1, \ldots, z_n) = \sum M_{1u}(x; z_{(1)}, M_{1,a_2}(y_1; z_{(2)}), z_{(3)}, \ldots, z_{(2r-1)}, M_{1,a_{2r}}(y_k, z_{(2r)}), z_{(2r+1)}),$$

where the sum is taking over all the words (possibly empty) such that the concatenation $z_{(1)} \ldots . z_{(2r+1)} = z_1 \ldots z_n$.

## Example

$M_{11}(M_{11}(x; y_1); z_1) = M_{11}(x, M_{11}(y_1; z_1)) + M_{12}(x; y_1, z_1) + M_{12}(x; z_1, y_1)$.
It is in particular right symmetric (pre-Lie), thus a Lie algebra.

# Motivation: Poincaré Birkoff-Witt Theorem

$$\mathrm{Br}_{\mathrm{a}lg} \underset{M}{\overset{U}{\rightleftarrows}} \mathrm{Dend}_{\mathrm{alg}} \quad (\perp)$$

## Theorem (Ronco)

$$\mathrm{Br}_{\mathrm{a}lg} \underset{\mathrm{Prim}}{\overset{U}{\rightleftarrows}} \mathrm{Dend}_{\mathrm{big}} \quad (\perp)$$

*is an equivalence of categories; furthermore,*

$$T^c(Br(V)) = Dend(V)$$

It amounts to say that: $Dend(V)$ has for basis 1-Dyck paths labelled by $V$ and $Br(V)$ prime 1-Dyck paths labelled by $V$.

## Definition

An $m$-**brace** algebra is a brace algebra $B$ equipped with operations $\bullet_i : B^{\otimes 2} \to B$ for $1 \leq i \leq m-1$, satisfying some relations

- Quadratic relations involving $\bullet_i$ and $\bullet_j$
- Relations of the form $M_{1n}(x \bullet_i y; z_1, \ldots, z_n) = \ldots$

## Towards a Poincaré-Birkoff-Witt Theorem

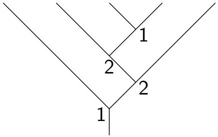
$$m - \mathrm{Br}_{alg} \underset{M}{\overset{U}{\rightleftarrows}} m - \mathrm{Dyck}_{\mathrm{alg}} \quad (\perp)$$

$$m - \mathrm{Br}_{alg} \underset{\mathrm{Prim}}{\overset{U}{\rightleftarrows}} m - \mathrm{Dyck}_{\mathrm{big}} \quad (\perp)$$

$T^c(Br_m(V)) = \mathcal{D}yck_m(V)$: there should be an $m$-brace structure on prime $m$-Dyck paths!

# prime $m$-Dyck paths and rooted planar binary trees

There is a bijection $\mathrm{PDy}_n^m \to Y_n^m$, where $Y_n^m$ is a subset of planar binary trees with $n$ leaves and vertices labelled by elements in $\{1, \ldots, m\}$.
$s \in Y_n^m \iff \forall t \subset s, t = t^l \vee_i t^r$, the root of $t^l$ has label $\geq i$.




$\mapsto \quad (2) \times_1 [((2) \times_2 ((2) \times_1 (2))) \times_2 (2)] = (2) \times_1 (2454) = (23565)$

## Proposition

*Let $X$ be a basis of a vector space $V$. Let $Y^m(X)$ be the vector space spanned by trees in $Y_n^m$ with leaves labelled by $X$. Then $Y^m(X)$ is the m-brace algebra generated by $V$.*

# prime $m$-Dyck paths and rooted planar binary trees

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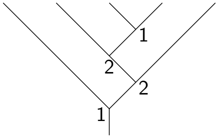


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## Exercise

Let

$$P_m(x) = \sum_{n\geq 1} p_{m,n} x^n$$

Prove that

$$P_m^{-1}(x) = \sum_{k=1}^{m+1} (-1)^{k-1} \binom{m}{k-1} x^k.$$

Thanks you for your attention!

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