Lett Math Phys (2012) 101:195–222
DOI 10.1007/s11005-012-0556-7

Open-Closed Homotopy Algebras and Strong Homotopy Leibniz Pairs Through Koszul Operad Theory

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Received: 25 May 2011 / Revised: 17 January 2012 / Accepted: 2 February 2012
Published online: 24 March 2012 –

Abstract. Open-closed homotopy algebras (OCHA) and strong homotopy Leibniz pairs (SHLP) were introduced by Kajiura and Stasheff in 2004. In an appendix to their paper, Markl observed that an SHLP is equivalent to an algebra over the minimal model of a certain operad, without showing that the operad is Koszul. In the present paper, we show that both OCHA and SHLP are algebras over the minimal model of the zeroth homology of two versions of the Swiss-cheese operad and prove that these two operads are Koszul. As an application, we show that the OCHA operad is non-formal as a 2-colored operad but is formal as an algebra in the category of 2-collections.

Mathematics Subject Classification. 18G55, 18D50.

Keywords. Koszul Operads, homotopy algebras, deformation theory.

1. Introduction

In the famous paper [7] of 1963, a Gerstenhaber bracket was built on the Hochschild complex $C(A, A)$ of an associative algebra A, leading to a Gerstenhaber structure on its cohomology $HH(A)$. Later on, Pierre Deligne conjectured that, since the homology of the little disk operad $\mathcal{D}_2$ acts on $HH(A)$, there should be an operad weakly equivalent to the singular chain complex of $\mathcal{D}_2$ acting on $C(A, A)$.

E. Hoefel is grateful to the Erwin Schrödinger Institute in Vienna for organizing the "2010 Higher Structures Conference" and to Sergei Merkulov for valuable conversations. M. Livernet thanks the hospitality of the Chern Institute at Nankai University, Tianjin, during the summer school and conference "Operads and universal algebra" and especially Bruno Vallette and Vladimir Dotsenko for fruitful discussions. The authors are grateful to Joan Milles for his explanations on linear-quadratic operads. This collaboration is funded by the MathAmSud program "OPECSHA" coordinated by M. Ronco and by Fundação Araucária (Brazil) Grant FA 490/6032.

The attempts to prove Deligne's conjecture resulted in significant advances in operad theory and deformation theory. Different proofs of it were given in the past decade by several authors: Batanin, Berger, Fresse, Kontsevich, Markl, McClure, Smith, Soibelman, Tamarkin and Voronov.

Shifting the Hochschild complex by one, the Gerstenhaber bracket gives a differential graded Lie algebra structure on $C(A, A)$ [1]. Given a Lie algebra L (concentrated in degree 0 with zero differential), the pair (L, A) together with a degree 0 morphism of dg Lie algebras $\phi : L \to C(A, A)$ [1] is called a Leibniz pair and has been studied by Flato, Gerstenhaber and Voronov in [4]. Following the general idea of deforming structure maps, Kajiura and Stasheff in [12] and [13], defined the notion of open-closed homotopy algebras (OCHA) inspired by Zwiebach's open-closed string field theory. On the way to the definition of OCHA, the notion of L_∞-algebras acting on A_∞-algebras, also known as Strong Homotopy Leibniz Pairs, appeared as a necessary step.

Let A be an A_∞-algebra. One can still define its deformation complex, which is bigraded by $C^{k,n}(A, A) = \prod_i \hom_{\mathbf{k}}((A^{\otimes k})_i, A_{i-n})$ and endowed with a differential of total degree -1, induced by the A_∞-structure. This complex is still endowed with the Gerstenhaber bracket $C^{k,n} \otimes C^{k',n'} \to C^{k+k'-1,n+n'}$ and one can shift the first degree by one in order to obtain a dg Lie algebra. We denote by $C^{>0}(A, A)$ the truncated complex, where $k > 0$. The shifted complex $C^{>0}(A, A)$ [1] is a Lie subalgebra of the shifted deformation complex. An SHLP (Strong Homotopy Leibniz Pair) consists of an L_∞-algebra L, an A_∞-algebra A and an L_∞-morphism $L \to C^{>0}(A, A)$ [1]. If we consider the full complex $C^*(A, A)$, then an L_∞ morphism $L \to C^*(A, A)$ [1] results in an OCHA.

Given an operad $\mathcal{P}$, one can define, under some assumptions, its Koszul cooperad $\mathcal{P}^{¡}$. When the operad $\mathcal{P}$ is Koszul one gets a quasi-isomorphism $\Omega(\mathcal{P}^{¡}) \to \mathcal{P}$, where Ω is the cobar construction for cooperads. The operad $\Omega(\mathcal{P}^{¡})$ is often denoted $\mathcal{P}_\infty$ and algebras over this operad are called *strong homotopy $\mathcal{P}$-algebras*. In the appendix of [12], Markl showed that SHLP correspond to algebras over the operad $\Omega(\mathcal{LP}^{¡})$, where $\mathcal{LP}$ is the operad for Leibniz pairs, but he did not prove that the operad is Koszul. In [3], Dolgushev showed that OCHA correspond to algebras over $\Omega(\mathcal{D})$, where $\mathcal{D}$ is a cooperad that will be specified later, but without proving that $\mathcal{D}$ is Koszul.

The aim of our paper is to prove that the operads involved are Koszul, justifying completely that SHLP and OCHA are "strong homotopy" algebras over operads. It is important to remark that, though the case of SHLP is rather classical in the theory of operads (binary quadratic colored operads), the case of OCHA is less classical because the operads involved are not quadratic. The techniques used are the ones employed by Galvez-Carillo, Tonks and Vallette in the case of BV-algebras in [6]. In order to prove those theorems we also use the relation established by the first author between the Swiss-cheese operad and OCHA in [10]. Such relation is analogous to the one between strong homotopy Lie algebras and the little disks operad.

1.1. PLAN OF THE PAPER

In Section 2, we review some facts about colored operads and Koszul duality that will be needed for the paper. In Section 3 we describe the topological operads involved, i.e., the two versions of the Swiss-cheese operad. Since one version appeared in a paper of Voronov [22] and the other one in a paper of Kontsevich [14], we sometimes refer to them as Voronov Swiss-cheese, denoted $\mathcal{SC}^{\mathrm{vor}}$ and Kontsevich Swiss-cheese, denoted $\mathcal{SC}$. The difference between the two versions is analogous to the difference between the truncated deformation complex of an associative algebra $C^{>0}(A, A)$ and the full deformation complex $C(A, A)$. Sections 4 and 5 are devoted to the study of SHLP and the zeroth homology of $\mathcal{SC}^{\mathrm{vor}}$. We define Leibniz pairs and its associated colored operad $\mathcal{LP}$ and prove that its Koszul dual operad is $H_0(\mathcal{SC}^{\mathrm{vor}})$. We prove that $\mathcal{LP}$ is a Koszul operad. Consequently SHLP's are algebras over the cobar construction of the Koszul dual cooperad of $\mathcal{LP}$ which is a minimal resolution of $\mathcal{LP}$. Sections 6 and 7 are devoted to the study of OCHA and the zeroth homology of $\mathcal{SC}$. One has that the operad $H_0(\mathcal{SC})$ is not quadratic but can be presented by linear-quadratic relations, so that we can apply results of [6].

We prove in Theorem 6.2.3 that $H_0(\mathcal{SC})$ is a Koszul operad. This means that one has a quasi-isomorphism $\Omega(H_0(\mathcal{SC})^{¡}) \to H_0(\mathcal{SC})$ where the left-hand side is not a minimal model since there is a linear differential coming from the one on $H_0(\mathcal{SC})^{¡}$. Nevertheless, taking the Koszul dual operad of $H_0(\mathcal{SC})$, denoted $H_0(\mathcal{SC})^!$ we prove that there is a quasi-isomorphism $\Omega(\mathcal{D}) \to H_0(\mathcal{SC})^!$, where $\mathcal{D}$ is a suspension of the cooperad $H_0(\mathcal{SC})^*$. The left-hand side of the quasi-isomorphism is minimal and algebras over it are OCHA as pointed out by Dolgushev in [3] (see Section 7.1). The result, however, is not totally satisfying because $H_0(\mathcal{SC})^!$ is a differential graded operad and one would like to replace it by its homology.

We prove in Section 7 that there is a sequence of quasi-isomorphisms $\Omega(\mathcal{D}) \to H_0(\mathcal{SC})^! \to H_*(H_0(\mathcal{SC})^!)$, not of operads but of algebras in the category of 2-collections. We use plainly that all these operads are multiplicative operads. We prove also that the last quasi-isomorphism cannot be a quasi-isomorphism in the category of operads. We interpret $H_*(H_0(\mathcal{SC})^!)$ as a suspension of the "top homology" of the Swiss-cheese operad, completing the analogy with the little disks operad $\mathcal{D}_2$.

1.2. NOTATION

We work on a ground field $\mathbf{k}$ of characteristic 0. The category **dgvs** is the category of lower graded $\mathbf{k}$-vector spaces together with a differential of degree -1. We will consider the category of vector spaces and the one of graded vector spaces as full subcategories of **dgvs**. The vector space $\hom_{\mathbf{k}}(V, W)$ denotes the $\mathbf{k}$-linear morphisms between two vector spaces V and W. When V and W are objects in **dgvs**, the differential graded vector spaces of maps from V to W is

$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_i(V, W)$, where $\mathrm{Hom}_i(V, W)=\prod_n \mathrm{hom}_{\mathbf{k}}(V_n, W_{n+i})$ together with the differential $(\partial f)(v)=\mathrm{d}_W(f(v))-(-1)^{|f|} f(d_V v)$.

The graded linear dual of V in **dgvs** is $V^* = \mathrm{Hom}(V, \mathbf{k})$, where $\mathbf{k}$ is concentrated in degree 0 with 0-differential. Consequently, $(V^*)_n = (V_{-n})^*$ and $\partial f(x) = -(-1)^n f(d_V x)$ for any $f \in (V^*)_n$ and $x \in V_{-n+1}$. Let V be in **dgvs**. The free non-unital commutative algebra generated by V is denoted by $S(V)$, whereas the free unital commutative algebra generated by V is denoted by $S^+(V)$ and similarly for the notation $T(V)$ and $T^+(V)$ for the free non-unital or unital associative algebra generated by V.

The symmetric group acting on n elements is denoted by S_n. An element $\sigma \in S_n$ will be denoted by its image notation $(\sigma(1)\ldots\sigma(n))$. An $\mathbb{S}$-module is a collection of objects $\{M(n)\}_{n\geq 0}$ in **dgvs** such that each $M(n)$ is endowed with an action of the symmetric group S_n. The Koszul dual cooperad of an operad $\mathcal{P}$, when it exists, is denoted by $\mathcal{P}^{¡}$ and its Koszul dual operad is denoted by $\mathcal{P}^!$. Similarly, the Koszul dual algebra of a coalgebra C, when it exists, is denoted by $C^{¡}$. The notation B stands for the bar construction (of operads or algebras) and Ω for the cobar construction (of cooperads or coalgebras). All these notation will be specified in the text.

2. Colored Operads, 2-Colored Operads

Here we give the definitions and theorems needed for the sequel. We refer to [21, 17] for the general theory of colored operads and to [5,16] for the general theory of Koszul operads.

2.1. COLORED OPERADS

DEFINITION 2.1.1 [21]. Let I be a set. The category $\mathbf{Fin}_I$ is the category whose objects $(X, x_0; i : X \to I)$ are pointed, non-empty finite sets together with a map i and whose morphisms are pointed bijections commuting with i. An I*-collection* is a contravariant functor from $\mathbf{Fin}_I$ to **dgvs**. An I*-colored operad* is an I-collection $\mathcal{P}$ together with natural transformations, called composition maps

$$\circ_x : \mathcal{P}(X, x_0; i_X)\otimes \mathcal{P}(Y, y_0; i_Y) \to \mathcal{P}(X \cup_x Y, x_0; i_X \cup_X i_Y)$$

for any $x \in X\backslash\{x_0\}$ such that $i_X(x)=i_Y(y_0)$, where $i_X \cup_x i_Y : X \cup_x Y \to I$ is the map induced by i_X and i_Y. These natural transformations are associative, that is, for any I-sets $(X, x_0; i_X)$, $(Y, y_0; i_Y)$ and $(Z, z_0; i_Z)$ and $\alpha \in \mathcal{P}(X, x_0; i_X)$, $\beta \in \mathcal{P}(Y, y_0; i_Y)$ and $\gamma \in \mathcal{P}(Z, z_0, i_Z)$ one has

$$(\alpha \circ_x \beta)\circ_y \gamma = \alpha \circ_x (\beta \circ_y \gamma), \quad \text{if } i_X(x)=i_Y(y_0) \text{ and } i_Y(y)=i_Z(z_0),$$
$$(\alpha \circ_x \beta)\circ_{x'} \gamma = (\alpha \circ_{x'} \gamma)\circ_x \beta, \quad \text{if } i_X(x)=i_Y(y_0) \text{ and } i_X(x')=i_Z(z_0).$$

Furthermore, for any I-set $\{x, x_0, i\}$ such that $i(x) = i(x_0)$, there exists a map $\mathbf{k} \to \mathcal{P}(\{x, x_0, i\})$ which is a right and left identity for the composition maps.

2.2. 2-COLORED OPERADS

In this article, we work with 2-colored operads only. Most of the time the colors will be denoted by c (for closed) and o (for open). Let us consider the category $\mathbf{Sk}_2$ whose objects are $(n, m; x)$ with $n, m \in \mathbb{N}$ and $x \in \{c, o\}$ and whose set of morphisms $\mathbf{Sk}_2((n, m; x); (n', m'; x'))$ is $S_n \times S_m$ if $n = n', m = m', x = x$ and $\emptyset$ otherwise.

A *2-collection* is a contravariant functor from $\mathbf{Sk}_2$ to $\mathbf{dgvs}$. There is an equivalence between the category of 2-collections and the category of $\{c, o\}$-collections as defined in Definition 2.1.1. The equivalence of categories goes as follows: let $\mathcal{P}$: $\mathbf{Fin}_{\{c,o\}} \to \mathbf{dgvs}$ be a contravariant functor. To the object $(n, m; x) \in \mathbf{Sk}_2$ one associates the object $J_{\{n,m;x\}} = \{0, 1, \ldots, n+m\} \in \mathbf{Fin}_{\{c,o\}}$, with the structure map

$$
\begin{aligned}
j : J_{\{n,m;x\}} &\to \{c, o\} \\
0 &\mapsto x \\
1 \leq k \leq n &\mapsto c \\
n+1 \leq k \leq n+m &\mapsto o.
\end{aligned}
$$

This defines a functor $J : \mathbf{Sk}_2 \to \mathbf{Fin}_{\{c,o\}}$. Note that the category $\mathbf{Sk}_2$ is the skeleton of the category $\mathbf{Fin}_{\{c,o\}}$. The functor $\mathcal{P} \mapsto \mathcal{P} \circ J$ gives one map of the equivalence between the two categories.

Conversely, let $\mathcal{P}$ be a 2-collection. Let $(X, x; i)$ be an object in $\mathbf{Fin}_{\{c,o\}}$ and let n be the number of elements in $i^{-1}(c)\backslash x$ and m be the number of elements in $i^{-1}(o)\backslash x$. The associated $\{c, o\}$-collection is given by

$$
\mathcal{P}(X, x; i) = \left(\bigoplus_{\mathbf{Fin}_{\{c,o\}}((X,x;i),(J_{\{n,m;x\}},j))} \mathcal{P}(n, m; x) \right)_{S_n \times S_m},
$$

where the coinvariants are taken under the simultaneous action of $S_n \times S_m$ on $\mathbf{Fin}_{\{c,o\}}((X, x; i), (J_{\{n,m;x\}}, j))$ and $\mathcal{P}(n, m; x)$.

A *2-colored operad* is a 2-collection together with composition maps which are equivariant with respect to the action of the symmetric group and identity maps $I_c : \mathbf{k} \to \mathcal{P}(1, 0; c)$ and $I_o : \mathbf{k} \to \mathcal{P}(0, 1; o)$ which are unit for the composition maps. We will use the following notation for the compositions: $\circ_i^c : \mathcal{P}(n, m; x) \otimes \mathcal{P}(n', m'; c) \to \mathcal{P}(n+n'-1, m+m'; x)$ for $1 \leq i \leq n$. The notation for $\circ_i^o$ is similar.

A 2-colored operad is *reduced* if $\mathcal{P}(0, 0; x) = 0$ and $\mathcal{P}(1, 0; c) = \mathbf{k} = \mathcal{P}(0, 1; o)$. Consider the 2-collection defined by $I(n, m; x) = \mathbf{k}$, if $(n, m; x) = (1, 0; c)$ or $(n, m; x) = (0, 1, o)$ and by $I(n, m; x) = 0$ otherwise, with the trivial 2-colored operad structure. A 2-colored operad $\mathcal{P}$ is *augmented* if there is a morphism of operads $\mathcal{P} \to \mathcal{I}$. We denote by $\overline{\mathcal{P}}$ the kernel of the augmentation map.

DEFINITION 2.2.1. In the 2-colored case, we will consider (colored) pairs of differential graded vector spaces $V=(V_c, V_o)$. Let $\mathcal{P}$ be a 2-colored operad. An *algebra over* $\mathcal{P}$ or a $\mathcal{P}$-*algebra* is a pair $V=(V_c, V_o)$ in **dgvs**, together with evaluation maps

$$\mathcal{P}(n, m; x) \otimes_{S_n \times S_m} (V_c^{\otimes n} \otimes V_o^{\otimes m}) \to V_x$$

satisfying associativity and unit conditions. Giving a $\mathcal{P}$-algebra is the same as giving a morphism of 2-colored operads $\mathcal{P} \to \mathrm{End}_V$, where End_V is the 2-colored operad $\mathrm{End}_V(n, m; x) = \mathrm{Hom}(V_c^{\otimes n} \otimes V_o^{\otimes m}, V_x)$ with the natural action of the symmetric group and the natural compositions maps. The forgetful functor from $\mathcal{P}$-algebras to pairs in **dgvs** admits a left adjoint, the free $\mathcal{P}$-algebra functor which takes the following form: let $V=(V_c, V_o)$ be a pair in **dgvs**. For $x \in \{c, o\}$, one has

$$\mathcal{P}(V)_x = \bigoplus_{n,m} \mathcal{P}(n, m; x) \otimes_{S_n \times S_m} (V_c^{\otimes n} \otimes V_o^{\otimes m}). \tag{1}$$

The forgetful functor from 2-colored operads to 2-collections also admits a left adjoint. In fact, the free 2-colored operad functor is denoted by $\mathcal{F}(E)$, for any 2-collection E. It can be described in terms of 2-colored trees (i.e., with colored edges) and has a natural weight grading $\mathcal{F}^{(w)}(E)$, where w is the number of vertices of the trees.

2.3. QUADRATIC 2-COLORED OPERAD

As pointed out in [17] and [21], to treat Koszul duality for I-colored operads it is more convenient to view colored operads as K-operads as was originally defined by Ginzburg and Kapranov in [9] by setting K to be the semi-simple algebra $K = \oplus_{c \in I} \mathbf{k}_c$. The vector space $\mathcal{P}(n) = \sum_{(X,x;i)||X|=n+1} \mathcal{P}(X, x; i)$ is then a $K - K^{\otimes n}$-bimodule. It is also an S_n-module and the collection $(\mathcal{P}(n))_n$ forms an $\mathbb{S}$-module. Thus the usual theory of Koszul operads applies in the colored context.

DEFINITION 2.3.1. A *quadratic* 2-*colored operad* is a 2-colored operad of the form $\mathcal{F}(E)/(R)$, where E is a 2-collection, R is a $\mathbb{S}$-submodule of $\mathcal{F}^{(2)}(E)$ and (R) is the ideal generated by R. There are analogous notions of 2-colored cooperads, of free 2-colored cooperads denoted by $\mathcal{F}^c(E)$ and of 2-colored cooperads cogenerated by a 2-collection V with correlation R denoted by $C(V, R)$. Any quadratic 2-colored operad $\mathcal{P} = \mathcal{F}(E)/(R)$ admits a *Koszul dual cooperad* given by $\mathcal{P}^{\text{¡}} = C(sE, s^2R)$, where sE denotes the suspension of the vector space E, that is, $(sE)_n = E_{n-1}$.

We say that the operad is binary quadratic if it is of the above form with E binary, that is, concentrated in arity 2. Namely, $E = E_c \oplus E_o$ with $E_x = E(c, c; x) \oplus E(c, o; x) \oplus E(o, o; x) \oplus E(o, o; x)$. The action of the symmetric group S_2 is internal in $E(c, c; x)$ and $E(o, o; x)$ and permutes the components $E(c, o; x)$ and $E(o, c; x)$. In particular, if E is finite dimensional, then $\dim(E(o, c; x))=\dim(E(c, o; x))$.

DEFINITION 2.3.2. If E is binary and of finite dimension, the *quadratic Koszul dual operad* of $\mathcal{P}=\mathcal{F}(E)/(R)$ is $\mathcal{P}^! := \mathcal{F}(E^\vee)/(R^\perp)$, where $E^\vee = E^* \otimes \mathrm{sgn}_2$, sgn denotes the sign representation and $R^\perp$ denotes the orthogonal of R under the induced pairing $\mathcal{F}^{(2)}(E) \otimes \mathcal{F}^{(2)}(E^\vee)$ as defined by Ginzburg and Kapranov (see also [16, Chapter 7]). Let Λ be the $\mathbb{S}$-module suspension which, for any $\mathbb{S}$-module V, is defined by $\Lambda(V)(k)=s^{1-k}V(k)\otimes \mathrm{sgn}_k$. Recall that if $\circ$ denotes the plethysm for $\mathbb{S}$-modules, then $\Lambda V \circ \Lambda W = \Lambda(V \circ W)$. Moreover, if $\mathcal{P}$ is an operad, then A is a $\mathcal{P}$-algebra if and only if sA is a $\Lambda\mathcal{P}$-algebra. One has the following relation: $(\mathcal{F}(E)/(R))^{¡}=(\Lambda(\mathcal{F}(E)/(R))^!)^*$.

In the non-binary case, the aforementioned definition generalizes. Let us define the *Koszul dual operad* $\mathcal{P}^!$ of a finite dimensional quadratic operad $\mathcal{P}$ as

$$\mathcal{P}^! := (\Lambda\mathcal{P}^{¡})^* \text{ or equivalently } \mathcal{P}^{¡}=(\Lambda\mathcal{P}^!)^*=\Lambda^{-1}((\mathcal{P}^!)^*). \tag{2}$$

Loday and Vallette have proven that if one starts with a quadratic operad $\mathcal{P}$, then $\mathcal{P}^!$ is again quadratic.

PROPOSITION 2.3.3 [16, Proposition 724]. *Let $\mathcal{P}=\mathcal{F}(E)/(R)$ be a quadratic operad. The operad $\mathcal{P}^!$ is quadratic and $\mathcal{P}^!=\mathcal{F}(s^{-1}\Lambda^{-1}E^*)/(R^\perp)$.*

Convention 2.3.4. For a 2-colored operad $\mathcal{P}$, when dealing with $\mathcal{P}$-algebras we will prefer the notation of Section 2.2, that is, the notation $\mathcal{P}(n,m;x)$ and compositions $\circ_i^x$. When dealing with the Koszul duality of operads we will prefer the notation of Section 2.1 and 2.3. Here is an example of notation: $\mathcal{P}(c, c, o, c; o)$ and compositions $\circ_i$.

Remark 2.3.5. All the aforementioned definitions make sense if we replace the category **dgvs** by the category of topological spaces, except for the equivalence of category between 2-collections and $\{c, o\}$-collections, where one replaces

$$\mathcal{P}(X,x;i)=\left(\bigoplus_{\mathbf{Fin}_{\{c,o\}}((X,x;i),(J_{\{n,m;x\}},j))} \mathcal{P}(n,m;x)\right)_{S_n\times S_m}$$

by $\mathcal{P}(X,x;i)=\mathbf{Fin}_{\{c,o\}}((X,x;i),(J_{\{n,m;x\}},j))\times_{(S_n\times S_m)}\mathcal{P}(n,m;x)$.

Given a 2-colored operad in topological spaces, its singular chain complex with coefficients in $\mathbf{k}$, or its cellular chain complex, if the topological operad is an operad in CW-complexes, is an operad in **dgvs**. Moreover, its homology with coefficients in $\mathbf{k}$ is an operad in graded vector spaces. And its degree 0 homology with coefficients in $\mathbf{k}$ is an operad in vector spaces.

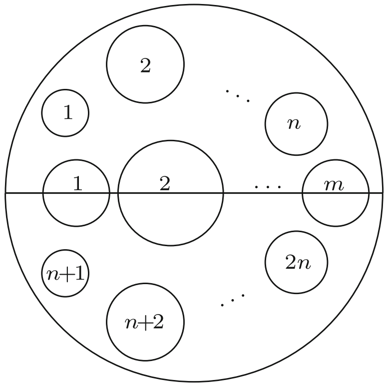


Figure 1. A configuration in $\mathcal{SC}(n, m; o)$.

3. Two Versions of the Swiss-Cheese Operad

Here we provide two definitions for the Swiss-cheese operad. The difference between the two will be related to the difference between OCHA and SHLP.

3.1. LITTLE DISKS

Let D^2 denote the standard unit disk in the complex plane $\mathbb{C}$. By a configuration of n disks in D^2 we mean a map $d: \coprod_{1 \leq s \leq n} D_s^2 \to D^2$ from the disjoint union of n numbered standard disks $D_1^2, \ldots, D_n^2$ to D^2 such that d, when restricted to each disk, is a composition of translations and dilations and such that the images of the different components are disjoint. The image of each such restriction is called a little disk. The space of all configurations of n disks is denoted by $\mathcal{D}_2(n)$ and is topologized as a subspace of $(\mathbb{R}^2 \times \mathbb{R}^+)^n$ containing the coordinates of the center and radius of each little disk. The symmetric group acts on $\mathcal{D}_2(n)$ by renumbering the disks. For $n = 0$, we define $\mathcal{D}_2(0) = \emptyset$. The $\mathbb{S}$-module $\mathcal{D}_2 = \{\mathcal{D}_2(n)\}_{n \geq 0}$ admits a well-known structure of operad given by gluing configurations of disks into little disks, see [18] for the original definition.

DEFINITION 3.1.1. For $m, n \geq 0$ such that $m + n > 0$, let us define $\mathcal{SC}(n, m; o)$ as the space of those configurations $d \in \mathcal{D}_2(2n + m)$ such that its image in D^2 is invariant under complex conjugation and exactly m little disks are left fixed by conjugation. A little disk that is fixed by conjugation must be centered at the real line; in this case it is called *open*. Otherwise, it is called *closed*. The little disks in $\mathcal{SC}(n, m; o)$ are labeled according to the following rules:

(i) Open disks have labels in $\{1, \ldots, m\}$ and closed disks have labels in $\{1, \ldots, 2n\}$.
(ii) Closed disks in the upper half plane have labels in $\{1, \ldots, n\}$. If conjugation interchanges the images of two closed disks, their labels must be congruent modulo n (Figure 1).

There is an action of $S_n \times S_m$ on $\mathcal{SC}(n,m;\, o)$ extending the action of $S_n \times \{e\}$ on pairs of closed disks having modulo n congruent labels and the action of $\{e\} \times S_m$ on open disks.

DEFINITION 3.1.2 (*Swiss-cheese operad*). The 2-colored operad $\mathcal{SC}$ is defined as follows: For $m, n \geq 0$ with $m+n>0, \mathcal{SC}(n,m;\, o)$ is the configuration space defined above and $\mathcal{SC}(0,0;\, o)=\emptyset$. For $n\geq 0$, $\mathcal{SC}(n,0;\, c)$ is defined as $\mathcal{D}_2(n)$ and $\mathcal{SC}(n,m;\, c)=\emptyset$ for $m\geq 1$. The operad structure in $\mathcal{SC}$ is given by

$$\circ_i^c : \mathcal{SC}(n,m;\, x) \times \mathcal{SC}(n',0;\, c) \to \mathcal{SC}(n+n'-1,m;\, x), \quad \text{for } 1\leq i\leq n$$
$$\circ_i^o : \mathcal{SC}(n,m;\, x) \times \mathcal{SC}(n',m';\, o) \to \mathcal{SC}(n+n',m+m'-1;\, x), \quad \text{for } 1\leq i\leq m$$

When $x=c$ and $m=0, \circ_i^c$ is the usual gluing of little disks in $\mathcal{D}_2$. If $x=o, \circ_i^c$ is defined by gluing each configuration of $\mathcal{SC}(n',0;\, c)$ in the little disk labeled by i and then taking the complex conjugate of the same configuration and gluing the resulting configuration in the little disk labeled by $i+n$. Since $\mathcal{SC}(n,m;\, c)=\emptyset$ for $m\geq 1, \circ_i^o$ is only defined for $x=o$ and is given by the usual gluing operation of $\mathcal{D}_2$.

DEFINITION 3.1.3. There is a suboperad $\mathcal{SC}^{\mathrm{vor}}$ of $\mathcal{SC}$ defined by $\mathcal{SC}^{\mathrm{vor}}(n,m;x)=\mathcal{SC}(n,m;x)$, if $x=c$ or $m\geq 1$ and by $\mathcal{SC}^{\mathrm{vor}}(n,m;x)=\emptyset$, otherwise.

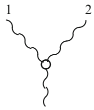
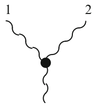
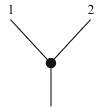
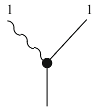
Remark 3.1.4. The aforementioned definition says that $\mathcal{SC}^{\mathrm{vor}}$ coincides with $\mathcal{SC}$ except for $m=0$ and $x=o$, where $\mathcal{SC}^{\mathrm{vor}}(n,0,\, o)=\emptyset$ for any $n\geq 0$. The operad $\mathcal{SC}^{\mathrm{vor}}$ is equivalent to the one defined by Voronov in [22], while $\mathcal{SC}$ coincides with the one defined by Kontsevich in [14].

4. The Operad $H_0(\mathcal{SC}^{\mathrm{vor}})$ and Leibniz Pairs

In this section, we prove that $H_0(\mathcal{SC}^{\mathrm{vor}})$ is a quadratic Koszul operad whose Koszul dual is the operad of Leibniz pairs.

4.1. THE HOMOLOGY OF $\mathcal{SC}^{\mathrm{vor}}$

For presenting a 2-colored operad using trees, we need colored edges. We will adopt the colors *wiggly* $\sim\!\sim\!\sim$ and *straight* ——— corresponding, respectively, to closed and open. The space $\mathcal{SC}^{\mathrm{vor}}(2,0;\, c)$ is homotopy equivalent to S^1, the action of S_2 being the $\mathbb{Z}/2\mathbb{Z}$-action on S^1 via the antipodal map; hence $H(\mathcal{SC}^{\mathrm{vor}}(2,0;\, c))$ has one commutative generator in degree 0 denoted by f_2 and another commutative one, in degree 1 denoted by g_2. The space $\mathcal{SC}^{\mathrm{vor}}(0,2;\, o)$ is homotopy equivalent to 2 points and the action of S_2 is free. The planar tree denoted by $e_{0,2}$ is a generator of the S_2-module $H_0(\mathcal{SC}^{\mathrm{vor}}(0,2;\, o))=H(\mathcal{SC}^{\mathrm{vor}}(0,2;\, o))$. The space $\mathcal{SC}^{\mathrm{vor}}(1,1;\, o)$ is contractible and the only generator of $H(\mathcal{SC}^{\mathrm{vor}}(1,1;\, o))$ will be denoted by $e_{1,1}$. In terms of trees, all the above-described generators will be denoted as follows:

{ $f_2=$  ; $g_2=$  ; $e_{0,2}=$  ; $e_{1,1}=$  }. (3)

The above elements generate the operad $H(\mathcal{SC}^{\mathrm{vor}})$. Using the result of Cohen [2] about algebras over $H(\mathcal{D}_n)$, Voronov has proven the following Theorem:

THEOREM 4.1.1 (Voronov [22]). *An algebra over $H(\mathcal{SC}^{\mathrm{vor}})$ is a pair (G, A), where G is a Gerstenhaber algebra and A is an associative algebra over the commutative ring G.*

An algebra over the commutative ring G corresponds to a map $\rho: G\otimes A\to A$ satisfying

$$\rho(cc', a)=\rho(c, \rho(c', a))=\rho(c', \rho(c, a)) \quad \text{and} \quad \rho(c, aa')=\rho(c, a)a'=a\rho(c, a'). \tag{4}$$

COROLLARY 4.1.2 (Free $H_0(\mathcal{SC}^{\mathrm{vor}})$-algebras). *Let $V=(V_c, V_o)$ be a pair of graded vector spaces. The free $H_0(\mathcal{SC}^{\mathrm{vor}})$-algebra generated by the vector space V has the following form:*

$$H_0(\mathcal{SC}^{\mathrm{vor}})(V)_c=S(V_c) \quad \textit{and} \quad H_0(\mathcal{SC}^{\mathrm{vor}})(V)_o=S^+(V_c)\otimes T(V_o), \tag{5}$$

where the associative structure on $H_0(\mathcal{SC}^{\mathrm{vor}})(V)_o$ is obtained as the usual tensor product of associative algebras. The module structure is given by

$$\begin{aligned} S(V_c)\otimes(S^+(V_c)\otimes T(V_o)) &\to S^+(V_c)\otimes T(V_o)\\ x\otimes(u\otimes v) &\mapsto xu\otimes v \end{aligned}$$

COROLLARY 4.1.3 (Operad description). *The operad $H_0(\mathcal{SC}^{\mathrm{vor}})$ has a binary quadratic presentation $\mathcal{F}(E_v)/(R_v)$ where*

$$E_v=\underbrace{kf_2}_{=E_v(c,c;c)} \oplus \underbrace{k[S_2]e_{0,2}}_{=E_v(o,o;o)} \oplus \underbrace{k[S_2]e_{1,1}}_{=E_v(c,o;o)\oplus E_v(o,c;o)}.$$

The action of the symmetric group on f_2 is the trivial action. The representation $k[S_2]$ is the regular representation. The element $e_{1,1}$ forms a basis of $E_v(c, o; o)$ and $e_{1,1}\cdot(21)$ a basis of $E_v(o, c; o)$. The vector space R_v is the sub S_3-module of $\mathcal{F}^{(2)}(E_v)$ generated by the relations:

- *associativity of* $f_2: f_2 \circ_2 f_2 - f_2 \circ_1 f_2$,
- *associativity of* $e_{0,2}: e_{0,2} \circ_2 e_{0,2} - e_{0,2} \circ_1 e_{0,2}$,
- $e_{1,1}$ *is an action:*
  $e_{1,1} \circ_1^o e_{1,1} - e_{1,1} \circ_1^c f_2$,
  $e_{1,1} \circ_1^o e_{0,2} - e_{0,2} \circ_1^o e_{1,1}$ *and* $e_{1,1} \circ_1^o e_{0,2} - (e_{0,2} \circ_2^o e_{1,1}) \cdot (213)$.

Remark 4.1.4. The generator g_2, corresponding to the Gerstenhaber bracket in Theorem 4.1.1, does not show up in the aforementioned results because it does not belong to the zeroth homology.

4.2. LEIBNIZ PAIRS

A *Leibniz pair* consists in the following data: a Lie algebra L, an associative algebra A and a Lie morphism $L \to \mathrm{Der}(A)$.

From the definition of a Leibniz pair one can build the colored operad $\mathcal{LP}$ which is binary quadratic of the form $\mathcal{F}(E_{lp})/(R_{lp})$ so that $\mathcal{LP}$-algebras are Leibniz pairs. The collection E_{lp} has the following description:

$$E_{lp} = \underbrace{k[\mathrm{sgn}]\mathfrak{l}_2}_{=E_{lp}(c,c;c)} \oplus \underbrace{k[S_2]\mathfrak{n}_{0,2}}_{=E_{lp}(o,o;o)} \oplus \underbrace{k[S_2]\mathfrak{n}_{1,1}}_{E_{lp}(c,o;o) \oplus E_{lp}(o,c;o)},$$

where sgn denotes the signature representation. The S_3-submodule $R_{lp} \in \mathcal{F}^{(2)}(E_{lp})$ is generated by the following relations:

(J) The Jacobi relation: $(\mathfrak{l}_2 \circ_1^c \mathfrak{l}_2) \cdot (\mathrm{id} + (231) + (312))$,
(A) The associativity relation: $\mathfrak{n}_{0,2} \circ_1^o \mathfrak{n}_{0,2} - \mathfrak{n}_{0,2} \circ_2^o \mathfrak{n}_{0,2}$,
(D) The derivation relation: $\mathfrak{n}_{1,1} \circ_1^o \mathfrak{n}_{0,2} - (\mathfrak{n}_{0,2} \circ_2^o \mathfrak{n}_{1,1}) \cdot (213) - \mathfrak{n}_{0,2} \circ_1^o \mathfrak{n}_{1,1}$,
(M) The Lie algebra morphism relation: $\mathfrak{n}_{1,1} \circ_1^c \mathfrak{l}_2 - (\mathfrak{n}_{1,1} \circ_1^o \mathfrak{n}_{1,1}) \cdot (\mathrm{id} - (213))$.

LEMMA 4.2.1. *The Koszul dual operad of* $\mathcal{LP}$ *is the operad* $H_0(\mathcal{SC}^{\mathrm{vor}})$.

Proof. The operad $\mathcal{LP}$ is binary quadratic and we can use Definition 2.3.2. The Koszul dual operad of $\mathcal{LP}$ is $\mathcal{LP}^! = \mathcal{F}(E_{lp}^\vee)/(R_{lp}^\perp)$. From $E_{lp} = k[\mathrm{sgn}]\mathfrak{l}_2 \oplus k[S_2]\mathfrak{n}_{0,2} \oplus k[S_2]\mathfrak{n}_{1,1}$, one gets $E_{lp}^\vee = kf_2 \oplus k[S_2]e_{0,2} \oplus k[S_2]e_{1,1} = E_v$. The pairing between E_v and E_{lp} induces a pairing between $\mathcal{F}^{(2)}(E_{lp})$ and $\mathcal{F}^{(2)}(E_v)$ (see [16, Chapter 7]). One gets that $R_{lp}^\perp(c, c, c; c)$ is the orthogonal of the Jacobi relation for $\mathfrak{l}_2$, that is, the associativity relation for f_2. Similarly $R_{lp}^\perp(o, o, o; o)$ is the orthogonal of the associativity relation for $\mathfrak{n}_{0,2}$, that is, the associativity relation for $e_{0,2}$.

The space $\mathcal{F}(E_{lp})(c, c, o; o)$ has dimension 3 and $R_{lp}(c, c, o; o)$ has dimension 1 which is relation (M). As a consequence, the dimension of $R_{lp}^\perp(c, c, o; o)$ is 2 and corresponds to the first relation presented in (4). On the other hand, $\mathcal{F}(E_{lp})(c, o, o; o)$ has dimension 6 and $R_{lp}(c, o, o; o)$ has dimension 2 which is relation (D). Hence the dimension of $R_{lp}^\perp(c, o, o; o)$ is 4 and corresponds to the second relation in (4). □

LEMMA 4.2.2. *Let $V=(V_c, V_o)$ be a pair of graded vector spaces. The free $\mathcal{LP}$-algebra generated by V has the following form: $\mathcal{LP}(V)_c=\mathcal{L}ie(V_c)$, and $\mathcal{LP}(V)_o=T(T^+(V_c)\otimes V_o)$. The action by derivation of $\mathcal{LP}(V)_c$ on $\mathcal{LP}(V)_o$ is uniquely determined by the action of V_c on $T^+(V_c)\otimes V_o$ and is induced by the concatenation $V_c\otimes T^+(V_c)\to T^+(V_c)$.*

Proof. We prove that the structure defined above satisfies the universal property with respect to the Leibniz pair structure. Let (L, A) be a Leibniz pair. Let us denote by $l: L\otimes L\to L$ the Lie bracket, by $\mu: A\otimes A\to A$ the associative product and by $\alpha: L\otimes A\to A$ the action of L on A. Let $\phi: V_c\to L$ and $\psi: V_o\to A$ be two linear maps. One has to build a unique pair $(\tilde\phi: \mathcal{L}ie(V_c)\to L, \tilde\psi: \mathcal{LP}(V)_o\to A)$ such that $\tilde\phi$ is a morphism of Lie algebras extending ϕ, such that $\tilde\psi$ is a morphism of associative algebras extending ψ and such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{LP}(V)_c\otimes\mathcal{LP}(V)_o \arrow[r, "\mathfrak{n}_{1,1}"] \arrow[d, "\tilde\phi\otimes\tilde\psi"'] & \mathcal{LP}(V)_o \arrow[d, "\tilde\psi"] \\
L\otimes A \arrow[r, "\alpha"] & A.
\end{tikzcd}$$

Since $\mathcal{LP}(V)_c$ is the free Lie algebra generated by V_c, $\tilde\phi$ is the unique morphism of Lie algebras extending ϕ. Since $\mathcal{LP}(V)_o$ is the free associative algebra generated by $T^+(V_c)\otimes V_o$ the morphism $\tilde\psi$ is uniquely determined by its value on $T^+(V_c)\otimes V_o$. Besides $T^+(V_c)\otimes V_o$ is the free $\mathcal{L}ie(V_c)$-module generated by V_o for the universal enveloping algebra of $\mathcal{L}ie(V_c)$ is $T(V_c)$ and the action on $T^+(V_c)\otimes V_o$ is defined as the free action. Consequently $\tilde\psi$ is uniquely determined by its value on $1\otimes V_o$ which is ψ. The commutativity of the diagram follows. $\square$

4.3. THE OPERAD $\mathcal{LP}$ IS KOSZUL

Let $\mathcal{P}$ be a binary quadratic operad and A be a $\mathcal{P}$-algebra. The $\mathcal{P}$-homology of A, denoted by $H_*^{\mathcal{P}}(A)$, is the homology of the complex $((\mathcal{P}^!)^c(sA), \partial)$, where $(\mathcal{P}^!)^c(sA)$ is the free $\mathcal{P}^!$-coalgebra cogenerated by the suspension of A, and ∂ is the unique coderivation of $\mathcal{P}^!$-coalgebra extending the $\mathcal{P}$-algebra structure of A. Thus, given a Leibniz pair (L, A), the complex computing $H_*^{\mathcal{LP}}(L, A)$ is

$$C_*(L, A)=(\mathcal{LP}^{!^c}(s(L, A)), \partial)=(H_0(\mathcal{SC}^{\mathrm{vor}})^c(s(L, A)), \partial).$$

In Proposition 4.3.2 we recognize the complex introduced in [4] and in Theorem 4.3.3 we prove that the operad $\mathcal{LP}$ is Koszul.

The first step of our study is to describe the inverse map of the bijection

$$\mathrm{Coder}(H_0(\mathcal{SC}^{\mathrm{vor}})^c(V))\to\mathrm{Hom}(H_0(\mathcal{SC}^{\mathrm{vor}})^c(V), V),$$

when $V=(V_c, V_o)$ is a pair of vector spaces. Recall that $H_0(\mathcal{SC}^{\text{vor}})^c(V)_c = S^c(V_c)$ and $H_0(\mathcal{SC}^{\text{vor}})^c(V)_o = (S^c)^+(V_c) \otimes T^c(V_o)$, where S^c denotes the free cocommutative functor and T^c the free coassociative functor. Denote by $\bar{v}_{[n]}$ the element $\overline{v_1 \otimes \cdots \otimes v_n}$ in $S^c(V)$. For any subset $A=\{a_1, \ldots, a_k\} \subset \{1, \ldots, n\}$ the element $\overline{v_{a_1} \otimes \cdots \otimes v_{a_k}} \in S^c(V)$ is denoted by $\bar{v}_A$. For $w_1, \ldots, w_m \in W$ and $1 \le k \le l \le m$, the element $w_k \otimes \cdots \otimes w_l$ of $T^c(W)$ is denoted by $w_{(k;l)}$.

LEMMA 4.3.1. *Let $V=(V_c, V_o)$ be a pair of vector spaces. The inverse map of the bijection*

$$\operatorname{Coder}(H_0(\mathcal{SC}^{\text{vor}})^c(V)) \to \operatorname{Hom}(H_0(\mathcal{SC}^{\text{vor}})^c(V), V)$$

has the following form: For $\psi: H_0(\mathcal{SC}^{\text{vor}})^c(V)_c \to V_c$ and $\phi: H_0(\mathcal{SC}^{\text{vor}})^c(V)_o \to V_o$ the $H_0(\mathcal{SC}^{\text{vor}})^c(V)$-coderivations $\tilde{\psi}: H_0(\mathcal{SC}^{\text{vor}})^c(V)_c \to H_0(\mathcal{SC}^{\text{vor}})^c(V)_c$ and $\tilde{\phi}: H_0(\mathcal{SC}^{\text{vor}})^c(V)_o \to H_0(\mathcal{SC}^{\text{vor}})^c(V)_o$ are described by

$$\tilde{\psi}(\bar{v}_{[p]}) = \sum_{\substack{A \sqcup B = \{1,\ldots,p\},\\ A \neq \emptyset}} \psi(\bar{v}_A)\bar{v}_B, \tag{6}$$

$$\begin{aligned}\tilde{\phi}(\bar{v}_{[p]} \otimes w_{(1;q)}) = &\sum_{\substack{A \sqcup B = \{1,\ldots,p\},\\ A \neq \emptyset}} \psi(\bar{v}_A)\bar{v}_B \otimes w_{(1;q)} \\ &+ \sum_{\substack{A \sqcup B = \{1,\ldots,p\},\\ 0 \le i < j \le q}} \bar{v}_A \otimes (w_{(1;i)}\phi(\bar{v}_B \otimes w_{(i+1;j)})w_{j+1;q}).\end{aligned} \tag{7}$$

Proof. The projection of $\tilde{\psi}$ onto V_c is ψ and the projection of $\tilde{\phi}$ onto V_o is ϕ. Hence it is enough to prove that $(\tilde{\psi}, \tilde{\phi})$ is a coderivation of the 2-colored coalgebra $(H_0(\mathcal{SC}^{\text{vor}})^c(V))$. The map $\tilde{\psi}$ is a coderivation of the cofree cocommutative coalgebra $S^c(V_c)$. Dualizing Corollary 4.1.2, one gets that the coassociative coproduct Δ_o on $(H_0(\mathcal{SC}^{\text{vor}})^c(V))_o = (S^c)^+(V_c) \otimes T^c(V_o)$ is given by $\Delta_o(\bar{v}_{[p]} \otimes w_{(1;q)}) = \sum_{A \sqcup B = [p], 1 \le i \le q-1} (\bar{v}_A \otimes w_{(1;i)}) \otimes (\bar{v}_B \otimes w_{(i+1;q)})$, while the comodule structure is given by $\gamma_{c,o}(\bar{v}_{[p]} \otimes w_{(1;q)}) = \sum_{A \sqcup B = [p], A \neq \emptyset} \bar{v}_A \otimes (\bar{v}_B \otimes w_{(1;q)})$. The map $\tilde{\phi}$ is a coderivation of the coassociative coalgebra $(H_0(\mathcal{SC}^{\text{vor}})^c(V))_o$ because $\Delta_o\tilde{\phi} = (\tilde{\phi} \otimes 1 + 1 \otimes \tilde{\phi})\Delta_o$. The pair $(\tilde{\phi}, \tilde{\psi})$ is a coderivation of comodule because $\gamma_{c,o}\tilde{\phi} = (\tilde{\psi} \otimes 1 + 1 \otimes \tilde{\phi})\gamma_{c,o}$. $\square$

PROPOSITION 4.3.2. *Let (L, A) be a Leibniz pair and let $C_*(L, A)$ be the complex computing the homology $H_*^{\mathcal{LP}}(L, A)$. The closed component $C_*(L, A)_c$ is the Chevalley–Eilenberg complex $C_*^{CE}(L; \mathbf{k})$. The open component $C_*(L, A)_o$ is the bicomplex $C_{p,q}(L, A)_o = C_p^{CE}(L, C_q^{\text{Hoch}}(A))$, where $C_q^{\text{Hoch}}(A)$ is the Hochschild complex computing the Hochschild homology of the non-unital associative algebra A with coefficients in $\mathbf{k}$.*

Proof. Denote by $[-,-]$ the Lie bracket of L as well as the Lie action on A. The Leibniz pair structure on $V=(L,A)$ induces a natural map $H_0(\mathcal{SC}^{\text{vor}})^c(sV)\to sV$ which is given componentwise by $\psi: S^c(sL)\to sL$, with $\psi(\overline{sl_1\otimes sl_2\ldots\otimes sl_n})=s[l_1,l_2]$ if $n=2$ and is zero if $n\neq 2$ and $\phi:(S^c)^+(sL)\otimes T^c(sA)\to sA$, with $\phi(\overline{sl_1\otimes\ldots\otimes sl_n}\otimes sa_1\otimes\ldots sa_k)=s[l_1,a_1]$, if $n=k=1$, and is $s(a_1\cdot a_2)$ if $n=0$ and $k=2$ and is 0 otherwise.

Since L and A are concentrated in degree 0, sL and sA are concentrated in degree 1. Hence one can use the isomorphism $S^n(sL)\simeq\Lambda^n(L)$. Formula (6) gives the differential on the closed component of the complex $d_c: C_n(L,A)_c=\Lambda^n(L)\to\Lambda^{n-1}(L)$, by

$$d_c(l_1\wedge\ldots\wedge l_n)=\sum_{1\le i<j\le n}(-1)^{i+j-1}[l_i,l_j]\wedge l_1\ldots\wedge\hat{l}_i\wedge\ldots\wedge\hat{l}_j\wedge\ldots\wedge l_n$$

which is the complex computing the Chevalley–Eilenberg homology of the Lie algebra L with trivial coefficients.

Formula (7) gives the differential on the open component of the complex $d_o: C_n(L,A)_o=\oplus_{p+q=n}\Lambda^p(L)\otimes T^q(A)\to C_{n-1}(L,A)_o$ by

$$\begin{aligned}
&d_o(l_1\wedge\ldots\wedge l_p\otimes a_1\otimes\cdots\otimes a_q)\\
&\quad=\sum_{1\le i<j\le p}(-1)^{i+j-1}[l_i,l_j]\wedge l_1\ldots\wedge\hat{l}_i\wedge\ldots\wedge\hat{l}_j\wedge\ldots\wedge l_p\otimes a_{(1;q)}\\
&\qquad+\sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{p-i+j-1}l_1\cdots\wedge\hat{l}_i\wedge\cdots\wedge l_p\otimes(a_1\otimes\cdots a_{j-1}\otimes[l_i,a_j]\otimes\cdots\otimes a_q)\\
&\qquad+\sum_{i=1}^{q-1}(-1)^{p+i}l_{[p]}\otimes a_1\otimes\cdots\otimes a_i\cdot a_{i+1}\otimes\cdots\otimes a_q.
\end{aligned}$$

□

THEOREM 4.3.3. *The operad $\mathcal{LP}$ is Koszul and so is the operad $H_0(\mathcal{SC}^{\text{vor}})$.*

Proof. Consider the free $\mathcal{LP}$-algebra generated by $V=(V_c,V_o)$ given in Lemma 4.2.2: $\mathcal{LP}(V)_c=\mathcal{L}ie(V_c)$ and $\mathcal{LP}(V)_o=T(T^+(V_c)\otimes V_o)$. The homology of the complex $C_*(\mathcal{LP}(V))_c$ is concentrated in degree 0 with value V_c, because it is the Chevalley–Eilenberg homology of the free Lie algebra $\mathcal{L}ie(V_c)$. Let us compute the homology of the bicomplex

$$C_{p,q}(\mathcal{LP}(V))_o=C_p^{CE}(\mathcal{L}ie(V_c),C_q^{\text{Hoch}}(\mathcal{LP}(V)_o)).$$

Since $\mathcal{LP}(V)_o$ is a free associative algebra its Hochschild homology is concentrated in degree 0 with value $M=T^+(V_c)\otimes V_o$. The induced Lie-module structure is the free Lie module structure. Consequently $H_*^{CE}(\mathcal{L}ie(V_c);T^+(V_c)\otimes V_o)$ is concentrated in degree 0 with value V_o. The spectral sequence converging to $H_*(\mathcal{LP}(V))_o$

collapses at page 2 and $H_0(\mathcal{LP}(V))_o = V_o$, $H_n(\mathcal{LP}(V))_o = 0, n > 0$. Since $H_0(\mathcal{SC}^{\mathrm{vor}})$ is the Koszul dual operad associated with $\mathcal{LP}$ it is also Koszul. □

Remark 4.3.4. Johan Alm has proven in [1] that $\mathcal{LP}$ is Koszul. He introduced the operad $\mathcal{NCG}$ and used rewriting methods to prove that it is Koszul. Since $\mathcal{NCG} = \Lambda_c\mathcal{LP}$, where Λ_c is the c-suspension (see Definition 7.2.2), the koszulity of $\mathcal{LP}$ follows.

5. Strong Homotopy Leibniz Pairs

Since $\mathcal{LP}$ is Koszul and $\mathcal{LP}^! = H_0(\mathcal{SC}^{\mathrm{vor}})$, it follows that the cobar construction of the cooperad $\mathcal{LP}^{¡} = (\Lambda H_0(\mathcal{SC}^{\mathrm{vor}}))^*$ (see Definition 2.3.2) is quasi-isomorphic to $\mathcal{LP}$. We will denote the operad $\Omega[(\Lambda H_0(\mathcal{SC}^{\mathrm{vor}}))^*]$ by $\mathcal{LP}_\infty$. Algebras over this operad are called *Strong Homotopy Leibniz Pairs* (or SHLP for short).

5.1. DESCRIPTION OF $\mathcal{LP}_\infty$ IN TERMS OF TREES

From Corollary 4.1.2 one gets that the 2-collection structure of $\overline{H_0(\mathcal{SC}^{\mathrm{vor}})}$ is such that $H_0(\mathcal{SC}^{\mathrm{vor}})(n, 0; c)$ is the trivial representation $\mathbf{k}$ of S_n, for $n > 1$ and $H_0(\mathcal{SC}^{\mathrm{vor}})(p, q; o)$ is the tensor product $\mathbf{k} \otimes \mathbf{k}[S_q]$ of the trivial representation of S_p and the regular representation of S_q, for $p + q > 1$ and $q > 0$.

A *Partially Planar Tree* is an isotopy class of trees embedded in the Euclidean 3-dimensional space $\mathbb{R}^3$ such that the straight (or planar) edges are constrained to be in a fixed plane, say the xy-plane. The planar edges will also be called *open* while the wiggly (or spatial) edges will be called *closed*.

For illustrations of the partially planar corollae $\mathfrak{l}_n$ and $\mathfrak{n}_{p,q}$, we refer the reader to [10,12]. Since $\mathcal{LP}_\infty$ is the free operad generated by $s^{-1}(\Lambda \overline{H_0(\mathcal{SC}^{\mathrm{vor}})})^*$, it follows that $\mathcal{LP}_\infty$ is generated by the corollae $\mathfrak{l}_n$ and $\mathfrak{n}_{p,q}$ for all $n \geq 2$, $p + q \geq 2$ and $q > 0$, of degrees $n - 2$ and $p + q - 2$. For each tree $T \in \mathcal{LP}_\infty$, its degree is $|T| = \#l - \#i - 2$, where $\#l$ denotes the number of leaves and $\#i$ denotes the number internal edges of T. Notice that $|T \circ_i^x S| = |T| + |S|$ for any label i and color x.

The differential on $\mathcal{LP}_\infty$ is the unique derivation extending the cooperad structure of $(\Lambda H_0(\mathcal{SC}^{\mathrm{vor}}))^*$. It is not difficult to check that it coincides with the vertex expansion operator on partially planar trees:

$$\mathrm{d}T = \sum_{T = T'/e} \pm T', \tag{8}$$

where the signs $\pm$ are given by the operadic suspension Λ. For an explicit description of the differential d on trees, we refer the reader to [12]. An $\mathcal{LP}_\infty$-algebra structure on a pair (L, A) is given by a morphism $\mathcal{LP}_\infty \to \mathrm{End}_{(L,A)}$ which must respect the differential d and the operad structure. Respecting the differential d is equivalent to verifying the relations presented in Theorem 2 of [12]. Hence an

SHLP is a pair (L, A) where L is an L_∞-algebra and A is an A_∞-algebra with an L_∞-morphism $L \to C^{>0}(A, A)[1]$, where $C^{>0}(A, A)$, the truncated deformation complex of A has been defined in the introduction and in [12]. Consequently, SHLP is the minimal model of Leibniz pairs as suggested by Martin Markl in the appendix of [12].

5.2. CODERIVATIONS ON $S^c(L)$ AND $(S^c)^+(L) \otimes T^c(A)$

Using the description of coderivations in Lemma 4.3.1, one can describe SHLP in terms of square zero 2-colored operad coderivations. Here we give another description in the spirit of a paper written by the first author [11], i.e., a description of strong homotopy Leibniz pairs in terms of coderivations with respect to the coassociative coalgebra structures of $S^c(L)$ and $(S^c)^+(L) \otimes T^c(A)$ as opposed to the description in terms of 2-colored operad coderivations.

It is well known that an L_∞-algebra structure on L is equivalent to a degree -1 coderivation $\mathcal{D}$ in $\mathrm{Coder}(S^c(sL))$ such that $[\mathcal{D}, \mathcal{D}] = 0$, where the bracket denotes the commutator of coderivations. We will give an analogous description for SHLPs using $\mathrm{Coder}(S^c(sL))$ and $\mathrm{Coder}((S^c)^+(sL) \otimes T^c(sA))$.

NOTATION 5.2.1. *We will denote by L^p the subspace of weight p elements in $(S^c)^+(L)$ and by $L^{\wedge p}$ the subspace of weight p elements in the exterior coalgebra $(\Lambda^c)^+(L)$. In other words, the weight grading of those coalgebras will be denoted by $(S^c)^+(L) = \bigoplus_{p \geq 0} L^p$ and $(\Lambda^c)^+(L) = \bigoplus_{p\geq 0} L^{\wedge p}$.*

We first note that a map $g : L^p \otimes A^{\otimes q} \to A$ with $q \geq 1$, can be lifted to a coderivation in $\mathrm{Coder}((S^c)^+(L) \otimes T^c(A))$ as follows: For any $\bar{v}_{[n]} \otimes w_{(1;m)} \in (S^c)^+(L) \otimes T^c(A)$, we have that $\tilde{g}(\bar{v}_{[n]} \otimes w_{(1;m)})$ is zero if $n < p$ or $m < q$; otherwise it is given by

$$\tilde{g}(\bar{v}_{[n]} \otimes w_{(1;m)}) = \sum_{\substack{A \sqcup B = \{1,\ldots,n\},\ 0 \leq i < j \leq m \\ \#B = p,\ j-i=q}} \pm \bar{v}_A \otimes (w_{(1;i)} g(\bar{v}_B \otimes w_{(i+1;j)}) w_{(j+1;m)}),$$

the sign $\pm$ is given by the action of permutations on $\bar{v}_{[n]} \otimes w_{(1;m)}$ and each w_i in $w_{(1;m)} = w_1 \otimes \cdots \otimes w_m \in A^{\otimes m}$ is homogeneous of degree $|w_i|$.

We denote by $\mathrm{Coder}_A((S^c)^+(L) \otimes T^c(A))$ the subvector space of $\mathrm{Coder}((S^c)^+(L) \otimes T^c(A))$ spanned by those coderivations obtained by lifting maps $g : (S^c)^+(L) \otimes T^c(A) \to A$. Hence, as vector spaces we have: $\mathrm{Coder}_A((S^c)^+(L) \otimes T^c(A)) \cong \mathrm{Hom}((S^c)^+(L) \otimes T^c(A), A)$.

Since, $\mathrm{Hom}((S^c)^+(L) \otimes T^c(A), A) = \mathrm{Hom}((S^c)^+(L), \mathrm{Hom}(T^c(A), A))$ and $\mathrm{Hom}(T^c(A), A) \cong \mathrm{Coder}(T^c(A))$, we have the following isomorphism of vector spaces:

$$\mathrm{Coder}_A((S^c)^+(L) \otimes T^c(A)) \cong \mathrm{Hom}((S^c)^+(L), \mathrm{Coder}(T^c(A))) \tag{9}$$

Let l_G denote the Lie bracket in $\mathrm{Coder}(T^c(A))$ given by the commutator of coderivations, i.e., the Gerstenhaber bracket. Since $(S^c)^+(L)$ is a cocommutative coalgebra with coproduct Δ, the convolution bracket given by $[f,g]=l_G\circ(f\otimes g)\circ\Delta$, defines a graded Lie algebra structure on $\mathrm{Hom}((S^c)^+(L),\mathrm{Coder}(T^c(A)))$. Then the isomorphism (9) is an isomorphism of graded Lie algebras.

5.2.1. *Semi-Direct Product*

A natural Lie algebra representation is defined by $\rho(\phi)f=f\circ\phi$. for any $\phi\in\mathrm{Coder}(S^c(L))$ and $f\in\mathrm{Hom}((S^c)^+(L),\mathrm{Coder}(T^c(A)))$. One can check that ρ is a representation by derivations; hence, in view of the isomorphism (9), there is a Lie algebra action by derivations of $\mathrm{Coder}(S^c(L))$ on $\mathrm{Coder}_A((S^c)^+(L)\otimes T^c(A))$.

THEOREM 5.2.2. *An SHLP structure on a pair (L,A) is equivalent to a degree -1 element $\mathcal{D}\in\mathrm{Coder}(S^c(sL))\ltimes\mathrm{Coder}_A((S^c)^+(sL)\otimes T^c(sA))$ such that $[\mathcal{D},\mathcal{D}]=0$.*

Proof. Using the description of $\mathcal{LP}_\infty$ in terms of trees, an algebra over $\mathcal{LP}_\infty$ consists of two families of maps $\{l_n:L^{\wedge n}\to L\}_{n\geq 2}$ and $\{n_{p,q}:L^{\wedge p}\otimes A^{\otimes q}\to A\}_{p+q\geq 2}$. The maps l_n and $n_{p,q}$ correspond, respectively, to the corollae $\mathfrak{l}_n$ and $\mathfrak{n}_{p,q}$. So, their degrees must be given by $|l_n|=n-2$ and $|n_{p,q}|=p+q-2$ and they must verify the identities corresponding to the definition of the differential operator of $\mathcal{LP}_\infty$ given by formulas (8) which in terms of maps becomes two sets of equations. The first one gives precisely the relations defining L_∞-algebras while the second one gives, up to signs, the relations in Theorem 2 of [12].

Let $\mathcal{D}\in\mathrm{Coder}(S^c(sL))\ltimes\mathrm{Coder}_A((S^c)^+(sL)\otimes T^c(sA))$ be a degree -1 coderivation:

$$\mathcal{D}=\sum_{p\geq 1}\tilde{l}_p+\sum_{p\geq 0,q\geq 1}\tilde{n}_{p,q}.$$

Since suspension converts the symmetric algebra into the exterior algebra and the coderivation $\mathcal{D}$ has degree -1 on the suspension (sL,sA), it follows that the components of $\mathcal{D}$ can be viewed as maps $l_p:L^{\wedge p}\to L$ and $n_{p,q}:L^{\wedge p}\otimes A^{\otimes q}\to A$ of degrees $|l_p|=p-2$ and $|n_{p,q}|=p+q-2$. Denoting $\mathcal{D}_L=\sum_{p\geq 1}\tilde{l}_p$ and $\mathcal{D}_A=\sum_{p\geq 0,q\geq 1}\tilde{n}_{p,q}$, we have that $\mathcal{D}=\mathcal{D}_L+\mathcal{D}_A$. Using the definition of the bracket in the semi-direct product $\mathrm{Coder}(S^c(sL))\ltimes\mathrm{Coder}_A((S^c)^+(sL)\otimes T^c(sA))$, it follows that

$$[\mathcal{D},\mathcal{D}]=[\mathcal{D}_L+\mathcal{D}_A,\mathcal{D}_L+\mathcal{D}_A]=[\mathcal{D}_L,\mathcal{D}_L]+2[\mathcal{D}_L,\mathcal{D}_A]+[\mathcal{D}_A,\mathcal{D}_A],$$

where $[\mathcal{D}_L,\mathcal{D}_L]\in\mathrm{Coder}(S^c(sL))$ and $[\mathcal{D}_L,\mathcal{D}_A]+[\mathcal{D}_A,\mathcal{D}_A]\in\mathrm{Coder}((S^c)^+(sL)\otimes T^c(sA))$. The equation $[\mathcal{D},\mathcal{D}]=0$ is equivalent to $[\mathcal{D}_L,\mathcal{D}_L]=0$ and $2[\mathcal{D}_L,\mathcal{D}_A]+[\mathcal{D}_A,\mathcal{D}_A]=0$. Up to a factor of 2 the first identity gives the first set of relations in a $\mathcal{LP}_\infty$-algebra and the second identity gives the second set of relations. $\square$

6. The Operad $H_0(\mathcal{SC})$ is Koszul

In this section we follow closely the article by Imma Galvez-Carrillo, Andy Tonks and Bruno Vallette [6] to prove that the operad $H_0(\mathcal{SC})$ is Koszul. In fact, that operad is not quadratic, and we need first to describe it in terms of generators and relations that are quadratic and linear. By projecting the relations onto the quadratic part, we obtain an operad $qH_0(\mathcal{SC})$ which turns out to be Koszul. By definition [6, Appendix A.3], one has that $H_0(\mathcal{SC})$ is Koszul.

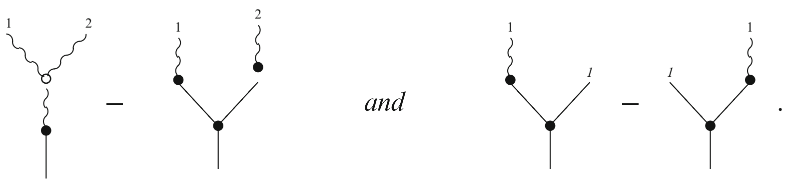
6.1. THE HOMOLOGY OF $\mathcal{SC}$

We say that an associative algebra A over a commutative ring R is *unital* if the R-module structure of A is given by a central ring monomorphism $u: R \to A$. Note that being unital as an R-algebra does not imply that A is unital as a $\mathbf{k}$-algebra. The following Theorem is an easy consequence of both F. Cohen and Voronov's computation:

THEOREM 6.1.1. *An algebra over $H(\mathcal{SC})$ is a pair (G, A), where G is a Gerstenhaber algebra and A is a unital associative algebra over the commutative algebra G.*

For our purposes, it is useful to present the degree zero homology $H_0(\mathcal{SC})$ using trees. The trees involved in this case are f_2 and $e_{0,2}$ already presented in (3) and the tree $e_{1,0} =$ , which generates the vector space $H_0(\mathcal{SC}(1, 0; \mathfrak{c}))$.

COROLLARY 6.1.2. *The operad $H_0(\mathcal{SC})$ can be presented as $\mathcal{F}(e_{0,2}, f_2, e_{1,0})/R$, where R is the ideal generated by the associator of $e_{0,2}$, the associator of f_2 and by the terms*



It follows from the above Theorem that an $H_0(\mathcal{SC})$-algebra is a pair (C, A), where A is an associative unital algebra over the commutative algebra C.

6.2. $H_0(\mathcal{SC})$ IS A QUADRATIC-LINEAR OPERAD

Let us first recall the theory explained in [6] for quadratic-linear operads. A *quadratic-linear* operad is of the form $\mathcal{F}(E)/(R)$ with $R \subset \mathcal{F}^{(1)}(E) \oplus \mathcal{F}^{(2)}(E)$. Such an R is called quadratic-linear. We also ask the presentation to satisfy

$$\text{(ql1): } R\cap E=\{0\} \quad \text{and} \quad \text{(ql2): } (R\otimes E+E\otimes R)\cap\mathcal{F}^{(2)}(E)\subset R\cap\mathcal{F}^{(2)}(E).$$

Let q denote the projection $\mathcal{F}(E)\twoheadrightarrow\mathcal{F}^{(2)}(E)$ and let qR be the image of R under this projection. A quadratic-linear operad $\mathcal{F}(E)/(R)$ satisfying (ql1) and (ql2) is said to be *Koszul* if $\mathcal{F}(E)/(qR)$ is a quadratic Koszul operad. Its Koszul dual cooperad is $(\mathcal{F}(E)/(qR))^{¡}$ *together with a differential* that will be explained in the next section.

In order to apply the theory, one needs to express $H_0(\mathcal{SC})$ as a quadratic-linear operad, which is not the presentation given in Corollary 6.1.2. We add a new generator in the description of the operad $H_0(\mathcal{SC})$ to replace the quadratic-cubical relation by quadratic-linear relations. This new generator $e_{1,1}$, will correspond at the level of algebras to the operation $\rho(c;a):=f(c)a$. Consequently, we introduce new relations in the operad corresponding to the relations $f(c)a=af(c)=\rho(c;a)$ and $\rho(c;f(c'))=f(cc')=f(c)f(c')$ that are present in the algebra setting.

PROPOSITION 6.2.1. *The operad $H_0(\mathcal{SC})$ has a presentation $\mathcal{F}(E)/(R)$, where*

$$E=\underbrace{kf_2}_{=E(c,c;c)}\oplus\underbrace{k[S_2]e_{0,2}}_{=E(o,o;o)}\oplus\underbrace{k[S_2]e_{1,1}}_{=E(c,o;o)\oplus E(o,c;o)}\oplus\underbrace{k\alpha}_{=E(c;o)}.$$

The action of the symmetric group on f_2 is the trivial action and $k[S_2]$ denotes the regular representation. The element $e_{1,1}$ forms a basis of $E(c,o;o)$ and $e_{1,1}\cdot(21)$ a basis of $E(o,c;o)$.

The space of relations R is the submodule of $\mathcal{F}^{(1)}(E)\oplus\mathcal{F}^{(2)}(E)$ defined by $R=R_v\oplus R(\alpha)$, where R_v, defined in Corollary 4.1.3, describes the relations in the presentation of the operad $H_0(\mathcal{SC}^{\mathrm{vor}})$ and $R(\alpha)$ is the S_2-submodule of $\mathcal{F}(E)$ generated by the following relations:

- *two quadratic-linear relations: $e_{0,2}\circ_1\alpha-e_{1,1}$ and $(e_{0,2}\cdot(21))\circ_1\alpha-e_{1,1}$,*
- *a new quadratic relation: $e_{1,1}\circ^o\alpha-\alpha\circ_1 f_2$.*

Moreover this presentation satisfies (ql1) *and* (ql2).

The projection of $R=R_v\oplus R(\alpha)$ onto $\mathcal{F}^{(2)}(E)$ is $qR=R_v\oplus R'(\alpha)$, where $R'(\alpha)$ is the submodule of $\mathcal{F}^{(2)}(E)$ generated by the relations $e_{0,2}\circ_1\alpha$ and $e_{0,2}\circ_2\alpha$ and $e_{1,1}\circ_2\alpha-\alpha\circ_1 f_2$. Consequently, an algebra over this operad is an algebra (C,A) over the operad $H_0(\mathcal{SC}^{\mathrm{vor}})$ together with a linear map $f:C\to A$ satisfying $f(c)a=af(c)=0$ for all $c\in C, a\in A$ and $\rho(c;f(c'))=f(cc')$ for all $c,c'\in C$. As in [6], the operad $qH_0(\mathcal{SC})$ is obtained as the result of a distributive law between the operad $H_0(\mathcal{SC}^{\mathrm{vor}})$ and $P(\alpha)$, where $P(\alpha)$ is a free colored operad generated by a 1-dimensional vector space V with basis $\alpha\in V(c;o)$. The distributive law is given by

$$\begin{aligned}&H_0(\mathcal{SC}^{\mathrm{vor}})\circ P(\alpha)\to P(\alpha)\circ H_0(\mathcal{SC}^{\mathrm{vor}})\\&e_{0,2}\circ_1\alpha,\ e_{0,2}\circ_2\alpha\mapsto 0\\&\qquad\qquad e_{1,1}\circ_2\alpha\mapsto\alpha\circ_1 f_2.\end{aligned}\tag{10}$$

PROPOSITION 6.2.2. *The operad $qH_0(\mathcal{SC})$ is identical to the operad $P(\alpha)\circ H_0(\mathcal{SC}^{\mathrm{vor}})$, with composition given by the distributive law* (10).

THEOREM 6.2.3. *The operad $H_0(\mathcal{SC})$ is Koszul.*

Proof. From [16, Chapter 8], one has that $qH_0(\mathcal{SC}) = P(\alpha)\circ H_0(\mathcal{SC}^{\mathrm{vor}})$ is Koszul since $H_0(\mathcal{SC}^{\mathrm{vor}})$ and $P(\alpha)$ are Koszul colored operads. □

The Koszul dual cooperad of $qH_0(\mathcal{SC}) = P(\alpha)\circ H_0(\mathcal{SC}^{\mathrm{vor}})$ is $(qH_0(\mathcal{SC}))^{¡} = H_0(\mathcal{SC}^{\mathrm{vor}})^{¡}\circ P(\alpha)^{¡}$, where $H_0(\mathcal{SC}^{\mathrm{vor}})^{¡} = (\Lambda\mathcal{LP})^*$ and $P(\alpha)^{¡}$ is the cofree 2-colored cooperad cogenerated by an element α of degree 1 in $P(\alpha)^{¡}(c;o)$.

6.3. THE KOSZUL DUAL OF $H_0(\mathcal{SC})$ AND ITS KOSZUL RESOLUTION

In the proof of Theorem 6.2.3 we have considered the operad $qH_0(\mathcal{SC}) = \mathcal{F}(E)/qR$, where qR is the image of R by the projection $q:\mathcal{F}(E)\twoheadrightarrow\mathcal{F}^{(2)}(E)$. Furthermore, we have seen that algebras over the operad $qH_0(\mathcal{SC})$ are $H_0(\mathcal{SC}^{\mathrm{vor}})$-algebras (C,A) endowed with a map $f:C\to A$ such that $f(c)a = af(c) = 0$ and $\rho(c, f(c')) = f(cc')$.

Let $\varphi: qR\to E$ defined by

$$\begin{cases}\varphi(e_{0,2}\circ_1\alpha) = \varphi((e_{0,2}\cdot(21))\circ_1\alpha) = e_{1,1},\\ \varphi(R_v) = 0,\\ \varphi(e_{1,1}\circ_2\alpha - \alpha\circ_1 f_2) = 0.\end{cases}\tag{11}$$

The Koszul dual cooperad of $qH_0(\mathcal{SC})$ is $qH_0(\mathcal{SC})^{¡} = C(sE, s^2qR)$, with the notation of Definition 2.3.1. To φ is associated the composite map $qH_0(\mathcal{SC})^{¡}\twoheadrightarrow s^2qR\xrightarrow{s^{-1}\varphi} sE$. There exists a unique coderivation $\widetilde{d_\varphi}: qH_0(\mathcal{SC})^{¡}\to\mathcal{F}^c(sE)$ which extends this map. Moreover, $\widetilde{d_\varphi}$ induces a square zero coderivation d_φ on the Koszul dual cooperad $qH_0(\mathcal{SC})^{¡}$. The Koszul dual cooperad of $H_0(\mathcal{SC})$ is by definition $H_0(\mathcal{SC})^{¡} = (C(sV, s^2qR), d_\varphi)$. The Koszulity of the operad $H_0(\mathcal{SC})$ implies the quasi-isomorphism (see [6, Theorem 38]):

$$\Omega(H_0(\mathcal{SC})^{¡})\xrightarrow{\sim} H_0(\mathcal{SC}).\tag{12}$$

Note that $\Omega(H_0(\mathcal{SC})^{¡})$ is not a minimal model of $H_0(\mathcal{SC})$ since its differential has a linear part coming from the differential on $H_0(\mathcal{SC})^{¡}$. In order to understand the differential in the cooperad $H_0(\mathcal{SC})^{¡}$ it is usually more convenient to understand the Koszul dual operad $H_0(\mathcal{SC})^!$. Recall from (2) that $qH_0(\mathcal{SC})^! = (\Lambda(qH_0(\mathcal{SC})^{¡}))^* = \mathcal{F}(s^{-1}\Lambda^{-1}E^*)/(qR)^\perp$, with a derivation δ_φ which is the unique derivation of operads extending the map ${}^t\varphi: s^{-1}\Lambda^{-1}E^*\to\mathcal{F}(s^{-1}\Lambda^{-1}E^*)$ where ${}^t\varphi$ is a combination of transpose and signed suspension of φ. Namely, $H_0(\mathcal{SC})^!$ is a differential graded operad and we have the following Proposition:

PROPOSITION 6.3.1. *An algebra over $H_0(\mathcal{SC})^!$ consists in a DG Lie algebra $(L, [,], d_L)$, a DG associative algebra (A, d_A), a Lie algebra action by derivations $\rho : L \otimes A \to A$ and a degree -1 map $f : L \to A$ such that, for all $l \in L, a \in A$, we have $d_A(f(l)) = -f(d_L l)$ and*

$$f([l,l']) = (-1)^{|l|}\rho(l, f(l')) - (-1)^{|l||l'|+|l'|}\rho(l', f(l)) \tag{13}$$

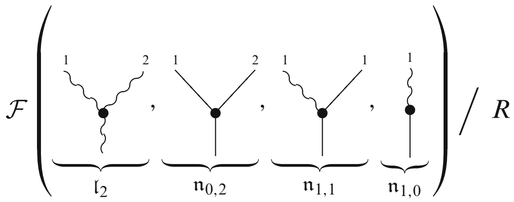
$$d_A\rho(l,a) = \rho(d_L l, a) + (-1)^{|l|}\rho(l, d_A a) + f(l)a - (-1)^{|a|(|l|+1)}af(l). \tag{14}$$

Proof. From $qH_0(\mathcal{SC})^{¡} = H_0(\mathcal{SC})^{¡} \circ \mathcal{P}(\alpha)^{¡}$, one obtains that a graded algebra over $H_0(\mathcal{SC})^!$ is a graded Leibniz pair (L, A, ρ) together with a degree -1 map $f : L \to A$. Relation (13) comes from the transpose of the distributive law (10). The derivation δ_φ is non-zero only when ρ is involved, as seen in the definition of φ in (11). If $\mathcal{P}$ is a differential graded operad and A is a differential graded $\mathcal{P}$-algebra then

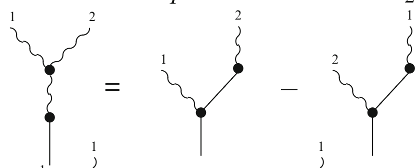
$$\mathrm{d}_A(\mu(a_1, \dots, a_n)) = (\mathrm{d}_{\mathcal{P}}(\mu))(a_1, \dots, a_n) + \sum \pm\mu(a_1, \dots, \mathrm{d}_A a_i, \dots, a_n).$$

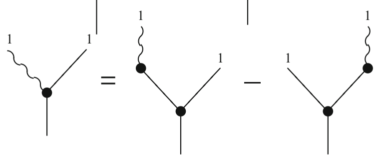
As a consequence if $\mathrm{d}_{\mathcal{P}}(\mu) = 0$ then μ preserves the differential on A. This is the reason why L is a differential graded Lie algebra, A is a differential associative algebra and f preserves the differential. Relation (14) comes from the definition of δ_φ. □

PROPOSITION 6.3.2. *The DG operad $H_0(\mathcal{SC})^!$ can be presented as*



where $\mathfrak{n}_{1,0}$ has degree -1, all the others generators have degree 0 and $\mathfrak{l}_2$ is anti-symmetric. The ideal R is generated by the following relations: $\mathfrak{l}_2$ satisfies Jacobi identity; $\mathfrak{n}_{0,2}$ satisfies associativity; $\mathfrak{n}_{1,1}$ satisfies the Leibniz rule with respect to the product $\mathfrak{n}_{0,2}$ and the Lie action condition with respect to the bracket $\mathfrak{l}_2$.

The relation called "The Eye Law"  *also holds,*

and the differential satisfies d  *and vanishes on the other generators.*

The definition of the differential $d: H_0(\mathcal{SC})^! \to H_0(\mathcal{SC})^!$ says that the map $f: L \to A$ corresponding to $\mathfrak{n}_{1,0}$ is central up to homotopy having the map $\rho: L \otimes A \to A$ corresponding to $\mathfrak{n}_{1,1}$ as the homotopy operator. For a geometrical description of the above relations in terms of the Kontsevich compactification [15], we refer the reader to [10,13]. The next Proposition is immediate but very helpful.

PROPOSITION 6.3.3. *The differential graded operad $H_0(\mathcal{SC})^!$ is the operad $\mathcal{LP} \circ \mathcal{F}(\mathfrak{n}_{1,0})$, where $\mathfrak{n}_{1,0}$ has degree -1 and is a generator of $\mathcal{F}(\mathfrak{n}_{1,0})(c; o)$, with the operadic composition given by the distributive law $\mathcal{F}(\mathfrak{n}_{1,0}) \circ \mathcal{LP} \to \mathcal{LP} \circ \mathcal{F}(\mathfrak{n}_{1,0})$ defined as $\mathfrak{n}_{1,0} \circ_1 \mathfrak{l}_2 \mapsto \mathfrak{n}_{1,1} \circ_2 \mathfrak{n}_{1,0} - (\mathfrak{n}_{1,1} \circ_2 \mathfrak{n}_{1,0}) \cdot (21)$ and differential given by $d(\mathfrak{n}_{1,1}) = \mathfrak{n}_{0,2} \circ_1 \mathfrak{n}_{1,0} - (\mathfrak{n}_{1,1} \circ_2 \mathfrak{n}_{1,0}) \cdot (21)$ and vanishing elsewhere.*

7. Open-Closed Homotopy Algebras

In this section we will show that the operad $\Omega[(\Lambda H_0(\mathcal{SC}))^*]$, which coincides with the OCHA operad, is a resolution of the differential graded operad $H_0(\mathcal{SC})^!$ (see Theorem 7.1.1). We compute its homology and prove that the OCHA operad is formal in the category of algebras in 2-collections (see Theorem 7.2.5). The non-formality of the OCHA operad in the category of 2-colored operads is also proven.

7.1. THE OCHA OPERAD

The OCHA operad $\mathcal{OC}_\infty$ is generated by the partially planar corollae $\mathfrak{l}_n$ and $\mathfrak{n}_{p,q}$ with $n \geq 2$ and $2p + q \geq 2$. Those corollae are exactly the same we used for the operad $\mathcal{LP}_\infty$. The only difference is that in the case of $\mathcal{OC}_\infty$ we have generators of the form $\mathfrak{n}_{p,0}$. The operadic composition, symmetric group action and differential of $\mathcal{OC}_\infty$ are the usual ones [12]. It is not difficult to check that $\mathcal{OC}_\infty = \Omega(\Lambda H_0(\mathcal{SC})^*)$.

THEOREM 7.1.1. *There is a quasi-isomorphism of DG operads $\mathcal{OC}_\infty \to H_0(\mathcal{SC})^!$.*

Proof. Since $H_0(\mathcal{SC})$ is Koszul, one has the quasi-isomorphism (12): $\Omega(H_0(\mathcal{SC})^{¡}) \xrightarrow{\sim} H_0(\mathcal{SC})$ where $H_0(\mathcal{SC})^{¡}$ is a differential graded cooperad related to $H_0(\mathcal{SC})^!$ by: $H_0(\mathcal{SC})^{¡} = (\Lambda H_0(\mathcal{SC})^!)^*$. There is an adjunction (Ω, B) between cooperads and operads, where B denotes the bar construction of an operad and Ω the cobar construction of a cooperad. The unit of the adjunction is a quasi-isomorphism and the bar construction B preserves quasi-isomorphism (see, e.g. [8]). Furthermore, in the case of finite dimensional operads one has $B\mathcal{P} = \Omega(\mathcal{P}^*)^*$. Combining these results with equation (12) one gets $\Lambda H_0(\mathcal{SC})^{¡} \xrightarrow{\sim} B\Omega(\Lambda H_0(\mathcal{SC})^{¡}) \xrightarrow{\sim} B\Lambda H_0(\mathcal{SC}) = \Omega(\Lambda H_0(\mathcal{SC})^*)^*$ and dualizing the quasi-isomorphism, we have

$$\Psi: \mathcal{OC}_\infty = \Omega(\Lambda H_0(\mathcal{SC})^*) \xrightarrow{\sim} (\Lambda H_0(\mathcal{SC}))^{¡})^* = H_0(\mathcal{SC})^!.$$

□

7.2. THE HOMOLOGY OF THE OCHA OPERAD

The aim of this section is to prove analogous results for the Swiss-cheese operad to the following one for the little disks operad $\mathcal{D}_2$

- The zeroth homology of $\mathcal{D}_2$ is Koszul dual to the top homology. More precisely $H_0(\mathcal{D}_2)$ is the operad $\mathcal{C}om$ for commutative algebras. For any n, $H_*(\mathcal{D}_2(n))$ has top homology in degree $n-1$. It is thus a suboperad of $H(\mathcal{D}_2)$ denoted by $H_{\mathrm{top}}(\mathcal{D}_2)$ and coincides with $\Lambda^{-1}\mathcal{L}ie$ where $\mathcal{L}ie$ is the operad for Lie algebras.
- One has that $\mathcal{L}ie_\infty = \Omega\mathcal{L}ie^{¡} = \Omega((\Lambda\mathcal{C}om)^*) = \Omega((\Lambda H_0(\mathcal{D}_2))^*) \to \mathcal{L}ie$ is a quasi-isomorphism.

These two items express the same theorem: $\mathcal{L}ie$ and $\mathcal{C}om$ are Koszul dual operads. But the first one gives a geometric interpretation of this fact, whereas the second one gives an interpretation in deformation theory.

We have seen in Theorem 7.1.1 that $H_0(\mathcal{SC})$ is Koszul and that there is a quasi-isomorphism of DG operads $\mathcal{OC}_\infty = \Omega((\Lambda H_0(\mathcal{SC}))^*) \to H_0(\mathcal{SC})^!$. That would give the desired analogy for the Swiss-cheese operad; however, $H_0(\mathcal{SC})^!$ is a differential graded operad and cannot be obtained as a suboperad of $H(\mathcal{SC})$. So we introduce the operad $\mathcal{OC}$ that plays the role of the top homology in $H_*(\mathcal{SC})$. We prove in Theorem 7.2.5 that, up to some suspension, the homology of $H_0(\mathcal{SC})^!$ is $\mathcal{OC}$. As a consequence, we have that $H_*(\mathcal{OC}_\infty) = \mathcal{OC}$. If one had a quasi-isomorphism of operads $\mathcal{OC}_\infty \to \mathcal{OC}$, then one would have the exact analogy with the little disks case. The non-formality proved in Proposition 7.2.7, states that such a quasi-isomorphism does not exist. Nevertheless, Corollary 7.2.6 shows that there is another structure, algebras in the category of 2-collections, such that $\mathcal{OC}_\infty \to \mathcal{OC}$ is a quasi-isomorphism of algebras.

The quasi-isomorphism $\mathcal{OC}_\infty \to \mathcal{OC}$ in the category of $\mathcal{L}_\infty$-modules has been studied by the first author in [10]. It relies on the fact that $H_*(\mathcal{OC}_\infty) = \mathcal{OC}$. The proof of that fact presented there contains a mistake. So, the present section provides the correct proof of that quasi-isomorphism.

DEFINITION 7.2.1. We will denote by $\mathcal{OC}$ the suboperad of $H(\mathcal{SC})$ generated by top dimensional homology classes, that is, by the generators of $H_1(\mathcal{SC}(2,0;c))$, $H_0(\mathcal{SC}(0,2;o))$ and $H_0(\mathcal{SC}(1,0;o))$.

DEFINITION 7.2.2. Given a 2-collection $\mathcal{P}$, the suspension with respect to the color c will be denoted by $\Lambda_c\mathcal{P}$ and is defined as follows:

$$\Lambda_c\mathcal{P}(n,m;x) = \begin{cases} s^{1-n}P(n,m;x) \otimes (\mathrm{sgn_n} \otimes \mathbf{k}) & \text{if } x = c, \\ s^{-n}P(n,m;x) \otimes (\mathrm{sgn_n} \otimes \mathbf{k}) & \text{if } x = o. \end{cases}$$

where $(\mathrm{sgn_n} \otimes \mathbf{k})$ is the one-dimensional representation of $S_n \times S_m$ given by the tensor product of the signature representation of S_n and the trivial representation of S_m.

LEMMA 7.2.3. *The operad* $\Lambda_c\mathcal{OC}$ *can be presented by the generators* $\tilde{\mathfrak{l}}_2, \tilde{\mathfrak{n}}_{0,2}$ *and* $\tilde{\mathfrak{n}}_{1,0}$*, where* $\tilde{\mathfrak{l}}_2$ *and* $\tilde{\mathfrak{n}}_{0,2}$ *have degree* 0*, while* $\tilde{\mathfrak{n}}_{1,0}$ *has degree* -1 *verifying the following conditions:*

(*a*) *The generator* $\tilde{\mathfrak{l}}_2$ *is antisymmetric;*
(*b*) *The generator* $\tilde{\mathfrak{n}}_{0,2}$ *satisfies associativity and* $\tilde{\mathfrak{l}}_2$ *satisfies Jacobi identity;*
(*c*) *The generator* $\tilde{\mathfrak{n}}_{1,0}$ *is central with respect to the associative product* $\tilde{\mathfrak{n}}_{0,2}$.

Remark 7.2.4. $\tilde{\mathfrak{l}}_2$ is the desuspension of the Gerstenhaber bracket g_2 in (3).

THEOREM 7.2.5. *The homology of the operad* $H_0(\mathcal{SC})^!$ *is the operad* $\Lambda_c\mathcal{OC}$.

Proof. Let $\mathcal{P}$ denote the differential graded operad $H_0(\mathcal{SC})^!$. The proof goes in several steps. The first step consists in proving that there is a map of operads $\phi:\Lambda_c\mathcal{OC}\to H_*(\mathcal{P})$ inducing a map of algebras in the category of $\{c, o\}$-collections. In the second step, we identify the algebra $\mathcal{P}$ with the cobar construction of a coalgebra C in the category of $\{c, o\}$-collections. The third step consists in proving that C is a Koszul coalgebra, inducing that the algebra map $\psi:\mathcal{P}=\Omega C\to C^{¡}$ is a quasi-isomorphism, where $C^{¡}$ is the Koszul dual algebra associated to C, following ideas of Priddy in [19]. In the fourth step we identify $C^{¡}$ with $\Lambda_c\mathcal{OC}$ and prove that $H_*(\psi)\phi$ is the identity morphism. Consequently, ϕ is an isomorphism of algebras and thus an isomorphism of operads.

First step. Because $d\mathfrak{l}_2=d\mathfrak{n}_{0,2}=d\mathfrak{n}_{0,1}=0$ in $\mathcal{P}$, one has a well-defined morphism of operads from $\mathcal{F}(\tilde{\mathfrak{l}}_2, \tilde{\mathfrak{n}}_{0,2}, \tilde{\mathfrak{n}}_{0,1})\to H_*(\mathcal{P})$. Because $d\mathfrak{n}_{1,1}=\mathfrak{n}_{0,2}\circ_1\mathfrak{n}_{0,1}-\mathfrak{n}_{0,2}\cdot(21)\circ_1\mathfrak{n}_{0,1}$, this morphism of operads is well defined on the quotient by the ideal of relations in $\Lambda_c\mathcal{OC}$, yielding an operad morphism $\phi:\Lambda_c\mathcal{OC}\to H_*(\mathcal{P})$. It is clear also that the closed part of ϕ is an isomorphism. So in the sequel we focus on the open part ϕ_o of the morphism. We will prove that given any object $\alpha:(X, o)\to\{c, o\}$ in $\mathbf{Fin}_{\{c,o\}}$, the map $\phi_{(X,o)}$ is an isomorphism. Let us denote the triple (X, o, α) by (I, J) where $I=\alpha^{-1}(c)$, $J=\alpha^{-1}(o)$. Contravariant functors from the full subcategory of $\mathbf{Fin}_{\{c,o\}}$ generated by these objects will be called *open* 2*-collections*.

Algebras in the category of open 2*-collections*. The category of open 2-collections is endowed with a coproduct $(F\oplus G)(I, J)=F(I, J)\oplus G(I, J)$ and a symmetric tensor product

$$(F\otimes G)(I, J)=\bigoplus_{\substack{I_1\sqcup I_2=I,\\ J_1\sqcup J_2=J}} F(I_1, J_1)\otimes G(I_2, J_2)$$

with the unit $U(I, J)=\mathbf{k}$ if $I=J=\emptyset$ and is 0 elsewhere. This tensor product is bilinear with respect to the coproduct. Consequently, it makes sense to consider associative algebras, commutative algebras, coassociative coalgebras in this category. In the sequel we will use the terminology associative algebras, coassociative coalgebras for associative algebras and coalgebras in the category of open 2-collections. Furthermore given an open 2-collection V one can define an open 2-collection $T(V)$ and $T^+(V)$ as $T(V)=\oplus_{n\geq 1}V^{\otimes n}$ and $T^+(V)=U\oplus T(V)$. The open 2-collection $T(V)$ is endowed with the concatenation product so that it becomes the free associative algebra generated by V. The operads $\Lambda_c\mathcal{OC},\mathcal{P}$ and $H_*(\mathcal{P})$ are associative algebras in the category of 2-collections, for $\tilde{\mathfrak{n}}_{0,2}$ and $\mathfrak{n}_{0,2}$ are associative elements. The morphism ϕ is then a morphism of associative algebras.

Second step. Recall from Proposition 6.3.3 that $\mathcal{P}=\mathcal{LP}\circ\mathcal{F}(\mathfrak{n}_{1,0})$. Let $\mu_n\in\mathcal{LP}(0,n;o)$ denote the $(n-1)$-th composite of $\mathfrak{n}_{0,2}$ (no matter the way we compose since $\mathfrak{n}_{0,2}$ is an associative operation). Let $\gamma_k\in\mathcal{LP}(k,1;o)$ be defined by induction as $\gamma_0=1_{o;o}$ and $\gamma_{k+1}=\gamma_k\circ^o\mathfrak{n}_{1,1}$. From Lemma 4.2.2, the open part of the free Leibniz pair generated by a pair $V=(V_c,V_o)$ is $\mathcal{LP}(V)_o=T(T^+(V_c)\otimes V_o)$. Consequently, any element in $\mathcal{LP}(I,J;o)$ writes $\mu_n(\gamma_{l_1},\ldots,\gamma_{l_n})\cdot(\sigma,\tau)$ where $n=|J|,l_1+\cdots+l_n=|I|$, and σ and τ are bijections between I and $\{1,\ldots,|I|\}$ and between J and $\{1,\ldots,n\}$, respectively. Let $\kappa_{k+1}=\gamma_k\circ^o\mathfrak{n}_{0,1}\in\mathcal{P}(k+1,0;o)$. Then any element in $\mathcal{P}(I,J;o)$ writes $\mu_n(\epsilon_{l_1},\ldots,\epsilon_{l_n})\cdot(\sigma,\tau)$, where $\epsilon_k\in\{\gamma_k,\kappa_k\}$ and $|J|=|\{r|\epsilon_{l_r}=\gamma_{l_r}\}|$ and $|I|=l_1+\cdots+l_n$. We recall that γ_k and μ_n are of degree 0 while κ_k is of degree -1. Let X_c be the 2-collection defined by $X_c(I,J)=\mathbf{k}$ in degree 0, if $|I|=1,|J|=0$ and 0 elsewhere. Let X_o be the 2-collection defined by $X_o(I,J)=\mathbf{k}$ of degree 1 if $|I|=0,|J|=1$ and 0 elsewhere.

With this notation, the algebra $\mathcal{P}$ is the free associative algebra generated by the 2-collection $s^{-1}\overline{C}$ with $C=(T^+(X_c)\oplus T^+(X_c)\otimes X_o)$. The differential on $\mathcal{P}$ is a derivation of the operad $\mathcal{P}$. Consequently it is a derivation of algebras: $\mathrm{d}(\mu_n(\epsilon_{l_1},\ldots,\epsilon_{l_n}))=\sum_{i=1}^n(-1)^{r_i}\mu_n(\epsilon_{l_1},\ldots,\mathrm{d}\epsilon_{l_i},\ldots,\epsilon_{l_n})$, where r_i is the sum of the degrees of ϵ_{l_k} for $k<i$. Let us describe $\mathrm{d}\gamma_n$ and $\mathrm{d}\kappa_n$.

For $I=\{1,\ldots,n\}$, an element in $T(X_c)(I,\emptyset)$ is uniquely determined by a sequence $i_{[n]}=(i_1,\ldots,i_n)$ of degree 0. One has $s^{-1}i_{[n]}=\kappa_n\cdot\sigma$ in $\mathcal{P}(n,0;o)$, where σ is the permutation $j\mapsto i_j$. Similarly, an element in $(T^+(X_c)\otimes X_o)(I,\{x\},o)$ writes $(i_{[n]};x)=(i_1,\ldots i_n;x)$ of degree 1. One has $s^{-1}(i_{[n]};x)=\gamma_n\cdot(\sigma,1)$ in $\mathcal{P}(n,1;o)$. For a set $A=\{a_1<\cdots<a_k\}\subset\{1,\ldots,n\}$ we denote by i_A the sequence $(i_{a_1},\ldots,i_{a_k})$ and $(i_A;x)$ the sequence $(i_{a_1},\ldots,i_{a_k};x)$.

We have:

$$\mathrm{d}(s^{-1}i_{[n]})=\bigoplus_{\substack{A\sqcup B=[n]\\ A,B\neq\emptyset}}s^{-1}i_A\otimes s^{-1}i_B \tag{15}$$

$$\mathrm{d}(s^{-1}(i_{[n]};x))=\bigoplus_{\substack{A\sqcup B=[n]\\ A\neq\emptyset}}s^{-1}i_A\otimes s^{-1}(i_B;x)-s^{-1}(i_B;x)\otimes s^{-1}i_A. \tag{16}$$

The proof is by induction on n. For $i_{[n]}=(1,\ldots,n)$, we have $s^{-1}i_{[n]}=\kappa_n$ and $s^{-1}(i_{[n]};x)=\gamma_n$. For $n=2$, one has $\mathrm{d}\kappa_1=\mathrm{d}\mathfrak{n}_{0,1}=0$ and hence (15) is proved. One has $\mathrm{d}\gamma_1=\mathrm{d}\mathfrak{n}_{1,1}=\mathfrak{n}_{0,2}\circ_1\mathfrak{n}_{0,1}-\mathfrak{n}_{0,2}\cdot(21)\circ_1\mathfrak{n}_{0,1}$ which writes $\mathrm{d}s^{-1}(i_{[1]};x)=i_{[1]}\otimes(\emptyset;x)-(\emptyset;x)\otimes i_{[1]}$, proving (16). Now assume we have proven the second relation for $n-1$. The relation $\gamma_n = \gamma_1 \circ^o \gamma_{n-1}$ and $\gamma_1 \circ^o \mathfrak{n}_{0,2} = \mathfrak{n}_{0,2} \circ_2 \gamma_1 \cdot (213) + \mathfrak{n}_{0,2} \circ_1 \gamma_1$ implies that $\mathrm{d}s^{-1}(i_{[n]}; x)$ is given by

$$\begin{aligned}
&\mathrm{d}(s^{-1}(i_{[1]}; x)) \circ_x s^{-1}(i_{\{2,\dots,n\}}; x) \\
&\quad + s^{-1}(i_{[1]}; x) \circ_x \sum_{A \sqcup B = \{2,\dots,n\}} s^{-1} i_A \otimes s^{-1}(i_B; x) - s^{-1}(i_B; x) \otimes s^{-1} i_A \\
&= s^{-1}(i_{[n]}) \otimes s^{-1}(\emptyset; x) - s^{-1}(\emptyset; x) \otimes s^{-1}(i_{[n]}; x) \\
&\quad + \sum_{A \sqcup B = \{2,\dots,n\}} s^{-1} i_{\{1\}\cup A} \otimes s^{-1}(i_B; x) + s^{-1} i_A \otimes s^{-1}(i_{\{1\}\cup B}; x) \\
&\quad - \sum_{A \sqcup B = \{2,\dots,n\}} s^{-1}(i_{\{1\}\cup B}; x) \otimes s^{-1} i_A - s^{-1}(i_B; x) \otimes s^{-1} i_{\{1\}\cup A}.
\end{aligned}$$

which is equal to $\bigoplus_{\substack{A \sqcup B = [n] \\ A \neq \emptyset}} s^{-1} i_A \otimes s^{-1}(i_B; x) - s^{-1}(i_B; x) \otimes s^{-1} i_A$. This proves (16). Using $\kappa_n = \gamma_n \circ^o \kappa_1$ one gets (15). Since $\mathrm{d}^2 = 0$, C is a coassociative coalgebra with $\Delta(i_{[n]}) = \bigoplus_{A \sqcup B = [n]} i_A \otimes i_B$ and $\Delta(i_{[n]}; x) = \bigoplus_{A \sqcup B = [n]} i_A \otimes (i_B; x) + (i_B; x) \otimes i_A$ and $\mathcal{P} = \Omega C = (T(s^{-1}\bar{C}), \mathrm{d})$, where d is the unique derivation that lifts the coproduct on C.

Third step: C is a Koszul coalgebra. From the definition of Δ, one sees that $C = D \otimes M$ where $D = T(X_c)$ is the unshuffle coalgebra and $M = U \oplus X_o$ is the coalgebra with coproduct $\Delta(1) = 1 \otimes 1$ and $\Delta(x) = 1 \otimes x + x \otimes 1$. Following Priddy in [19], if D and M are Koszul coalgebras, so is C and $C^{¡} = D^{¡} \otimes M^{¡}$. From Stover in [20], D is the enveloping algebra of the free Lie algebra $\mathcal{L}ie(X_c)$ which is a free cocommutative coalgebra cogenerated by $\mathcal{L}ie(X_c)$. But this is a Koszul coalgebra and its Koszul dual is $S(s^{-1}\mathcal{L}ie(X_c))$. Hence D is Koszul. Following the notation of [16, section 3], if $C(V, R)$ denotes the quadratic coalgebra cogenerated by V with correlation R, then $C(V, R)^{¡} = A(s^{-1}V, s^{-2}R)$ is the associative algebra generated by $s^{-1}V$ with relations $s^{-2}R$. Since $M = C(X_o, 0)$ $M^{¡} = A(s^{-1}X_o, 0)$ is the free associative algebra generated by $s^{-1}X_o$, which is also Koszul. Consequently, C is Koszul and its dual Koszul algebra is $C^{¡}(I, J) = S(s^{-1}\mathcal{L}ie(X_c))(I, \emptyset) \otimes T(s^{-1}X_o)(\emptyset, J)$, with the concatenation product. The Koszul duality implies that the algebra map $\Omega C \to C^{¡}$ is a quasi-isomorphism.

Fourth step. Recall that the operad $\Lambda_c\mathcal{OC}$ is generated by $\tilde{\mathfrak{n}}_{0,2}, \tilde{\mathfrak{l}}_2$ of degrees 0 and $\tilde{\mathfrak{n}}_{0,1}$ of degree -1 subject to the associativity relation for $\tilde{\mathfrak{n}}_{0,2}$, the Jacobi relation for $\tilde{\mathfrak{l}}_2$ and the centrality relation for $\tilde{\mathfrak{n}}_{0,1}$. Any element in $\Lambda_c\mathcal{OC}([n], [l]; o)$ writes $\tilde{\mu}_p(l_{i_1}, \dots, l_{i_p}) \otimes (\tilde{\mu}_l \cdot \tau)$ with $\sqcup_{1 \le j \le k} I_j = [n], \tau \in S_l$ and $l_{i_j} = \tilde{\mathfrak{n}}_{0,1} \circ \tilde{\mathfrak{l}}_{i_j} \in s^{-1}\mathcal{L}ie(I_j)$, where $\tilde{\mathfrak{l}}_{i_j}$ is an iterated composition of $\tilde{\mathfrak{l}}_2$ combined with permutations, which is called a Lie element in $\Lambda_c\mathcal{OC}$.

Furthermore, the centrality of $\tilde{\mathfrak{n}}_{0,1}$ implies that the product $\tilde{\mu}_p$ is invariant under the action of the symmetric group. The algebra structure follows and one has $\Lambda_c\mathcal{OC} = C^{¡}$ as an algebra. Consequently, $\psi : H_0(\mathcal{SC})^{¡} = \Omega C \to C^{¡} = \Lambda_c\mathcal{OC}$ is a quasi-isomorphism of algebras and $H_*(\psi) : H_*(\mathcal{P}) \to \Lambda_c\mathcal{OC}$ is an isomorphism.

The map $\phi : \Lambda_c\mathcal{OC} \to H_*(\mathcal{P})$ is an algebra map. It suffices to prove that $H_*(\psi)\circ\phi$ is the identity on a Lie element of type $\tilde{\mathfrak{n}}_{0,1}\circ\tilde{\mathfrak{l}}_{i_j}$. Because of the eye relation in $\mathcal{P}$ this Lie element is sent exactly to a Lie element in $\mathcal{L}ie(X_c)\subset\Omega s^{-1}S^c\mathcal{L}ie(X_c)$ and hence to $\tilde{\mathfrak{n}}_{0,1}\circ\tilde{\mathfrak{l}}_{i_j}$ via $H_*(\psi)$. □

COROLLARY 7.2.6. *The operads $\mathcal{OC}_\infty$, $H_0(\mathcal{SC})^!$ and $\Lambda_c\mathcal{OC}$ are multiplicative operads and hence are associative algebras in the category of* 2*-collections. The maps $\mathcal{OC}_\infty \to H_0(\mathcal{SC})^! \to \Lambda_c\mathcal{OC}$ are quasi-isomorphisms of associative algebras.*

PROPOSITION 7.2.7. *The OCHA operad $\mathcal{OC}_\infty$ is non-formal.*

Proof. Since $\mathcal{OC}_\infty$ is a minimal operad whose homology is $\Lambda_c\mathcal{OC}$, its formality would imply the existence of an operad morphism $\mathcal{OC}_\infty\to\Lambda_c\mathcal{OC}$ inducing an isomorphism in homology. Note that $\mathfrak{n}_{1,1}\in\mathcal{OC}_\infty$ is an element of degree 0. But all elements in $\Lambda_c\mathcal{OC}(1,1;o)$ have degree -1 because they can only be obtained by composing $\tilde{\mathfrak{n}}_{1,0}$ and $\tilde{\mathfrak{n}}_{0,2}$. Hence, any quasi-isomorphism $\mathcal{OC}_\infty\to\Lambda_c\mathcal{OC}$ would take $\mathfrak{n}_{1,1}$ to 0. On the other hand, since $\mathfrak{l}_2$ and $\mathfrak{n}_{1,0}$ are generators of the homology of $\mathcal{OC}_\infty$ they cannot be taken to zero by any quasi-isomorphism. The "Eye Law" says that $\mathfrak{n}_{1,0}\circ_1\mathfrak{l}_2=\mathfrak{n}_{1,1}\circ_2\mathfrak{n}_{1,0}+(\mathfrak{n}_{1,1}\circ_2\mathfrak{n}_{1,0})\cdot(21)$, a contradiction. □

References

1. Alm, J.: Two-colored noncommutative Gerstenhaber formality and infinity Duflo isomorphism. arXiv:1104.2194v1 (2011, preprint)
2. Cohen, F.R.: The homology of C_{n+1}-spaces, $n\geq 0$. In: The Homology of Iterated Loop Spaces. Lecture Notes in Math., vol. 533, pp. 207–351. Springer-Verlag, Berlin (1976)
3. Dolgushev, V.: Formality theorem for Hochschild cochains via transfer. Lett. Math. Phys. **97**(2), 109–149 (2011)
4. Flato, M., Gerstenhaber, M., Voronov, A.A.: Cohomology and deformation of Leibniz pairs. Lett. Math. Phys. **34**(1), 77–90 (1995)
5. Fresse, B.: Koszul duality of operads and homology of partition posets. In: Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory. Contemp. Math., vol. 346, pp. 115–215. Amer. Math. Soc., Providence, RI (2004)
6. Galvez-Carrillo, I., Tonks, A., Vallette, B.: Homotopy Batalin-Vilkovisky algebras. arXiv:0907.2246 (2009, preprint)
7. Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Maths **78**(2), 267–288 (1963)
8. Getzler, E., Jones, J.D.S.: A_∞-algebras and the cyclic bar complex. Ill. J. Math. **34**(2), 256–283 (1990)
9. Ginzburg, V., Kapranov, M.: Koszul duality for operads. Duke Math. J. **76**(1), 203–272 (1994)
10. Hoefel, E.: OCHA and the Swiss-cheese operad. J. Hom. Relat. Struct. **4**(1), 123–151 (2009)
11. Hoefel, E.: On the coalgebra description of OCHA. J. Pure Appl. Algebra **216**(3), 734–741 (2012)

12. Kajiura, H., Stasheff, J.: Homotopy algebras inspired by classical open-closed string field theory. Commun. Math. Phys. **263**(3), 553–581 (2006)
13. Kajiura, H., Stasheff, J.: Open-closed homotopy algebra in mathematical physics. J. Math. Phys. **47**, 28 (2006). doi:10.1063/1.2171524
14. Kontsevich, M.: Operads and motives in deformation quantization. Lett. Math. Phys. **48**(1), 35–72 (1999) [Moshé Flato (1937–1998)]
15. Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**(3), 157–216 (2003)
16. Loday, J.-L., Vallette, B.: Grundlehren der mathematischen Wissenschaften. In: Algebraic Operads, vol. 346. Springer-Verlag, Berlin (2012, to appear)
17. Markl, M.: Homotopy algebras are homotopy algebras. Forum Math. **16**(1), 129–160 (2004)
18. May, J.P.: The Geometry of Iterated Loop Spaces. Lectures Notes in Mathematics, vol. 271. Springer-Verlag, Berlin (1972)
19. Priddy, S.B.: Koszul resolutions. Trans. Am. Math. Soc. **152**, 39–60 (1970)
20. Stover, C.R.: The equivalence of certain categories of twisted Lie and Hopf algebras over a commutative ring. J. Pure Appl. Algebra **86**(3), 289–326 (1993)
21. van der Laan, P.: Operads, Hopf algebras and coloured Koszul duality. PhD thesis Universiteit Utrecht (2004)
22. Voronov, A.A.: The Swiss-cheese operad. In: Homotopy invariant algebraic structures (Baltimore, MD, 1998). Contemp. Math., vol. 239, pp. 365–373. Amer. Math. Soc., Providence, RI (1999)