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A_∞-actions and recognition of relative loop spaces

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ABSTRACT

We show that relative loop spaces are recognized by A_∞-actions. A certain version of the 2-sided bar construction is used to prove such recognition theorem. The operad Act_∞ of A_∞-actions is presented in terms of the Boardman–Vogt resolution of the operad Act. We exhibit an operad homotopy equivalence between such resolution and the 1-dimensional Swiss-cheese operad $\mathcal{SC}_1$.

1. Introduction

Given an A_∞-space X and a topological space P, an A_∞-action of X on P is, by definition, an A_∞-map $X \to \mathrm{End}(P)$. In this paper, we show that any A_∞-action is weakly equivalent to the A_∞-action of a loop space on a relative loop space. It is also shown that the weak equivalence is A_∞-equivariant. In other words, we show that spaces admitting an A_∞-action are recognized as relative loop spaces. As one would naturally expect, those results do not hold without the assumption that the A_∞-space X is grouplike, i.e., $\pi_0(X)$ is a group.

The history of the notion of homotopy actions goes somewhat like the following. Starting in 1895 with Poincaré's definition of the fundamental group of a topological space [14], the concept of homotopy

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associativity became well established, but the higher homotopies had to wait some six decades to be recognized [18]. Meanwhile, the corresponding notion of a homotopy action (or action up to homotopy) was at least implicit in the action of the fundamental group $\pi_1(X)$ on the set $\pi_1(X, A)$ for $A \subset X$. The next step was taken by Hilton[1] [5] in considering the long exact sequence associated to a fibration $F \to E \to B$,

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

ending with

$$\cdots \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B).$$

Of course, exactness is very weak at the end since the last three are in general only sets, but exactness at $\pi_0(F)$ is in terms of the action of $\pi_1(B)$ on $\pi_0(F)$. This passage to homotopy classes obscures the 'action' of ΩB on F. Initially, this was referred to as a *homotopy action*, meaning only that $\lambda(\mu f)$ was homotopic to $(\lambda\mu)f$ for $f \in F$ and $\lambda, \mu \in \Omega B$.

If X is a topological monoid, Nowlan [13] developed the notion of an A_∞-action of X fibrewise on a fibration $p : P \to B$ and hence of an A_∞-principal fibration when X is the fiber of p, building on previous work by the third author [16]. Nowlan's results were generalized to the case in which X is itself an A_∞-space by Iwase and Mimura [6]. Given an A_∞-space X and a map $h : X \to P$, they introduced the notion of A_∞-action along h.

In the present paper, the map h as well as its higher analogous maps $h_n : X^{\times n} \to P$ are part of an Act_∞-algebra structure on the pair (X, P), where the operad Act_∞ is defined in section 3.2. We also show that the structure of an Act_∞-algebra on the pair (X, P) is naturally equivalent to the existence of an A_∞-map $X \to \mathrm{End}(P)$.

The homotopy orbit space $P/\!\!/X$, defined originally by Iwase and Mimura, will be presented in terms of an A_∞-version of the 2-sided bar construction and will play a crucial role in the present paper. That version of the 2-sided bar construction uses the associahedra and an A_∞-space X with two A_∞-actions on spaces Q and P. It is denoted by $B(Q, X, P)$, where X acts from the right on Q and from the left on P. The space $B(X, X, *)$ where X acts canonically on X and trivially on $\{*\}$ has been used by the third author in [15]. The appropriate identifications in $B(Q, X, P)$ are presented in terms of planar trees by using the parametrization of the associahedra introduced by Boardman and Vogt [2].

After having defined the homotopy orbit space as $P/\!\!/X := B(*, X, P)$, our main result (see Theorem 2.22) states that:

$$P \simeq \Omega_M(CP/\!\!/X, P/\!\!/X),$$

where $\Omega_M(CP/\!\!/X, P/\!\!/X)$ denotes the space of relative Moore loops on the pair $(CP/\!\!/X, P/\!\!/X)$ and CP denotes the unreduced cone over P. Moreover, the above weak equivalence is A_∞-equivariant with respect to the usual A_∞-map $X \to \Omega_M(BX)$. The concepts of A_∞-action and A_∞-equivariance are treated in detail in section 2. The operad Act_∞ is defined in section 3 in terms of the Boardman–Vogt resolution of Act. In section 4 we discuss the relations between A_∞-actions and the 1-dimensional Swiss-cheese operad $\mathcal{SC}_1$.

1.1. Conventions and notation

In this paper, all spaces are assumed to be compactly generated, Hausdorff and to have non-degenerate base points. All products and spaces of maps are taken with respect to the compactly generated topology.

[1] The third author learned it as a graduate student from Hilton's *Introduction to Homotopy Theory* [5], the earliest textbook on the topic, published in 1953.

Throughout this paper, a *weak equivalence* will always mean a weak homotopy equivalence between topological spaces. All the operads considered in this paper are non-symmetric. Concerning colored operads, we will follow the definitions and notation of [1]. Most of the colored operads involved are 2-colored, i.e., the set of colors $\{c_1, c_2\}$ has two elements. An algebra over a 2-colored operad consists of a pair of spaces (E_{c_1}, E_{c_2}). Such pairs are not to be confused with the topological pairs (Y, B) where B is a subspace of Y. Whenever it is clear from the context that the color c_1 comes before the color c_2, given any pair (X, P) one can assume that X is the space of color c_1 while P has color c_2. For simplicity we will use the notation:

$$x_{i,\dots,j} = (x_i, \dots, x_j) \in X^{\times(j-i+1)}$$

for any set X and any integers $i < j$.

2. Recognition of relative loop spaces

In this section we introduce the notion of A_∞-actions and of A_∞-equivariance. We will prove a recognition theorem according to which any A_∞-action is weakly equivalent to the action of a loop space on a relative loop space in an A_∞-equivariant way. The proof involves the 2-sided bar construction for A_∞-actions.

2.1. A_∞-spaces and A_∞-maps

We begin by recalling the definitions of A_∞-spaces and A_∞-maps. When studying A_∞-maps between A_∞-spaces X and Y, we will concentrate in the case where Y is a topological monoid with unit.

We will recall the definition of *metric trees*, i.e., trees such that each internal edge has a length in $[0, 1]$. It will be convenient to first introduce the *edge-labeled trees* as an assignment of a real number in $[0, 1]$ to each of its internal edge. Then the space of metric trees is naturally defined as a quotient of the space of edge-labeled trees.

2.1.1. Metric trees and edge-labeled trees

Let us consider planar oriented finite trees such that some edges are external, i.e., half-open segments attached to a single vertex. Hence our trees are non-compact, albeit finite. We assume that each vertex has at least two incoming edges and that the trees are rooted in the sense that there is only one external edge with an outward orientation. The other external edges are inward oriented and called leaves.

For each tree T, its number of leaves will be denoted by $|T|$. The set of internal edges in T will be denoted by $i(T)$ while its cardinality is denoted by $|i(T)|$.

The corolla with k leaves is denoted by δ_k.

Definition 2.1. A tree T endowed with a function $\ell : i(T) \to [0, 1]$ will be called an *edge-labeled* tree. The function ℓ will be called an *edge-labeling* on T.

For a fixed tree T, the space of edge-labelings on T can be identified with $[0, 1]^{\times|i(T)|}$. We denote by $\mathcal{T}_n$ the space of edge-labeled trees with n leaves, it is topologized as the disjoint union:

$$\mathcal{T}_n = \bigsqcup_{|T|=n} [0, 1]^{\times|i(T)|} \quad .$$

Given two edge-labeled trees T and S and $r \in [0, 1]$, the edge-labeled tree $T \circ_i^r S$ is obtained by grafting the root of the tree S on the ith leaf of T and by assigning the length r to the newly created edge. This operation of grafting trees satisfies the usual operadic relations:

$$U \circ_i^q (T \circ_j^r S) = (U \circ_i^q T) \circ_{j+i-1}^r S, \qquad 1 \leqslant i \leqslant |U|, 1 \leqslant j \leqslant |T|, \tag{1}$$

$$(U \circ_i^q T) \circ_j^r S = (U \circ_j^r S) \circ_{i+|S|-1}^q T, \qquad 1 \leqslant j < i \leqslant |U|. \tag{2}$$

In order to define metric trees, we introduce a natural identification $\sim$ on $\mathcal{T}_n$ that is defined as follows. If some internal edge is labeled by 0, then the tree is identified to the tree obtained by collapsing that edge into a single vertex.

Definition 2.2 *(Metric trees).* We define the space of metric trees with n-leaves as the quotient $\mathcal{T}_n /\!\sim$, where $\sim$ is the above defined relation. The topology on $\mathcal{T}_n /\!\sim$ is the quotient topology. Any element of $\mathcal{T}_n /\!\sim$ will be called a *metric tree* with n-leaves.

2.1.2. A_∞-spaces

Let us recall the definition of A_∞-spaces as algebras over the operad $\mathcal{K} = \{K_n\}_{n \geqslant 0}$ defined by the associahedra. We will use the Boardman and Vogt [2] parametrization of K_n by planar metric trees, i.e., we will think of the associahedra K_n as the space $\mathcal{T}_n /\!\sim$ of metric trees with n-leaves.

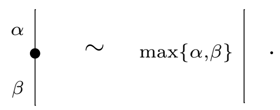
The grafting operation on $\mathcal{T}_n$ is well defined on $K_n = \mathcal{T}_n /\!\sim$ and it induces the operad structure of $\mathcal{K}$:

$$\circ_i : K_n \times K_m \to K_{n+m-1}, \quad 1 \leqslant i \leqslant n,$$

where $T \circ_i S = T \circ_i^1 S$.

The components K_1 and K_0 are defined as one point spaces. The only point of K_1 corresponds to the tree with only one edge and no vertices. It plays the role of the identity in the operad $\mathcal{K}$. The only point δ_0 of K_0 induces the degeneracy maps s_i in the sense that grafting δ_0 to the ith leaf of a tree in K_n is equal to erasing that leaf.

There is one subtlety in the definition of the degeneracy maps s_i that can be solved as follows. Suppose that the metric tree T has some vertex with exactly two incoming edges and one of those edges is the ith leaf. After erasing that leaf, the adjacent vertex will only have one incoming edge. For the map s_i to be well defined, we need to include one further relation. The adjacent vertex having only one incoming edge will be erased and the resulting edge will have length equal to the maximum of the lengths of the previous adjacent edges:



Now we can see that the degeneracy maps

$$s_i : K_n \to K_{n-1}, \quad 1 \leqslant i \leqslant n,$$

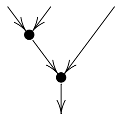
given by $s_i(\tau) = \tau \circ_i \delta_0$, are well defined.

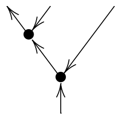
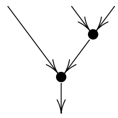
There is another operation on edge-labeled trees that will be useful in this paper. Such operations will be called *shift* and *deshift*. The shift (resp. deshift) operation will be denoted by $T \mapsto \overrightarrow{T}$ (resp. $T \mapsto \overleftarrow{T}$) and is characterized by the following properties for any trees S and T and for any $r \in [0,1]$:

$$\overrightarrow{\delta_k} = \delta_k; \quad \overrightarrow{S \circ_i^r T} = \overrightarrow{S} \circ_{i+1}^r T; \quad \overrightarrow{S \circ_{|S|}^r T} = \overrightarrow{T} \circ_1^r \overrightarrow{S}, \quad \text{for } 1 \leqslant i < |S|. \tag{3}$$

$$\overleftarrow{\delta_k} = \delta_k; \quad \overleftarrow{S \circ_j^r T} = \overleftarrow{S} \circ_{j-1}^r T; \quad \overleftarrow{S \circ_1^r T} = \overleftarrow{T} \circ_{|T|}^r \overleftarrow{S}, \quad \text{for } 1 < j \leqslant |S|. \tag{4}$$

One can check that this definition does not depend on the decomposition of a tree as in (1) and (2). The shift $\overrightarrow{T}$ can be understood by reversing the orientation of all edges joining the last leaf to the root. On the other hand, the deshift $\overleftarrow{T}$ can be understood by reversing the orientation of all edges joining the first leaf to the root.

Let us illustrate the deshift operation. Given a tree $T =$  , its deshift is:

$\overleftarrow{T}$ =  =  .

One can see that the deshift (resp. shift) transform the first (resp. last) leaf into the root. The root, in turn, is transformed into the last (resp. first) leaf. One can see that a shift followed by a deshift is the identity.

Definition 2.3 *(A_∞-spaces).* An A_∞-space is a topological space X endowed with the structure of an algebra over the operad $\mathcal{K} = \{K_n\}_{n\geqslant 0}$.

A $\mathcal{K}$-algebra structure on X is equivalent to the existence of a family of maps

$$M_n : K_n \times X^n \to X, \quad n \geqslant 0,$$

such that for any $\rho \in K_n$ and $\tau \in K_m$, the following relation holds:

$$M_{n+m-1}(\rho \circ_i \tau; x_1, \dots, x_{n+m-1}) = M_n(\rho; x_1, \dots, M_m(\tau; x_i, \dots, x_{i+m-1}), \dots, x_{n+m-1}),$$

and such that M_1 acts as the identity.

For $\delta_0 \in K_0$, since the degeneracies are given by $s_i(\rho) = \rho \circ_i \delta_0$, we have in particular:

$$M_{n-1}(s_i(\rho); x_1, \dots, x_{n-1}) = M_n(\rho; x_1, \dots, x_{i-1}, e_X, x_i, \dots, x_{n-1}),$$

where $1 \leqslant i \leqslant n$ and $e_X = M_0(\delta_0)$ is the only point of X in the image of the map $M_0 : K_0 \to X$. The element e_X will be called the unit of the A_∞-structure on X.

In the present paper, unless otherwise stated, all A_∞-spaces are assumed to have a strict unit as above. For A_∞-spaces with homotopy units, we refer the reader to [12,11,7].

2.1.3. A_∞-maps

Later in this section we will recall the fact that grouplike spaces admitting an A_∞-structure are precisely those spaces having the weak homotopy type of loop spaces. Similarly, maps between A_∞-spaces having the weak homotopy type of loop maps are called A_∞-maps (see [17], Theorem 8.12).

In the next definition we recall the notion of A_∞-maps $f : X \to Y$ where X is an A_∞-space and Y is a topological monoid (i.e., an associative H-space with unit e_Y). It is possible to define A_∞-maps between two A_∞-spaces in full generality by taking advantage of the W-construction introduced by Boardman and Vogt. For any operad $\mathcal{P}$, let $\mathcal{P}_{b\to w}$ be the operad whose algebras are pairs of $\mathcal{P}$-algebras with a morphism between them (for details, see: [10,8]). An A_∞-map between two A_∞-spaces is then defined as the structure of an algebra over the operad $W(\mathsf{Ass}_{b\to w})$ compatible with the given A_∞-structures.

Definition 2.4 *(A_∞-maps).* Let Y be a monoid and X be an A_∞-space. We say that a map $f : X \to Y$ is an A_∞-map if there exists maps $\{f_i : K_{i+1} \times X^i \to Y\}_{i\geqslant 1}$ such that $f(x) = f_1(\delta_2; x)$, for all $x \in X$ and:

$$f_k(\rho \circ_{j+1} \tau; x_{1,\dots,k}) = f_{|\rho|-1}(\rho; x_{1,\dots,j-1}, M_{|\tau|}(\tau; x_{j,\dots,j+|\tau|-1}), x_{j+|\tau|,\dots,k}), \tag{5}$$

$$f_k(\rho \circ_1 \tau; x_{1,\dots,k}) = f_{|\tau|-1}(\tau; x_{1,\dots,|\tau|-1}) \cdot f_{|\rho|-1}(\rho; x_{|\tau|,\dots,k}), \tag{6}$$

for $1 \leqslant j < |\rho|$, where ρ and τ are metric trees such $|\rho|, |\tau| \geqslant 2$ and $k = |\rho| + |\tau| - 2$.

If moreover the family satisfies:

$$f_{n-1}(s_{j+1}(\rho); x_1, \dots, x_{n-1}) = f_n(\rho; x_1, \dots x_{j-1}, e_X, x_j, \dots, x_{n-1}), \quad \text{for } n \geqslant 1,$$

where $f_0(\delta_1) = e_Y$, then the A_∞-map is called *unital*.

Notice that for unital A_∞-maps we have: $f(e_X) = e_Y$. There is a sequence of polyhedra parameterizing the higher homotopies of an A_∞-map, such polyhedra are known as multiplihedra. The correspondence between the multiplihedra and $W(\mathsf{Ass}_{b\to w})$ is not immediate and involves the *level trees* introduced in [2]. A nice description of such relations, can be found in [4,20]. The multiplihedra can also be obtained as a compactification of a certain class of configuration spaces, as shown by Merkulov in [9].

In the following definition, the monoid structure given by the opposite product defined on a topological monoid M is denoted by M^{op}.

Definition 2.5 *(A_∞-action).* Given an A_∞-space X and a space P, a left A_∞-action of X on P is defined as a unital A_∞-map $X \to \mathrm{End}(P)$. On the other hand, a right A_∞-action of X on a space Q is defined as a unital A_∞-map $X \to \mathrm{End}(Q)^{\mathrm{op}}$.

For any A_∞-space, if P admits a left (resp. right) A_∞-action, we will say simply that P is a left (resp. right) X-space. The structure of left X-space on P will be denoted by $X \circlearrowright_\infty P$, while right X-space structures will be denoted by $Q \circlearrowleft_\infty X$.

Proposition 2.6. *Let X be an A_∞-space and P be a topological space. An A_∞-action structure $X \circlearrowright_\infty P$ is equivalent to a family of maps $\{N_i : K_i \times X^{i-1} \times P \to P\}_{i\geqslant 1}$ such that $N_1 = \mathrm{Id}_P$ and the following conditions are satisfied:*

i) $N_k(\rho \circ_j \tau; x_{1,\dots,k-1}, p) = N_{|\rho|}(\rho; x_{1,\dots,j-1}, M_{|\tau|}(\tau; x_{j,\dots,j+|\tau|-1}), x_{j+|\tau|,\dots,k-1}, p)$, *for* $1 \leqslant j \leqslant |\rho| - 1$, $|\tau| \geqslant 0$;
ii) $N_k(\rho \circ_{|\rho|} \tau; x_1, \dots, x_{k-1}, p) = N_{|\rho|}(\rho; x_1, \dots, x_{|\rho|-1}, N_{|\tau|}(\tau; x_{|\rho|}, \dots, x_{k-1}, p))$,

where ρ and τ are arbitrary metric trees and $k = |\rho| + |\tau| - 1$.

Proof. Assuming that a family of maps N_i is given as above, we use the deshift operation (4) on trees to define the unital A_∞-map from X to $\mathrm{End}(P)$:

$$f_{n-1}(T; x_{1,\dots,n-1})(p) = N_n(\overleftarrow{T}; x_{1,\dots,n-1}, p), \qquad \text{for } n \geqslant 2.$$

To see that f is unital, we take $T = \rho \circ_i \tau$ with $|\tau| = 0$ and use condition *i)*. For $n = 1$ we observe that $p = N_1(\delta_1; p) = N_2(\delta_2; e_X, p)$. Hence:

$$f(e_X)(p) = f_1(\delta_2; e_X)(p) = N_2(\delta_2; e_X, p) = p$$

in other words: $f(e_X) = \mathrm{Id}_P$.

Let us check the compatibility with the operadic composition $\circ_1$:

$$\begin{aligned} f_{n-1}(T \circ_1 S; x_{1,\dots,n-1})(p) &= N_n(\overleftarrow{T \circ_1 S}; x_{1,\dots,n-1}, p) \\ &= N_n(\overleftarrow{S} \circ_{|S|} \overleftarrow{T}; x_{1,\dots,n-1}, p) = N_{|S|}(\overleftarrow{S}; x_{1,\dots,|S|-1}, N_{|T|}(\overleftarrow{T}; x_{|S|,\dots,n-1}, p)) \\ &= f_{|S|-1}(S; x_{1,\dots,|S|-1}) f_{|T|-1}(T; x_{|S|,\dots,n-1})(p), \end{aligned}$$

where $n = |S| + |T| - 1$. Checking the compatibility for the others $\circ_i{}'s$ is similar.

We left to the reader the proof of the converse by using the shift $\overrightarrow{T}$. $\square$

Observation 2.7. *Analogous results hold for right A_∞-actions. For $\tau = \delta_0 \in K_0$, relation i) is: $N_{r-1}(s_j(\rho); x_1, \dots, x_{r-2}, p) = N_r(\rho; x_1, \dots, x_{j-1}, e_X, x_j, \dots, x_{r-2}, p)$.*

2.2. Moore loops and Moore paths

Let $\mathbb{R}^+$ be the space of non-negative real numbers. The space of Moore paths on a topological space B is defined as follows:

$$\mathcal{P}_M(B) = \{(\gamma, r) \in B^{\mathbb{R}^+} \times \mathbb{R}^+ : \gamma(t) = \gamma(r), \ \ \forall t \geqslant r\}$$

For any $(\gamma, r) \in \mathcal{P}_M(B)$, the number r will be referred to as the *length* of the Moore path (γ, r). By abuse of notation, we will often denote a Moore path simply by γ and its length by $|\gamma|$; such abuse of notation must be understood as: $|(\gamma, r)| = r$.

The space of based Moore paths on a based topological space $(B, *)$ is defined as follows:

$$\mathcal{P}^*_M(B) = \{(\gamma, r) \in \mathcal{P}_M(B) : \gamma(0) = *\}.$$

The space of Moore loops on $(B, *)$ is defined as:

$$\Omega_M(B) = \{(\gamma, r) \in \mathcal{P}_M(B) : \gamma(0) = * \text{ and } \gamma(r) = *\}.$$

The juxtaposition $(\gamma_1 \cdot \gamma_2, r_1 + r_2)$ of paths (γ_1, r_1) and (γ_2, r_2) such that $\gamma_1(r_1) = \gamma_2(0)$ is defined by:

$$(\gamma_1 \cdot \gamma_2)(t) = \begin{cases} \gamma_1(t), & 0 \leqslant t \leqslant r_1, \\ \gamma_2(t - r_1), & r_1 \leqslant t. \end{cases}$$

Such operation defines the structure of a monoid action $\Omega_M(B) \curvearrowright \mathcal{P}^*_M(B)$. Given an inclusion $A \subseteq B$, the space of relative Moore loops is defined as:

$$\Omega_M(B, A) = \{(\gamma, r) \in \mathcal{P}^*_M(B) : \gamma(r) \in A\}.$$

Again there is a monoid action $\Omega_M(B) \curvearrowright \Omega_M(B, A)$ given by juxtaposition.

The operation of cutting a path is defined as follows. Given $0 \leqslant a < b$ and an element $(\gamma, r) \in \mathcal{P}_M(B)$ the path $(\gamma_{[a,b]}, b - a)$ is defined by

$$\gamma_{[a,b]}(t) = \begin{cases} \gamma(t + a), & 0 \leqslant t \leqslant b - a, \\ \gamma(b), & t \geqslant b - a. \end{cases}$$

2.3. Two-sided bar construction for A_∞-actions

Let us begin by defining the two-sided bar construction. In our particular case, the definition will be given in terms of the Associahedra. Given an A_∞-space X along with right and left X-spaces, the 2-sided bar construction amounts to considering planar metric trees whose first leaf is labeled by the elements of the right X-space, the last leaf is labeled by elements of the left X-space and the remaining leaves are labeled by the elements of the A_∞-space X.

The identifications are essentially those of Boardman–Vogt's M-construction whose elements are called *cherry trees*. Except that when the first (resp. last) leaf is involved, the identification must take into account the right (resp. left) A_∞-actions on the corresponding spaces.

Definition 2.8 *(2-sided bar construction).* Let $(X, \{M_k\}_{k\geqslant 0})$ be an A_∞-space and let P and Q be left and right X-spaces respectively, with structure maps given by: $N_k : K_k \times X^{\times k-1} \times P \to P$ and $R_k : K_k \times Q \times X^{\times k-1} \to Q$, for $k \geqslant 1$. We define $B(Q, X, P)$ as the following quotient.

$$B(Q,X,P) = \left(\coprod_{n\geqslant 0} K_{n+2} \times Q \times X^{\times n} \times P \right) \Big/ \equiv$$

where $\equiv$ is the relation defined by:

$$(\rho \circ_{i+1} \tau; x_0, x_1, \ldots, x_{n+1}) \equiv (\rho; x_0, \ldots, x_{i-1}, E_{|\tau|}(\tau; x_{i,\ldots,i+|\tau|-1}), x_{i+|\tau|}, \ldots, x_{n+1})$$

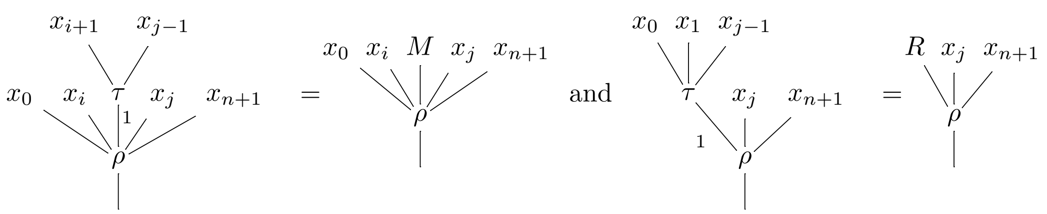
where $x_0 \in Q$, $x_{n+1} \in P$ and $|\tau| \geqslant 1$ with $0 \leqslant i \leqslant |\rho| - 1$. In this definition, E_k is just a symbol that must be replaced by R_k, N_k or M_k according to the position (left, right or middle) it occupies in the formula.

The topology of $B(Q, X, P)$ is the limit topology of the sequence $B_{n-1}(Q, X, P) \subseteq B_n(Q, X, P)$, where $B_n(Q, X, P)$ denotes the image of

$$K_n(Q,X,P) = \coprod_{k=0}^{n} K_{k+2} \times Q \times X^{\times k} \times P,$$

under the quotient map and each $B_n(Q, X, P)$ is given the quotient topology.

In terms of metric trees, the identifications in the above definition occur when some internal edge has length 1. For instance, we have the following identifications:



where $M = M_{j-i-1}(\tau; x_{i+1,\ldots,j-1})$ and $R = R_j(\tau; x_0, x_{1,\ldots,j-1})$. The above trees are simplified since only part of their leaves are shown.

Given an A_∞-space X, both X and the one point space $*$ are X-spaces with the canonical and trivial actions respectively.

Definition 2.9. Let X be an A_∞-space with a left A_∞-action $X \circlearrowright P$.

i) The classifying space of X is defined as $B(*, X, *)$ and denoted by BX.
ii) The homotopy orbit space of the A_∞-action $X \circlearrowright P$ is defined as $B(*, X, P)$ and denoted by $P/\!\!/X$.
iii) The space $B(X, X, *)$ will be denoted by EX.

The fact that the natural projection $EX \to BX$ is a quasi-fibration will be mentioned in Theorem 2.18 at page 140 and the contractibility of EX will be proved in Theorem 2.12 at page 136.

Notation. *For each element* $(T; q, x_{[n]}, p) \in K_{n+2} \times Q \times X^n \times P$*, its class in* $B(Q, X, P)$ *will be denoted by* $[T; q, x_{[n]}, p]$.

2.4. Fundamental properties of the two-sided bar construction

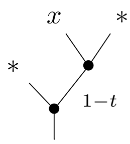
There is an important map

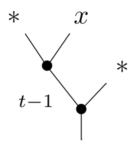
$$f : X \to \Omega_M(BX)$$

that can be defined as follows. For each $x \in X$, the loop $f(x) \in \Omega_M(BX)$ of length 2 is defined by:

$$f(x)(t) = \begin{cases} [\delta_2 \circ_2^{1-t} \delta_2; *, x, *], \ 0 \leqslant t \leqslant 1, \\ [\delta_2 \circ_1^{t-1} \delta_2; *, x, *], \ 1 \leqslant t \leqslant 2. \end{cases}$$

The map f will be called the *usual map*. The following picture illustrates the map f. The idea is that x is sitting over a leaf which is sliding along the edges of the tree. It defines a closed loop because the action on $\{*\}$ is trivial.



for $0 \leqslant t \leqslant 1$ and


for $1 \leqslant t \leqslant 2$. (7)

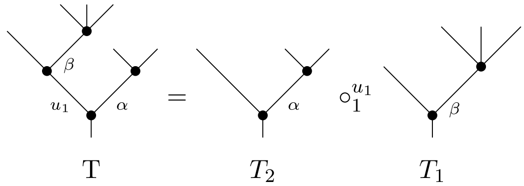
We have chosen not to use simplicial constructions. Hence, our constructions are directly based on the associahedra. This choice allows us to build an A_∞-map $f : X \to \Omega BX$ explicitily. Such explicit A_∞-map is necessary in our proof of the A_∞-equivariance of our main Theorem 2.22. Since we are not using simplicial constructions, the degeneracies maps are not needed here. That is the reason why we are not using the reduced bar construction. Indeed, even with the reduced bar construction $\overline{B}X$, the A_∞-map $f : X \to \Omega\overline{B}X$ would not be a unital A_∞-map.

In this section, we prove that P is a deformation retract of $B(X, X, P)$ and that the usual map $X \to \Omega_M(BX)$ is an A_∞-weak equivalence provided X is grouplike. It follows that $B(X, X, P)$ provides a model for P which is appropriate for the study of the A_∞-equivariance of A_∞-actions. The idea behind the proofs is the process of sliding edges of trees associated with the homotopy extension property.

For the next theorems we need to introduce further notation: an edge-labeled tree T is called comb-reducible if it decomposes as $T = L \circ_1^u R$ for some trees L, R and some $u \in [0, 1]$, and comb-irreducible otherwise. Thus any edge-labeled tree T decomposes uniquely as follows:

$$T = T_{k+1} \circ_1^{u_k} T_k \circ_1^{u_{k-1}} \cdots \circ_1^{u_1} T_1 \tag{8}$$

where the trees T_i are comb-irreducibles. We will call such a decomposition the comb decomposition of an edge-labeled tree. Below is a picture of a comb-decomposition.



Let us now show how to associate a Moore path to a tree T. Since each Moore path must have a length, we first define the length of an edge-labeled tree T as: $l(T) = 2(1 + \sum_{r=1}^{k} u_r)$.

Note that:

$$l(T \circ_1^r S) = l(T) + l(S) + 2r - 2.$$

The length is well defined on the quotient space K_n. Note that one can extend the length to K_1 by $l(\delta_1) = 0$.

Theorem 2.10. *The usual map $X \to \Omega_M(BX)$ is an A_∞-map.*

Proof. We need to exhibit maps $f_n : K_{n+1} \times X^n \to \Omega_M(BX)$ for $n \geqslant 1$, such that relations (5) and (6) are verified and f_1 coincides with the usual map $X \to \Omega_M(BX)$. To that end, we use a family of maps $\sigma_{n+1} : K_{n+1} \times \mathbb{R}^+ \to K_{n+2}$, for $n \geqslant 0$. The existence of a similar family of maps was originally proven by the third author in [15, Proposition 25]. See Lemma 2.11 for a proof of their existence.

For simplicity, we omit the subscript and for any $T \in K_{n+1}$, define $\sigma(T)$ to be a Moore path $\sigma(T) : \mathbb{R}^+ \to K_{n+2}$ of length $l(T)$, defined above. The properties of σ that are relevant for the A_∞-structure are the following:

$$\sigma(T)(0) = \delta_2 \circ_2 \overleftarrow{T} \quad \text{and} \quad \sigma(T)(l(T)) = \delta_2 \circ_1 T; \tag{9a}$$

$$\sigma(T_1 \circ_i T_2) = \sigma(T_1) \circ_i T_2, \qquad \text{for} \qquad 1 < i \leqslant |T_1|; \tag{9b}$$

$$\sigma(T_1 \circ_1 T_2) = (\sigma(T_2) \circ_{|T_2|+1} \overleftarrow{T_1}) \cdot (\sigma(T_1) \circ_1 T_2), \tag{9c}$$

where $\overleftarrow{T}$ is the deshift (4) and the trees T_1 and T_2 are such that $|T_1|, |T_2| \geqslant 2$. In addition we ask that $\sigma(\delta_2)$ coincides with the sliding of edges described in (7).

Finally, the maps $f_n : K_{n+1} \times X^n \to \Omega_M(BX)$ are defined by:

$$f_n(T; x_{[n]})(t) = [\sigma(T)(t); *, x_{[n]}, *].$$

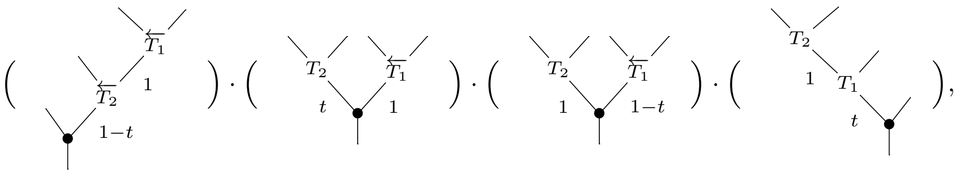
The fact that such a family verifies the required conditions follows from the properties of σ and the definition of the 2-sided bar construction.

In fact, since $\sigma(\delta_2)$ is the path described in (7), it follows that $f_1 : X \to \Omega_M(BX)$ is the usual map. From (9a) we have that $f_n(T; x_{[n]})$ is indeed a loop because of the trivial action of X on $\{*\}$. On the other hand, equation (9b) along with the identifications in $B(*, X, *)$ implies that the family of maps $\{f_n\}$ satisfies the condition (5) in the definition of A_∞-maps.

To finish this proof, we just need to check equation (6) in the definition of A_∞-maps. We observe that equation (9c) implies that:

$$f_n(T_1 \circ_1 T_2; x_{[n]}) = f_i(T_2; x_{1,\dots,i}) \cdot f_j(T_1; x_{i+1,\dots,n}),$$

where $i + j = n$ and the loop $f_i(T_2; x_{1,\dots,i})$ given by the movement of the tree T_2 is followed by the loop $f_j(T_1; x_{i+1,\dots,n})$ given by the movement of T_1, as schematically illustrated (i.e. not showing the true number of leaves) in the following picture:



where $t \in [0, 1]$ for each tree, and $\cdot$ denotes the juxtaposition of paths. □

Lemma 2.11. *There exists a family of maps $\sigma_n : K_n \times \mathbb{R}^+ \to K_{n+1}$ for $n \geqslant 2$ satisfying the properties (9) presented above.*

Proof. We use a recursive argument involving edge-labeled trees.

For any tree $T \in \mathcal{T}_n$ the path $\sigma(T) : \mathbb{R}^+ \to K_{n+1}$ has the following property:

$$\sigma(T)(t) = \begin{cases} \delta_2 \circ_2^{1-t} \overleftarrow{T}, & 0 \leqslant t \leqslant 1, \\ \delta_2 \circ_1^{t-l(T)+1} T, & l(T) - 1 \leqslant t \leqslant l(T). \end{cases} \tag{10}$$

This property defines the path $\sigma(\delta_n)$ for any corolla for $l(\delta_n) = 2$.

Let us assume that σ is defined for edge-labeled trees with $k < n$ leaves and satisfies (10). Now consider a tree T such that $T = T_1 \circ_i^r T_2$

When $1 < i$, define: $\sigma(T) = \sigma(T_1) \circ_i^r T_2$.

When $i = 1$, define $\sigma(T)$ as the following juxtaposition of paths

$$(\sigma(T_2)_{[0,l(T_2)+r-1]} \circ^r_{|T_2|+1} \overleftarrow{T}_1) \cdot (\sigma(T_1)_{[1-r,l(T_1)]} \circ_1^r T_2).$$

Note that this definition is consistent thanks to (10). It is not hard to check that these definitions are independent of the decomposition of a tree as in (1) and (2), that σ satisfies (10), that σ is continuous with respect to T and that it passes to the quotient space K_n. Moreover, properties (9) follow directly from the construction of σ. □

The intuitive description of the paths $\sigma(T)$ is in terms of *sliding subtrees along edges in a coherent way*, as illustrated in equation (25) at page 142.

Theorem 2.12. *For any A_∞-action $X \circlearrowright P$, the space P is a deformation retract of $B(X, X, P)$.*

Proof. The embedding $P \hookrightarrow B(X, X, P)$ is defined by $p \hookrightarrow [\delta_2; e, p]$. Now consider the map $B(X, X, P) \to P$ given by $[T; x, x_{[n]}, p] \mapsto N_{n+2}(T; x, x_{[n]}, p)$, where the $\{N_i\}'_{i \geqslant 1}s$ are the structure maps of the A_∞-action $X \circlearrowright P$. This map is well defined because the equivalence relations in the definition of $B(X, X, P)$ involve the A_∞-structure maps. The composition $P \to B(X, X, P) \to P$ is the identity on P.

Let us show that $B(X, X, P) \to P \to B(X, X, P)$ is homotopic to the identity. This last map, can be explicitly described as follows:

$$[T; x, x_{[n]}, p] \mapsto [\delta_2 \circ_2 T; e, x, x_{[n]}, p].$$

The proof is divided in several steps. We first build in Lemma 2.14 a path $\gamma_T : [0,1] \to K_{n+1}$, for any tree $T \in K_n$, such that $\gamma_T(0) = T \circ_1 \delta_2$ and $\gamma_T(1) = \delta_2 \circ_2 T$ yielding a homotopy $\Gamma : K(X,X,P) \times I \to B(X,X,P)$. Unfortunately this homotopy does not pass to the quotient space $B(X,X,P)$. However, given two elements $x, y \in K(X,X,P)$ such that $x \equiv y$ we will prove in Lemma 2.15 that $\Gamma(x)$ is homotopic to $\Gamma(y)$. In Lemma 2.16, we prove, using the homotopy extension property, that we can replace Γ by $\tilde{\Gamma}$, homotopic to Γ and satisfying $\tilde{\Gamma}(x) = \tilde{\Gamma}(y)$ whenever $x \equiv y$, yielding the required homotopy. □

Notation 2.13. *Given an edge-labeled tree $T \in \mathcal{T}_n$ and its comb decomposition (8)*

$$T = T_{r+1} \circ_1^{u_r} T_r \circ_1^{u_{r-1}} \cdots \circ_1^{u_1} T_1,$$

we define:

- $T = R_i \circ_1^{u_i} L_i$, *where* $R_i = T_{r+1} \circ_1^{u_r} T_r \circ_1^{u_{r-1}} \cdots \circ_1^{u_{i+1}} T_{i+1}$ *and* $L_i = T_i \circ_1^{u_{i-1}} T_r \circ_1^{u_r} \cdots \circ_1^{u_1} T_1$ *for any* $1 \leqslant i \leqslant r$,
- $l(T) = 2(1 + \sum_{k=1}^{r} u_k)$,
- $l_i(T) = 1 + 2\sum_{k=1}^{i-1} u_k + u_i$ *for* $1 \leqslant i \leqslant r$.

When there is no confusion, we will write l, l_i instead of $l(T), l_i(T)$.

Lemma 2.14. *There exists a map*

$$\Gamma : K(X,X,P) \times I \to B(X,X,P)$$

satisfying

$$\begin{aligned} \Gamma(T; x; x_{[n]}, p)(0) &= [T; x; x_{[n]}, p], \\ \Gamma(T; x; x_{[n]}, p)(1) &= [\delta_2 \circ_2 T; e, x, x_{[n]}, p], \\ \Gamma(T \circ_{i+1} S; x, x_{[n]}, p)(t) &= \Gamma(T; x, x_{1,\ldots,i-1}, E_{|S|}(S; x_{i,\ldots,i+|S|-1}), x_{i+|S|,\ldots,n}, p)(t),\ i \geq 1 \end{aligned} \tag{11}$$

and

$$\Gamma(T \circ_1 S, x, x_{[n]}, p)(t) = \begin{cases} [T; M_{|S|}(S; x, x_{1,\ldots,|S|-1}), x_{|S|,\ldots,n}, p], & 0 \leqslant t \leqslant \frac{l(S)}{l}, \\ \Gamma(T, M_{|S|}(S; x, x_{1,\ldots,|S|-1}), x_{|S|,\ldots,n}, p)\left(\frac{t - \frac{l(S)}{l}}{1 - \frac{l(S)}{l}}\right), & \frac{l(S)}{l} \leqslant t \leqslant 1. \end{cases} \tag{12}$$

Proof. Let T be a tree and consider its comb decomposition as in Notation 2.13. We build the path $\gamma_T : [0,1] \to K_{n+1}$ by sliding continuously δ_2 along the path from the left most leaf of T to its root as follows

$$\gamma_T(t) = \begin{cases} T \circ_1^{1-lt} \delta_2, & 0 \leqslant t \leqslant \frac{1}{l}, \\ R_k \circ_1^{u_k} \delta_2 \circ_2^{lt-(l_k - u_k)} L_k, & \frac{l_k - u_k}{l} \leqslant t \leqslant \frac{l_k}{l},\ 1 \leqslant k \leqslant r, \\ R_k \circ_1^{l_k + u_k - lt} \delta_2 \circ_2^{u_k} L_k, & \frac{l_k}{l} \leqslant t \leqslant \frac{l_k + u_k}{l},\ 1 \leqslant k \leqslant r, \\ \delta_2 \circ_2^{lt-(l_r+u_r)} T, & \frac{l_r+u_r}{l} \leqslant t \leqslant 1. \end{cases}$$

A direct inspection shows that it is well defined, continuous and passes to the quotient space K_n. Furthermore, the following relations hold, $\forall T, S, u \in [0,1]$ and $i \geqslant 2$:

$$\gamma_{T\circ_i^u S} = \gamma_T \circ_{i+1}^u S, \tag{13}$$

$$\gamma_{T\circ_1^u S}(t) = \begin{cases} T \circ_1^u \gamma_S\left(\frac{l}{l(S)}t\right), & 0 \leqslant t \leqslant \frac{l(S)+u-1}{l}, \\ \gamma_T\left(\frac{t-\frac{l(S)+2u-2}{l}}{1-\frac{l(S)+2u-2}{l}}\right) \circ_2^u S, & \frac{l(S)+u-1}{l} \leqslant t \leqslant 1. \end{cases} \tag{14}$$

In addition, $\forall T \in K_n, \forall t \in [0,1]$,

$$M_{n+1}(\gamma_T(t); e, x_{[n]}) = M_n(T; x_{[n]}). \tag{15}$$

The paths γ_T define the map

$$\begin{array}{rccc} \Gamma : & K(X,X,P) \times I & \to & B(X,X,P) \\ & (T; x; x_{[n]}, p) \times t & \mapsto & [\gamma_T(t); e, x, x_{[n]}, p] \end{array}$$

satisfying Conditions (11). It implies that Γ passes to the quotient by the equivalence relation of Definition 2.8 when $i \geqslant 1$. From equation (14) and relation (15) one gets Relation (12). □

The next lemma needs some technical facts whose proofs are left to the reader. For $\alpha_1, \alpha_2, \alpha_3 : I \to Y$ such that $\alpha_1(1) = \alpha_2(0)$ and $\alpha_2(1) = \alpha_3(0)$ and for $0 < a < b < 1$, define:

$$(\alpha_1 *^a \alpha_2)(u) = \begin{cases} \alpha_1(\frac{u}{a}), & 0 \leqslant u \leqslant a, \\ \alpha_2(\frac{u-a}{1-a}), & a \leqslant u \leqslant 1, \end{cases} \quad \text{and} \quad (\alpha_1 *^a \alpha_2 *^b \alpha_3)(u) = \begin{cases} \alpha_1(\frac{u}{a}), & 0 \leqslant u \leqslant a, \\ \alpha_2(\frac{u-a}{b-a}), & a \leqslant u \leqslant b, \\ \alpha_3(\frac{u-b}{1-b}), & b \leqslant u \leqslant 1. \end{cases}$$

The following relation holds:

$$(\alpha_1 *^{\frac{a}{b}} \alpha_2) *^b \alpha_3 = \alpha_1 *^a \alpha_2 *^b \alpha_3 = \alpha_1 *^a (\alpha_2 *^{\frac{b-a}{1-a}} \alpha_3).$$

For $0 < a < b < c$, let $x_{c;a,b} := \frac{a(c-b)}{b(c-a)} \in]0,1[$. For any $0 < a_1 < a_2 < a_3 < c$,

$$(\alpha_1 *^{x_{c;a_1,a_2}} \alpha_2) *^{x_{c;a_2,a_3}} \alpha_3 = \alpha_1 *^{x_{c;a_1,a_3}} \alpha_2 *^{x_{c;a_2,a_3}} \alpha_3 = \alpha_1 *^{x_{c;a_1,a_3}} (\alpha_2 *^{x_{c-a_1;a_2-a_1,a_3-a_1}} \alpha_3) \tag{16}$$

Lemma 2.15. *Let T and S be labeled trees, let $x, x_1, \ldots, x_n \in X$ and $p \in P$; the map*

$$\begin{aligned} & H_{(T\circ_1 S,T)}(x, x_{[n]}, p)(t,u) \\ & = \begin{cases} [T; M_{|S|}(S; x, x_{1,\ldots,|S|-1}), x_{|S|,\ldots,n}, p], & 0 \leqslant t \leqslant (1-u)\frac{l(S)}{l}, \\ \Gamma(T, M_{|S|}(S; x, x_{1,\ldots,|S|-1}), x_{|S|,\ldots,n}, p)\left(\frac{t-(1-u)\frac{l(S)}{l}}{1-(1-u)\frac{l(S)}{l}}\right), & (1-u)\frac{l(S)}{l} \leqslant t \leqslant 1, \end{cases} \end{aligned}$$

defines a homotopy from $\Gamma(T \circ_1 S; x, x_{[n]}, p)$ to $\Gamma(T, M_{|S|}(S; x, x_{1,\ldots,|S|-1}), x_{|S|,\ldots,n}, p)$. Furthermore this homotopy satisfies, for any trees T, S, V and $i \geqslant 1$,

$$\begin{aligned} & H_{(T\circ_1 S,T)}(x, x_{1,\ldots,i-1}, E_{|V|}(V; x_{i,\ldots,i+|S|-1}), x_{i+|S|,\ldots,n}, p) \\ & = \begin{cases} H_{((T\circ_1 S)\circ_{i+1} V,T)}(x, x_{[n]}, p), & 1 \leqslant i+1 \leqslant |S|, \\ H_{((T\circ_1 S)\circ_{i+1} V, T\circ_{i+1-|S|} V)}(x, x_{[n]}, p), & |S|+1 \leqslant i+1 \leqslant |T|+|S|-1, \end{cases} \end{aligned} \tag{17}$$

and

$$
\begin{aligned}
&H_{(T,T_3)}(x, x_{[n]}, p)(t) \\
&\quad = H_{(T,T_3\circ_1 T_2)}(x, x_{[n]}, p)(t) *^{x_{l;l_1,l_1+l_2}} H_{(T_3\circ_1 T_2,T_3)}(M_{|T_1|}(x, x_{1,\dots,|T_1|-1}), x_{|T_1|,\dots,n}, p)(t),
\end{aligned} \tag{18}
$$

where $T = T_3 \circ_1 T_2 \circ_1 T_1$, $l_1 = l(T_1)$, $l_2 = l(T_2)$, $l = l(T)$ and where the composition of paths is relative to the variable u in the definition of H.

Proof. By direct inspection. □

Last step: construction of $\tilde{\Gamma}$.

Lemma 2.16. *There exist maps $\Psi_{n+2} : K_n(X, X, P) \times I \times I \to B(X, X, P)$ satisfying:*

$$
\begin{aligned}
\Psi_{n+2}(T; x, x_{[n]}, p)(t, 0) &= \Gamma(T; x, x_{[n]}, p)(t), \\
\Psi_{n+2}(T; x, x_{[n]}, p)(0, u) &= [T; x, x_{[n]}, p], \\
\Psi_{n+2}(T; x, x_{[n]}, p)(1, u) &= [\delta_2; e, N_{n+2}(T; x, x_{[n]}, p)],
\end{aligned} \tag{19}
$$

$$
\Psi_{n+2}(T \circ_{i+1} S; x, x_{[n]}, p) = \Psi_{|T|}(T; x, x_{1,\dots,i-1}, E_{|S|}(S; x_{i,\dots,i+|S|-1}), x_{i+|S|,\dots,n}, p), \tag{20}
$$

for $i > 0$ and

$$
\Psi_{n+2}(T \circ_1 S, x, x_{[n]}, p)(t, 1) = \Psi_{|T|}(T; M_{|S|}(x, x_{1,\dots,|S|-1}), x_{|S|,\dots,n}, p)(t, 1). \tag{21}
$$

The path $\tilde{\Gamma} : K(X, X, P) \times I \to B(X, X, P)$ defined by

$$
\tilde{\Gamma}(T; x, x_{[n]}, p)(t) = \Psi_{|T|}(T; x, x_{[n]}, p)(t, 1)
$$

is well defined on $B(X, X, P)$ and provides the deformation retract of Theorem 2.12.

Proof. Let us build Ψ_n by induction. We start by $\Psi_2(\delta_2; x, p)(t, u) = \Gamma(\delta_2; x, p)(t)$. Let us assume that Ψ_k is defined for $k \leqslant n + 1$ and satisfies the above conditions.

Since the inclusion of the boundary ∂K_n into K_n is a cofibration, then the map

$$
\partial K_{n+2} \times X^{n+1} \times P \times I \cup K_{n+2} \times X^{n+1} \times P \times \partial I \to K_n(X, X, P) \times I
$$

is also a cofibration. Then using the homotopy extension property, it is enough to define Ψ on

$$
Y_n = (\partial K_{n+2} \times X^{n+1} \times P \times I \cup K_{n+2} \times X^{n+1} \times P \times \partial I) \times I \cup K_n(X, X, P) \times I \times \{0\}.
$$

Equations (19), and (20) implies that Ψ is defined by induction on Y_n except for the elements of the form $(T \circ_1 S, x, x_{[n]}, p)$. We then define

$$
\begin{aligned}
&\Psi_{n+2}(T \circ_1 S, x, x_{[n]}, p)(t) \\
&\quad = H_{(T\circ_1 S,T)}(x, x_{[n]}, p)(t) *^{x_{l;l(S),l-1}} \Psi_{|T|}(T; M_{|S|}(S; x, x_{1,\dots,|S|-1}), x_{|S|,\dots,n}, p)(t).
\end{aligned}
$$

Since $l = l(T) + l(S)$ and $l(T) \geqslant 2$, then $l(S) < l - 1 < l$ and the composition is well defined because

$$
\begin{aligned}
H_{(T\circ_1 S,T)}(x, x_{[n]}, p)(t, 1) &= \Psi_{|T|}(T; M_{|S|}(S; x, x_{1,\dots,|S|-1}), x_{|S|,\dots,n}, p)(t, 0) \\
&= \Gamma(T; M_{|S|}(S; x, x_{1,\dots,|S|-1}), x_{|S|,\dots,n}, p)(t).
\end{aligned}
$$

Moreover, Relations (16) and (18) imply that this definition is independent of the decomposition of a tree $T = T_1 \circ_1 S_1 = T_2 \circ_1 S_2$. Similarly, Relations (20) and (17) imply that this definition is independent of the decomposition of a tree $T = T_1 \circ_1 S_1 = U \circ_{i+1} V$, with $i > 0$. Finally, one can check that this definition satisfies (19) and (21). $\square$

The following corollary is obtained by projecting the inverse path of $\tilde{\Gamma}$ onto $B(*, X, P)$.

Corollary 2.17. *For any A_∞-action $X \circlearrowright_\infty P$, there exists a map*

$$\alpha_P : B(X, X, P) \to \mathcal{P}_M(B(*, X, P))$$

such that

$$\alpha_P([T; x; x_{[n]}, p])(0) = [\delta_2; *, N_{n+2}(T; x; x_{[n]}, p)]$$
$$\alpha_P([T; x; x_{[n]}, p])(1) = [T; *; x_{[n]}, p]$$
$$\alpha_P([\delta_2, x, p])(t) = \begin{cases} [\delta_2 \circ_2^{1-2t} \delta_2; *, x, p], & 0 \leqslant t \leqslant \frac{1}{2}, \\ [\delta_2 \circ_1^{2t-1} \delta_2; *, x, p], & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Let us recall that an H-space X is called *grouplike* [19] when its product induces a group structure on $\pi_0(X)$.

When the H-space in question is an A_∞-space, then the grouplike property implies that all possible translations

$$M_n(T; x_{1,\dots,i-1}, \_, x_{i+1,\dots,n}) : X \to X$$

are homotopy equivalences, for all $n \geqslant 2, T \in K_n$ and $x_j \in X$, as well as

$$N_n(T; x_{1,\dots,n-1}, \_) : P \to P,$$

if $X \circlearrowright_\infty P$. As a consequence, the same argument used in [17, Section 7] can be used to prove the following theorem.

Theorem 2.18. *If X is grouplike, then $B(X, X, P) \to B(*, X, P)$ is a quasi-fibration.*

Theorem 2.19. *If X is grouplike, then the usual map $f : X \to \Omega_M(BX)$ is a weak equivalence.*

Proof. Corollary 2.17 applied to $P = *$ endowed with the trivial A_∞-action implies that $\alpha_* : B(X, X, *) = EX \to \mathcal{P}^*_M(BX)$ is a weak equivalence since EX is contractible as is $\mathcal{P}^*_M(BX)$. Moreover α_* restricted to X has values in $\Omega_M(BX)$ and is homotopic to the usual map. Finally $B(X, X, *) \to B(*, X, *)$ is a quasi-fibration and the usual five lemma argument shows that the usual map $X \to \Omega_M(BX)$ is a weak equivalence. $\square$

2.5. A_∞-equivariance

Let us now describe the notion of A_∞-equivariance for maps between spaces admitting A_∞-actions. Our approach will be analogous to the one given for A_∞-maps in section 2.1.

Definition 2.20. Consider an A_∞-space $(X, \{M_n\}_{n\geqslant 0})$ and an X-space P with structure maps $\{N_j : K_j \times X^{j-1} \times P \to P\}_{j\geqslant 1}$ and a monoid Y with a monoid action $Y \curvearrowright Q$. We say that a map $F : P \to Q$ is

A_∞-equivariant with respect to an A_∞-map $\{f_i : K_{i+1} \times X^i \to Y\}_{i \geqslant 1}$ if there is a family $\{F_n : K_{n+1} \times X^{n-1} \times P \to Q\}_{n \geqslant 1}$ such that $F_1 = F$ and, for any $\tau \in K_{i+1}$ and $\rho \in K_{j+1}$, the following conditions are satisfied:

$$F_n(\tau \circ_k \rho; x_{[n-1]}, p) = \begin{cases} F_i(\tau; x_{1,\dots,k-2}, M_{j+1}(\rho; x_{k-1,\dots,k+j-1}), x_{k+j,\dots,n-1}, p), & 2 \leqslant k \leqslant i; \\ F_i(\tau; x_{1,\dots,i-1}, N_{j+1}(\rho; x_{i,\dots,n-1}, p)), & k = i+1; \\ f_j(\rho; x_{1,\dots,j}) \cdot F_i(\tau; x_{j+1,\dots,n-1}, p), & k = 1, \end{cases}$$

where $i + j = n$ and the action $Y \curvearrowright Q$ is denoted by: $y \cdot q$.

Any A_∞-action $X \circlearrowright P$ can be canonically extended to an action on the (unreduced) cone CP. Since the action $X \circlearrowright P$ in general does not fix the base point $p_0 \in P$, we need to work with the unreduced cone. The natural embedding $P \hookrightarrow \Omega_M(CP /\!\!/ X, P /\!\!/ X)$ is defined by

$$p \mapsto \gamma_p(t) = [\delta_2; *, [p, t]]\,,$$

where we recall that $P /\!\!/ X := B(*, X, P)$. We will show that the natural embedding is A_∞-equivariant with respect to $X \to \Omega_M B(*, X, CP)$, the A_∞-map obtained as the composition:

$$X \to \Omega_M B(*, X, *) \hookrightarrow \Omega_M B(*, X, CP). \tag{22}$$

where $X \to \Omega_M BX$ is the usual map of Theorem 2.10.

Proposition 2.21. *The natural embedding $P \hookrightarrow \Omega_M(CP /\!\!/ X, P /\!\!/ X)$ sending p to γ_p is A_∞-equivariant with respect to the A_∞-map $X \to \Omega_M(CP /\!\!/ X)$ defined by (22).*

Proof. The proof will follow the lines of the proof of Theorem 2.10. We will exhibit a family of maps

$$F_n : K_{n+1} \times X^{n-1} \times P \to \mathcal{P}_M{}^*(CP /\!\!/ X), \quad \text{for} \quad n \geqslant 1.$$

To any $(T; x_{[n-1]}, p) \in K_{n+1} \times X^{n-1} \times P$, the map F_n associates the following juxtaposition of paths:

$$\Lambda(T; x_{[n-1]}, p) \cdot \left[T; *, x_{[n-1]}, [p, t]\right]$$

where Λ defined by $\Lambda(T; x_{[n-1]}, p)(t) = \left[\lambda(T, t); *, x_{[n-1]}, [p, 0]\right]$, with λ defined as a family of maps $\lambda_n : K_n \to \mathcal{P}_M(K_n)$ whose construction is well tailored to ensure $\{F_n\}_{n \geqslant 1}$ has the properties of Definition 2.20.

More precisely, $\lambda(T)$ satisfies:

$$\begin{aligned} &\lambda(T)(0) = \delta_2 \circ_2 s_{|T|}(\overleftarrow{T}) \ \text{ and } \ \lambda(T)(|\lambda(T)|) = T; \\ &\lambda(T_1 \circ_i T_2) = \lambda(T_1) \circ_i T_2, \quad \text{for} \quad 1 < i \leqslant |T_1|; \\ &\lambda(T_1 \circ_1 T_2) = (\sigma(T_2) \circ_{|T_2|+1} s_{|T_1|}(\overleftarrow{T}_1)) \cdot (\lambda(T_1) \circ_1 T_2). \end{aligned} \tag{23}$$

These conditions ensure that $\{F_n\}_{n \geqslant 1}$ has the properties of Definition 2.20.

As in the proof of Lemma 2.11, we will construct the family $\lambda_n : K_n \to \mathcal{P}_M(K_n)$ recursively. However, the case in which the arity (i.e. the number of incoming edges) of the root vertex is 2 will be treated separately.

First case: We define $\lambda(T)$ for trees T whose arity of the root vertex is > 2. In this case, the Moore path $\lambda(T)$ has length $l(T) - 1$ and is such that:

$$\lambda(T, t) = \delta_2 \circ_2^{1-t} s_{|T|} \overleftarrow{T}, \quad 0 \leqslant t \leqslant 1. \tag{24}$$

In particular, this defines $\lambda(\delta_n)$ for any corolla δ_n with $n \geqslant 3$, since $|\lambda(\delta_n)| = l(\delta_n) - 1 = 1$.

Let us assume that λ is defined for edge-labeled trees with $k < n$ leaves and whose number of incoming edges at the root is > 2, and that it satisfies (24). Let us now consider a tree T such that $T = T_1 \circ_i^r T_2$.

For $1 < i$, we define: $\lambda(T) = \lambda(T_1) \circ_i^r T_2$. On the other hand, for $i = 1$ the path $\lambda(T)$ is defined as the following juxtaposition of paths:

$$\big(\sigma(T_2)_{[0,l(T_2)+r-1]} \circ^r_{|T_2|+1} s_{|T_1|}(\overleftarrow{T}_1)\big) \cdot \big(\lambda(T_1)_{[1-r,l(T_1)-1]} \circ_1^r T_2\big),$$

where we have used the operation $\gamma_{[a,b]}$ defined in subsection 2.2.

It is not hard to check that these definitions are independent of the decomposition of a tree as in (1) and (2), that λ satisfies (24), that λ is continuous with respect to T and that it passes to the quotient space K_n. The properties (23) follow directly from the construction of λ.

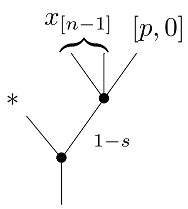
Second case: We define $\lambda(T)$ where the arity of the root vertex of T is 2. Let us first define $\lambda(\delta_2) = \delta_2$ as the constant path with length 0.

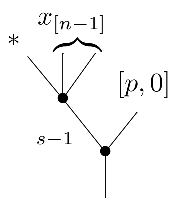
Now, let us consider $T = (\delta_2 \circ_2^v T_2) \circ_1^u T_1$ and define $\lambda(T)$ as the following juxtaposition:

$$\big(\sigma(T_1)_{[0,l(T_1)+u-1]} \circ^{Max(u,v)}_{|T_1|+1} T_2\big) \cdot \big((\delta_2 \circ_2^{Max(u,v)-t} T_2)_{[0,Max(u,v)-v]} \circ_1^u T_1\big)$$

Notice that in particular we have: $\lambda(\delta_2 \circ_1^u T_1) = \lambda((\delta_2 \circ_2^1 \delta_1) \circ_1^u T_1) = \sigma(T_1)_{[0,l(T_1)+u-1]}$ and $\lambda(\delta_2 \circ_2^v T_2) = \lambda((\delta_2 \circ_2^v T_2) \circ_1^1 \delta_1) = (\delta_2 \circ_2^{1-t} T_1)_{[0,1-v]}$. It is necessary to check that this construction is well defined on the quotient K_n. In particular one has to check that it is compatible with the first case when u or v tends to 0. The properties (23) must also be checked. The verification of these facts are long but straightforward. $\square$

For the $(n+1)$-corolla $\delta_{n+1} \in K_{n+1}$, it follows that $F_n(\delta_{n+1}; x_{[n-1]}, p)$ is defined by first "sliding subtrees along edges" as follows:



for $0 \leqslant s \leqslant 1$ and


for $1 \leqslant s \leqslant 2,$ (25)

and ending with the path $[\delta_2; *; [p,t]]$, taking the base point to $[\delta_2; *; [p,1]]$.

Theorem 2.22 *(Recognition theorem of relative loop spaces). For any A_∞-action $X \circlearrowright P$ by a grouplike A_∞-space X, there is a weak equivalence $P \to \Omega_M(CP/\!\!/X, P/\!\!/X)$ that is A_∞-equivariant with respect to the natural map $X \to \Omega_M(CP/\!\!/X)$.*

Proof. The A_∞-equivariance is the statement of Proposition 2.21. Let us show that the map $P \to \Omega_M(CP/\!\!/X, P/\!\!/X)$ is a weak equivalence. Let us consider $\beta : B(X,X,P) \to \Omega_M(CP/\!\!/X, P/\!\!/X)$ obtained as the composition of paths

$$\beta([T; x, x_{[n]}, p]) = \big[\delta_2; *, [N_{n+2}(x, x_{[n]}, p), t]\big] \cdot \iota\alpha_P([T; x, x_{[n]}, p])$$

where ι is the embedding $\mathcal{P}_M(P/\!\!/X) \hookrightarrow \mathcal{P}_M(CP/\!\!/X)$ and α_P has been defined in Corollary 2.17. Now β induces a map of quasi-fibrations:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook] & \mathrm{ev}^{-1}([p_0,1]) \arrow[d, hook] \\
B(X,X,P) \arrow[r, "\beta"] \arrow[d] & \Omega_M(CP/\!\!/X, P/\!\!/X) \arrow[d, "\mathrm{ev}"] \\
P/\!\!/X \arrow[r, equal] & P/\!\!/X,
\end{tikzcd}$$

where $\mathrm{ev} : \Omega_M(CP/\!\!/X, P/\!\!/X) \to P/\!\!/X$ is the evaluation map that takes each relative loop to its final point in $P/\!\!/X$.

Since CP is contractible, projecting onto its tip gives a weak equivalence $\mathrm{ev}^{-1}([p_0,1]) \to \Omega_M(BX)$. The top map followed by this weak equivalence is homotopic to the usual map $X \hookrightarrow \Omega_M(BX)$ which is a weak equivalence. Hence the top map is a weak equivalence and the usual five lemma applies to show that the middle map is a weak equivalence. As a consequence, the weak equivalence β composed with the usual inclusion $P \hookrightarrow B(X,X,P)$ which is known to be a homotopy equivalence (see Theorem 2.12) is a weak equivalence. The map so obtained is given by

$$p \mapsto \underbrace{([\delta_2;*,[p,t]])}_{\gamma_p(t)} \cdot \underbrace{\left(\left[\delta_2 \circ_2^{1-2t} \delta_2;*,e_X,[p,1]\right]\right) \cdot \left(\left[\delta_2 \circ_1^{2t} \delta_2;*,e_X,[p,1]\right]\right)}_{\rho_p}.$$

It is then enough to prove that ρ_p is homotopic to a constant path in order to prove that the map $p \mapsto \gamma_p$ is a weak equivalence. The path ρ_p lifts to a map $\tilde{\rho}_p : I \to B(X,X,P)$ replacing $*$ by e_X, the unit of the A_∞-structures on X.

The map $H_p : I \times I \to B(*,X,P)$ defined by $H_p(t,u) = \alpha_P(\tilde{\rho}_p(t),u)$ is a homotopy from the constant path at $[\delta_2,*,p]$ to ρ_p thanks to Corollary 2.17. $\square$

3. The 2-colored operad of A_∞-actions

In this section, we study the approach to A_∞-actions by means of the Boardman–Vogt W-construction $W(\mathsf{Act})$ on the operad of monoid actions Act. We also compare our definition with the one given by Iwase and Mimura in [6].

3.1. The operad Act

The 2-colored non-symmetric operad of monoid actions on spaces will be denoted by Act, i.e., it is the operad whose algebras consist of a topological monoid X and a left X-space P. The colors in this case will be called closed and open and denoted by $\{\mathfrak{c},\mathfrak{o}\}$. The color $\mathfrak{c}$ is assigned to X while the color $\mathfrak{o}$ is assigned to P.

Such choice of colors comes from Theorem 2.22 where the elements of the monoid correspond to loops while the elements of the X-space correspond to relative loops. The latter are not necessarily closed.

Definition 3.1 *(Operad of left actions).* Let us define Act as the $\{\mathfrak{c},\mathfrak{o}\}$-colored operad such that $\mathsf{Act}(p,0;\mathfrak{c})$ and $\mathsf{Act}(p,1;\mathfrak{o})$ are singletons for any $p \geqslant 0$, and $\mathsf{Act}(p,q;\mathrm{x})$ is the empty space otherwise.

An algebra over Act consists of a pair (X,P) such that X is a topological monoid and P is a left X-space. The unit e is the base point of X. The action of X on P is given by the only point of $\mathsf{Act}(1,1;\mathfrak{o})$.

The product on X is given by the only point of $\mathsf{Act}(2,0;\mathfrak{c})$, its associativity follows from the fact that each $\mathsf{Act}(n,0;\mathfrak{c})$ is a single point space and the identity element exists because of the degeneracy induced by $\mathsf{Act}(0,0;\mathfrak{c})$. The usual axioms satisfied by the action follow from the fact that $\mathsf{Act}(n,1;\mathfrak{o})$ is a singleton. For the same reason, the action determined by Act is unital in the sense that $e_X \cdot p = p, \quad \forall p \in P$.

The operad of non-unital actions will be denoted by $\overline{\mathsf{Act}}$. It coincides with Act except when $p = q = 0$, in which case it is defined as the empty space: $\overline{\mathsf{Act}}(0,0;\mathfrak{c}) = \emptyset$.

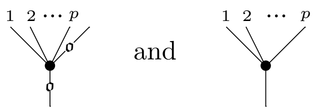
3.2. The operad Act_∞

Let us now introduce the operad Act_∞ whose algebras are precisely A_∞-actions $X \curvearrowright_\infty P$. We will give a definition of Act_∞ in terms of the Boardman and Vogt's W-construction. In the next section we will relate the operad Act_∞ to the 1-dimensional Swiss-cheese operad $\mathcal{SC}_1$.

Since in this work the A_∞-actions are strictly unital, in the following definition we use the W-construction on the operad of non-unital actions $\overline{\mathsf{Act}}$ and augment it by including the strict unit, i.e., by defining $\mathsf{Act}_\infty(0,0;\mathfrak{c})$ as the one point space K_0.

Definition 3.2. The operad Act_∞ is defined as follows: $\mathsf{Act}_\infty(p,q;\mathrm{x}) = W\left(\overline{\mathsf{Act}}\right)(p,q;\mathrm{x})$, for $p+q>0$, and $\mathsf{Act}_\infty(0,0;\mathfrak{c}) = K_0$.

Since $\mathsf{Act}(p,0;\mathfrak{c})$ and $\mathsf{Act}(p,1;\mathfrak{o})$ are spaces with a single point, the trees in Act_∞ can be viewed as planar metric trees with colored edges. The edges of color $\mathfrak{c}$ (closed) will be represented by straight line segments. When the color is open, this will be indicated by the symbol $\mathfrak{o}$ over the edge. For instance, we have the trees



belonging respectively to $\mathsf{Act}_\infty(p,1;\mathfrak{o})$ and $\mathsf{Act}_\infty(p,0;\mathfrak{c})$.

Since the open leaf of a tree in $\mathsf{Act}_\infty(p,1;\mathfrak{o})$ is the last leaf of a planar tree, and since $\mathsf{Act}_\infty(0,0;\mathfrak{c}) = K_0$ will induce the degeneracy for closed leaves, it follows from Proposition 2.6 that Act_∞-algebras are precisely A_∞-actions.

Iwase and Mimura [6] have defined A_∞-actions along a given map as follows.

Definition 3.3 *(A_∞-action after Iwase–Mimura).* Given an A_∞-space $(X,\{M_s\})$, a space P and a map $h: X \to P$, an (left) A_∞-action of X on P along h is a sequence of maps $\{N_i : K_i \times X^{i-1} \times P \to P\}_{i\geqslant 1}$, such that $N_1 = Id_P$ and:

i) $N_k(\rho \circ_j \tau; x_1,\dots,x_{k-1},p) = N_r(\rho; x_1,\dots,x_{j-1}, M_s(\tau; x_j,\dots,x_{j+s-1}),\dots,x_{k-1},p)$, for $1 \leqslant j \leqslant r-1$.
ii) $N_k(\rho \circ_r \sigma; x_1,\dots,x_{k-1},p) = N_r(\rho; x_1,\dots,x_{k-s}; N_s(\sigma; x_{k-s+1},\dots,x_{k-1},p))$.
iii) $N_k(\mu; x_1,\dots,x_{k-1},p_0) = h(M_{k-1}(s_k(\mu); x_1,\dots,x_{k-1}))$.

where $k = r+s-1$, $\rho \in K_r$, $\tau,\sigma \in K_s$, $\mu \in K_k$ and p_0 is the base point of P.

From Proposition 2.6, it follows that any family of maps as in the above definition gives rise to an A_∞-action. The converse is true up to A_∞-equivariant weak equivalence because of Theorem 2.22. Indeed, let (B,A) be a pair of topological spaces and consider the action $\Omega_M(B) \curvearrowright \Omega_M(B,A)$. By choosing a path $p_0 \in \Omega_M(B,A)$ as the base point, the map $h: \Omega_M(B) \to \Omega_M(B,A)$ can be defined as $h(\gamma) = \gamma \cdot p_0$. Since it is an action in the strict sense, it follows immediately that it is an action along h as defined above.

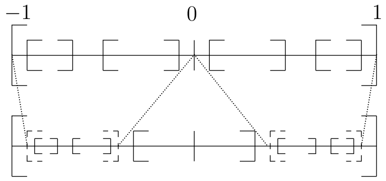


Fig. 1. Illustration of $\circ_1^{\mathfrak{c}} : \mathcal{SC}_1(1,1;\mathfrak{o}) \times \mathcal{SC}_1(2,0;\mathfrak{c}) \to \mathcal{SC}_1(2,1;\mathfrak{o})$.

Let us close this section by observing that a $W\left(\overline{\mathsf{Act}}\right)$-algebra structure on a pair (X,P) is equivalent to the existence of a family of maps as in Definition 3.3 satisfying only relations *i)* and *ii)* with the condition $s \geqslant 1$. Such structures will be called *non-unital A_∞-actions*.

4. The one-dimensional Swiss-cheese operad

In this section we will explore the relations between the operad of A_∞-actions and the one dimensional Swiss-cheese operad. Such comparison will imply in particular the existence of an A_∞-action $\Omega(B) \circlearrowright_\infty \Omega(B,A)$ in the space of usual loops and usual paths endowed with the Poincaré product of loops and juxtaposition of paths.

4.1. The operad $\mathcal{SC}_1$

The one dimensional Swiss-cheese operad is a 2-colored operad. The elements of color *open* consist of those configurations of little intervals inside the radius 1 interval centered at the origin: $[-1,1]$ that are invariant under the map $x \mapsto -x$. On the other hand, the elements color *closed* are the usual little intervals in $[0,1]$.

4.1.1. Composition law

The space of usual little intervals configurations will be denoted by $\mathcal{SC}_1(n,0;\mathfrak{c})$ while the space of open little intervals configurations will be denoted by $\mathcal{SC}_1(n,u;\mathfrak{o})$, where $u=0$ when there is no little interval in $[-1,1]$ that is centered at the origin and $u=1$ for those open symmetric configurations of little intervals in $[-1,1]$ having a little interval centered at the origin.

The composition law for $\mathcal{SC}_1(n,0;\mathfrak{c})$ is the usual composition law of the little intervals operad. The composition law for $\mathcal{SC}_1(n,u;\mathfrak{o})$ goes as follows. First the composition

$$\circ^{\mathfrak{o}} : \mathcal{SC}_1(p,1;\mathfrak{o}) \times \mathcal{SC}_1(q,u;\mathfrak{o}) \to \mathcal{SC}_1(p+q,u;\mathfrak{o}) \tag{26}$$

is given by the usual gluing operation in the little interval centered at the origin, and

$$\circ_i^{\mathfrak{c}} : \mathcal{SC}_1(k,u;\mathfrak{o}) \times \mathcal{SC}_1(n,0;\mathfrak{c}) \to \mathcal{SC}_1(k+n-1,u;\mathfrak{o}) \tag{27}$$

is given by first symmetrizing the configuration in $[0,1]$ into a symmetric configuration in $[-1,1]$ and then gluing the resulting positive and negative little intervals of the new symmetric configuration accordingly into the corresponding little intervals of the ith pair of conjugated little intervals in $\mathcal{SC}_1(k,u;\mathfrak{o})$, as illustrated by Fig. 1 where the gluing is from the bottom to the top.

Note also that one can interpret $\mathcal{SC}_1(n,1,\mathfrak{o})$ as the configuration of $n+1$ little intervals such that the last little interval contains 1. We will use this interpretation in the sequel.

4.1.2. Example

That the classical $[0,1]$-parameterized based loop space admits an A_∞-structure is a straightforward generalization of the standard proof of homotopy associativity, which involves only a homotopy defined on

$[0, 1]$. The specific pentagonal higher homotopy illustrated in [17] involves precisely *dyadic* homeomorphisms [3] of $[0, 1]$ to itself as do the higher homotopies.

The $[0, 1]$-parameterized space of relative loops on a pair $A \subseteq B$ is defined as:

$$\Omega(B, A) = \{\gamma : [0, 1] \to B : \gamma(0) = * \text{ and } \gamma(1) \in A\}.$$

To see that the pair $(\Omega B, \Omega(B, A))$ is an $\mathcal{SC}_1$-algebra, we observe that the closed part of $\mathcal{SC}$ is isomorphic to the little intervals operad and it acts on ΩB in the usual way. For the open part of $\mathcal{SC}$, the action on $(\Omega B, \Omega(B, A))$ is given by:

$$\rho : \mathcal{SC}_1(n, 1, \mathfrak{o}) \times \Omega(B)^{\times n} \times \Omega(B, A) \to \Omega(B, A)$$

where the path $\rho(d; \ell_1, \dots, \ell_n, \gamma)$ is obtained by running each loop ℓ_i through the corresponding little interval, as one usually does in loop spaces and then the path γ through the subinterval of d containing 1. The remaining points of $[0, 1]$ are taken to the base point of B. From the definition of $\mathcal{SC}_1(n, 1, \mathfrak{o})$, the configuration d will always have a little interval containing 1, hence if γ is a path that is not a loop, then $\rho(d; \ell_1, \dots, \ell_n, \gamma)$ is also a path that is not a loop, i.e., the map ρ respects the color.

4.2. Homotopy equivalence

There exists an operad morphism $\theta : W\left(\overline{\mathsf{Act}}\right) \to \mathcal{SC}_1$ so that $\theta^{\mathfrak{c}}(n, 0) : W\left(\overline{\mathsf{Act}}\right)(n, 0; \mathfrak{c}) \to \mathcal{SC}_1(n, 0; \mathfrak{c})$ and $\theta^{\mathfrak{o}}(n, 1) : W\left(\overline{\mathsf{Act}}\right)(n, 1; \mathfrak{o}) \to \mathcal{SC}_1(n, 1; \mathfrak{o})$ are homotopy equivalences for any $n \geqslant 1$.

It is possible to exhibit θ in an elementary way as follows. Each corolla in $W\left(\overline{\mathsf{Act}}\right)(n, 0; \mathfrak{c})$ is taken to a configuration of little intervals in $\mathcal{SC}_1(n, 0; \mathfrak{c})$ where each subinterval has diameter $\frac{1}{n}$, while each corolla in $\mathcal{SC}_1(n, 1; \mathfrak{o})$ is taken to a configuration of little intervals in $\mathcal{SC}_1(n, 1; \mathfrak{o})$ where each subinterval has diameter $\frac{1}{n+1}$, thus the configuration contains 1. The compatibility with the operad structure is obtained via a convex combination. To be more precise we define θ by double induction, on the number of leaves of a tree T and then on the number of internal edges. If the latter number is zero, T is a corolla and θ has been defined on corollas. One defines for any tree $T = X \circ_i^r Y$ in $W\left(\overline{\mathsf{Act}}\right)$ the configuration $\theta(T)$ as $(1 - r)\theta(X) \circ_i \theta(Y) + r\theta(X \circ_i^0 Y)$.

From the existence of the above operad homotopy equivalence, there follows the existence of a non-trivial $W\left(\overline{\mathsf{Act}}\right)$-algebra structure on the pair: $(\Omega B, \Omega(B, A))$. In other words, there is a *non-unital A_∞-action* of ΩB on $\Omega(B, A)$.

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