HOMOTOPY THEORY OF SPECTRAL SEQUENCES

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Abstract

Let R be a commutative ring with unit. We consider the homotopy theory of the category of spectral sequences of R-modules with the class of weak equivalences given by those morphisms inducing a quasi-isomorphism at a certain fixed page. We show that this admits a structure close to that of a category of fibrant objects in the sense of Brown and in particular the structure of a partial Brown category with fibrant objects. We use this to compare with related structures on the categories of multicomplexes and filtered complexes.

1. Introduction

Spectral sequences are important tools for computing homological and homotopical invariants. Many categories of interest have associated functorial spectral sequences, generally via an associated filtered chain complex.

The category of spectral sequences has a hierarchy of notions of weak equivalence. For $r \geqslant 0$, we have E_r-quasi-isomorphism, that is morphisms which are isomorphisms from the $r + 1$ page onwards. In this paper we explore underlying homotopy theories with these weak equivalences.

Various categories with associated functorial spectral sequences, such as filtered complexes or multicomplexes, can be endowed with an r-model category structure, in which the weak equivalences are the maps inducing an isomorphisms from the $r + 1$ page of the associated spectral sequence onwards [**3, 4**]. This motivates a study of the corresponding homotopy theory in the category of spectral sequences itself.

After some preliminary definitions and discussion in Section 2, we introduce the category of spectral sequences in Section 3. We study some basic properties, noting that this category is neither complete nor cocomplete. Therefore we cannot have model category structures and we will work with a weaker setting for homotopy theory.

Many such settings, intermediate between a category with weak equivalences and a model category, have appeared in the literature. Examples include Waldhausen categories [**9**], Cartan-Eilenberg categories [**6**] and categories of fibrant objects. The latter were introduced and studied by Brown in [**1**]. A summary of this theory can

be found in [**5**, I.9]. That setting is the most relevant for us, but it is not precisely what we need.

In Section 4, we introduce the notion of an *almost Brown category*. As the name suggests this is a structure closely related to Brown's notion of category of fibrant objects. Like that setting, ours involves two distinguished classes of morphisms, weak equivalences and fibrations, satisfying certain axioms. We explore the connections as well as the relationship to the notion of partial Brown category in the sense of Horel [**7**].

We show in Theorem 5.5 that, for each $r \geqslant 0$, the category of spectral sequences admits the structure of an almost Brown category with E_r-quasi-isomorphims as weak equivalences and with fibrations characterised by surjectivity conditions. In particular, this means that we have a partial Brown category with fibrant objects, in the sense of Horel [**7**]. Indeed, we have a version with functorial path objects.

These results provide a context in which we can compare the homotopy theoretic structure of the category of spectral sequences with previous results establishing such structures for filtered complexes [**3**] and for multicomplexes [**4**]. We make a start on such comparisons in Section 6.

Acknowledgments

2.   Preliminaries

In this preliminary section, we collect the main definitions that we will use. We begin with bigraded modules and r-bigraded complexes as these are underlying definitions for spectral sequences. Then we cover filtered complexes and multicomplexes, these being the main categories to be compared with spectral sequences later on.

Throughout this paper, we let R denote a commutative ring with unit.

2.1.   Bigraded complexes

In this section we let $r \geqslant 0$ be an integer.

Definition 2.1. A *bigraded R-module A* is a collection of R-modules $A = \{A^{p,q}\}$ with $p, q \in \mathbb{Z}$.

Definition 2.2. An *r-bigraded complex* is a bigraded R-module $A = \{A^{p,q}\}$ together with maps of R-modules $\delta_r \colon A^{p,q} \to A^{p-r,q+1-r}$, called differentials, such that $\delta_r^2 = 0$. A *morphism of r-bigraded complexes* is a map of bigraded modules commuting with the differentials.

We denote by $r\text{-bC}_R$ the category of r-bigraded complexes. The homology of an r-bigraded complex is a bigraded R-module and the category of r-bigraded modules has a natural class of quasi-isomorphisms, namely morphisms inducing isomorphisms on homology.

2.2. Filtered complexes

We consider unbounded complexes of R-modules endowed with increasing filtrations indexed by the integers.

Definition 2.3. A *filtered R-module* (A, F) is an R-module A together with a family of submodules of A denoted $\{F_p A\}_{p \in \mathbb{Z}}$ indexed by the integers such that for all $p \in \mathbb{Z}$, $F_{p-1}A \subseteq F_p A$. A *morphism of filtered modules* is a morphism $f \colon A \to B$ of R-modules which is compatible with filtrations: $f(F_p A) \subseteq F_p B$ for all $p \in \mathbb{Z}$.

Definition 2.4. A *filtered complex* (A, d, F) is an unbounded cochain complex (A, d) together with a filtration F of each R-module A^n such that $d(F_p A^n) \subseteq F_p A^{n+1}$ for all $p, n \in \mathbb{Z}$. Note in particular that $(F_p A, d|_{F_p})$ is a subcomplex of (A, d). Denote by $\mathbf{FC}_R$ the category of filtered complexes of R-modules. Its morphisms are given by morphisms of complexes compatible with filtrations.

Definition 2.5. Let $f, g \colon A \to B$ be two morphisms of filtered complexes. An *r-homotopy from f to g* is a morphism $h \colon A \to B$ of graded R-modules of degree -1, such that $dh + hd = g - f$ and $h(F_p A) \subseteq F_{p+r} B$ for all p. We write $h \colon f \underset{r}{\simeq} g$.

Every filtered complex A has an associated spectral sequence $\{E_r(A), \delta_r\}_{r \geqslant 0}$. The r-page $E_r(A)$ is an r-bigraded complex and may be written as the quotient

$$E_r^{p,q}(A) \cong Z_r^{p,q}(A)/B_r^{p,q}(A),$$

where the *r-cycles* are given by

$$Z_r^{p,n+p}(A) := F_p A^n \cap d^{-1}(F_{p-r}A^{n+1})$$

and the *r-boundaries* are given by $B_0^{p,n+p}(A) = Z_0^{p-1,n+p-1}(A)$ and

$$B_r^{p,n+p}(A) := Z_{r-1}^{p-1,n+p-1}(A) + dZ_{r-1}^{p+r-1,n+p+r-2}(A) \text{ for } r \geqslant 1.$$

Given an element $a \in Z_r(A)$, we denote by $[a]_r$ its image in $E_r(A)$. For $[a]_r \in E_r(A)$, we have $\delta_r([a]_r) = [da]_r$.

2.3. Multicomplexes

Definition 2.6. A *multicomplex* or *∞-multicomplex* A is a bigraded R-module $A = \{A^{p,q}\}_{p,q \in \mathbb{Z}}$ endowed with a family of maps $\{d_i \colon A \to A\}_{i \geqslant 0}$ of bidegree $(-i, 1-i)$ satisfying for all $l \geqslant 0$,

$$\sum_{i+j=l} (-1)^i d_i d_j = 0. \tag{1}$$

Let $n \geqslant 1$ be an integer. An *n-multicomplex* is a multicomplex with $d_i = 0$ for all $i \geqslant n$.

For $1 \leqslant n \leqslant \infty$, a *(strict) morphism* of n-multicomplexes is a map f of bigraded R-modules of bidegree $(0,0)$ satisfying $d_i f = f d_i$ for all $i \geqslant 0$. We denote by $n\text{-mC}_R$ the category of n-multicomplexes and strict morphisms.

3.   The category of spectral sequences

3.1.   Definitions and basic properties

Definition 3.1. A *spectral sequence* (A, ψ) is a family of r-bigraded complexes (A_r, d_r), for $r \geqslant 0$, together with a family of isomorphisms of bigraded R-modules

$$\psi_r \colon H_*(A_r) \to A_{r+1} \quad \text{for} \quad r \geqslant 0,$$

called *characteristic maps*.

A *morphism of spectral sequences* is a family of morphisms $f_r \colon A_r \to B_r$ of r-bigraded complexes, for $r \geqslant 0$, which is compatible with characteristic maps. We denote by $\mathbf{SpSe}_R$ the category of spectral sequences.

We will often omit the characteristic maps in the notation.

Note that a morphism of spectral sequences $f \colon (A, \psi) \to (B, \varphi)$ is completely determined by the 0-page, $f_0 \colon A_0 \to B_0$, since $f_{i+1} = \varphi_i H_*(f_i) \psi_i^{-1}$, for all $i \geqslant 0$. Furthermore, it is clear that the following proposition holds.

Proposition 3.2. *The category of spectral sequences is an additive category.*   □

Remark 3.3. There are various conventions for spectral sequences. We have chosen ours to be compatible with the conventions for filtered complexes and multicomplexes in previous work on related model category structures in [**3, 4**]. The differential on the r-page has bidegree $(-r, 1-r)$. Of course, it is straightforward to translate our results to the standard setting of a homological spectral sequence where the corresponding bidegree is $(-r, r-1)$ or that of a cohomological spectral sequence where it is $(r, 1-r)$.

Remark 3.4. The category of spectral sequences $\mathbf{SpSe}_R$ is neither complete nor cocomplete. Indeed, as in the following examples, cokernels and kernels do not exist in general in $\mathbf{SpSe}_R$. Thus it is not a pre-abelian category.

We write $R^{p,n}$ for the ring R in bidegree (p, n). We denote by $R(p, n)$ the spectral sequence with the ring R concentrated in bidegree (p, n) and all differentials zero.

Example 3.5. Let S be the spectral sequence given by $S_0 = R^{0,0} \oplus R^{1,0}$ with $d_0 = 0$, $S_1 = R^{0,0} \oplus R^{1,0}$ with $d_1 = 1_R \colon R^{1,0} \to R^{0,0}$ and $S_{\geqslant 2} = 0$. The morphism of spectral sequences $f \colon R(0, 0) \to S$ determined by $f_0^{0,0} = f_1^{0,0} = 1_R \colon R^{0,0} \to R^{0,0}$ has no cokernel.

Example 3.6. Let T be the spectral sequence given by $T_0 = R^{0,0} \oplus R^{0,1}$ with $d_0 = 1_R \colon R^{0,0} \to R^{0,1}$ and $T_{\geqslant 1} = 0$. The morphism of spectral sequences $\pi \colon T \to R(0, 0)$ such that $\pi_0^{0,0} = 1_R \colon R^{0,0} \to R^{0,0}$ and $\pi_i = 0$ for $i > 0$ has no kernel.

3.2.   Pullback of surjections

Definition 3.7. A morphism of spectral sequences $f \colon A \to B$ is called a *surjection* if the morphism f_r is bidegreewise surjective for every $r \geqslant 0$. We write $\mathcal{S}ur$ for the class of surjective morphisms in $\mathbf{SpSe}_R$.

Lemma 3.8. *The category of spectral sequences $\mathbf{SpSe}_R$ admits pullbacks of surjections along any map and this preserves surjections. Moreover, such pullbacks are computed pagewise.*

Proof. Let

$$
\begin{array}{c}
A \\
\downarrow{\scriptstyle p} \\
U \xrightarrow{\ g\ } B
\end{array}
$$

be a diagram of spectral sequences where p is a surjection. For $m \geqslant 0$, let X_m be the pullback in the category of m-bigraded complexes of the m-page of the spectral sequence. Since the category of m-bigraded complexes is abelian, and p_m is surjective we have a short exact sequence of m-bigraded complexes

$$
0 \longrightarrow X_m \longrightarrow U_m \oplus A_m \xrightarrow{\ g_m - p_m\ } B_m \longrightarrow 0
$$

which yields a long exact sequence in homology. The map $H_*(g_m - p_m)$ is isomorphic to the map $(g_{m+1} - p_{m+1}) \colon U_{m+1} \oplus A_{m+1} \to B_{m+1}$, hence surjective, so that $H_*(X_m)$ is isomorphic to $\mathrm{Ker}(g_{m+1} - p_{m+1}) = X_{m+1}$. Therefore the collection $X = (X_m)_{m \geqslant 0}$ is a spectral sequence.

We claim that this is the pullback of the diagram in the category of spectral sequences. For the universal property, given maps of spectral sequences $Y \to A$, $Y \to U$ making the diagram commute, we get a unique map of m-bigraded complexes $f_m \colon Y_m \to X_m$ making the diagram of m-pages commute, because X_m is the pullback on the m-page. Noting that the forgetful functor from m-bigraded complexes to bigraded modules preserves pullbacks, we see that $H(f_m) \cong f_{m+1}$ so that $f = (f_m)$ is the required unique map of spectral sequences.

Note that

$$
X_m = \{(u, a) \mid g_m(u) = p_m(a), u \in U_m, a \in A_m\},
$$

so that the induced map $X \to U$ is a surjection. $\qquad\square$

Remark 3.9. The category $\mathbf{SpSe}_R$ does not admit general pullbacks of epimorphisms, as shown by the following proposition and example.

Proposition 3.10. *A morphism $f \colon A \to B$ of spectral sequences where $f_0 \colon A_0 \to B_0$ is surjective is an epimorphism.*

Proof. Let $i, j \colon B \to X$ be morphisms of spectral sequences such that $if = jf$. In particular, we have $i_0 f_0 = j_0 f_0$ and since f_0 is surjective, we have $i_0 = j_0$. Thus $i = j$. $\qquad\square$

Example 3.11. Let us consider the morphism $\pi \colon T \to R(0,0)$ of Example 3.6. It is an epimorphism because π_0 is surjective, but the pullback of π along the map $0 \to R(0,0)$ does not exist, because π does not admit a kernel.

4. Homotopy theory without model category structures

The goal of this paper is to describe the homotopy theory of spectral sequences with respect to E_r-quasi-isomorphism (see Definition 5.1). We cannot expect to have a model category structure on the category of spectral sequences with this class of

maps as the class of weak equivalences since we have seen that this category is neither complete nor cocomplete.

Thus we will work with a weaker structure. Many variants are available in the literature; we will work with something close to what is known as a *Brown category*. In this section we introduce the homotopy theoretic material needed to achieve our goal.

4.1. Almost Brown categories

In this section we assume that the reader is familiar with the language of model categories, in particular with the notion of acyclic fibrations and fibrant objects.

Definition 4.1. An *almost Brown category* is a category $\mathcal{C}$ with finite products and a final object e together with two distinguished classes of maps called *weak equivalences* ($\mathcal{W}$) and *fibrations* ($\mathcal{F}ib$), satisfying the following axioms.

(A) Let f and g be composable morphisms. If any two of f, g and gf are weak equivalences, then so is the third. (That is, the class of weak equivalences satisfies the two-out-of-three property.) All isomorphisms are weak equivalences.

(B) The composite of two fibrations is a fibration. All isomorphisms are fibrations.

(C) The pullback of an acyclic fibration along any map exists and is an acyclic fibration.

(D) Any morphism $u \colon X \to Y$ in $\mathcal{C}$ can be factored $u = pi$ with p a fibration and i right inverse to an acyclic fibration.

(E) Any object of $\mathcal{C}$ is fibrant.

In addition, if axiom (D) holds functorially, we will say that $\mathcal{C}$ is an *almost Brown category with functorial factorization*.

Definition 4.2. A functor $F \colon \mathcal{C} \to \mathcal{D}$ between almost Brown categories is called *left exact* if it preserves finite products, the class of fibrations, the class of acyclic fibrations and pullback of acyclic fibrations.

We recall that given an object B of $\mathcal{C}$, a *path space* for B is an object B^I together with maps

$$B \xrightarrow{\;\iota\;} B^I \xrightarrow{(\partial_-, \partial_+)} B \times B$$

where ι is a weak equivalence, (∂_-, ∂_+) a fibration and the composite is the diagonal map. Note that axiom (D) of Definition 4.1 implies the existence of a path space for any object in an almost Brown category.

We next recall the original definition of Brown in [**1**].

Definition 4.3. A *Brown category* is a category with finite products and a final object e together with two distinguished classes of maps called *weak equivalences* ($\mathcal{W}$) and *fibrations* ($\mathcal{F}ib$), satisfying the following axioms.

(A) Let f and g be composable morphisms. If any two of f, g and gf are weak equivalences, then so is the third. All isomorphisms are weak equivalences.

(B) The composite of two fibrations is a fibration. All isomorphisms are fibrations.

(C') The pullback of a fibration along any map exists and is a fibration. The pullback of an acyclic fibration along any map exists and is an acyclic fibration.

(D') For any object B there exists at least one path space B^I.

(E) Every object is fibrant.

Remark 4.4. In a Brown category $\mathcal{C}$, axiom (D') is equivalent to axiom (D). This is due to the factorization lemma, which is proved by using the axiom that the pullback of a fibration along any map exists and is a fibration. Concretely, any morphism $u\colon A \to B$ factorizes as

$$A \xrightarrow{(1_A, \iota u)} A \times_B B^I \xrightarrow{\partial_+ \pi_2} B$$
$$\underbrace{\phantom{A \xrightarrow{(1_A, \iota u)} A \times_B B^I \xrightarrow{\partial_+ \pi_2} B}}_{u}$$

where the object $A \times_B B^I$ is called the mapping path space of u, the first map is a weak equivalence right inverse to an acyclic fibration and the second map is a fibration.

The following corollary is a direct consequence of the remark above.

Corollary 4.5. *A Brown category is an almost Brown category.* $\qquad \square$

Remark 4.6. If $\mathcal{C}$ is a model category, then the subcategory $\mathcal{C}^f$ of fibrant objects of $\mathcal{C}$ is a Brown category. If all objects of $\mathcal{C}$ are fibrant, then it is a Brown category, hence an almost Brown category. A right Quillen functor between two model categories whose objects are all fibrant is a left exact functor in the sense of Definition 4.2.

Unfortunately, in our examples we do not have all the axioms of a Brown category, since we usually do not have pullbacks of fibrations, however we have path objects and the factorization induced by them, that is the mapping path space of a morphism.

4.2. Comparison with partial Brown categories of fibrant objects

In [7], Horel introduced the notion of a partial Brown category of cofibrant objects and in Remark 2.4 of loc. cit. it is noted that all the results dualize to the case of interest for us. This gives a setting for homotopy theory closely related to the one we have presented above and we compare the two here.

We start by making explicit the dual to Horel's Definition 2.2. The notation $\mathcal{C}^{[1]}$ denotes the arrow category of $\mathcal{C}$.

Definition 4.7. A *partial Brown category of fibrant objects* is a category $\mathcal{C}$, with two subcategories $w\mathcal{C}$ and $f\mathcal{C}$ whose maps are called respectively the *weak equivalences* and *acyclic fibrations* such that the following axioms are satisfied.

1. Both $w\mathcal{C}$ and $f\mathcal{C}$ contain the isomorphisms of $\mathcal{C}$ and $f\mathcal{C}$ is contained in $w\mathcal{C}$.

2. The weak equivalences satisfy the two-out-of-three property.

3. The pullback of an acyclic fibration along any map exists and is an acyclic fibration.

4. There are three functors f, w, s from $w\mathcal{C}^{[1]} \to w\mathcal{C}^{[1]}$ such that for each weak equivalence g we have $g = f(g) \circ w(g)$, $s(g) \circ w(g) = 1$ and $f(g)$ and $s(g)$ are in $f\mathcal{C}^{[1]}$.

The following proposition is immediate.

Proposition 4.8. *If $\mathcal{C}$ is an almost Brown category with functorial factorization, then $\mathcal{C}$ is a partial Brown category of fibrant objects.* $\square$

Definition 4.9. Let $\mathcal{C}$ and $\mathcal{D}$ be partial Brown categories of fibrant objects. A functor $\mathcal{C} \to \mathcal{D}$ is called *left exact* if it preserves weak equivalences, acyclic fibrations and pullbacks of acyclic fibrations.

Proposition 4.10. *If $\mathcal{C}$ and $\mathcal{D}$ are almost Brown categories with functorial factorization, then a left exact functor $F \colon \mathcal{C} \to \mathcal{D}$ is also left exact as a functor of partial Brown categories.*

Proof. We need to check that F preserves weak equivalences. Let u be a weak equivalence and factorize this as $u = pi$ with p an acyclic fibration and i right inverse to an acyclic fibration. Then $F(u) = F(p)F(i)$ and since F preserves acyclic fibrations, $F(p)$ is an acyclic fibration and $F(i)$ is right inverse to an acyclic fibration. This implies that $F(i)$ is a weak equivalence and thus so is $F(u)$. $\square$

5.   Almost Brown category structures on spectral sequences

In this section we again fix an integer $r \geqslant 0$.

5.1. E_r-quasi-isomorphisms and r-fibrations

Definition 5.1. A morphism of spectral sequences $f \colon A \to B$ is called an *E_r-quasi-isomorphism* if the morphism $f_r \colon A_r \to B_r$ is a quasi-isomorphism of r-bigraded complexes, or equivalently if the morphisms f_k are isomorphisms for $k > r$.

A morphism of spectral sequences $f \colon A \to B$ is called an *r-fibration* if the morphisms f_k are surjective for $0 \leqslant k \leqslant r$.

We denote by $\mathcal{E}_r$ the class of E_r-quasi-isomorphisms of $\mathbf{SpSe}_R$. This class contains all isomorphisms of $\mathbf{SpSe}_R$ and satisfies the two-out-of-three property.

We denote by $\mathcal{F}ib_r$ the class of r-fibrations of $\mathbf{SpSe}_R$. This class contains all isomorphisms and is stable under composition.

Note that acyclic fibrations are those maps that are surjective at the k-page of the spectral sequence for $k \leqslant r$ and isomorphisms for $k > r$. In particular the class of acyclic fibrations coincides with the class of surjective E_r-quasi-isomorphisms, that is $\mathcal{E}_r \cap \mathcal{F}ib_r = \mathcal{E}_r \cap \cap_s \mathcal{F}ib_s = \mathcal{E}_r \cap \mathcal{S}ur$.

It is clear that we have inclusions:

$$\mathcal{E}_r \subset \mathcal{E}_{r+1}, \qquad \mathcal{F}ib_{r+1} \subset \mathcal{F}ib_r \qquad \text{and} \qquad \mathcal{E}_r \cap \mathcal{F}ib_r \subset \mathcal{E}_{r+1} \cap \mathcal{F}ib_{r+1}$$

for all $r \geqslant 0$.

5.2.   Mapping path space construction for spectral sequences

In this section we define a functorial r-path and an explicit r-mapping path space for any morphism in the category of spectral sequences.

Definition 5.2. Let Λ_r be the spectral sequence $Re_- \oplus Re_+ \oplus Ru$ where $e_{\pm}$ are in bidegree $(0,0)$ and u is in bidegree $(-r, 1-r)$, with all differentials zero except at the r-page of the spectral sequence where $d_r(e_-) = -u$, $d_r(e_+) = u$. The $r+1$-page of the spectral sequence is then concentrated in bidegree $(0,0)$ with a single R-module, free of rank 1, generated by $e_+ + e_-$.

Note that we can consider $\Lambda_r \otimes A$ for any spectral sequence A and that this is again a spectral sequence.

We next define a collection of functorial paths indexed by an integer $r \geqslant 0$ on the category of spectral sequences, giving rise to the corresponding notions of r-homotopy.

Definition 5.3. The r-*path of a spectral sequence* A is the spectral sequence $P(r; A) = \Lambda_r \otimes A$. Explicitly, the pages of the spectral sequence $P(r; A)$ are given by

$$P_m(r; A)^{p,q} := \begin{cases} A_m^{p,q} \oplus A_m^{p+r,q+r-1} \oplus A_m^{p,q}, & \text{if } 0 \leqslant m \leqslant r \\ A_m^{p,q}, & \text{if } m > r \end{cases}$$

with the differentials $D_m \colon P_m(r; A) \to P_m(r; A)$ of bidegree $(-m, 1-m)$ given by

$$D_m := \begin{pmatrix} d_m & 0 & 0 \\ 0 & (-1)^{m+r+1} d_m & 0 \\ 0 & 0 & d_m \end{pmatrix} \text{ for } m < r, \; D_r := \begin{pmatrix} d_r & 0 & 0 \\ -1 & -d_r & 1 \\ 0 & 0 & d_r \end{pmatrix}$$

and $D_m = d_m$ for $m > r$.

We have a factorisation of the diagonal map

$$R \xrightarrow{\;\iota\;} \Lambda_r \xrightarrow{(\partial_-, \partial_+)} R \times R$$

and thus morphisms of spectral sequences

$$A \xrightarrow{\;\iota_A\;} P(r; A) \overset{\partial_A^+}{\underset{\partial_A^-}{\rightrightarrows}} A \;\; ; \;\; \partial_A^\pm \circ \iota_A = 1_A,$$

given by $\partial_A^-(x, y, z) = x$, $\partial_A^+(x, y, z) = z$ and $\iota_A(x) = (x, 0, x)$ on the m-page of the spectral sequence for $m \leqslant r$ and by the identity maps for $m > r$. We will often omit the subscripts of these maps when there is no danger of confusion.

The use of the term r-path is justified below. In particular, ι_A is an E_r-quasi-isomorphism. Furthermore $(\partial_A^+, \partial_A^-) \colon P(r; A) \to A \times A$ is an r-fibration. In addition ∂_A^+ and ∂_A^- are acyclic r-fibrations.

Definition 5.4. The r-*path of a morphism* $f \colon (A, d_m^A) \to (B, d_m^B)$ of spectral sequences is the morphism of spectral sequences $P(r; f) \colon (P(r; A), D_m^A) \to (P(r; B), D_m^B)$ given by

$$P(r; f)_m := (f_m, (-1)^m f_m, f_m),$$

for $m \leqslant r$ and $P(r; f) = f_m$ for $m > r$.

The above definitions give rise to a functorial path $P(r; -) \colon \mathbf{SpSe}_R \to \mathbf{SpSe}_R$ in the category of spectral sequences.

We would like to use this for the factorization of any morphism $u \colon A \to B$ of spectral sequences in the spirit of Remark 4.4. We remark that the r-mapping path

space $\overline{P}(r;u) := A \times_B P(r;B)$ of u exists and takes the following form

$$\overline{P}(r;u)_m^{p,q} := \begin{cases} A_m^{p,q} \oplus B_m^{p+r,q+r-1} \oplus B_m^{p,q}, & \text{if } 0 \leqslant m \leqslant r, \\ A_m^{p,q}, & \text{if } m > r, \end{cases}$$

with differentials $D_m \colon \overline{P}(r;u)_m \to \overline{P}(r;u)_m$ of bidegree $(-m, -m+1)$ given by

$$D_m = \begin{pmatrix} d_m^A & 0 & 0 \\ 0 & (-1)^{m+r+1}d_m^B & 0 \\ 0 & 0 & d_m^B \end{pmatrix} \text{ for } m < r, \quad D_r = \begin{pmatrix} d_r^A & 0 & 0 \\ -u_r & -d_r^B & 1_B \\ 0 & 0 & d_r^B \end{pmatrix}$$

and D_m is induced by d_m^A for $m > r$. The factorization of u as

$$A \xrightarrow{\ i\ } \overline{P}(r;u) \xrightarrow{\ p\ } B$$

takes the following form

$$i_m(a) = \begin{cases} (a, 0, u_m(a)), & \text{if } 0 \leqslant m \leqslant r, \\ a, & \text{if } m > r \end{cases} \quad \text{and}$$

$$p_m(a, b', b) = \begin{cases} b, & \text{if } 0 \leqslant m \leqslant r, \\ u_m(a), & \text{if } m > r. \end{cases}$$

The map i is right inverse to an acyclic fibration, namely the projection of $\overline{P}(r;u)$ onto A. It is clear from the formulas that p is an r-fibration and that the factorization is functorial.

5.3. Homotopy theory of spectral sequences

Theorem 5.5. *The category of spectral sequences together with the class $\mathcal{E}_r$ of E_r-quasi-isomorphisms and the class $\mathcal{F}ib_r$ of r-fibrations is an almost Brown category with functorial factorization. Hence it is a partial Brown category of fibrant objects.*

Proof. Axioms $(A), (B)$ and (E) are clearly satisfied. Let us show axiom (C). Let $p \colon C \to B$ be an acyclic r-fibration and let $u \colon A \to B$ be a morphism in $\mathbf{SpSe}_R$. Any acyclic r-fibration is surjective, and by Lemma 3.8, the pullback of p along u exists and is the spectral sequence X whose m-page is described by

$$X_m = \{(a, c) \in A_m \times C_m \mid u_m(a) = p_m(c)\}$$

with induced map $\pi_1 \colon X \to A$ the projection to the first factor. The proof of that lemma shows that if p_r is a quasi-isomorphism so is the r-page $(\pi_1)_r$ of π_1. Axiom (D) follows from the mapping path space construction of Section 5.2. $\quad\square$

Notation 5.6. We write $(\mathbf{SpSe}_R)_r$ for the almost Brown category of spectral sequences with the structure specified in Theorem 5.5.

Recall that we have inclusions:

$$\mathcal{E}_r \subset \mathcal{E}_{r+1}, \qquad \mathcal{F}ib_{r+1} \subset \mathcal{F}ib_r \qquad \text{and} \qquad \mathcal{E}_r \cap \mathcal{F}ib_r \subset \mathcal{E}_{r+1} \cap \mathcal{F}ib_{r+1}$$

for all $r \geqslant 0$. Thus, for $r \leqslant s$, the identity functor $\mathrm{Id} \colon (\mathbf{SpSe}_R)_r \to (\mathbf{SpSe}_R)_s$ preserves weak equivalences and acyclic fibrations, but not fibrations. Therefore it is not left exact as a functor of almost Brown categories, but it is left exact when viewed as a functor of the corresponding partial Brown categories.

5.4. r-homotopies

As in classical homotopy theory, the functorial path $P(r; -) \colon \mathbf{SpSe}_R \to \mathbf{SpSe}_R$ yields a natural notion of homotopy: for $f, g \colon A \to B$ two morphisms of spectral sequences, an *r-homotopy from f to g* is given by a morphism of spectral sequences $h \colon A \to P(r; B)$ such that $\partial_B^- \circ h = f$ and $\partial_B^+ \circ h = g$. We use the notation $h \colon f \underset{r}{\cong} g$. An *$r$-homotopy equivalence* is a morphism of spectral sequences $f \colon A \to B$ such that there exists a morphism $g \colon B \to A$ satisfying $f \circ g \underset{r}{\cong} 1_B$ and $g \circ f \underset{r}{\cong} 1_A$.

Proposition 5.7. *The notion of r-homotopy defines an equivalence relation on the set of morphisms between two given spectral sequences, which is compatible with the composition.*

Proof. Unravelling the definition we have that if $f, g \colon (A, d_m^A) \to (B, d_m^B)$ are two morphisms of spectral sequences, then $f \underset{r}{\cong} g$ if and only if there exists a collection of morphisms $\widehat{h}_m \colon A_m \to B_m$ of bigraded modules, of bidegree $(r, r-1)$, for every $0 \leqslant m \leqslant r$, satisfying

$$\begin{cases} (-1)^{m+r+1} d_m^B \widehat{h}_m - \widehat{h}_m d_m^A = 0, & \text{if } 0 \leqslant m < r, \\ -d_r^B \widehat{h}_r - \widehat{h}_r d_r^A = f_r - g_r, \end{cases}$$

and $H_*(\widehat{h}_m) = \widehat{h}_{m+1}$ for $0 \leqslant m < r$. The proposition then follows. $\qquad\square$

Denote by $\mathcal{S}_r$ the class of r-homotopy equivalences of $\mathbf{SpSe}_R$. This class is closed under composition and contains all isomorphisms. In addition, we have $\mathcal{S}_r \subseteq \mathcal{S}_{r+1}$ and $\mathcal{S}_r \subseteq \mathcal{E}_r$, for all $r \geqslant 0$.

Proposition 5.8. *The localized category $\mathbf{SpSe}_R[\mathcal{S}_r^{-1}]$ is canonically isomorphic to the quotient category $\mathbf{SpSe}_R / \underset{r}{\cong}$.*

Proof. The proof is classical, and requires that the map from a spectral sequence A to its path space $\iota_A \colon (A, d_m) \longrightarrow (P(r; A), D_m)$, is not only an E_r-quasi-isomorphism but also an r-homotopy equivalence. We prove this statement. Recall that $(\iota_A)_m(x) = (x, 0, x)$ for $m \leqslant r$ and $(\iota_A)_m = 1_A$ for $m > r$. Since $\partial_A^- \iota_A = 1_A$, it suffices to define an r-homotopy from $1_{P(r;A)}$ to $\iota_A \partial_A^-$. Consider the morphism $\widehat{h}_m \colon P(r; A) \to P(r; A)$ of bidegree $(r, r-1)$ defined by $\widehat{h}_m(x, y, z) = (0, 0, -y)$ for $0 \leqslant m \leqslant r$. It is clear that for $m < r$ we have $(-1)^{m+r-1} D_m \widehat{h}_m - \widehat{h}_m D_m = 0$, $H_*(\widehat{h}_m) = \widehat{h}_{m+1}$ and

$$(-D_r \widehat{h}_r - \widehat{h}_r D_r)(x, y, z) = (0, y, d_r y) + (0, 0, -d_r y - x + z) = (x, y, z) - (x, 0, x).$$

As in the proof of Proposition 5.7 this implies that ι_A is an r-homotopy equivalence. $\qquad\square$

In section 6, we will compare this notion of homotopy for spectral sequences with notions for filtered complexes and multicomplexes.

5.5. Generation of r-fibrations and acyclic r-fibrations

This section is devoted to the description of r-fibrations and acyclic r-fibrations as maps having the right lifting property with respect to a set of morphisms in $\mathbf{SpSe}_R$. We adopt the language of model categories. For I a class of maps in $\mathbf{SpSe}_R$, we

say that a morphism of spectral sequences f is I-injective if it has the right lifting property with respect to I.

To describe the generating sets, we first introduce some basic objects.

Definition 5.9. Let $p, n \in \mathbb{Z}$. For all $r \geqslant 0$, let $\mathcal{D}_r(p, n)$ be the spectral sequence defined as follows:

$$\begin{cases} \mathcal{D}_r(p, n)_i = R^{p,n} \oplus R^{p-r,n+1-r}, & d_i = 0, \text{ for } 0 \leqslant i < r, \\ \mathcal{D}_r(p, n)_r = R^{p,n} \xrightarrow{1} R^{p-r,n+1-r}, \\ \mathcal{D}_r(p, n)_i = 0, & \text{for } i > r. \end{cases}$$

For all $r \geqslant 1$ define

$$\mathcal{S}_r(p, n) := \mathcal{D}_{r-1}(p - 1, n - 1) \oplus \mathcal{D}_{r-1}(p + r - 1, n + r - 2).$$

For all $r \geqslant 1$ define a morphism of spectral sequences

$$\varphi_r \colon \mathcal{D}_r(p, n) \longrightarrow \mathcal{S}_r(p, n)$$

via the identity on R whenever it is bigradedly defined.

Definition 5.10. Let (A, φ) be a spectral sequence.

1. A sequence of elements $(a_0^{p,n}, \ldots, a_{m+1}^{p,n})$ with $a_i^{p,n} \in A_i^{p,n}$ is said to be *compatible* if for every $0 \leqslant i \leqslant m$, $d_i a_i^{p,n} = 0$ and $a_{i+1}^{p,n} = \varphi_i([a_i^{p,n}])$ where $[a_i^{p,n}]$ is the class of $a_i^{p,n}$ in $H_*(A_i)$.

2. Denote by $D_r^{p,n}(A)$ the R-submodule of $A^{\times(2r+2)}$ consisting of pairs

$$(a_0^{p,n}, \ldots, a_r^{p,n}); (b_0^{p-r,n+1-r}, \ldots, b_r^{p-r,n+1-r})$$

of compatible sequences satisfying $d_r a_r^{p,n} = b_r^{p-r,n+1-r}$. This yields a functor, denoted by $D_r^{p,n} \colon \mathbf{SpSe}_R \to \mathrm{Mod}_R$.

The following proposition is a direct consequence of the definitions.

Proposition 5.11. *Let (A, φ) be a spectral sequence.*

1. *There is a one-to-one correspondence between infinite compatible sequences*

$$(a_0^{p,n}, \ldots, a_m^{p,n}, \ldots)$$

and morphisms of spectral sequences $R(p, n) \to A$.

2. *We have*

$$D_r^{p,n} = \mathrm{Hom}_{\mathbf{SpSe}_R}(\mathcal{D}_r(p, n), -),$$

that is, $D_r^{p,n}$ is represented by $\mathcal{D}_r(p, n)$. $\qquad\square$

Definition 5.12. Let I_r and J_r be the sets of morphisms of $\mathbf{SpSe}_R$ given by

$$I_r := \{\varphi_{r+1} \colon \mathcal{D}_{r+1}(p, n) \longrightarrow \mathcal{S}_{r+1}(p, n)\}_{p,n \in \mathbb{Z}} \text{ and } J_r := \{0 \longrightarrow \mathcal{D}_r(p, n)\}_{p,n \in \mathbb{Z}}.$$

Let I_r' and J_r' be the sets of morphisms of $\mathbf{SpSe}_R$ given by

$$I_r' := \cup_{k=0}^{r-1} J_k \cup I_r \text{ and } J_r' := \cup_{k=0}^{r} J_k.$$

Proposition 5.13. *A morphism of spectral sequences is an r-fibration if and only if it has the right lifting property with respect to J_r'.*

Proof. Let $f\colon (A,\varphi) \to (B,\psi)$ be a morphism of spectral sequences. It is clear that if f is J_r-injective then f_r is bidegreewise surjective. It is also clear that f is J_0-injective if and only if f_0 is bidegreewise surjective. Assume that for every $0 \leqslant i \leqslant r$ we have that f_i is bidegreewise surjective. Let $(b_0, \ldots, b_r; b'_0, \ldots, b'_r)$ in $D_r^{p,n}(B)$. Since f_r is surjective, there exists $a_r \in A_r$ such that $f_r(a_r) = b_r$ hence $f_r(d_r a_r) = b'_r$. We set $a'_r = d_r a_r$. Pick u_{r-1} a cycle in A_{r-1} such that $\varphi([u_{r-1}]) = a_r$. Hence

$$\psi([f_{r-1}(u_{r-1})]) = f_r(a_r) = b_r = \psi[b_{r-1}]$$

and there exists $y \in B_{r-1}$ such that $f_{r-1}(u_{r-1}) = b_{r-1} + d_{r-1}y$. And $y = f_{r-1}(x)$ for some x since f_{r-1} is surjective. Hence $b_{r-1} = f_{r-1}(u_{r-1} - d_{r-1}x)$ and $a_{r-1} = u_{r-1} + d_{r-1}(x)$ satisfies the required conditions. By induction, we obtain that there exists $(a_0, \ldots, a_r; a'_0, \ldots, a'_r)$ in $D_r^{p,n}(A)$ such that for all i we have $f_i(a_i) = b_i$ and $f_i(a'_i) = b'_i$, giving the required lift. $\square$

Note that this proposition can be stated as $f\colon A \to B$ is an r-fibration if and only if for every $0 \leqslant k \leqslant r$ and for every $p, n \in \mathbb{Z}$, $D_k^{p,n}(f)$ is surjective.

Proposition 5.14. *A morphism of spectral sequences is an acyclic r-fibration if and only if it has the right lifting property with respect to I'_r.*

Proof. Assume first that $f\colon A \to B$ is I'_r-injective. Then f_k is bidegreewise surjective for every $0 \leqslant k \leqslant r-1$. Let us show that f_r is surjective on cycles. Let $b_r \in B_r^{p+r,n+r-1}$ be such that $d_r(b_r) = 0$. One can then build a sequence

$$(b_0, \ldots, b_{r-1}, b_r; 0, \ldots, 0)$$

in $D_r^{p+r,n+r-1}(B)$ which yields a commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}_{r+1}(p,n) & \xrightarrow{\;0\;} & A \\
{\scriptstyle \varphi_{r+1}}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{S}_{r+1}(p,n) & \xrightarrow[\;b_r\;]{} & B
\end{array}
$$

This diagram admits a lift, giving an element $(a_0, \ldots, a_r; 0, \ldots, 0)$ in $D_r^{p+r,n+r-1}(A)$ satisfying $d_r a_r = 0$ and $f_i(a_i) = b_i$ for $0 \leqslant i \leqslant r$. This proves that f_r is surjective on cycles, and thus that f_{r+1} is surjective.

Let us show that f_r is surjective. Let $b_r \in B_r^{p+r,n+r-1}$. One can choose compatible sequences: $(b_0, \ldots, b_r; b'_0, \ldots, b'_r)$ in $D_r^{p+r,n+r-1}(B)$ and from the first part a lift $a' = (a'_0, \ldots, a'_r)$ of $(b'_0, \ldots, b'_r)$ since $d_r b'_r = d_r d_r b_r = 0$, which yields again a commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}_{r+1}(p,n) & \xrightarrow{\;a'\;} & A \\
{\scriptstyle \varphi_{r+1}}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{S}_{r+1}(p,n) & \xrightarrow[\;b_r\;]{} & B
\end{array}
$$

admitting a lift. This gives an element $a_r \in A_r$ such that $d_r a_r = a'_r$ and $f_r(a_r) = b_r$. As a consequence f is J'_r-injective, and thus an r-fibration.

We have proved that f_{r+1} is surjective. Let us show that f_{r+1} is injective. Let $a_{r+1} \in A_{r+1}$ be such that $f_{r+1}(a_{r+1}) = 0$ and $(a_0, \ldots, a_{r+1}; a'_0, \ldots, a'_{r+1}) \in D_{r+1}(A)$ which represents a_{r+1}. Since $f_{r+1}(a_{r+1}) = 0$, we have $f_{r+1}(a'_{r+1}) = 0$ and there exist $b_r, c_r \in B_r$ such that $d_r(b_r) = f_r(a_r)$ and $d_r(c_r) = f_r(a'_r)$. This induces the following diagram denoted $(*)$

$$
\begin{array}{ccc}
\mathcal{D}_{r+1}(p, n) & \xrightarrow{a_{r+1}} & A \\
{\scriptstyle \varphi_{r+1}} \downarrow & & \downarrow {\scriptstyle f} \\
\mathcal{S}_{r+1}(p, n) & \xrightarrow[(c_r, b_r)]{} & B
\end{array}
$$

which admits a lift. In particular, there exists α_r with $d_r \alpha_r = a_r$. Hence $[a_r] = [0]$, that is $a_{r+1} = 0$.

Conversely, assume f is an acyclic r-fibration. Consider the diagram $(*)$. Since $f_{r+1}(a_{r+1}) = 0$ and f_{r+1} is an isomorphism we deduce that $a_{r+1} = 0$ so that a_r is a boundary, as well as a'_r. We conclude that a lift exists using the fact that $D_r(f)$ is surjective. □

6.   Comparisons with filtered complexes and multicomplexes

In this section we compare $(\mathbf{SpSe}_R)_r$ with corresponding structures on filtered complexes and multicomplexes. In previous work [3] we have established model category structures on these categories where the weak equivalences are the E_r-quasi-isomorphisms. Thus we are able to compare the underlying almost Brown category structures with $(\mathbf{SpSe}_R)_r$.

6.1.   Filtered complexes

Let $\mathbf{FC}_R$ be the category of filtered complexes and let $(\mathbf{FC}_R)_r$ denote this category with the r-model structure of [3, Theorem 3.16]. We consider it as an almost Brown category where the weak equivalences are the E_r-quasi-isomorphisms and the fibrations are the maps such that $Z_0(f)$ is surjective and $E_i(f)$ is surjective for $0 \leqslant i \leqslant r$.

Proposition 6.1. *The spectral sequence functor $E \colon (\mathbf{FC}_R)_r \to (\mathbf{SpSe}_R)_r$ preserves weak equivalences and is a left exact functor of almost Brown categories.*

Proof. It is clear that E preserves finite products, weak equivalences, fibrations and acyclic fibrations. For the pullback condition, consider the diagram:

$$
\begin{array}{ccc}
 & & A \\
 & & \downarrow {\scriptstyle p} \\
U & \xrightarrow{g} & B
\end{array}
$$

in $(\mathbf{FC}_R)_r$ where p is an acyclic fibration. The pullback in $\mathbf{FC}_R$ is

$$X = \mathrm{Ker}(p - g \colon U \oplus A \to B) = \{(u, a) \mid g(u) = p(a), u \in U, a \in A\},$$

with $d(u, a) = (du, da)$, since $\mathbf{FC}_R$ is a pre-abelian category. Using the surjectivity of $E_i(p)$ for all i, the same proof as in Lemma 3.8 shows that the associated spectral

sequence has

$$E(X)_i = \{(u, a) \mid E(g)(u) = E(p)(a), u \in E(U)_i, a \in E(A)_i\}.$$

That is, it has the pagewise pullback of i-bigraded complexes as its i-page and, by Lemma 3.8, this is the pullback in $\mathbf{SpSe}_R$. $\qquad\square$

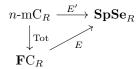Recall the notion of r-homotopy between morphisms of filtered complexes from Definition 2.5.

Proposition 6.2. *The spectral sequence functor* $E\colon (\mathbf{FC}_R)_r \to (\mathbf{SpSe}_R)_r$ *preserves* r*-homotopy.*

Proof. The notion of r-homotopy between morphisms $f, g\colon A \to B$ of filtered complexes can be formulated in terms of a version Λ_r^{FC} of Λ_r in filtered complexes. Let $\Lambda_r^{FC} = Re_- \oplus Re_+ \oplus Ru$ where e_-, e_+ are in degree 0 and filtration 0 and u is in degree 1 and filtration $-r$. The differential is determined by $d(e_-) = -u, d(e_+) = u$. And we have morphisms $\partial^-, \partial^+\colon \Lambda_r^{FC} \to R$ given by projection to Re_- and Re_+ respectively. Then giving an r-homotopy from f to g is equivalent to giving a morphism of filtered complexes $h\colon A \to \Lambda_r^{FC} \otimes B$ such that $\partial_B^- \circ h = f$ and $\partial_B^+ \circ h = g$.

The associated spectral sequence $E(\Lambda_r^{FC})$ is Λ_r as in Definition 5.2 and more generally $E(\Lambda_r^{FC} \otimes A) \cong \Lambda_r \otimes E(A)$. Thus an r-homotopy h between f and g gives rise to an r-homotopy $E(h)$ between $E(f)$ and $E(g)$. $\qquad\square$

6.2. Multicomplexes

Recall that we write $n\text{-mC}_R$ for the category of n-multicomplexes and strict morphisms. Here $2 \leqslant n \leqslant \infty$, where the case $n = \infty$ is the category of multicomplexes. An n-multicomplex has an associated functorial spectral sequence, described explicitly in [**8**]. Indeed there is a totalization functor to filtered complexes and then we take the associated spectral sequence. That is, we have a commutative diagram:

$$
\begin{array}{ccc}
n\text{-mC}_R & \xrightarrow{\;E'\;} & \mathbf{SpSe}_R \\
{\scriptstyle \text{Tot}}\big\downarrow & \nearrow_{E} & \\
\mathbf{FC}_R & &
\end{array}
$$

Note that we write $E' = E \circ \text{Tot}$ for the composite functor, but we will often drop the dash and just write E_i for the pages of the spectral sequence associated to a multicomplex.

We write $(n\text{-mC}_R)_r$ for the category of n-multicomplexes and strict morphisms with the r-model structure of [**4**, Theorem 3.30]. We use the same notation for the corresponding almost Brown category where the weak equivalences are the E_r-quasi-isomorphisms and the fibrations are the maps f such that $E_i(f)$ is surjective for $0 \leqslant i \leqslant r$.

Proposition 6.3. *The spectral sequence functor* $E'\colon (n\text{-mC}_R)_r \to (\mathbf{SpSe}_R)_r$ *preserves weak equivalences and is a left exact functor of almost Brown categories.*

Proof. It is clear that E' preserves finite products, weak equivalences and fibrations. For the pullback condition, consider the diagram:

$$
\begin{array}{c}
A \\
\downarrow p \\
U \xrightarrow{\ g\ } B
\end{array}
$$

in $n\text{-mC}_R$. The pullback in $n\text{-mC}_R$ exists and it is

$$X = \mathrm{Ker}(p - g\colon U \oplus A \to B) = \{(u,a) \mid g(u) = p(a), u \in U, a \in A\},$$

with $d_i(u,a) = (d_i u, d_i a)$ for all $i \geqslant 0$. Indeed, the category $n\text{-mC}_R$ has a description as a module category given in [**4**, Proposition 4.4] and so it is abelian.

Let Y_i denote the pullback in i-bigraded complexes of

$$
\begin{array}{c}
E_i(A) \\
\downarrow E_i(p) \\
E_i(U) \xrightarrow[E_i(g)]{} E_i(B)
\end{array}
$$

and note that $E_0(X) \cong Y_0$ as 0-bigraded complexes.

Now suppose that p is an acyclic fibration, in particular $E_i(p)$ is surjective for all i, and assume that $E_n(X) \cong Y_n$ as n-bigraded complexes.

As in Lemma 3.8, we have $Y_{n+1} \cong \ker(E_{n+1}(p) - E_{n+1}(g))$ and the argument of that proof also shows that we have an isomorphism of underlying bigraded R-modules $E_{n+1}(X) \cong Y_{n+1}$. It remains to check that this can be upgraded to an isomorphism of $(n+1)$-bigraded complexes and this can be seen from the explicit description of the differentials in the spectral sequence of a multicomplex in [**8**].

Then $E'(X)$ has the pagewise pullback of i-bigraded complexes as its i-page and, by Lemma 3.8, this is the pullback in $\mathbf{SpSe}_R$. $\qquad\square$

Remark 6.4. Note that the proof shows that E' preserves pullbacks along any map of any map p such that $E_i(p)$ is surjective for all i.

Remark 6.5. Note that in this multicomplex case E' also reflects the weak equivalences and fibrations.

Proposition 6.6. *For $n = \infty$, the spectral sequence functor*

$$E'\colon (n\text{-mC}_R)_r \to (\mathbf{SpSe}_R)_r$$

preserves the r-path.

Proof. The r-path for multicomplexes was defined in [**2**, Definition 3.14]. From the explicit description of the spectral sequence of a multicomplex, it is straightforward to see that the spectral sequence corresponding to the multicomplex $P_r(A)$ is $P(r; E(A))$. We have $E(\iota_A) = \iota_{E(A)}$, $E(\delta_B^-) = \delta_{E(B)}^-$ and $E(\delta_B^+) = \delta_{E(B)}^+$. $\qquad\square$

Proposition 6.7. *The spectral sequence functor*

$$E'\colon (n\text{-mC}_R)_r \to (\mathbf{SpSe}_R)_r$$

preserves r-homotopy.

Proof. We start with the case $n = \infty$. Here an r-homotopy between morphisms of multicomplexes $f, g \colon C \to D$ is defined in [**2**, Definition 3.16] as an ∞-morphism of multicomplexes $h \colon C \to P_r(D)$ such that $\partial_D^- \circ h = f$ and $\partial_D^+ \circ h = g$.

We write tC_R for the category of multicomplexes with ∞-morphisms. By [**2**, Theorem 3.8], we have a totalisation functor $\mathrm{Tot} \colon \mathrm{tC}_R \to \mathbf{F}\mathrm{C}_R$. We can refine the commutative diagram given earlier to

$$n\text{-mC}_R \xrightarrow{\ i_n\ } \infty\text{-mC}_R \xrightarrow{\ i\ } \mathrm{tC}_R \xrightarrow{\ \tilde{E}\ } \mathbf{SpSe}_R$$

with $\mathrm{Tot} \downarrow$ and E into $\mathbf{F}\mathrm{C}_R$

where i_n and i are inclusions of subcategories and the Tot discussed earlier can be obtained as the composite of the inclusions and the Tot on tC_R.

Thus, using Proposition 6.6, we have a morphism of spectral sequences

$$E(h) \colon E(C) \to E(P_r(D)) = P(r; E(D)).$$

Since $E(\partial_D^-) = \partial_{E(D)}^-$ and $E(\partial_D^+) = \partial_{E(D)}^+$, it follows from Section 5.4 that $E(h)$ is an r-homotopy between $E(f)$ and $E(g)$.

For $n < \infty$, an r-path object for n-multicomplexes was given in [**4**, Definition 5.5], giving rise to a notion of r-homotopy. Let us write P_r^n for the r-path in n-multicomplexes, in order to distinguish it from P_r, the r-path in multicomplexes. These r-paths can be expressed in the form $P_r^n(C) = \Lambda_r^n \otimes C$ and $P_r(C) = \Lambda_r \otimes C$. The two can be compared in the category of multicomplexes, since there is a natural transformation $P_r^n \to P_r$ such that $P_r^n(C) = \Lambda_r^n \otimes C \to P_r(C) = \Lambda_r \otimes C$ is $\alpha \otimes 1_C$, where α is the identity in bidegrees where this is possible and zero otherwise.

Let $h \colon C \to P_r(D)$ be an r-homotopy from f to g in n-mC$_R$. Then $(\alpha \otimes 1_D) \circ h$ gives an r-homotopy from f to g in multicomplexes. In other words, the inclusion of n-multicomplexes into multicomplexes preserves homotopy. $\square$

Remark 6.8. The inclusion i_n of n-multicomplexes into multicomplexes also reflects homotopy. Indeed $i(P_r^n)$ gives another functorial path for multicomplexes and so gives rise to an equivalent notion of homotopy.

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