Level algebras and Steenrod operations

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Level Algebra

Definition : A level algebra $(A, *)$ is a commutative algebra (non associative) satisfying

$$(a*b)*(c*d) = (a*c)*(b*d)$$

Fundamental example : Carlsson unstable module over $\mathcal{A}_2$:

$K = \mathbb{F}_2[\widehat{x}_i, i \in \mathbb{Z}], \qquad ||\widehat{x}_i|| = (2^{-i}, 1).$

$K(i)^j = \mathbb{F}_2 < \prod \widehat{x}_k^{\alpha_k} \mid \sum \frac{\alpha_k}{2^k} = i,\ \sum \alpha_k = j > .$

$\mathrm{Sq}^1(\widehat{x}_i) = \widehat{x}_{i+1}^2 +$ Cartan formula.

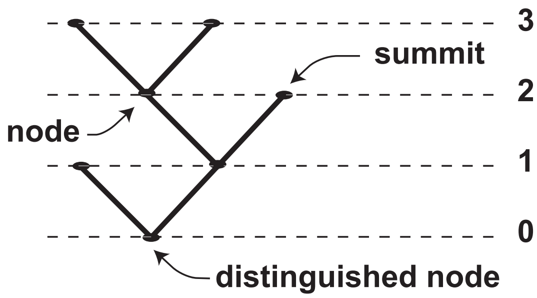
Proposition : (K(i),*) is a level algebra

$$\left(\prod \widehat{x}_k^{\alpha_k}\right) * \left(\prod \widehat{x}_l^{\alpha_l}\right) = \prod \widehat{x}_{k+1}^{\alpha_k} \widehat{x}_{l+1}^{\alpha_l}.$$

Action of $\mathcal{A}_2$ satisfies Cartan.

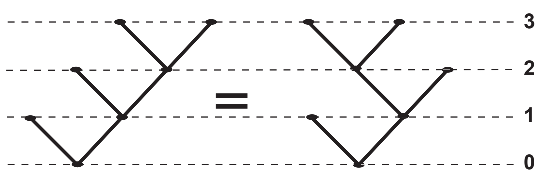
Level trees

Planar rooted binary tree

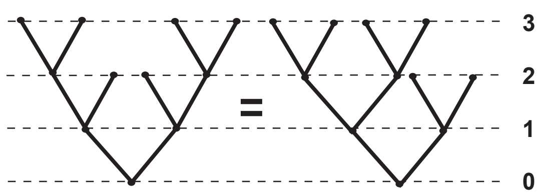


Equivalence relation : exchanging part above same level

Ex 1



Ex 2



Proposition : (Davis-Miller)

$K(1)^n = < \prod \hat{x}_k^{\alpha_k} | \sum \frac{\alpha_k}{2^k} = 1, \sum \alpha_k = n >$ $\longleftrightarrow$ {n-level trees}

Operads

Definition : Operad $\mathcal{P} = (\mathcal{P}(n))_{n\geq 1}$

– Σ_n-action on $\mathcal{P}(n)$.

– $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1)$

$\mathcal{P}$-algebra A : $\mathcal{P}(n) \otimes A^{\otimes n} \to A$.

$$\{\text{Operads}\} \underset{\mathcal{U}}{\overset{\mathcal{F}\text{ree}}{\overset{\longleftarrow}{\longrightarrow}}} \{\Sigma - \text{modules}\}$$

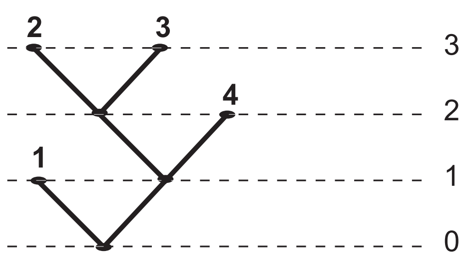
Type of algebra	Product	Relation
Commutative	2, com	3, as
Associative	2	3, as
Level	2, com	4, (a*b)*(c*d) =(a*c)*(b*d)

Type $\mathcal{P}$	Operad $\mathcal{P}$	$\mathcal{P}(n)$
$\mathcal{C}$om	$\mathcal{F}\text{ree}(k)/R_{\mathcal{C}\text{om}}$	k
$\mathcal{A}s$	$\mathcal{F}\text{ree}(k[\Sigma_2])/R_{\mathcal{A}s}$	$k[\Sigma_n]$
Lev	$\mathcal{F}\text{ree}(k)/R_{\text{Lev}}$	?

The operad Lev

Definition : A n-labelled level tree is a n-level tree where the summits are labelled.

Remark : $\{n$-labelled level trees$\}$ is in 1-to-1 correspondance with the set of all ordered partition of $\{1,\ldots,n\}$, $I=(I_j)_{j\geq 0}$ such that $\sum \frac{|I_j|}{2^j}=1$.



$$I=(\emptyset,\{1\},\{4\},\{2,3\})$$

Theorem : (L., Schwartz) The collection of vector spaces generated by $\{n$-labelled level trees$\}$ defines an operad. This operad is Lev.

Corollary : $K(1)$ is the free level algebra on one generator $\widehat{x}_0$.

E_∞-operads over $\mathbb{F}$

Definition : a E_∞-operad is a $(\mathcal{F}\text{ree}(V), d)$ with

- $V(n)$ is a $\mathbb{F}[\Sigma_n]$-projective module
- $(\mathcal{F}\text{ree}(V), d) \xrightarrow{\sim} \mathcal{C}\text{om}$.

A E_∞-algebra is a $\mathcal{F}\text{ree}(V)$-algebra.

Importance of E_∞-operads :

- $C^*(X)$ is a E_∞-algebra.
- (Mandell) This structure determines the homotopy type of X.
- (Chataur) There exists a structure of E_∞-algebra on $H^*(X; \mathbb{F})$ inducing the usual commutative structure.

Problem : Understand the combinatorics of E_∞-operads to make computation, e.g. find a "nice" E_∞-operad (McClure-Smith, Berger-Fresse).

Steenrod operations in E_∞-operads

(May) A E_∞-algebra is an algebra over $\mathcal{B}_2$.

– Where can we read Steenrod operations ?
  via $\cup_i$-products in $\mathcal{E}(2)$
– Where can we read Adem and Cartan relations ?
  A priori in $\mathcal{E}(4)$, or using a resolution of level operads
– Can we find higher order cohomology operations ?

A major ingredient

$(\mathcal{P}, d)$ is a dg operad ;
(A, ∂_A) is a dg $\mathcal{P}$-algebra, means the evaluation map

$$\begin{array}{rcl} \mathcal{P}(n) \otimes A^{\otimes n} & \longrightarrow & A \\ p \otimes a_1 \otimes \ldots \otimes a_n & \mapsto & p(a_1, \ldots, a_n) \end{array}$$

satisfies the Leibniz rule

$$\partial_A(p(a_1, \ldots, a_n)) = (dp)(a_1, \ldots, a_n) + \sum_{i=1}^{n} \pm p(a_1, \ldots, \partial_A a_i, \ldots, a_n).$$

If $\partial_A = 0$, then $(dp)(a_1, \ldots, a_n) = 0$.

This gives a relation between elements in A.

Application :
$\cup_i$-products and Steenrod operations

Let $\mathcal{E}$ be an E_∞-operad ; since
$\mathcal{E}(2) \xrightarrow{\sim} \mathcal{C}\mathrm{om}(2) = \mathbb{F}_2$, we can choose

$$\begin{cases} \mathcal{E}(2)_{-i} = \mathbb{F}_2[\Sigma_2] = \mathbb{F}_2 < e_i, \tau e_i > \\ d e_i = e_{i-1} + \tau e_{i-1} \end{cases}$$

Corollary : A a E_∞-algebra, $x, y \in A$.
$x \cup_i y := e_i(x, y)$; $\mathrm{Sq}^i(x) := x \cup_{|x|-i} x$.

1. $\partial_A = 0$ implies $x \cup_i y = y \cup_i x$
2. $\mathrm{Sq}^i(x) = 0$ if $i > |x|$
3. $\mathrm{Sq}^{|x|}(x) = x^2$.

Adem relations, Cartan formula

Adem relations : $i < 2j$

$$\mathsf{Sq}^i \mathsf{Sq}^j x = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathsf{Sq}^{i+j-k} \mathsf{Sq}^k x.$$

equality

$$e_l(e_k(x,x), e_k(x,x)) = \mathsf{Sq}^{2|x|-l-k} \mathsf{Sq}^{|x|-k}(x)$$

implies this formula reads in $\mathcal{E}$
$R_{Ad}(x) = (dp)(x,x,x,x)$ for $p \in \mathcal{E}(4)$

Cartan formula :

$$\mathsf{Sq}^n(xy) = \sum_{i+j=n} \mathsf{Sq}^i(x) \mathsf{Sq}^j(y),$$

$$e_m(e_0(x,y), e_0(x,y)) = \sum_{k+l=m} e_0(e_k(x,x), e_l(y,y))$$

implies this formula reads in $\mathcal{E}$
$R_{Ca}(x) = (dq)(x,x,y,y)$ for $q \in \mathcal{E}(4)$.

Using the level operad

Main idea : Associativity of the product is not involved in Cartan and Adem relations.
$\Rightarrow$ replace $\mathcal{C}$om by Lev and find a resolution of it.

Theorem : (Chataur,L.) there exists a cofibrant operad Lev^{AC} satisfying

1. Lev$^{AC}(2) = \mathcal{E}(2)$
2. there is a fibration f : Lev$^{AC} \to$ Lev s.t. f induces an iso $H^0($Lev$^{AC}) \simeq$ Lev.

and boundaries in Lev^{AC} give
$\to$ Explicit operadic description of Cartan formula
$\to$ Explicit operadic description of Adem relations

Relation with E_∞-algebras

Since Lev $\to \mathcal{C}$om then Lev$^{AC} \to \mathcal{C}$om can be lifted to Lev$^{AC} \to \mathcal{E}$.

Corollary : If A is an E_∞-algebra, with $\partial_A = 0$, then A is an algebra over the extended Steenrod algebra $\mathcal{B}_2$.

Secondary cohomological operations

We have an evaluation map

$$\mathsf{Lev}^{AC}(4) \otimes H^*(X, \mathbb{F}_2)^{\otimes 4} \to H^*(X, \mathbb{F}_2)$$

Theorem : there are elements $G_n^m \in \mathsf{Lev}^{AC}(4)$ such that for $x \in H^*(X, \mathbb{F}_2)$, $G_n^m(x, x, x, x)$ coincides with the stable secondary cohomological operations of Adams.

Idea of construction of Lev^{AC}

Construction by adjoining cells to $\mathcal{F}\mathrm{ree}(\mathcal{E})(4)$.
Let $u_{m,n} \in \mathcal{F}\mathrm{ree}(\mathcal{E})(4)$ defined by

$$\begin{aligned} u_{m,n} = & \sum_k e_m(e_k, \tau^k e_{n-k}) \\ = & \quad e_m(\Delta e_n) \end{aligned}$$

First step : Attach G^1_n in degree $-n$ with the relation

$$dG^1_{n+1} = (id + (12)(34))G^1_n + u_{0,n} + (23)u_{n,0}.$$

Proposition : If A is a Lev^{AC}-algebra, with $\partial_A = 0$, then A satisfies the Cartan formula.

Proof : $(dG^1_{n+1})(x,x,y,y) = 0$ reads

$$\sum_k e_0(e_k(x,x), e_{n-k}(y,y)) = e_n(e_0(x,y), e_0(x,y))$$

Idea of construction of Lev^{AC}

Higher steps : by induction, we attach some cells G_n^m for $m \geq 1$ in degree $-n$ with $n \geq m$.

Proposition : If A is a Lev^{AC}-algebra, with $\partial_A = 0$, then A satisfies the Adem relation.

Proof : $(dG_{n+1}^m)(x,x,x,x) = 0$ gives an Adem relation.

Prospects

$\rightarrow$ Determine a diagonal in Lev^{AC}

$\rightarrow$ Operadic description of higher order cohomology operations

$\rightarrow$ Problem of realisability