ICRA 2026 SUMMER SCHOOL: ∞-CATEGORIES

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Abstract. This notes are based on a three-hour introductory mini-course on ∞-categories given at the workshop *Representations of Algebras*, held in Grenoble in June 2026.

1. Introduction

There are already so many excellent introductory notes on ∞-categories, (see e.g. the notes by Martin Gallauer or the slides by Marco Robalo) that the aim of these notes is not to offer yet another introduction to the subject. Rather, the aim is to present the minimal amount of material (definitions, propositions, examples), together with references, needed to understand, at least, some statement in mathematics involving ∞-categories (see the final section on DG categories and stable ∞-categories). I have also tried to avoid as much as possible the language of Quillen model categories, although it remains a very efficient tool for constructing different models for the theory of ∞-categories. Instead, these notes focus on the model introduced by Joyal and developed by Lurie, namely *quasicategories*. Among quasicategories are *Kan complexes* which lie at the crossroads of topological spaces and groupoids. Among Kan complexes are *contractible Kan complexes* which lie at the crossroads of contractible topological spaces and contractible groupoids.

The philosophy of ∞-categories that we would like to emphasize is that working with them is much like working in an ordinary 1-categorical setting except that sets are replace by Kan complexes. More precisely, whenever one encounters a set S of solutions to a given problem in ordinary category theory, in the ∞-world one instead has a Kan complex of solutions.

From this perspective, ordinary category theory seeks

existence + uniqueness.

In contrast, ∞-category theory replaces uniqueness by

contractibility of the Kan complex of solutions.

The guiding principle throughout these notes is therefore that uniqueness statements in ordinary category theory should be replaced by contractibility statements in ∞-category theory.

Following Lurie ∞-categories will always mean $(\infty, 1)$-categories.

Warning. We will not discuss set-theoretic size issues here, so we will never specify whether a category is small or locally small. For a rigorous theory, it is essential to take this into account, and we refer the reader to [11, Section 1.2.15] for the necessary background.

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Thanks. I would like to thank the organizers and the scientific committee of the workshop for giving me the opportunity to deliver this mini-course. Special thanks go to Bernhard Keller for our breakfast discussions, and to Yann Palu for explaining (and drawing on the board) the diagram relating topological and algebraic enhancements of extriangulated categories. Over the years, I have also benefit greatly from the patience of Yonatan Harpaz and Geoffroy Horel who have answered my many questions on ∞-categories. Finally, a conversation with Jordan Levin just before I left for the workshop was particularly inspiring.

Contents

2. A very useful construction in category theory

Let $\mathbf{A}$ be a small category and $\mathbf{C}$ be a cocomplete category. Denote the category of functors from $\mathbf{A}^{\mathrm{op}}$ to $\mathbf{Set}$ by $\hat{\mathbf{A}}$. The Yoneda functor

$$よ : \mathbf{A} \to \widehat{\mathbf{A}}$$

sends an element $a \in \mathbf{A}$ to the representable functor $\mathrm{Hom}_{\mathbf{A}}(-, a)$. Any functor $F : \mathbf{A} \to \mathbf{C}$ induces, by left Kan extension, an adjunction

$$\begin{tikzcd} \widehat{\mathbf{A}} \arrow[r, shift left, "|-|_F"] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, shift left, "N_F"] \end{tikzcd} \tag{1}$$

where

$$N_F(c)(a) = \mathrm{Hom}_{\mathbf{C}}(F(a), c) \text{ and } |\mathrm{Hom}_{\mathbf{A}}(-, a)|_F = F(a).$$

In here this adjunction appears many times:

- An adjunction between simplicial sets and categories, Section 3.1
- An adjunction between simplicial sets and topological spaces, Section 3.3
- An adjunction between simplicial sets and DG categories, Section 6.1
- An adjunction between simplicial sets and simplicial categories, Section 6.2

3. Simplicial sets

For this section, one can refer to kerodon (Sections 1.1 to 1.3).

3.1. **The nerve construction.** Let $\mathbf{\Delta}$ be the simplex category. Its set of objects is in bijection with $\mathbb{N}$. An object is denoted $[n]$ and is thought as the set $\{0, 1, \ldots, n\}$. Morphisms in $\mathbf{\Delta}$ are non-decreasing maps. There is a functor $\mathbf{\Delta} \to \mathbf{Cat}$ sending $[n]$ to the linearly ordered set $0 < 1 < \ldots < n$, seen as a category. This functor is a full embedding. In particular, the category $\mathbf{\Delta}$ can be defined as the full subcategory of $\mathbf{Cat}$ generated by the categories $0 < 1 < \ldots < n$, for $n \in \mathbb{N}$. By the construction (1), this yields an adjunction

$$\widehat{\mathbf{\Delta}} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{Cat} \tag{2}$$

The category $\widehat{\mathbf{\Delta}}$ is called the category of *simplicial sets* and is denoted $\mathbf{sSet}$ and the functor N is called the *nerve functor*.

3.2. **Simplicial sets.** The category $\mathbf{\Delta}$ is generated by the morphisms $\delta_i : [n-1] \to [n]$ and $\sigma_i : [n+1] \to [n]$, for $0 \leq i \leq n$ where δ_i "forgets" i and σ_i "repeats" i. It results to an alternative definition of simplicial sets.

Definition 3.2.1. A *simplicial set* X is a collection of sets $(X_n)_{n\geq 0}$ together with maps $d_i : X_n \to X_{n-1}$ called *face maps* and $s_i : X_n \to X_{n+1}$ called *degeneracy maps*, for all $n \in \mathbb{N}$ and all $0 \leq i \leq n$, that satisfy some relations. The elements in X_n are called *n-simplices* and the elements in $X_n \setminus (\cup_{i=0}^{n-1} s_i X_{n-1})$ are called *non-degenerate n-simplices*. The set of those elements is denoted $\mathrm{ND}(X_n)$.

Note that a simplicial set X gives rise to a quiver $(X_0, X_1, s = d_1, t = d_0)$.

Notation 3.2.2. The representable functor $\mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$ is denoted Δ^n. We have

$$(\Delta^n)_k = \{(a_0, \ldots, a_k), 0 \leq a_0 \leq \ldots \leq a_k \leq n\} \text{ and}$$

$$\mathrm{ND}((\Delta^n)_k) = \{(a_0, \ldots, a_k), 0 \leq a_0 < \ldots < a_k \leq n\} = \{S \subset [n], \mathrm{card}(S) = k+1\}.$$

In particular Δ^n has $\binom{n+1}{k+1}$ non-degenerate k-simplices. Note that Δ^0 is the terminal object of $\mathbf{sSet}$.

- The *boundary of* Δ^n is its simplicial subset generated by all its non-degenerate k-simplices for $0 \leq k < n+1$ and is denoted $\partial\Delta^n$.
- For $0 \leq i \leq n$, the *i-horn of* Δ^n is its simplicial subset generated by all its non-degenerate k-simplices with $0 \leq k \leq n$ except the non-degenerate n-simplex $[n] \setminus \{i\}$ and is denoted Λ^n_i .
- An *inner horn* is an i-horn Λ^n_i with $0 < i < n$.
- The *spine* of Δ^n is its 1-skeletton. It is the simplicial subset generated by all k-simplices with $0 \leq k \leq 1$. It is denoted I_n. Note that $I_1 = \Delta^1$ and $I_2 = \Lambda^2_1$.

The Yoneda lemma implies that

$$\mathrm{Hom}_{\mathbf{\Delta}}(\Delta^n, X) \cong X_n.$$

We will use indiscriminately the notation $\sigma : \Delta^n \to X$ or $\sigma \in X_n$.

We will write $a \xrightarrow{f} b$ or $f : a \to b$ for an element $f \in X_1$ such that $d_0 f = b$ and $d_1 f = a$. Such an element is called an *arrow* or a *morphism*, depending on the context. Similarly given a triangle $\partial\Delta^2 \to X$ depicted as follows

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

we say that the *triangle is commutative* if there exists $\sigma : \Delta^2 \to X$ such that $d_0\sigma = g, d_1\sigma = h$ and $d_2\sigma = f$. Such a commutative triangle is depicted as follows

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & {} & a_2 \end{tikzcd}$$

Maps from horns to a simplicial set are depicted in the following pictures:

$$\Lambda^2_1 \to X \qquad\qquad \Lambda^2_0 \to X \qquad\qquad \Lambda^2_2 \to X$$

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] & & a_2 \end{tikzcd} \quad \begin{tikzcd} & a_1 & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd} \quad \begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[rr, "h"'] & & a_2. \end{tikzcd}$$

Since the category **sSet** is a category of functors, it is complete and cocomplete, where limits and colimits are computed in **Set** (pointwise). It is also cartesian closed.

Proposition 3.2.3. *The category* **sSet** *is cartesian closed, with internal hom sets denoted* Map(X, Y)*, for X and Y simplicial sets, and defined as*

$$\mathrm{Map}(X, Y)_n = \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y).$$

Remark 3.2.4. Lurie uses the notation Y^X and Cisinski the notation $\underline{\mathrm{Hom}}(X, Y)$ for Map(X, Y).

3.3. **Simplicial sets and topological spaces.** The following definition provides a functor $\mathbf{\Delta} \to$ **Top** where **Top** is the category of topological spaces.

Definition 3.3.1. The *standard n-simplex* $|\Delta^n|$ is the topological subspace of the euclidian space $\mathbb{R}^{n+1}$ defined by

$$|\Delta^n| = \left\{ (t_0, \ldots, t_n) \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1. \right\}$$

For $\varphi : [n] \to [m] \in \mathbf{\Delta}$ define $|\varphi| : |\Delta^n| \to |\Delta^m|$ by

$$\varphi(t_0, \ldots, t_n) = (u_0, \ldots, u_m), \text{ with } u_j = \sum_{k|\varphi(k)=j} t_k$$

By Construction (1) one gets an adjunction

$$\begin{tikzcd} \widehat{\mathbf{\Delta}} \arrow[r, shift left=1.5, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, shift left=1.5, "\mathrm{Sing}(-)"] \end{tikzcd} \tag{3}$$

The left adjoint is called the *geometric realization*. The right adjoint sends a topological space X to its *singular simplicial set* Sing(X) whose set of n-simplices is

$$\mathrm{Sing}(X)_n = \{f : |\Delta^n| \to X, \text{ continuous }\}.$$

3.4. **The fundamental property of the nerve construction.** Let us first describe the functors involved in (2).

Description of the nerve functor. Let $\mathbf{C}$ be a category. We have

$$N(\mathbf{C})_n = \{(f_1, \dots, f_n) \mid f_i \in \mathrm{Hom}_{\mathbf{C}}(a_{i-1}, a_i), \text{ for some objects } a_0, \dots, a_n \text{ of } \mathbf{C}\},$$

with

$$d_i(f_1, \dots, f_n) = \begin{cases} (f_2, \dots, f_n) & i = 0 \\ (f_1, \dots f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, \dots f_n) & 1 \leq i \leq n-1 \\ (f_1, \dots, f_{n-1}) & i = n \end{cases}$$

and

$$s_i(f_1, \dots, f_n) = (f_1, \dots, f_i, \mathrm{id}_{a_i}, f_{i+1}, \dots, f_n),\ 0 \leq i \leq n.$$

Note that, by construction,

$$N(\{0 < 1 < \dots < n\}) = \Delta^n.$$

Example 3.4.1. Let J be the groupoid generated by two objects 0 and 1 and an isomorphism between them. It is the category with two morphisms $u : 0 \to 1$ and $v : 1 \to 0$ such that $v \circ u = \mathrm{id}_0$ and $u \circ v = \mathrm{id}_1$. For each n the set of non-degenerate n-simplices of the nerve $N(J)$ has exactly two elements: $(u, v, u, v, \dots)$ and $(v, u, v, u, \dots)$. We have that the geometric realization $|N(J)|$ is homeomorphic to the infinite sphere $\mathbb{S}^\infty$. By abuse of notation we will write also J for the simplicial set $N(J)$. It is sometimes called the *walking isomorphism* and we have a morphism $u : \Delta^1 \to J$ in **sSet**. This morphism will play a role when we will define invertible morphisms in a quasicategory, in Definition 4.3.2.

Description of the functor τ. Let X be a simplicial set. The category $\tau(X)$ is the category freely generated by the quiver $(X_0, X_1, s = d_1, t = d_0)$ mod out by the relations,

$$\begin{aligned} [s_0 x] &= \mathrm{id}_x, & x &\in X_0, \\ [d_1\sigma] &= [d_0\sigma] \circ [d_2\sigma], & \sigma &\in X_2. \end{aligned}$$

where $[f]$ denotes the morphism in the category associated to $f \in X_1$.

Example 3.4.2. For X a topological space, $\tau(\mathrm{Sing}(X))$ is the fundamental groupoid of X.

Proposition 3.4.3. *The counit of the adjunction (τ, N) is an isomorphism. Equivalently, the nerve functor N :* **Cat** $\to$ **sSet** *is fully faithful.*

Recognizing the nerve of a category. The following theorem provides an important characterization of nerves of categories as simplicial sets satisfying lifting properties, which can be found in [6, Prop 1.4.11 and Prop 1.4.13].

Theorem 3.4.4. *Let X be a simplicial set. The following properties are equivalent*

(a) There exists a category $\mathbf{C}$ *such that $X \cong N(\mathbf{C})$*
(b) For each $n \in \mathbb{N}$ and each $0 < k < n$, any map $f : \Lambda^n_k \to X$ in **sSet** *admits a unique extension $\Delta^n \to X$, that is, there exists a unique lift in the diagram*

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

(c) *For each $n \in \mathbb{N}$ and each $f : I_n \to X$ in* **sSet** *, there is a unique lift in the diagram*

$$\begin{tikzcd} I_n \arrow[r, "f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Remark 3.4.5. Item (b) amounts to saying that the restriction map

$$X_n \to \mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, X)$$

is a bijection in **Set**. Item (c) amounts to saying that the restriction map, called the *Segal map*

$$X_n \to \underbrace{X_1 \times_{X_0} \ldots \times_{X_0} X_1}_{n-\text{times}}$$

is a bijection in **Set**.

We will see in next section a model for ∞-categories known as quasicategories. They are simplicial sets where in Item (b) of Theorem 3.4.4 the lift in the diagram exists but is not necessarily unique. Condition in Item (c) of Theorem 3.4.4 can also be adapted and relaxed in the category of simplicial spaces, and gives rise to the notions of Segal categories and Segal spaces, which are other models for ∞-categories. We refer to the book [2] by J. Bergner for details on these models.

4. Quasicategories and Kan complexes

4.1. **Main definitions.** We first give the original definition of quasicategories (and Kan complexes), Kan fibrations and trivial fibrations. For this we will need some terminology borrowed from Quillen's theory of model category structure.

Definition 4.1.1. Let $g : X \to Y$ be a morphism in **sSet**, and S a class of morphisms in **sSet**. We say that g has the *right lifting property* with respect to S if any diagram of the form below, with $f : A \to B$ in S admits a lift

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "b"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

that is, there exists $h : B \to X$ with $hf = a$ and $gh = b$. By abuse of terminology we say that X has the right lifting property with respect to S if the unique morphism $X \to \Delta^0$ has the right lifting property with respect to S.

Remark 4.1.2. Note that the class of morphisms having the right lifting property with respect to S is stable under pullbacks.

Definition 4.1.3. A simplicial set X is

(a) a *quasicategory* if X has the right lifting property with respect to the set

$$\left\{\Lambda^n_k \hookrightarrow \Delta^n, n \in \mathbb{N}, 0 < k < n\right\}$$

(b) a *Kan complex* if X has the right lifting property with respect to the set

$$\left\{\Lambda^n_k \hookrightarrow \Delta^n, n \in \mathbb{N}, 0 \leq k \leq n\right\}$$

(c) a *contractible Kan complex* if X has the right lifting property with respect to all monomorphisms.

In particular, any contractible Kan complex is a Kan complex and any Kan complex is a quasicategory.

Example 4.1.4. Here is a small list of examples and non-examples.

- The nerve of a category $\mathbf{C}$ is a quasicategory. It is a Kan complex if and only if $\mathbf{C}$ is a groupoid. Hence Δ^n is a quasicategory. It is a Kan complex if and only if $n = 0$. The walking isomorphism J is a Kan complex.
- For X a topological space, the singular simplicial set $\mathrm{Sing}(X)$ is a Kan complex.
- Any simplicial group, that is, a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Group}$ is a Kan complex.
- The simplicial sets Λ^2_0 and Λ^2_2 are quasicategories, whereas the simplicial set Λ^2_1 is not.

Definition 4.1.5. A morphism of simplicial sets $f : X \to Y$ is

(a) a *Kan fibration* if f has the right lifting property with respect to the set of morphisms

$$\left\{\Lambda^n_k \hookrightarrow \Delta^n, n \in \mathbb{N}, 0 \leq k \leq n\right\}$$

(b) a *trivial fibration* if f has the right lifting property with respect to the class of monomorphisms.

Remark 4.1.6. A simplicial set K is a contractible Kan complex if and only if $K \to \Delta^0$ is a trivial fibration. If K is a contractible Kan complex then its geometric realization $|K|$ is a contractible topological space. Indeed we will see later a characterization of contractible Kan complexes via J-homotopy equivalences (see 4.4.3). There is a model category structure, the *Kan-Quillen model category structure* on $\mathbf{sSet}$ whose cofibrations are monomorphisms, fibrations are Kan fibrations, and weak equivalences are weak homotopy equivalences (defined through geometric realization). Every object is cofibrant and the fibrant-cofibrant objects are Kan complexes. In Quillen's framework, the Whitehead theorem states that any weak equivalence between Kan complexes is a J-homotopy equivalence.

We now state two key propositions in the theory of ∞-categories.

Proposition 4.1.7. *Let X be a simplicial set. If Y is a quasicategory, then so is* $\mathrm{Map}(X, Y)$*. If Y is a Kan complex, then so is* $\mathrm{Map}(X, Y)$*.*

Corollary 4.1.8. *Let* $\mathbf{QCat}$ *and* $\mathbf{Kan}$ *be the full subcategories of* $\mathbf{sSet}$ *of quasicategories and Kan complexes respectively. These categories are cartesian closed.*

For next proposition we refer to [6, Cor 3.2.9 and Cor 3.7.6]

Proposition 4.1.9. *The following are equivalent*

(a) X is a quasicategory.
(b) The morphism of simplicial sets (induced by restriction) $\mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$ *is a trivial fibration.*
(c) The morphism of simplicial sets (induced by restriction) $\mathrm{Map}(\Delta^n, X) \to \mathrm{Map}(I_n, X)$ *is a trivial fibration.*

4.2. **Quasicagories are "almost" categories.** Let X be a quasicategory and $f : a \to b, g : b \to c$ two elements in X_1. By Definition 4.1.3, there exists $\sigma \in X_2$ such that the following triangle is commutative

$$\begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\ a \arrow[ur, "f"] \arrow[rr, "d_1\sigma"'] & {} & c \end{tikzcd}$$

Hence, $d_1\sigma$ can be seen as a composition of f and g and may be denoted "$g \circ_\sigma f$". There is a simplicial set $\mathrm{Comp}(f, g)$ of all possible compositions defined as the pullback in **sSet** of the diagram

$$\begin{tikzcd} \mathrm{Comp}(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^2, X) \arrow[d] \\ \Delta^0 \arrow[r, "(f,g)"] & \mathrm{Map}(\Lambda^2_1, X) \end{tikzcd}$$

From Proposition 4.1.9 and Remark 4.1.2 we deduce that $\mathrm{Comp}(f, g)$ is a contractible Kan complex. This fact illustrates the general philosophy: in the 1-categorical setting, there is a single possible composition, and in the ∞-world, there is a contractible Kan complex of possible compositions. One also says that composition is *essentially unique*. Moreover, given a family $(f_i : a_{i-1} \to a_i)_{1 \leq i \leq n}$ of elements in X_1, then $\mathrm{Comp}(f_1, \ldots, f_n)$ defined as the pullback of the diagram

$$\begin{tikzcd} \mathrm{Comp}(f_1,\ldots,f_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^n, X) \arrow[d] \\ \Delta^0 \arrow[r, "(f_1,\ldots,f_n)"] & \mathrm{Map}(I_n, X) \end{tikzcd}$$

is a contractible Kan complex. Hence, not only do we have a contractible Kan complex of choices for the composition of two morphisms, but we also have a contractible Kan complex of choices for the composition of n morphisms. It follows that composition is associative and unital up to a contractible Kan complex.

4.3. **The homotopy category of a quasicategory.** In [3], Boardman and Vogt introduced the notion of a quasicategory (under the name *weak Kan complex*) and associated to each quasicategory X a category $\mathrm{ho}(X)$, whose hom-sets are obtained as the quotient of X_1 by a suitable equivalence relation. They called $\mathrm{ho}(X)$ the fundamental category of X; today it is usually referred to as the *homotopy category of* X, and it coincides with the category $\tau(X)$.

More precisely, let $f, g : a \to b$ be two elements of X_1 with same source and target. By [3, Definition 4.10], f is *homotopic* to g if there is a commutative triangle

$$\begin{tikzcd} & b \arrow[dr, "s_0 b"] \arrow[d, phantom, "\sigma"] & \\ a \arrow[ur, "f"] \arrow[rr, "g"'] & {} & b \end{tikzcd}$$

By [3, Lemma 4.11] this relation is an equivalence relation if X is a quasicategory, and by [3, Proposition 4.12] it induces a category $\mathrm{ho}(X)$, whose objects are the 0-simplices of X and morphisms are homotopy classes of elements in X_1. For

$f : a \to b \in X_1$ we denote by $[f]$ the corresponding element in $\mathrm{Hom}_{\mathrm{ho}(X)}(a, b)$. The following can be found in e.g. kerodon, Section 1.4.5.

Proposition 4.3.1. *Let X be a quasicategory. The categories $\tau(X)$ and $\mathrm{ho}(X)$ are isomorphic.*

In the sequel, we will write $\mathrm{ho}(X)$ instead of $\tau(X)$ when X is a quasicategory, in order to emphasize that the morphisms in the homotopy category are obtained by quotienting out an equivalence relation. This notation also has the advantage that invertible morphisms and Kan complexes can be characterized in terms of the homotopy category, as we now explain.

Definition 4.3.2. Let X be a quasicategory. An element $f \in X_1$ is an *invertible morphism* if $[f]$ is an isomorphism in $\mathrm{ho}(X)$.

Proposition 4.3.3. [10, Corollary 1.4] *A quasicategory X is a Kan complex if and only if its homotopy category $\mathrm{ho}(X)$ is a groupoid.*

Remark 4.3.4. If X is a Kan complex, then it is clear that $\mathrm{ho}(X)$ is a groupoid, since it has the right lifting property with respect to the horn inclusions $\Lambda^2_0 \to \Delta^2$ and $\Lambda^2_2 \to \Delta^2$. The more difficult part of the proposition is the converse: if X is a quasicategory such that $\mathrm{ho}(X)$ is a groupoid, then X has the right lifting property with respect to the horn inclusions $\Lambda^n_k \to \Delta^n$, for every $n \in \mathbb{N}$ and every $k \in 0, n$.

The following proposition is an important tool for detecting invertible morphisms in quasicategories. Recall that J denotes the walking isomorphism introduced in Example 3.4.1.

Proposition 4.3.5. [10, Corollary 1.6] *An element $f \in X_1$ is an invertible morphism if and only if the morphism $\Delta^1 \to X$ which represents f can be extended along the inclusion $u : \Delta^1 \to J$.*

4.4. **Comparing quasicategories.** This section is inspired by Dugger and Spivak, especially [7, Section 2], entitled ”A quick introduction to quasicategories”. There, J is denoted by E_1.

By analogy with the 1-categorical setting, a *functor* $F : X \to Y$ between two quasicategories is a morphism of simplicial sets. Note that it is a 0-simplex in the quasicategory $\mathrm{Map}(X, Y)$ (see Proposition 4.1.7).

Definition 4.4.1. Two functors $F, G : X \to Y$ between two quasicategories are *J-homotopic* if there exists $\rho : X \times J \to Y$ such that, for each $\sigma \in X_n$, $\rho(\sigma, 0) = F(\sigma)$ and $\rho(\sigma, 1) = G(\sigma)$.

Remark 4.4.2. The above definition amounts to the existence of an invertible morphism ρ in $\mathrm{Map}(X, Y)_1 = \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^1, Y)$ such that $d_1\rho = G$ and $d_0\rho = F$. Equivalently, since $\mathrm{Map}(X, Y)$ is a quasicategory by Proposition 4.1.7, there exists an isomorphism τ in $\mathrm{Hom}_{\mathrm{ho}(\mathrm{Map}(X,Y))}(F, G)$. This proves that the relation ”being J homotopic” is an equivalence relation on $\mathrm{Map}(X, Y)_0 = \mathrm{Hom}_{\mathbf{sSet}}(X, Y)$. We denote by $[X, Y]_J$ the quotient of $\mathrm{Hom}_{\mathbf{sSet}}(X, Y)$ by this relation. Note that to define this notion, we only need the hypothesis that Y is a quasicategory, and X can be any simplicial set.

Definition 4.4.3. An equivalence of ∞-categories between the quasicategories X and Y is a *J-homotopy equivalence* $F : X \to Y$, that is, there exists a functor $G : Y \to X$ such that $F \circ G : Y \to Y$ is J-homotopic to id_Y and $G \circ F : X \to X$ is J-homotopic to id_X.

Remark 4.4.4. If $X = N(\mathbf{C})$ and $Y = N(\mathbf{C}')$, then an equivalence of ∞-categories between X and Y is an equivalence of categories beween $\mathbf{C}$ and $\mathbf{C}'$. A Kan complex K is a contractible Kan complex if and only if $K \to \Delta^0$ is a J-homotopy equivalence.

The next proposition can be found in [6, Section 3.7.1], although it is presented differently, using tools that we have chosen to omit from our presentation.

Proposition 4.4.5. *Let X be a quasicategory and $x, y \in X_0$. The pullback $X(x, y)$ of the diagram*

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^1, X) \arrow[d] \\ \Delta^0 \arrow[r, "(x,y)"] & \mathrm{Map}(\partial\Delta^1, X) = X \times X \end{tikzcd}$$

is a Kan complex.

Remark 4.4.6. The 0-simplices of $X(x, y)$ are the morphisms $f : x \to y$ in X_1.

We are now in position to characterize equivalences of ∞-categories as one does in the 1-categorical setting.

Definition 4.4.7. Let $F : X \to Y$ be a functor between quasicategories. We say that the functor F is

(a) *fully faithful* if for every $x, y \in X_0$ we have $X(x, y) \to Y(Fx, Fy)$ is a J-homotopy equivalence between Kan complexes
(b) *essentially surjective* if for every $y \in Y_0$ there exists $x \in X_0$ and an invertible morphism $Fx \to y$ in Y_1.

Next theorem can be found in [6, Theorem 3.9.7]

Theorem 4.4.8. *$F : X \to Y$ is an equivalence of ∞-categories if and only if F is fully faithful and essentially surjective.*

4.5. **Conclusion of the section.** We have chosen to be as direct as possible in introducing quasicategories, a model for ∞-categories and have picked the main theorems and propositions from the literature so that one can work with quasicategories, "almost as they were 1-categories", following the philosophy presented in the introduction. The propositions stated here do not always appear in the same order as in the literature. The reason is that, in the original developments by Joyal, Lurie, and Cisinski, which proceedeed in different directions, the search for a model category structure on **sSet** describing the homotopy types of ∞-categories was one of the driving forces behind the constructions. The Kan-Quillen model category structure on **sSet** (see 4.1.6) served as the benchmark model. An excellent textbook on this model structure and beyond is the book by Goerss and Jardine [9]. We have decided not to introduce the language of model categories; however, this language does make certain constructions easier, and we reproduce Joyal's structure theorem as it is stated in [7, Theorem 2.13].

Theorem 4.5.1. *There exists a unique model category structure on* **sSet** *in which the cofibrations are the monomorphisms and the fibrant objects are the quasicategories, called the Joyal model category structure. In that case, weak equivalences are called Joyal weak equivalences and the class they form is denoted by $\mathcal{W}_{Joyal}$.*

In the Kan-Quillen model category structure, the cofibrations are the same as in Joyal's, but there are fewer fibrant objects (the Kan complexes) and therefore more weak equivalences, the *weak homotopy equivalences*, whose class is denoted $\mathcal{W}_{\mathrm{KQ}}$. Both model categories share the same class of acyclic fibrations: the trivial fibrations. Since for any quasicategory X (or Kan complex) there is a factorization of the codiagonal

$$X \times \partial\Delta^1 \to X \times J \to X$$

by a cofibration followed by an acyclic fibration, the theory of Quillen tells us that J-homotopy is an equivalence relation on $\mathrm{Hom}_{\mathbf{sSet}}(A, X)$ when X is a quasicategory (compare with 4.4.2) and A is any simplicial set.

In particular, there are strict inclusions

$$\text{J-hpy equivalence} \hookrightarrow W_{\mathrm{Joyal}} \hookrightarrow W_{\mathrm{KQ}}$$

The following corollary is a consequence of Quillen's theory.

Corollary 4.5.2. *Let $f : X \to Y$ be a morphism in* **sSet***.*

(a) f is a Joyal weak equivalence if and only if, for every quasicategory Z, the map $[Y, Z]_J \to [X, Z]_J$ is a bijection.
(b) If X, Y are quasicategories, then f is a Joyal weak equivalence if and only if f is a J-homotopy equivalence.
(c) f is a weak homotopy equivalence if and only if, for every Kan complex Z, the map $[Y, Z]_J \to [X, Z]_J$ is a bijection.
(d) If X, Y are Kan complexes, then f is a weak homotopy equivalence if and only if f is a J-homotopy equivalence.

5. Limits and localization

We are now ready to return to 1-categorical constructions and adapt them to the case of quasicategories, whilst continuing to follow the same philosophy.

5.1. Limits.

Let $F : D \to \mathbf{C}$ be a diagram in a category $\mathbf{C}$. Recall that a limit of F is a *final object* in the *slice* category $\mathbf{C}/F$. In order to adapt the notion of a limit we will need to define what a final object is in a quasicategory X. Moreover given a functor $F : D \to X$ where D is a simplicial set, we need to define the slice X/F. For the theory of final objects we refer to [6, Section 4.3] and for the theory of slices to [6, Section 3.4]. Here, we present definitions that appear as propositions in Cisinski's book, where the general theory is developed, since we are only interested in very specific cases.

Definition 5.1.1. Let X be a quasicategory. An object $x \in X_0$ is *final* if for every $w \in X_0$ the Kan complex $X(w, x_0)$ is a contractible Kan complex.

Proposition 5.1.2. [6, Corollary 4.3.13] *Let X be a quasicategory. The full simplicial subset of X generated by final objects is either empty or a contractible Kan complex.*

Definition 5.1.3. Let X be a simplicial set and $x \in X_0$. We define the slice X/x as the simplicial set

$$(X/x)_n = \left\{\sigma : \Delta^{n+1} \to X \mid \sigma(n+1) = x\right\}$$

with the evident faces and degeneracy maps. Denote by $\pi : X/x \to X$ the morphism in **sSet** sending σ to $\sigma|_{\Delta^n}$ where the map $(\Delta^n)_m \hookrightarrow (\Delta^{n+1})_m$ sends $0 \le a_0 \le \ldots \le a_m \le n$ to $a_0 \le \ldots \le a_m$.

Proposition 5.1.4. *If X is a quasicategory, then so is X/x. If $\mathbf{C}$ is a category and c is an object of $\mathbf{C}$, then*

$$N(\mathbf{C}/c) \cong N(\mathbf{C})/c.$$

Definition 5.1.5. Let $F : D \to X$ be a morphism in **sSet** with X a quasicategory. Define X/F as the pullback of the diagram

$$\begin{tikzcd} X/F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}(D,X)/F \arrow[d, "\pi"] \\ X \arrow[r, "c_D"] & \mathrm{Map}(D,X) \end{tikzcd}$$

where c_D is the constant functor.

Proposition 5.1.6. *Th simplicial set X/F is a quasicategory.*

Definition 5.1.7. Let $F : D \to X$ be a morphism in **sSet** with X a quasicategory. A *limit* of F is a final object in X/F.

Remark 5.1.8. A limit if it exists is essentially unique.

5.2. **Localization.** We refer to [6, Section 7.1] for more details on this section.

Let X be a quasicategory and W be a simplicial subset of X. For Y a quasicategory, denote by $\mathrm{Map}_W(X,Y)$ the full simplicial subset of $\mathrm{Map}(X,Y)$ generated by functors $F : X \to Y$ such that for any $f : a \to b \in W_1$, $F(w)$ is invertible in Y. We have that $\mathrm{Map}_W(X,Y)$ is a quasicategory.

Definition 5.2.1. A localization of X by W is a functor $X \to W^{-1}X$ such that:

(a) $W^{-1}X$ is a quasicategory.
(b) It sends maps in W to invertible maps in $W^{-1}X$.
(c) For any quasicategory Y the induced map $\mathrm{Map}(W^{-1}X, Y) \to \mathrm{Map}_W(X,Y)$ is an equivalence of ∞-categories.

Proposition 5.2.2. [6, Proposition 7.1.3] *The localisation of X by W always exists and is essentially unique.*

Remark 5.2.3. [6, Remark 7.1.6] If $\mathbf{C}$ is a category and W is a class of maps in $\mathbf{C}$ there exists a (1-categorical) localization of $\mathbf{C}$ by W that we denote $\mathbf{C}[W^{-1}]$. There is an equivalence of categories

$$\mathrm{ho}((NW)^{-1}N\mathbf{C}) \cong \mathbf{C}[W^{-1}]$$

but there is no reason that this is true at the ∞-categorical level. More specifically, there is a priori no equivalence of ∞-categories between $(NW)^{-1}N\mathbf{C}$ and $N(\mathbf{C}[W^{-1}])$.

We conclude this section by a very nice proposition.

Proposition 5.2.4. [6, Proposition 7.3.15] *Any quasicategory is the localization of (the nerve of) a category.*

6. DG categories and enhancements

As can be seen in any introduction to enhancements of triangulated categories, there are algebraic and topological enhancements. The former are pre-triangulated DG categories as defined by Bondal and Kapranov in [4] and the latter are stable ∞-categories as defined by Lurie in [12]. The aim of this section is to define these two concepts and to conclude by providing an overview of the recent developments in the field of enhancements of extriangulated categories.

6.1. **DG-categories and DG nerve.**

We denote by $\mathbf{Ch}(\mathbb{Z})$ the category of $\mathbb{Z}$-graded chain complexes of abelian groups. A DG-category is a category enriched in $\mathbf{Ch}(\mathbb{Z})$. We denote by **DGCat** the category of DG-categories. In here we follow closely [12, Section 1.3.1].

Definition 6.1.1. The DG nerve is a functor from **DGCat** to **sSet**. Given a DG-category $\mathbf{C}$ the set of n-simplices $N_{\mathrm{dg}}(\mathbf{C})_n$ is the set of all ordered pairs $(\{X_i\}_{0\leq i\leq n}, \{f_I\})$, where

(a) For $0 \leq i \leq n$, X_i is an object of $\mathbf{C}$.
(b) For every subset $I = \{i_- < i_m < \ldots < i_1 < i_+\}$ of $[n]$, with $m \geq 0$, f_I is an element of $\mathbf{C}(X_{i_-}, X_{i_+})_m$ and satisfies

$$df_I = \sum_{j=1}^{m-1} (-1)^j (f_{I\setminus\{i_j\}} - f_{i_j<\cdots<i_1<i_+} \circ f_{i_-<i_m\cdots<i_j}).$$

The faces and degeneracy maps can be found in [12, Construction 1.3.1.6].

Example 6.1.2. The analysis of the low-dimensional simplices gives:

- $N_{\mathrm{dg}}(\mathbf{C})_0 = \mathrm{Ob}(\mathbf{C})$.
- $N_{\mathrm{dg}}(\mathbf{C})_1 = \{f : X_0 \to X_1 \text{ of degree 0 s.t. } df = 0\}$.
- A 2-simplex in $N_{\mathrm{dg}}(\mathbf{C})_2$ is depicted as follows

$$\begin{tikzcd} & X_1 \arrow[dr, "f_{12}"] \arrow[d, phantom, "f_{012}"] & \\ X_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & {} & X_2 \end{tikzcd}$$

with f_{ij} a cycle of degree 0 and $f_{012} \in \mathbf{C}(X_0, X_2)_1$ satisfies $df_{012} = f_{12} \circ f_{01} - f_{02}$.

Theorem 6.1.3. *For $\mathbf{C}$ a DG category, the DG nerve $N_{\mathrm{dg}}(\mathbf{C})$ is a quasicategory and its homotopy category coincides with the homotopy category of the DG category $\mathbf{C}$*

$$\mathrm{ho}(N_{\mathrm{dg}}(\mathbf{C}))(X, Y) \cong H_0(\mathbf{C}(X, Y)).$$

In particular, the homotopy category of the DG category is a pre-additive category.

Remark 6.1.4. The DG nerve functor is right adjoint to a functor obtained as a left Kan extension of a functor $\mathbf{\Delta} \to \mathbf{DGCat}$ as in Construction (1) as we will see in Section 6.2.3.

6.2. **The rigidification functor of Lurie.**

6.2.1. *Lurie's construction.* Let $\mathbf{Cat}_\Delta$ be the category of categories enriched in simplicial sets and enriched functors. Lurie defines a rigidification functor from $\mathbf{sSet}$ to $\mathbf{Cat}_\Delta$ which is obtained via Construction (1) and yields an adjunction

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, shift left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathbf{Cat}_\Delta \arrow[l, shift left, "N^\Delta"] \end{tikzcd} \tag{4}$$

The functor $\mathfrak{C}$ is called the *rigidification functor* and the right adjoint N^Δ the *homotopy coherent nerve*. Let us describe briefly the functor $\mathbf{\Delta} \to \mathbf{Cat}_\Delta$ inducing this adjunction.

Let **PosCat** be the category of categories enriched in partial ordered sets and enriched functors. For all $n \geq 0$, define $P_n \in \mathbf{PosCat}$ as:

- objects of P_n are $0, 1, \ldots, n$
- for $0 \leq i \leq j \leq n$, $P_n(i, j) = \{A \subset [n] \mid \{i, j\} \subset A \subset [[i, j]]\}$, ordered by inclusion
- for $i > j$, $P_n(i, j) = \emptyset$
- the composition $P_n(j, k) \times P_n(i, j) \to P_n(i, k)$ is defined by $(A, B) \mapsto A \cup B$.

Note that for every $i < j$, we have $P_n(i, j) = \{0 < 1\}^{\times j-i-1}$.

The nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ being strong monoidal, applying it to each hom-poset yields an enriched category in simplicial sets: $\tilde{P}_n = N(P_n)$. Note that every $\varphi : [n] \to [m] \in \mathbf{\Delta}$ induces an enriched functor $\varphi_* : P_n \to P_m$. It yields a functor $\mathbf{\Delta} \to \mathbf{Cat}_\Delta$ and thus adjunction (4).

Let $\mathbf{C}$ be a simplicial category and denote by $\mathbf{C}_0$ its underlying category, that is, $\mathbf{C}_0(x, y)$ is the set of 0-simplices of $\mathbf{C}(x, y)$. The homotopy coherent nerve of $\mathbf{C}$ satisfies

$$N^\Delta(\mathbf{C})_n = \mathrm{Hom}_{\mathbf{Cat}_\Delta}(\tilde{P}_n, \mathbf{C}).$$

Its low dimensional simplices are described below.

- $N^\Delta(\mathbf{C})_0 = \mathrm{Ob}(\mathbf{C}) = N(\mathbf{C}_0)_0$
- $N^\Delta(\mathbf{C})_1 = N(\mathbf{C}_0)_1$
- An element in $N^\Delta(\mathbf{C})_2$ consists in the following data: a triangle $(f, g, h) : \partial\Delta_2 \to N(\mathbf{C}_0)$ of arrows in $\mathbf{C}_0$, namely $f : a \to b, g : b \to c, h : a \to c$, together with an element $H \in \mathbf{C}(a, c)_1$ such that $d_0 H = g \circ f,\ d_1 H = h$.

6.2.2. *Another model for ∞-categories.* J. Bergner has shown that there is a model category structure on $\mathbf{Cat}_\Delta$ and we refer to [2] for more details. What is important to us is that fibrant objects in this model are categories enriched in Kan complexes and that the homotopy coherent nerve of a category enriched in Kan complexes is a quasicategory. In addition, the adjunction in (4) is a Quillen equivalence, which means that the model of Bergner is another model for ∞-categories.

6.2.3. *DG nerve and homotopy coherent nerve.* There is a (lax) monoidal functor $C_* : \mathbf{sSet} \to \mathbf{Ch}(\mathbb{Z})$ which associates to a simplicial set its normalized chain complex. It implies that by setting $\Lambda_n(i, j) = C_*(\tilde{P}_n(i, j))$, one gets a category Λ_n enriched in chain complexes and a functor $\Lambda : \mathbf{\Delta} \to \mathbf{DGCat}$. It is an easy exercise to prove that

$$N_{\mathrm{dg}}(\mathbf{C})_n = \mathrm{Hom}_{\mathbf{DGCat}}(\Lambda_n, \mathbf{C}).$$

Therefore, by Construction (1), the functor Λ yields the adjunction

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{DGCat} \arrow[l, shift left, "N_{\mathrm{dg}}"] \end{tikzcd}$$

6.3. **Stable ∞-categories.** In section 5.1 we have seen the notions of final objects and limits in a quasicategory and of initial objects and colimits as dual notions. In this section only the notions of pullback and pushout are necessary. To understand the definition of stable ∞-categories, we need first to introduce squares.

Definition 6.3.1. Let X be a quasicategory. A square in X is a morphism of simplicial sets $\Delta^1 \times \Delta^1 \to X$.

Giving a square in X amounts to giving two commutative triangles in X that can be glued along their diagonal. More precisely a square in X is a couple (σ, τ) of elements in X_2 such that $d_1\sigma = d_1\tau$ and is depicted as

$$\begin{tikzcd} x_{00} \arrow[r, "h_0"] \arrow[d, "v_0"'] \arrow[dr, "\sigma", "\tau"'] & x_{10} \arrow[d, "v_1"] \\ x_{01} \arrow[r, "h_1"'] & x_{11} \end{tikzcd}$$

A square in X yields a commutative diagram in $\mathrm{ho}(X)$: with the above notation, in the homotopy category $\mathrm{ho}(X)$, we have $[v_1] \circ [h_0] = [h_1] \circ [v_0]$.

- A 0 *object* in a quasicategory is an object 0 that is both final and initial.
- A *fiber* of a morphism $f : x \to y$ in X_1 is a limit of the diagram $0 \to y \xleftarrow{f} x$.
- A *cofiber* of the map f is a colimit of the diagram $y \xleftarrow{f} x \to 0$.

We are now in position to understand the following definition.

Definition 6.3.2. [12, Definition 1.1.9] A *stable ∞-category* is a quasicategory X satisfying:

(a) It has a 0 object.
(b) Every morphism in X_1 admits a fiber and a cofiber.
(c) A square is a pushout if and only if it is a pullback.

Theorem 6.3.3. *If* $\mathbf{C}$ *is a stable ∞-category, then* $ho(\mathbf{C})$ *is triangulated.*

The following theorem is due to Faonte in [8].

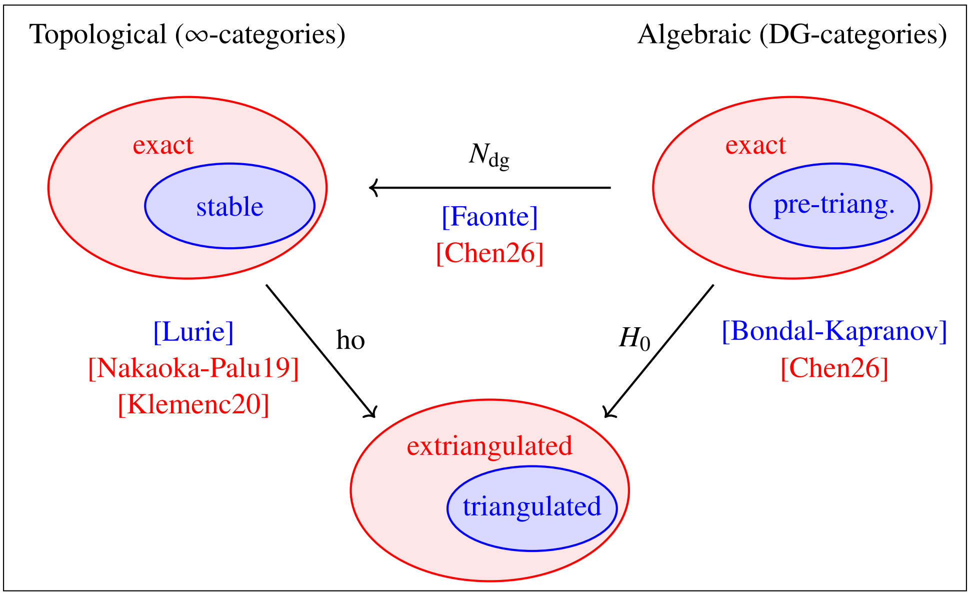
Theorem 6.3.4. *If* A *is a pretriangulated DG category, then* $N_{\mathrm{dg}}(A)$ *is a stable ∞-category.*

6.4. **Overview of recent developments.** The diagram below was explained to me by Yann Palu, and the explanation, which we reproduce here, is taken from the introduction in Xiofa Chen's PhD thesis, see also [5], in which he defines and develops the theory of *exact DG categories*. The notion of exact DG category is

- a dg version of Barwick's notion of exact ∞-category developed in [1] .
- a simultaneous generalization of the notions of exact category in the sense of Quillen and of pretriangulated dg category in the sense of Bondal–Kapranov [4], and
- a dg enhancement of the notion of extriangulated category in the sense of Nakaoka–Palu [13] in analogy with Bondal–Kapranov's dg enhancement of the notion of triangulated category.

Xiofa Chen proved a generalization of Faonte's Theorem 6.3.4

Theorem 6.4.1. *For a DG Category A such that $H_0(A)$ is additive, there is a bijection between the class of exact structures on $N_{dg}(A)$ and the class of exact structures on the additive category $H_0(A)$.*



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