A nested family of homotopy theories arising from spectral sequences

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Joint work with Sarah Whitehouse

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The main picture

$$E^\bullet_{\mathcal{C}} : \mathcal{C} \to \mathbf{SpSe}$$

Definition ($\mathcal{E}_r$)

Let f be a morphism in $\mathcal{C}$.

$$f \in \mathcal{E}_r \Longleftrightarrow E^r_{\mathcal{C}}(f) \text{ is a quasi-isomorphism}$$
$$\Longleftrightarrow E^s_{\mathcal{C}}(f) \text{ is an isomorphism}, \forall s > r$$

Questions

- Can we describe the homotopy category $\mathcal{C}[\mathcal{E}_r^{-1}]$?
- Can we compare them when r varies?
- For $F : \mathcal{C} \to \mathcal{D}$, what can we say on the induced $\mathcal{C}[\mathcal{E}_r^{-1}] \to \mathcal{D}[\mathcal{E}_r^{-1}]$?

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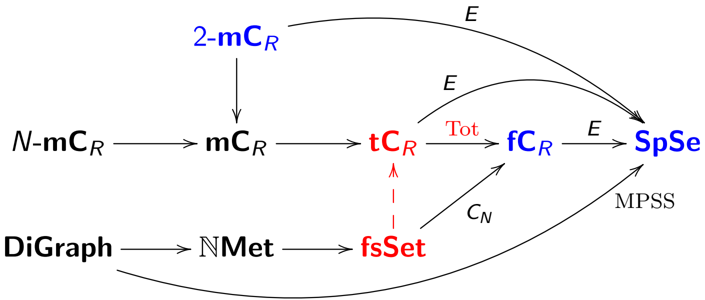
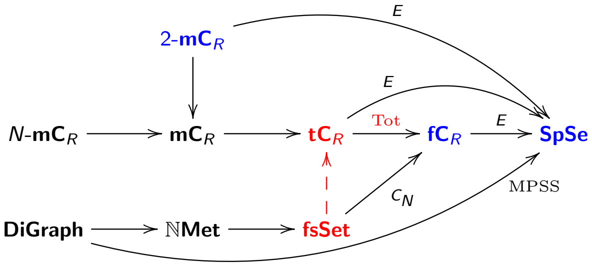


Notation

- $\mathbf{fC}_R$ filtered complexes; **fsSet** filtered simplicial sets
- $\mathbf{mC}_R$ multicomplexes and $\mathbf{tC}_R$ twisted complexes
- N-$\mathbf{mC}_R$ truncated versions of multicomplexes
- **DiGraph** directed graphs; $\mathbb{N}$**Met** metric spaces

Plan of the talk



1. The first line of the diagram
2. Homotopy theory
3. Homological invariants of directed graphs

Section 1

$\mathbf{mC}_R$, $\mathbf{fC}_R$ and **SpSe**

Spectral sequences

R is the underlying ring.

Definition

A *spectral sequence* (A, ψ) is

- a family of complexes of R-modules $\{(A_r, d_r)\}_{r\geq 0}$,
- a family of isomorphisms of R-modules $\{\psi_r : H_*(A_r) \leftarrow A_{r+1}\}_{r\geq 0}$ called *characteristic maps*

A *morphism of spectral sequences* is a family of morphisms of complexes $\{f_r : A_r \to B_r\}_{r\geq 0}$ which is *compatible with characteristic maps*.

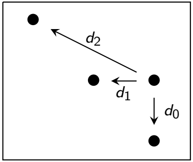
SpSe is the category of spectral sequences.

Spectral sequences: convention and properties

$$\{(A_r, d_r)\}_{r\geq 0} \text{ a spectral sequence.}$$

In practice: A_r is bigraded (over $\mathbb{Z}\times\mathbb{Z}$) and d_r has bidegree $(\pm r, \pm r \pm 1)$.

Today:
d_r is of bidegree $(-r, r-1)$
(A_r, d_r) is called
an *r-bigraded complex*



Proposition

The category **SpSe** *is*

- *an additive category*
- *not complete, not cocomplete*

The categories $\mathbf{mC}_R$ and $\mathbf{tC}_R$

- Same class of objects
- $\mathbf{mC}_R$ has strict morphisms and $\mathbf{tC}_R$ has ∞-morphisms

A *multicomplex or twisted complex* (A, d) is

- a bigraded R-module A
- for every $i \geq 0$ a map $d_i : A \to A$ of bidegree $(-i, i-1)$ such that

$$\sum_{i+j=l} \pm d_i d_j = 0, \ \forall l \geq 0.$$

Morphisms in $\mathbf{tC}_R$ are families of maps $f_i : (A, d) \to (A', d')$ of bidegree $(-i, i)$ satisfying for all $l \geq 0$,

$$\sum_{i+j=l} d'_i f_j = \sum_{i+j=l} \pm f_j d_i$$

Morphisms in $\mathbf{mC}_R$ are *strict morphisms* : $f_i = 0, i > 0$.

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Morphisms in $\mathbf{mC}_R$ are *strict morphisms* : $f_i = 0, i > 0$.

The categories $\mathbf{mC}_R$ and N-$\mathbf{mC}_R$

An *N-multicomplex* is a multicomplex (A, d) such that $d_i = 0, i \geq N$. The category N-$\mathbf{mC}_R$ is the full subcategory of $\mathbf{mC}_R$ of N-multicomplexes.

Example

- $N = 1$: vertical bicomplexes (bigraded)
- $N = 2$: bicomplexes.
- $N = 4$:

$$
\begin{aligned}
d_0^2 = d_3^2 &= 0 \\
d_0 d_1 - d_1 d_0 = d_3 d_2 - d_2 d_3 &= 0 \\
d_0 d_2 - d_1^2 + d_2 d_0 = d_3 d_1 - d_2^2 + d_1 d_3 &= 0 \\
d_0 d_3 - d_1 d_2 + d_2 d_1 - d_3 d_0 &= 0
\end{aligned}
$$

Symmetry $d_i \leftrightarrow d_{N-1-i}$

The functor $\mathrm{Tot} : \mathbf{tC}_R \to \mathbf{fC}_R$

Theorem (Cirici, Egas Santander, L., Whitehouse, 2018)

$$\mathrm{Tot} : \mathbf{tC}_R \to \mathbf{fC}_R$$

is an equivalence of categories when restricted to its essential image.

$$\mathrm{Tot}(A)^n := \prod_{i \leq 0} A^{i,n-i} \oplus \bigoplus_{i>0} A^{i,n-i} \qquad F_p\mathrm{Tot}(A)^n := \prod_{i \leq p} A^{i,n-i}$$

$$d(a)_i := \sum_{m \geq 0} \pm d_m(a_{i+m}), \qquad (\mathrm{Tot}(f)(a))_i := \sum_{m \geq 0} \pm f_m(a_{i+m})$$

There is an enriched version in vertical bicomplexes (1-$\mathbf{mC}_R$).

Recognizing multicomplexes

Proposition

- *If a filtered complex C has a decomposition*

$$C \cong \oplus F_p C / F_{p-1} C$$

  then it is a multicomplex.

- *(Lapin D_∞-module) if in a complex (C, d) we have*

$$d = \sum_{i \geq 0} d_i$$

  and we can put an extra grading so that d_i has bidegree $(-i, i-1)$ then it is a multicomplex.

Examples of multicomplexes in the litterature

- C.T.C. Wall, Resolutions for extensions of groups, Proc. Camb. Philos. Soc. (1961)
- J.-P. Meyer, Acyclic models for multicomplexes, Duke Math. J. (1978)
- A. Banyaga, D.E. Hurtubise, Morse-Bott homology, TAMS (2010)
- J. Cirici, S. Wilson, Dolbeault cohomology for almost complex manifolds, adv. in math 2021.

From $\mathbf{tC}_R$ to $\mathbf{SpSe}$

$$\mathbf{tC}_R \xrightarrow{\mathrm{Tot}} \mathbf{fC}_R \xrightarrow{E} \mathbf{SpSe}, \qquad E_{\mathbf{tC}} : \mathbf{tC}_R \to \mathbf{SpSe}$$

Theorem (L., Whitehouse, Ziegenhagen 2020)

$$E^r_{\mathbf{tC}}(A) = (Z^r(A)/B^r(A),\, \Delta^r)$$

$$Z^r(A) := \Big\{a_0 \in A \mid \exists a_1, \dots, a_{r-1},\quad d_l a_0 = \sum_{\substack{i+j=l \\ j>0}} \pm d_i a_j \text{ , for all } 0 \le l \le r-1\Big\},$$

$$\Delta^r[a_0] = \left[d_r a_0 + \sum_{i=1}^{r-1} \pm d_{r-i} a_i\right]$$

Exercise

in $\mathbf{tC}_R$

E^1

E^2

Section 2

Homotopy Theory

Recall the main picture

$$E^{\bullet}_{\mathcal{C}} : \mathcal{C} \to \mathbf{SpSe}$$

Definition ($\mathcal{E}_r$)

Let f be a morphism in $\mathcal{C}$.

$$\begin{aligned} f \in \mathcal{E}_r \iff & E^r_{\mathcal{C}}(f) \text{ is a quasi-isomorphism} \\ \iff & E^s_{\mathcal{C}}(f) \text{ is an isomorphism}, \forall s > r \end{aligned}$$

Questions

- Can we describe the homotopy category $\mathcal{C}[\mathcal{E}_r^{-1}]$?
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Quillen model category structure

Definition (Quillen model category structure on $\mathcal{C}$)

$\mathcal{C}$ complete and cocomplete. It has 3 classes of morphisms: $\mathcal{C}\mathrm{of}(\hookrightarrow), \mathcal{F}\mathrm{ib}(\twoheadrightarrow)$ and $\mathcal{W}(\simeq)$ satisfying some axioms, among them

$$X \sqcup X \hookrightarrow \mathrm{Cyl}(X) \overset{\simeq}{\twoheadrightarrow} X \text{ (cylinder object)}$$

$$X \overset{\simeq}{\hookrightarrow} \mathrm{Path}(X) \twoheadrightarrow X \times X \text{ (path object)}$$

giving the notion of homotopies (left and right).

Theorem (Quillen)

If $\mathcal{C}$ is a model category, then

$$\mathcal{C}[\mathcal{W}^{-1}] \cong \mathcal{C}^{cf} / \sim$$

Comparing Quillen model category structures

Definition (Quillen adjunction)

An adjunction

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

between two model category structures is a Quillen adjunction, if

$$L(\hookrightarrow) \subset \hookrightarrow \text{ and } R(\twoheadrightarrow) \subset \twoheadrightarrow .$$

It is a Quillen equivalence if it induces an equivalence of categories

$$\mathcal{C}[\mathcal{W}_{\mathcal{C}}^{-1}] \cong \mathcal{D}[\mathcal{W}_{\mathcal{D}}^{-1}]$$

Model category structure is a useful tool to prove comparison theorems.

Model category structure on chain complexes

Theorem

The category of (unbounded) chain complexes admits a model category structure, where:

- *weak equivalences are quasi-isomorphisms*
- *fibrations are surjective morphisms*

The model category structures on $\mathbf{mC}_R$ and $\mathbf{fC}_R$

$$N\text{-}\mathbf{mC}_R \longrightarrow \mathbf{mC}_R \xrightarrow{\mathrm{Tot}} \mathbf{fC}_R \xrightarrow{E^\bullet} \mathbf{SpSe}$$

Theorem

Let $\mathcal{C}$ be the category N-$\mathbf{mC}_R$, $\mathbf{mC}_R$ or $\mathbf{fC}_R$ and $r \geq 0$. The category $\mathcal{C}$ admits a model category structure, denoted $\mathcal{C}_r$, where:

- *weak equivalences are morphisms in $\mathcal{E}_r$*
- *fibrations are morphisms satisfying surjectivity conditions ($E^i(f)$ is bidegree-wise surjective for $0 \leq i \leq r$, or closely related conditions).*

- filtered complexes, bicomplexes [Cirici–Egas-Santander–L–Whitehouse 2020]
- (truncated) multicomplexes [Fu–Guan–L–Whitehouse 2022]

Comparisons

Theorem (Cirici–Egas-Santander–L–Whitehouse 2020)

Via the Deligne adjunction "shift" and "decalage", the model category structures $(\mathbf{fC}_R)_r$ *and* $(\mathbf{fC}_R)_{r+1}$ *are Quillen equivalent.*

Theorem (X. Fu, A. Guan, L., Whitehouse, 2022)

For $2 \leq n \leq l \leq \infty$ *and* $r \geq 0$ *the adjunction*

$$(n\text{-}\mathbf{mC}_R)_r \underset{i_{l,n}}{\overset{p_{l,n}}{\leftrightarrows}} (l\text{-}\mathbf{mC}_R)_r$$

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The relative category $(\mathbf{SpSe}, \mathcal{E}_r)$

SpSe is not complete, nor cocomplete: no model category structure.

Theorem (L., Whitehouse, HHA 24)

$(\mathbf{SpSe}, \mathcal{E}_r)$ *is an almost Brown category of fibrant objects.*

Theorem (L., Whitehouse, arxiv 2406.02777)

- $(\mathbf{SpSe}, \mathcal{E}_0)$ *is a relative category which is homotopically full in a model category* $(\mathbf{ESpSe}, \mathcal{E}'_0)$.
  $\Rightarrow$ *the relative category* $(\mathbf{SpSe}, \mathcal{E}_0)$ *is an* $(\infty, 1)$*-category in the model of Barwick-Kan.*
- $(\mathbf{SpSe}, \mathcal{E}_r)$ *has* $(\mathbf{SpSe}, \mathcal{E}_0)$ *as fibrant replacement in the model of Barwick-Kan.*

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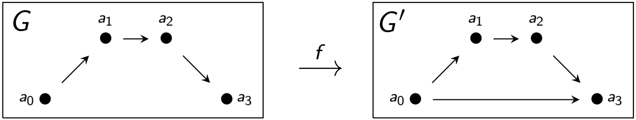
Section 3

Directed graphs and metric spaces.

Directed graphs and $\mathbb{N}$-metric spaces

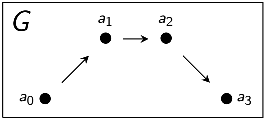
- **DiGraph**
  - Objects: a directed graph G is a pair of sets (V_G, E_G) with $\Delta_{V_G} \subset E_G \subset V_G \times V_G$
  - Morphisms: $f : V_G \to V_{G'}$ such that $(f \times f)(E_G) \subset E'_G$.

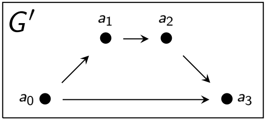


- $\mathbb{N}$**Met**
  - Objects: (X, d), $d : X \times X \to \mathbb{N} \cup \{\infty\}$, satisfying triangular inequality and $d(x, y) = 0 \Leftrightarrow x = y$.
  - Morphisms: 1-Lipschitz maps.

From Directed graphs to $\mathbb{N}$-metric spaces

$$
\begin{aligned}
X(-):\qquad \mathbf{DiGraph} &\longrightarrow \mathbb{N}\mathbf{Met}\\
G=(V,E) &\longmapsto (V, d_{\min})
\end{aligned}
$$





$d(a_1, a_0) = +\infty$
$d(a_0, a_3) = 3$

$d(a_1, a_2) = 1$
$d(a_0, a_3) = 1$

From $\mathbb{N}$-metric spaces to filtered simplicial sets

Definition (The nerve of a metric space (X, d))

$$F_pN(X)_n = \{(x_0, \ldots, x_n), \sum_{i=0}^{n-1} d(x_i, x_{i+1}) \leq p\}$$

$$\mathbf{DiGraph} \longrightarrow \mathbb{N}\mathbf{Met} \xrightarrow{N} \mathbf{fsSet} \xrightarrow{C_N} \mathbf{fC}_R \xrightarrow{E^\bullet} \mathbf{SpSe}$$

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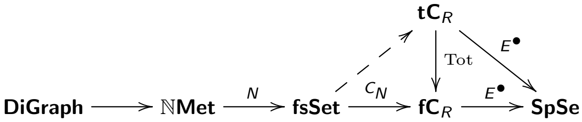

Key Lemma

The normalized chain complex C_N : **fsSet** $\to$ $\mathbf{fC}_R$ factorizes through $\mathrm{Tot} : \mathbf{tC}_R \to \mathbf{fC}_R$.

$\Rightarrow$ $E^\bullet(G), E^\bullet(X)$ via LWZ formulas.

A litlle bit of history

- Magnitude Homology: Richard Hepworth and Simon Willerton. *Categorifying the magnitude of a graph*. HHA, 2017.
- Path Homology: Alexander Grigor'yan, Yong Lin, Yuri Muranov, and Shing-Tung Yau. Pure Appl. Math. Q., 2014.
- The magnitude path spectral sequence: Yasuhiko Asao. *Magnitude homology and path homology*. Bull. Lond. Math. Soc., 2022. denoted $MPSS(G)$ $(= E^{\bullet}(G))$.
  - $E^1(G)$ magnitude homology
  - $E^2(G)_{p0}$ path homology
  - $E^2(G)$ bigraded path homology, after R. Hepworth and E. Roff

Results of R. Hepworth and E. Roff

Theorem (Mayer–Vietoris)

Let $i\colon A \hookrightarrow X$ be a cofibration, and $f : A \to B$ be any map in **DiGraph**. *Their pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \cup_A B \end{array}$$

gives rise to a long exact sequence on page 2 of the magnitude-path spectral sequence:

$$\to E^2_{p,q}(A) \to E^2_{p,q}(X) \oplus E^2_{p,q}(B) \to E^2_{p,q}(X \cup_A B) \to E^2_{p-2,q+1}(A) \to$$

Cofibration category?

Theorem (D.Carranza, B. Doherty, K. Kapulkin, M. Opie, M. Sarazola, and Liang Ze Wong, to appear in Algebr. Comb)

$(\textbf{DiGraph}, \mathrm{Cof}, \mathrm{PathHom.})$ *is a Brown category with cofibrant objects*

Question

Is there a notion of r-cofibration, such that $(\mathcal{C}, r\mathrm{-cof}, \mathcal{E}_r)$ is a Brown category of cofibrant objects where $\mathcal{C}$ is **DiGraph** or $\mathbb{N}$**Met**?