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Rational homotopy of Leibniz algebras

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Abstract. We construct a non-commutative rational homotopy theory by replacing the pair (Lie algebras, commutative algebras) by the pair (Leibniz algebras, Leibniz-dual algebras). Both pairs are Koszul dual in the sense of operads (Ginzburg–Kapranov). We prove the existence of minimal models and the Hurewicz theorem in this framework. We define Leibniz spheres and prove that their homotopy is periodic.

Introduction

In rational homotopy theory, Sullivan models ([Su]) deal with differential graded commutative algebras, whereas Quillen models ([Qu1]) deal with differential graded Lie algebras. The interest of these models lies in the fact that they both contain all rational homotopy and homology information of the simply connected space. The aim of this paper is to develop a similar theory in the non-commutative case. For that, we decide to replace Lie algebras by a non-commutative version, which are the *Leibniz algebras*. More precisely, a Leibniz algebra L is a vector space equipped with a bracket satisfying the identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \ \forall x,y,z \in L \text{ (see [Lo1]).}$$

If the antisymmetric relation is assumed, this identity is equivalent to the Jacobi relation. Hence, an antisymmetric Leibniz algebra is a Lie algebra. This raises the question of what will replace commutative algebras in order to construct a Sullivan-type model. From the work of Ginzburg and Kapranov ([G-K]), we know that Lie algebras and commutative algebras are algebras over Koszul operads which are dual to each other. This suggests to replace commutative algebras by *Leibniz-dual algebras* which are algebras over the dual operad defining the Leibniz algebras. More explicitly, a Leibniz-dual algebra M is a vector space together with a product satisfying the identity

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y), \ \forall x,y,z \in M.$$

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The main goal of this paper is to show that Leibniz algebras and Leibniz-dual algebras are suitable for a non-commutative rational homotopy theory. We define the homotopy and the homology of a differential graded Leibniz algebra and we prove that a construction of minimal models is valid in this framework. Moreover, these minimal models contain all the homotopy and homology information of the Leibniz algebra.

Our second goal is to see whether the classical theorems and constructions hold. We prove a Leibniz version of the Hurewicz theorem: if a differential graded Leibniz algebra is n-connected, then the Leibniz Hurewicz morphism is an isomorphism for $k \leq 2n$ (see theorem 4.3). We deduce immediately a Leibniz version of the Freudenthal suspension theorem (see theorem 4.6). We construct n-Leibniz spheres, which are differential graded Leibniz algebras whose cohomology is trivial except in degree n. We prove the uniqueness of such an object. Moreover, we compute its homotopy which turns out to be periodic of period $n-1$ (see theorem 5.3). We compare Leibniz spheres to classical ones and we obtain that the Lie algebra associated to the n-Leibniz sphere is exactly the Quillen model of the classical n-sphere (theorem 5.4). Finally, as the Quillen model of the classical sphere is a Lie algebra, it is a Leibniz algebra. We prove the periodicity of its Leibniz homology, in theorem 5.5.

Contents. All definitions and properties of Leibniz algebras and Leibniz-dual algebras used in this article are recalled in the first section. The second section is devoted to minimal models. In the third section, we define a pair of adjoint functors between the category of Leibniz algebras and the category of reduced Leibniz-dual coalgebras which allow us to define the homotopy and the homology of a Leibniz algebra. We prove two theorems linking minimal models and the homotopy and homology of a Leibniz algebra (see theorems 3.10 and 3.11). The Leibniz version of the Hurewicz theorem and the Freudenthal suspension theorem are the subject of the fourth section. The fifth section focus on spheres. We define Leibniz spheres and compare them to classical spheres. Finally, we compute the Leibniz homology of the classical sphere.

Notation. We work over a fixed field K of characteristic 0. Let V be a graded vector space. The suspension of V is $(sV)_n = V_{n-1}$ if V is lower graded, and $(sV)^n = V^{n+1}$ if V is upper graded. The graded vector space V is said to be *reduced* (resp. 2-*reduced*) if $V_0 = 0$ (resp. $V_0 = V_1 = 0$) or $V^0 = 0$ (resp. $V^0 = V^1 = 0$). The graded vector space V is said to be *finite dimensional* if its dimension is finite in every degree. The group of permutations on n elements is denoted by S_n.

1. Definitions and properties of Leibniz algebras and Leibniz-dual algebras

Leibniz algebras were introduced by J.-L. Loday (see [Lo1] for a survey). Leibniz algebras are algebras over a certain quadratic operad, denoted by $\mathcal{L}eib$. We refer to [G-K] concerning the theory of quadratic operads. The operad $\mathcal{L}eib$ admits a Koszul dual denoted by $\mathcal{L}eib^!$, and algebras over this operad are called Leibniz-dual algebras. In this section we recall the notion of graded algebras, differential graded algebras and prove that the operads $\mathcal{L}eib$ and $\mathcal{L}eib^!$ are Koszul operads.

Let V be a graded vector space and denote by $\overline{T}(V)$ the graded vector space $V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$. A tensor in $V^{\otimes n}$ is written either $(a_1, \cdots, a_n)$ or $a_1 \otimes \cdots \otimes a_n$. The symmetric group S_n acts on $V^{\otimes n}$ on the left by $\sigma \cdot (a_1 \otimes \cdots \otimes a_n) = \epsilon_K(\sigma) a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$, or on the right by $(a_1 \otimes \cdots \otimes a_n) \cdot \sigma = \epsilon_K(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$. The sign $\epsilon_K(\sigma)$ is the *Koszul sign* associated to σ. We say that the action on $V^{\otimes n}$ by the group S_n is the *signed action*, if it is the same action as before multiplied by the signature of the permutation.

Definition 1.1. *A* **graded Leibniz algebra** L *is a lower graded vector space together with a bracket of degree* 0 *satisfying the identity*

$$[x, [y, z]] = [[x, y], z] - (-1)^{|y||z|}[[x, z], y], \ \forall x, y, z \in L.$$

Let V be a graded vector space. By [Lo1], we know that $\overline{T}(V)$ has a unique structure of graded Leibniz algebra which may be described by

$$\begin{aligned}[x, (a_1, \cdots, a_n)] =& (x, (a_1 \otimes \cdots \otimes a_n) \cdot \mu_n), \\ & \forall x \in V^{\otimes m}, \ \forall (a_1, \cdots, a_n) \in V^{\otimes n}.\end{aligned}$$

The element $\mu_n \in K[S_n]$ is defined by induction as
$\mu_1 = Id$,
$\mu_{n+1} = \mu_n + (-1)^{n+1} \mu_n \circ \tau_{n+1}^{-1}$,
where τ_n is the cycle $(12 \cdots n)$ of S_n. Here, the action of S_n is the right signed action. Note that $a_1 \otimes \cdots \otimes a_k = [[\cdots [a_1, a_2], \cdots], a_k], \ \forall \ a_1 \otimes \cdots \otimes a_k \in V^{\otimes k}$. With this structure, the vector space $\overline{T}(V)$ is the free graded Leibniz algebra generated by V.

By reversing arrows in the definition of a Leibniz algebra we deduce the definition of a Leibniz coalgebra.

Definition 1.2. *A* **graded Leibniz coalgebra** C *is an upper graded vector space together with a comultiplication* Δ *of degree* 0 *satisfying the identity*

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id - (id \otimes T) \circ (\Delta \otimes id)) \circ \Delta,$$

where $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$.

The vector space $\overline{T}(V)$ can be equipped with a comultiplication Δ so that it becomes the free Leibniz coalgebra generated by V if $V^0 = 0$. More explicitly

$$\Delta(a_1, \cdots, a_n) = \sum_{k=1}^{n-1} (a_1, \cdots, a_k) \otimes \mu_{n-k}(a_{k+1}, \cdots, a_n),$$
$$\forall (a_1, \cdots, a_n) \in V^{\otimes n}, n \geq 1,$$

where the action of S_n is the left signed action.

Definition 1.3. *A* **graded Leibniz-dual algebra** *M is an upper graded vector space equipped with a multiplication of degree* 0 *satisfying the identity*

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) + (-1)^{|y||z|} x \cdot (z \cdot y), \ \forall x, y, z \in M.$$

Proposition 1.4. *Let M be a graded Leibniz-dual algebra. Denote by $*$ the product $x * y = x \cdot y + (-1)^{|x||y|} y \cdot x$. This product makes M into a graded associative and commutative algebra.*

Let V be a graded vector space. The free graded Leibniz-dual algebra generated by V is $\overline{T}(V)$ equipped with the following product

$$(v_1 \otimes \cdots \otimes v_p) \cdot (v_{p+1} \otimes \cdots \otimes v_{p+q}) =$$
$$(Id \otimes sh_{p-1,q})(v_1 \otimes \cdots \otimes v_{p+q}), \forall v_1, \cdots, v_{p+q} \in V,$$

where

$$sh_{p,q} = \sum_{\sigma \ = (p,q)-\text{shuffle}} \sigma \text{ and } S_n \text{ acts on the left.}$$

We refer to [Lo2] for the proof. Note that $v_1 \otimes \cdots \otimes v_k = v_1 \cdot (v_2 \cdot (\cdots (v_{k-1} \cdot v_k) \cdots))$.

By reversing arrows in the definition of a Leibniz-dual algebra we obtain the definition of a Leibniz-dual coalgebra.

Definition 1.5. *A* **graded Leibniz-dual coalgebra** *B is a lower graded vector space together with a comultiplication Δ of degree* 0 *satisfying the identity*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta + id \otimes (T \circ \Delta)) \circ \Delta.$$

The vector space $\overline{T}(V)$ can be equipped with a comultiplication Δ so that it becomes the free graded Leibniz-dual coalgebra on V if $V_0 = 0$ (see [O]):

$$\Delta(a_0, \cdots, a_p)$$
$$= \sum_{k=0}^{p-1} J_k(a_0, (a_1, \cdots, a_p) \cdot sh_{k,p-k}), \ \forall (a_0, \cdots, a_p) \in V^{\otimes p+1}, \ p \geq 1,$$

where S_n acts on the right. The map $J_k : \overline{T}(V) \to \overline{T}(V) \otimes \overline{T}(V)$ is defined by $J_k(a_0, a_1, \cdots, a_p) = (a_0, \cdots, a_k) \otimes (a_{k+1}, \cdots, a_p)$ if $0 \leq k \leq p-1$.

Definition 1.6. *A differential of graded Leibniz algebras ϕ is a morphism of degree -1 which is a derivation of graded Leibniz algebras. More explicitly:*

$$\phi([a,b]) = [\phi(a), b] + (-1)^{|a|}[a, \phi(b)].$$

A **differential** *of graded Leibniz coalgebras ϕ is a morphism of degree 1 which is a coderivation of graded Leibniz coalgebras. More explicitly, if one denotes by (C, Δ) a graded Leibniz coalgebra, then using Sweedler notation ($\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$)(cf. [Sw]),*

$$\Delta(\phi(x)) = \sum_{(x)} \phi(x_{(1)}) \otimes x_{(2)} + (-1)^{|x_{(1)}|} x_{(1)} \otimes \phi(x_{(2)}).$$

A **differential graded** *Leibniz algebra (resp. coalgebra) is a graded Leibniz algebra (resp. coalgebra) together with a differential. We have obviously the notion of a differential graded Leibniz-dual algebra and a differential graded Leibniz-dual coalgebra.*

Definition 1.7. *Let ϕ be a morphism between differential graded Leibniz algebras. If ϕ induces an isomorphism in homology, then ϕ is said to be a* **weak equivalence***.*

Since we have defined a certain type of algebras, we are interested in their homology. One can find their theoretic description in [G-K], or concerning the homology of Leibniz algebras in [Lo2], and the homology of Leibniz-dual algebras in [Bal]. The definition of the homology of a differential graded Leibniz algebra will be seen in definition 3.1.

Definition 1.8. *Let (M, d) be a reduced differential graded Leibniz-dual algebra. The* **homology** *of M, denoted by $HLD(M)$ is the cohomology of the total complex $(\mathcal{C}l(M), \partial) = (\overline{T}(sM), \partial_1 + \partial_2)$, where*

$$\begin{aligned}
\partial_1(sx_1 \otimes \cdots \otimes sx_n) &= \sum_{i=1}^{n} -(-1)^{u_{i-1}} sx_1 \otimes \cdots \otimes sdx_i \otimes \cdots \otimes sx_n \\
\partial_2(sx_1 \otimes \cdots \otimes sx_n) &= (-1)^{|sx_1|} s(x_1 \cdot x_2) \otimes \cdots \otimes sx_n \\
&\quad + \sum_{i=2}^{n-1} (-1)^{u_i} sx_1 \otimes \cdots \otimes s(x_i * x_{i+1}) \otimes \cdots \otimes sx_n
\end{aligned}$$

and $u_i = \sum_{j=1}^{i} |sx_j|$. The product $*$ is defined in proposition 1.4. Note that ∂ is a differential of degree 1.

Theorem 1.9. *Let $(M, d) = (\overline{T}(V), d)$ be a reduced differential free graded Leibniz-dual algebra. Its homology is the cohomology of the suspension of the indecomposable elements. More precisely,*

$$HLD_n(\overline{T}(V)) = H^{n+1}(V, \bar{d}),$$

where $\bar{d}$ is the differential induced by d on the indecomposable elements.

Proof. We define the bicomplex $C_{p,q} = (sM)^{\otimes p+1}_{q-p}$ with the differentials $\partial_1 : C_{p,q} \to C_{p,q+1}$ and $\partial_2 : C_{p,q} \to C_{p-1,q}$ defined in 1.8. The homology of M is the homology of the total complex C. We will prove that the homology of the $q-th$ row is concentrated in degree 0 and is exactly V^{q+1}. We fix q and set $D_p = C_{p,q}$ for $0 \le p \le q$ and $D_{-1} = V^{q+1}$. The differential on D is ∂_2 in degrees > 0 and the projection of $(s\overline{T}(V))^q$ on V^{q+1} in degree 0. There is a homotopy h between the identity and the zero morphism given by: $h_{-1} : D_{-1} \to D_0$ is the embedding of $(sV)^q$ into $(s\overline{T}(V))^q$, and $h_{n-1} : D_{n-1} \to D_n$ is given by $h_{n-1}(sa_1 \cdots a_k \otimes sx_2 \otimes \cdots \otimes sx_n) = (-1)^{|sa_1|} sa_1 \otimes sa_2 \cdots a_k \otimes sx_2 \otimes \cdots \otimes sx_n$ if $k > 1$, and is 0 otherwise. It is easy to check that $h_{n-1} \circ d + d \circ h_n = id_{D_n}$, $\forall n \ge 0$. $\square$

The definition 1.8 is exactly the one given by Ginzburg and Kapranov (in [G-K]) for the homology of an algebra over a quadratic operad, in the case of the operad $\mathcal{L}eib^!$. Hence, we have the following corollary.

Corollary 1.10. *The operad $\mathcal{L}eib^!$ and the operad $\mathcal{L}eib$ are Koszul operads.*

2. Minimal models of differential graded Leibniz algebras and Leibniz-dual algebras

The theory of minimal models of differential graded commutative algebras was first developped by Sullivan in [Su]. We refer to [Ta] or [Gr-M] for a survey. Later, it was proved that there exists a minimal model of a differential graded Lie algebra (see for instance [Ba-L] or [Ne]). In this section we prove that a differential graded Leibniz algebra or Leibniz-dual algebra, satisfying certain hypotheses, admits a minimal model unique up to isomorphism.

Definition 2.1. *A* **minimal differential graded Leibniz algebra** *is a differential free graded Leibniz algebra generated by a reduced vector space V, together with a decomposable differential, that is a differential d satisfying $d(V) \subset \overline{T}(V) \cdot \overline{T}(V)$.*

Definition 2.2. *A* **minimal differential graded Leibniz-dual algebra** *is a differential free graded Leibniz-dual algebra generated by a 2-reduced vector space V, together with a decomposable differential, that is a differential d satisfying $d(V) \subset \overline{T}(V) \cdot \overline{T}(V)$.*

Definition 2.3. *Let (M, d) be a differential graded Leibniz algebra (resp. Leibniz-dual algebra). A* **minimal model** *of M is a minimal differential graded Leibniz algebra (resp. Leibniz-dual algebra) (M', d') together with a weak equivalence $\phi : (M', d') \to (M, d)$.*

Theorem 2.4. *Any reduced differential graded Leibniz algebra (L, ∂) admits a minimal model unique up to isomorphism.*

Theorem 2.5. *Any differential graded Leibniz-dual algebra (M, d) whose cohomology is 2-reduced, admits a minimal model unique up to isomorphism.*

The proof of the theorem 2.5 is quite similar to the proof of the theorem 2.4. Let us prove theorem 2.4. The proof of the existence in the case of differential graded Lie algebras (see [Ne]) remains valid in our case. Uniqueness is proved using proposition 2.6 and the lifting lemma 2.9.

Proposition 2.6. *A weak equivalence between minimal differential graded Leibniz algebras is an isomorphism.*

Proof. Let $f : (X = \overline{T}(V), d) \to (Y = \overline{T}(W), d')$ be such a weak equivalence. Set $X[n] = \overline{T}(V_1 \oplus \cdots \oplus V_n)$, and $Y[n] = \overline{T}(W_1 \oplus \cdots \oplus W_n), \forall n \geq 1$. The vector space $X[n]$ (resp. $Y[n]$) is a sub-Leibniz algebra of X (resp. Y). We will prove, by induction on n, that f induces an isomorphism between $X[n]$ and $Y[n]$.

For $n = 1$, since V is reduced and d is decomposable, we have $d(V_2) = 0$. Hence $H_1(X) = V_1$, $H_1(Y) = W_1$ and $H_1(f) : V_1 \to W_1$ is an isomorphism. We deduce that f restricted to $X[1] = \overline{T}(V_1)$ is an isomorphism into $Y[1]$.

Assume that f induces an isomorphism between $X[n]$ and $Y[n]$. The short exact sequence associated to the inclusion $X[n] \to X$, yields the long exact sequence in cohomology

$$\begin{array}{ccccccccc}
H_{n+1}(X[n]) & \to & H_{n+1}(X) & \to & H_{n+1}(X/X[n]) & \to & H_n(X[n]) & \to & H_n(X) \\
\wr\downarrow H_{n+1}(f) & & \wr\downarrow H_{n+1}(f) & & \downarrow H_{n+1}(f) & & H_n(f)\downarrow\wr & & H_n(f)\downarrow\wr \\
H_{n+1}(Y[n]) & \to & H_{n+1}(Y) & \to & H_{n+1}(Y/Y[n]) & \to & H_n(Y[n]) & \to & H_n(Y).
\end{array}$$

Hence, applying the five lemma, we get that

$$H_{n+1}(f) : H_{n+1}(X/X[n]) \longrightarrow H_{n+1}(Y/Y[n])$$

is an isomorphism. But $H_{n+1}(X/X[n])$ is isomorphic to V_{n+1}, therefore f verifies the induction hypothesis at range $n + 1$. $\square$

Definition 2.7. *Let (Y, d) be a differential graded Leibniz algebra, and $\Lambda(t, dt)$ be the differential free graded commutative algebra generated by t in degree 0 and dt in degree -1, satisfying $d(t) = dt$, $d(dt) = 0$.*

The vector space $Y \otimes \Lambda(t,dt)$ is a differential graded Leibniz algebra: the bracket is given by $[y\otimes a, y'\otimes a'] = (-1)^{|a||y'|}[y,y']\otimes aa'$, and its differential is given by $d(y\otimes a) = dy\otimes a + (-1)^{|y|}y\otimes da$. We denote by $Y(t,dt)$ the differential graded Leibniz algebra $Y\otimes\Lambda(t,dt)$ in degrees $n\geq 1$ and $Y(t,dt)_0 = Ker(d : (Y\otimes\Lambda(t,dt))_0 \to (Y\otimes\Lambda(t,dt))_{-1}$. Define $p_0, p_1 : Y(t,dt)\to Y$ by $p_0(y\otimes(a(t)+b(t)dt)) = a(0)y$ and $p_1(y\otimes(a(t)+b(t)dt)) = a(1)y$. Two morphisms $f,g : (X,d)\to(Y,d')$ of differential graded Leibniz algebras are said to be **homotopic** *if there exists a morphism $h : X\to Y(t,dt)$ such that $p_0\circ h = f$ and $p_1\circ h = g$.*

Remark 2.8. Since p_0 and p_1 are weak equivalences, if f is homotopic to g, then $H(f) = H(g)$.

Lifting lemma 1. *Let $\pi : A\to B$ be a weak equivalence between differential graded Leibniz algebras. Let X be a minimal differential graded Leibniz algebra and $f : X\to B$ be a morphism. Then, there exists a morphism $\tilde f : X\to A$ such that $\pi\circ\tilde f$ is homotopic to f.*

Proof. We set $X = (\bar T(V), d)$ and $X[n] = \bar T(V_1\oplus\cdots\oplus V_n)$, if $n\geq 1$. We will construct, by induction on n, some morphisms $\tilde f : X[n]\to A$ and $G : X[n]\to B(t,dt)$ satisfying $p_0\circ G = \pi\circ\tilde f$ and $p_1\circ G = f$.

Assume $n = 1$ and fix $v\in V_1$. Since $df(v) = f(dv) = 0\in B_1$, there exists $y\in A_1$ such that $dy = 0$ and $[\pi(y)] = [f(v)]$ in $H_1(B)$. Hence, there exists $y\in A_1$ and $b\in B_2$ such that $\pi(y) = f(v)+db$. We set $\tilde f(v) = y$ and $G(v) = (f(v)+db)\otimes 1 - db\otimes t - b\otimes dt$, and check that these morphisms satisfy the hypothesis.

Assume that these morphisms are built for $k\leq n$. Fix $(x_\alpha)_{\alpha\in I}$ a basis of V_{n+1}. We want to extend $\tilde f$ to $X[n,x_\alpha] = \bar T(V_1\oplus\cdots\oplus V_n\oplus Kx_\alpha)$. Denote by c_α the element dx_α of $X[n]$. Since $d\tilde f(c_\alpha) = 0$, applying remark 2.8, we have the identity $[\pi\circ\tilde f](c_\alpha) = [f](c_\alpha) = 0$ in $H(B)$. By hypothesis π induces an isomorphism in homology, thus there exists $\eta\in A_{n+1}$ such that $\tilde f(c_\alpha) = d(\eta)$. We set $\tilde f(x_\alpha) = \eta$ and aim to extend G. Extending G is equivalent to the existence of a morphism $\bar G : X[n,x_\alpha]\to B(t,dt)$ making the following diagram commute

$$\begin{array}{ccc} X[n] & \xrightarrow{G} & B(t,dt) \\ \downarrow & & \downarrow {\scriptstyle \rho:=(p_0,p_1)} \\ X[n,x_\alpha] & \xrightarrow{r:=(\pi\circ\tilde f, f)} & B\times B. \end{array}$$

Claim 1. *The obstruction to extend G to $\bar G$ lies in $H_n(Ker\rho)$.*

Proof. As ρ is surjective in degrees ≥ 1, we may choose $a\in B(t,dt)$ such that $\rho(a) = r(x_\alpha)$, and we define $\theta = d(a) - G(c_\alpha)$. It is easy to check that $\theta\in Z_n(Ker\rho)$, and that its class in $H_n(Ker\rho)$, denoted by $[\theta]$, does not depend on the choice of a. Then G extends to $\bar{G}$ if and only if there exists $\nu \in B(t, dt)$, satisfying $\rho(\nu) = r(x_\alpha)$ and $d(\nu) = G(c_\alpha)$, so if and only if $[\theta] = 0$. $\square$

The element $a = \pi(\eta) + (f(x_\alpha) - \pi(\eta))t$ satisfies $\rho(a) = r(x_\alpha)$. We define $\theta = d(a) - G(c_\alpha) \in Z_n(Ker\rho)$. The next claim computes $[\theta]$.

Claim 2. *If $\theta \in Z_n(Ker\rho)$, then there exists $u \in Z_{n+1}B$ such that $[\theta] = [udt]$.*

Proof. Consider the following short exact sequence

$$0 \to Ker\rho \xrightarrow{i} B(t, dt) \xrightarrow{\rho} B \times B \to 0.$$

We have $H(B(t, dt)) = H(B)$, $H(B \times B) = H(B) \times H(B)$ and $H(\rho)$ is the diagonal map which is injective. Therefore, in the long exact sequence in homology associated to the previous sequence, we obtain $H(i) = 0$. Thus, the connecting morphism $\delta_* : H_{n+1}(B \times B) \to H_n(Ker\rho)$ is surjective and is given by $\delta_*([\alpha, \beta]) = [(-1)^{n+1}(\beta - \alpha)dt]$, $\forall(\alpha, \beta) \in Z_{n+1}(B \times B)$. $\square$

We are now able to finish the proof of the lifting lemma. We must modify $\tilde{f}(x_\alpha)$ so that there is no more obstruction to extend G. According to claim 2 and the hypothesis, there exists $u \in Z_{n+1}B$ and $\bar{u} \in Z_{n+1}(A)$ such that $[\pi(\bar{u})] = [u]$ and $[\theta] = [\pi(\bar{u})dt]$. We define $\tilde{f}(x_\alpha) = \eta + (-1)^{n+1}\bar{u}$. There exists a' such that $\rho(a') = (\pi(\eta) + (-1)^{n+1}\pi(\bar{u}), f(x_\alpha))$; for instance, we take $a' = a + (-1)^{n+1}\pi(\bar{u})(1 - t)$. Since $\theta' = d(a') - G(c_\alpha) = da - \pi(\bar{u})dt - G(c_\alpha) = \theta - \pi(\bar{u})dt$, we obtain $[\theta'] = 0$. Then, we conclude with claim 1. $\square$

Proof of uniqueness in theorem 2.4. Let $\phi : X \to M$ and $\psi : Y \to M$ be minimal models of (M, d). Since ϕ is a weak equivalence, by the lifting lemma 2.9, there exists a morphism $\tilde{\psi} : Y \to X$ such that $\phi \circ \tilde{\psi}$ is homotopic to ψ. Applying remark 2.8, we get $H(\phi) \circ H(\tilde{\psi}) = H(\psi)$, hence $H(\tilde{\psi})$ is an isomorphism. Proposition 2.6 allows us to conclude. $\square$

3. Relations between Leibniz algebras and Leibniz-dual algebras

In classical rational homotopy theory ([Qu1], [Ta]), differential graded Lie algebras are related to reduced differential graded cocommutative coalgebras through functors $\mathcal{C}$ and $\mathcal{L}$. We define adjoint functors $\mathcal{L}^!$ and $\mathcal{L}$ between the categories of differential graded Leibniz algebras and reduced differential graded Leibniz-dual coalgebras. We prove that these functors as well as the unit and the counit of the adjunction preserve weak equivalences. We then define the homotopy and the homology of a differential graded Leibniz algebra and prove that minimal models contain all the homotopy and the homology information of the Leibniz algebra.

Definition 3.1. *The functor $\mathcal{L}^!$: {differential graded Leibniz algebras} $\rightarrow$ {reduced differential graded Leibniz-dual coalgebras} is defined as follows: let (L, ∂) be a differential graded Leibniz algebra, then*

$$\mathcal{L}^!(L, \partial) := (\overline{T}(sL), d = d_1 + d_2),$$

where $\overline{T}(sL)$ is the free graded Leibniz-dual coalgebra on sL (see definition 1.5),

$$d_1(sx_1 \otimes \cdots \otimes sx_n) = \sum_{i=1}^{n} -(-1)^{\epsilon_i} sx_1 \otimes \cdots \otimes s\partial x_i \otimes \cdots \otimes sx_n,$$

$$d_2(sx_1 \otimes \cdots \otimes sx_n) = \sum_{1 \le i < j \le n} (-1)^{t_{i,j}} sx_1 \otimes \cdots \otimes s[x_i, x_j] \otimes \cdots \otimes sx_n,$$

$$\text{with } t_{i,j} = (\sum_{k=1}^{i} \mid sx_k \mid) + \mid sx_j \mid (\sum_{k=i+1}^{j-1} \mid sx_k \mid) \quad \text{and } \epsilon_i = \sum_{j=1}^{i-1} \mid sx_j \mid .$$

It is easy to check that d_1 and d_2 are differentials of graded Leibniz-dual coalgebra, and the structure of a Leibniz algebra on L implies the same statement for d. The definition 3.1 is exactly the one given by Ginzburg and Kapranov (in [G-K]) for the homology of an algebra over a quadratic operad, in the case of the operad $\mathcal{L}eib$. Since this operad is a Koszul operad (see corollary 1.10), we are able to compute the homology of a free object.

Theorem 3.2. *Let $(L, \partial) = (\overline{T}(V), \partial)$ be a differential free graded Leibniz algebra. The homology of $\mathcal{L}^!(L, \partial)$ is the homology of the suspension of the indecomposable elements. More precisely,*

$$H_n(\mathcal{L}^!(L, \partial)) = H_{n-1}(V, \bar{\partial}),$$

where $\bar{\partial}$ is the differential induced by ∂ on the indecomposable elements.

Definition 3.3. *The functor $\mathcal{L}$: {reduced differential graded Leibniz-dual coalgebras} $\rightarrow$ {differential graded Leibniz algebras} is defined as follows: for any reduced differential graded Leibniz-dual coalgebra (B, d),*

$$\mathcal{L}(B, d) := (\overline{T}(s^{-1}B), \partial = \partial_1 + \partial_2),$$

where $\overline{T}(s^{-1}B)$ is the free graded Leibniz algebra generated by $s^{-1}B$ (see definition 1.1), where ∂_1 is induced by d and ∂_2 is given by

$$\partial_2(s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_n) = \sum_{(x_1)} (-1)^{s^{-1}x_{1(1)}} s^{-1}x_{1(1)} \otimes s^{-1}x_{1(2)} \otimes \cdots \otimes s^{-1}x_n$$

$$+ \sum_{i=2}^{n} (-1)^{\sum_{j=1}^{i-1} |s^{-1}x_j|} \sum_{(x_i)} (-1)^{|s^{-1}x_{i(1)}|} s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_{i(1)} \otimes s^{-1}x_{i(2)} \otimes \cdots \otimes s^{-1}x_n$$

$$- \sum_{(x_i)} (-1)^{|s^{-1}x_{i(1)}| + |s^{-1}x_{i(1)}||s^{-1}x_{i(2)}|} s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_{i(2)} \otimes s^{-1}x_{i(1)} \otimes \cdots \otimes s^{-1}x_n.$$

Theorem 3.4. *The functor $\mathcal{L}$ is left adjoint to $\mathcal{L}^!$.*

Proof. For any differential graded Leibniz algebra (L, ∂) and any reduced differential graded Leibniz-dual coalgebra (B, d), we denote by $\epsilon : \mathcal{L}\mathcal{L}^!(L, \partial) \to (L, \partial)$ the counit and by $\eta : (B, d) \to \mathcal{L}^!\mathcal{L}(B, d)$ the unit of the adjunction. Since $\mathcal{L}\mathcal{L}^!(L, \partial) = \overline{T}(s^{-1}\mathcal{L}^!(L, \partial))$ is a free graded Leibniz algebra, it is sufficient to make ϵ explicit on $s^{-1}\mathcal{L}^!(L, \partial) = s^{-1}\,\overline{T}(sL)$. We set $\epsilon = 0$ on $s^{-1}\,\overline{T}(sL)^{\geq 2}$ and $\epsilon(s^{-1}sx) = x$ on $s^{-1}sL$. Similarly, applying the universal property for free Leibniz-dual coalgebras, η is the unique morphism from B to $\mathcal{L}^!\mathcal{L}(B, d)$ extending the map $\tilde{\eta} : B \to s\,\overline{T}(s^{-1}B)$ defined by $\tilde{\eta}(x) = ss^{-1}x$. □

Theorem 3.5. *The functor $\mathcal{L}^!$ preserves weak equivalences.*

Proof. Let $\psi : (L, \partial) \to (L', \partial')$ be a weak equivalence between differential graded Leibniz algebras. By definition 3.1, $\mathcal{L}^!(L, \partial) = (\overline{T}(sL), d = d_1 + d_2)$. There is a natural filtration on $\mathcal{L}^!$ given by $F^p = (\overline{T}(sL))^{\leq p} = \bigoplus_{k=1}^{p} (sL)^{\otimes k}$ and $F'^p = (\overline{T}(sL'))^{\leq p}$, $\forall p \geq 0$. Note that $F^0 = F'^0 = 0$, $F^1 = sL$, $F'^1 = sL'$, the pair (F^p, d) (resp. (F'^p, d')) is a sub-complex of $\mathcal{L}^!(L, \partial)$ (resp. $\mathcal{L}^!(L', \partial')$), and $\mathcal{L}^!(\psi)$ maps F^p to F'^p. Thus, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{p-1} & \longrightarrow & F^p & \longrightarrow & F^p/F^{p-1} & \longrightarrow & 0 \\ & & {\scriptstyle \mathcal{L}^!(\psi)}\downarrow & & & & \downarrow{\scriptstyle \overline{\mathcal{L}^!(\psi)}} & & \\ 0 & \longrightarrow & F'^{p-1} & \longrightarrow & F'^p & \longrightarrow & F'^p/F'^{p-1} & \longrightarrow & 0. \end{array}$$

Since the complex F^p/F^{p-1} is isomorphic to $((sL)^{\otimes p}, d_1)$ and d_1 coincides, up to sign, with ∂ on sL, then $H(F^p/F^{p-1}) \simeq (sH(L))^{\otimes p}$ and $H(\overline{\mathcal{L}^!(\psi)})$ is an isomorphism. Applying the long exact sequence in homology as well as the five lemma in the previous diagram, we deduce that, if the map $H\mathcal{L}^!(\psi) : H(F^{p-1}) \to H(F'^{p-1})$ is an isomorphism, then the map $H\mathcal{L}^!(\psi) : H(F^p) \to H(F'^p)$ is also an isomorphism. Because it is an isomorphism for $p = 0$ and $p = 1$ and because $H_k(\mathcal{L}^!(L, d)) = H_k(F^k)$, we get $H\mathcal{L}^!(\psi)$ is an isomorphism. □

Theorem 3.6. *The functor $\mathcal{L}$ preserves weak equivalences between* 2*-reduced differential graded Leibniz-dual coalgebras.*

Proof. This proof is similar to the previous one. Let $\psi : (B, d) \to (B', d')$ be a weak equivalence between 2-reduced differential graded Leibniz-dual coalgebras. Recall definition 3.3: $\mathcal{L}(B, d) = (\overline{T}(s^{-1}B), \partial = \partial_1 + \partial_2)$. The filtration on $\mathcal{L}$, given by $F^p = \overline{T}(s^{-1}B)^{\geq p}$, provides the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{p+1} & \longrightarrow & F^p & \longrightarrow & F^p/F^{p+1} & \longrightarrow & 0 \\ & & {\scriptstyle \mathcal{L}(\psi)}\downarrow & & & & \downarrow {\scriptstyle \overline{\mathcal{L}(\psi)}} & & \\ 0 & \longrightarrow & F'^{p+1} & \longrightarrow & F'^p & \longrightarrow & F'^p/F'^{p+1} & \longrightarrow & 0. \end{array}$$

This implies that, if the map $H(\mathcal{L}(\psi)) : H(F^{p+1}) \to H(F'^{p+1})$ is an isomorphism, then the map $H(\mathcal{L}(\psi)) : H(F^p) \to H(F'^p)$ is also an isomorphism. By hypothesis, B is 2-reduced, so $H_k(F^{p+1}) = H_k(F'^{p+1}) = 0,\ \forall k \leq p$ and we can conclude. $\square$

Theorem 3.7. *For any reduced differential graded Leibniz-dual coalgebra (B, d), the unit of the adjunction $\eta : (B, d) \to \mathcal{L}^!\mathcal{L}(B, d)$ is a weak equivalence. For any reduced differential graded Leibniz algebra (L, ∂), the counit of the adjunction $\epsilon : \mathcal{L}\mathcal{L}^!(L, \partial) \to (L, \partial)$ is a weak equivalence.*

Proof. The first part of the theorem is straightforward using theorem 3.2. We will prove the second part of the theorem in two steps. The first step consists in proving the weak equivalence for a minimal differential graded Leibniz algebra. The second step is the conclusion. Indeed, let (L, ∂) be a reduced differential graded Leibniz algebra. By theorem 2.4, (L, ∂) admits a minimal model $(\tilde{L}, \tilde{\partial})$. We have the following diagram

$$\begin{array}{ccc} \mathcal{L}\mathcal{L}^!(L, \partial) & \xrightarrow{\epsilon_L} & L \\ {\scriptstyle \mathcal{L}\mathcal{L}^!(\phi)}\uparrow & & \uparrow {\scriptstyle \phi} \\ \mathcal{L}\mathcal{L}^!(\tilde{L}, \tilde{\partial}) & \xrightarrow{\epsilon_{\tilde{L}}} & \tilde{L} \end{array}$$

Since ϕ and $\epsilon_{\tilde{L}}$ are weak equivalences as well as $\mathcal{L}\mathcal{L}^!(\phi)$ by theorems 3.5 and 3.6, we deduce that ϵ_L is a weak equivalence. Let's prove the first step. Let $(L, \partial) = (\overline{T}(V), \partial)$ be a minimal differential graded Leibniz algebra. We recall that $V_0 = 0$ and ∂ is decomposable. There is a natural filtration on L given by $F^p(L) = \overline{T}(V)^{\geq p}$. Since the differential is decomposable, $\partial(F^p(L)) \subset F^{p+1}(L)$. In the spectral sequence associated to this filtration, the E_0 term is $L' = (\overline{T}(V), 0)$. Since the filtration is bounded, the spectral sequence converges. We introduce a filtration on $\mathcal{L}\mathcal{L}^!(L, \partial) = \overline{T}(s^{-1}\,\overline{T}(sL))$. Fix an elementary element $y = y_1 \otimes \cdots \otimes y_p$ of $\mathcal{L}\mathcal{L}^!(L, \partial)$ where each y_i is of the form $s^{-1}sx_1^i \otimes \cdots \otimes sx_{q_i}^i$ and each $x_j^i \in V^{\otimes l_j^i}$. We define the degree of y_i by $\mathrm{Deg}(y_i) = \displaystyle\sum_{j=1}^{q_i} l_j^i$ and the degree of y by $\mathrm{Deg}(y) = \displaystyle\sum_{i=1}^{p} \mathrm{Deg}(y_i)$. Let A_p be the sub-vector space of $\mathcal{L}\mathcal{L}^!(L, \partial)$ generated by the elements of degree

p and we denote by $\tilde{F}^p(\mathcal{L}\mathcal{L}^!(L,\partial))$ the filtration $\bigoplus_{k\geq p} A_k$. The filtration is a filtration of complex and ϵ_L preserves the filtration. Using the definitions of the differential on $\mathcal{L}^!(L,\partial)$ (see definition 3.1) and on $\mathcal{L}\mathcal{L}^!(L,\partial)$ (see definition 3.3), we check that

$$\tilde{F}^p\mathcal{L}\mathcal{L}^!(L,\partial)/\tilde{F}^{p+1}\mathcal{L}\mathcal{L}^!(L,\partial) \simeq \tilde{F}^p\mathcal{L}\mathcal{L}^!(L')/\tilde{F}^{p+1}\mathcal{L}\mathcal{L}^!(L').$$

But we have the following isomorphism of complexes

$$\bigoplus_{k\geq 1} \tilde{F}^p\mathcal{L}\mathcal{L}^!(L')/\tilde{F}^{p+1}\mathcal{L}\mathcal{L}^!(L') \simeq \mathcal{L}\mathcal{L}^!(L').$$

Hence the E_0-term associated to the filtration $\tilde{F}$ is $\mathcal{L}\mathcal{L}^!(L')$, and the spectral sequence converges. Moreover, L' can be written $\mathcal{L}(C)$ where C is a trivial Leibniz-dual coalgebra with the zero differential. The first part of the theorem gives that $\eta : C \to \mathcal{L}^!\mathcal{L}(C)$ is a weak equivalence, and since C is 2-reduced and $\mathcal{L}$ preserves weak equivalence, we deduce that $\mathcal{L}\eta$ is a weak equivalence. But the composite

$$\mathcal{L}(C)\overset{\mathcal{L}\eta}{\longrightarrow}\mathcal{L}\mathcal{L}^!\mathcal{L}(C)\overset{\epsilon_{\mathcal{L}(C)}}{\longrightarrow}\mathcal{L}(C)$$

is the identity, hence $\epsilon_{\mathcal{L}(C)}$ is a weak equivalence. By Zeeman's theorem of comparison of spectral sequences, we deduce that ϵ_L is a weak equivalence. $\square$

Definition 3.8. *The* **homotopy** *of a differential graded Leibniz algebra* (L,∂)*, denoted by* $\pi\lambda(L)$*, is the graded vector space* $\pi\lambda_*(L) = H_*(sL,\partial)$*. The* **homology** *of* L*, denoted by* $H\lambda(L)$*, is the graded Leibniz-dual coalgebra* $H\lambda_*(L) = H_*(\mathcal{L}^!(L,\partial))$*. The* **cohomology** *of* L *is*

$$H\lambda^*(L) = H^*Hom(\mathcal{L}^!(L,\partial),K).$$

Note that since the linear dual of a differential graded Leibniz-dual coalgebra is a differential graded Leibniz-dual algebra, the cohomology of a differential graded Leibniz algebra has the structure of a graded Leibniz-dual algebra.

Definition 3.9. *A differential graded Leibniz algebra* L *is said to be* ***n*-connected** *if* $\pi\lambda_k(L) = 0,\ \forall k \leq n$.

In the next theorems, we show that the theory of minimal models developped in section 2, is strongly related to the homotopy and the homology of a differential graded Leibniz algebra.

Theorem 3.10. *Let (L, ∂) be a reduced differential graded Leibniz algebra and $(\overline{T}(V), \partial')$ be its minimal model. The homotopy of L is the homology of the suspension of its minimal model, and the homology of L is the suspension of the indecomposable elements of its minimal model. More precisely, we have*

$$\begin{array}{c} \pi\lambda_*(L) \simeq H_*(s\,\overline{T}(V), \partial') \\ H\lambda_*(L) \simeq V_{*-1} \end{array}.$$

Proof. The first part of the theorem comes from the definition of a minimal model. We have a weak equivalence $\phi : (\overline{T}(V), \partial') \to (L, \partial)$, and by theorem 3.5, the functor $\mathcal{L}^!$ preserves weak equivalences. Hence, $H_*(\mathcal{L}^!(\phi)) : H_*(\mathcal{L}^!\,\overline{T}(V), \partial') \to H\lambda_*(L)$ is an isomorphism. Hence the theorem 3.2 combined with the fact that the differential ∂' is decomposable allows us to conclude. □

Theorem 3.11. *Let (L, ∂) be a reduced finite dimensional differential graded Leibniz algebra. Let $(\overline{T}(V), d)$ be the minimal model of the differential graded Leibniz-dual algebra $Hom(\mathcal{L}^!(L, \partial), K)$. The cohomology of L is the cohomology of the complex $(\overline{T}(V), d)$, and the homotopy of L is the linear dual of the indecomposable elements of $\overline{T}(V)$. More precisely*

$$\begin{array}{c} H\lambda^*(L) \simeq H^*(\overline{T}(V), d) \\ \pi\lambda_*(L) \simeq Hom(V^*, K) \end{array}$$

Proof. The first part of the theorem comes from the definition of a minimal model. To prove the second part of the theorem, we use the functor $\mathcal{C}l$ defined in definition 1.8. In fact, for any finite dimensional 2-reduced differential graded Leibniz-dual algebra (M, d), we have

$$\mathcal{L}(Hom(M, K),^t d) \simeq Hom((\mathcal{C}l(M, d),^t \partial), K).$$

Since L is finite dimensional, we have a weak equivalence $\psi : \mathcal{L}^!(L, \partial) \to Hom(\overline{T}(V), K)$. But the functor $\mathcal{L}$ preserves weak equivalences between 2-reduced differential graded Leibniz-dual coalgebras, hence

$$\mathcal{L}(\psi) : \mathcal{L}\mathcal{L}^!(L, \partial) \to \mathcal{L}Hom(\overline{T}(V), K)$$

is a weak equivalence. Applying theorem 3.7, we deduce that the homology of the left member is $s^{-1}\pi\lambda_*(L)$. Using the previous remark, we deduce that the homology of the right member is $Hom(H^*(\mathcal{C}l(\overline{T}(V)), K)$, which is equal to $Hom(sV, K)$ by theorem 1.9. □

4. Hurewicz and Freudenthal theorems for Leibniz algebras

In the previous section, we have constructed the homology and the homotopy of a differential graded Leibniz algebra. We prove a theorem analogous to the classical Hurewicz theorem for Leibniz algebras. We give a definition of the suspension of a Leibniz algebra and prove a Freudenthal suspension theorem similar to the classical one.

Let (L, ∂) be a differential graded Leibniz algebra. We want to give an other description of the homotopy and the homology of L in terms of a bicomplex, denoted by $(C_{*,*}, d)$. Explicitly $C_{p,q} = (sL^{\otimes p+1})_{p+q},\ \forall p, q \geq 0$, and $d = d_1 + d_2$ where $d_1 : C_{p,q} \to C_{p,q-1}$ and $d_2 : C_{p,q} \to C_{p-1,q}$ are defined in definition 3.1.

Lemma 4.1. *With this new notation we have*

$$(\pi\lambda)_n(L) = H_n(C_{0,*}, d_1),$$
$$(H\lambda)_n(L) = H_n((TotC_{p,q})_*, d = d_1 + d_2).\square$$

Definition 4.2. *The embedding of complexes $(C_{0,*}, d_1) \to ((TotC_{p,q})_*, d)$ induces a morphism in homology $\phi\lambda_* : \pi\lambda_* \longrightarrow H\lambda_*$ called the* **Hurewicz morphism**.

Theorem 4.3. *Let (L, ∂) be a differential graded Leibniz algebra. If L is n-connected, then the Hurewicz morphism is an isomorphism for all $k \leq 2n$ and an epimorphism for $k = 2n + 1$. If L is 1-connected and $H\lambda_k(L) = 0,\ \forall k \leq n$, then the Hurewicz morphism is an isomorphism for all $k \leq 2n$ and an epimorphism for $k = 2n + 1$.*

Proof. The Künneth formula for chain complexes gives

$$H(sL^{\otimes k}) = (H(sL))^{\otimes k},\ \forall k \geq 1,$$

where the differential on sL is d_1.

Assume that $\pi\lambda_k = 0,\ \forall k < n + 1$. By lemma 4.1, we have $H_k(C_{0,*}) = 0,\ \forall k < n + 1$. By the Künneth formula we see that $H_s(sL^{\otimes r}) = 0,\ \forall s < (n + 1)r$. This gives us information about the homology of the columns of the bicomplex. Since $H_u(C_{r,*}) = H_{u+r}(sL^{\otimes r+1})$, we get $H_u(C_{r,*}) = 0$ if $u + r < (n + 1)(r + 1)$. For $r \geq 1$, if $u \leq 2n$, then $H_u(C_{r,*}) = 0$. As a result, the E_∞ term of the spectral sequence associated to the bicomplex C is 0 for $p \geq 1$ and $q \leq 2n$. We can conclude.

To prove the second statement, we use the previous result by induction. If $n = 1$, we apply the first part of the theorem. Assume that the result is true at range $n - 1$ and that $H\lambda_k = 0,\ \forall k \leq n$. Then by the induction hypothesis, $\pi\lambda_k = 0,\ \forall k \leq n - 1$ and, applying the first part of the proof, we obtain $H_u(C_{r,*}) = 0,\ \forall r \geq 1,\ u \leq 2(n - 1)$. Thus $\pi\lambda_n = H\lambda_n = 0$ and we go back to the first part. $\square$

Remark 4.4. This theorem implies a Leibniz version of the Hurewicz theorem: if L is $(n-1)$-connected, then $H\lambda_k(L) = 0,\ \forall k \leq n-1$ and $\phi\lambda_n$ is an isomorphism.

In classical rational homotopy theory, the suspension of a topological space S, denoted by ΣS, has for Quillen model $\pi_*(\Omega\Sigma S)\otimes\mathbb{Q}$ which is a free graded Lie algebra on $\mathbb{Q}$ equipped with the trivial differential. Moreover $H_n(S)$ is isomorphic to $H_{n+1}(\Sigma S)$. The Freudenthal suspension theorem states that if S is n-connected, then the suspension morphism $\Sigma_r : \pi_r(S) \to \pi_{r+1}(\Sigma S)$ is an isomorphism for $1 \leq r \leq 2n$ and an epimorphism for $r = 2n+1$. We prove that we can define the suspension of a Leibniz algebra such that an analogous theorem holds.

Definition 4.5. *Let (L, ∂) be a differential graded Leibniz algebra. The* **suspension** *of L, denoted by $\Sigma(L, \partial)$, is the differential graded Leibniz algebra $(\overline{T}(H\lambda_*(L, \partial)), 0)$. Of course we have*

$$H\lambda_{n+1}(\Sigma(L, \partial)) = H_{n+1}(\mathcal{L}^!(\Sigma(L, \partial)), d = d_2)$$

which is equal to $(H\lambda)_n(L, \partial)$ by theorem 3.2. Note that $\pi\lambda_{n+1}(\Sigma L) = \overline{T}(H\lambda_*(L, \partial))_n$. The **Freudenthal suspension morphism***, denoted by $\Sigma\lambda$, is the composite*

$$\Sigma\lambda_n : \pi\lambda_n(L) \xrightarrow{\phi\lambda_n} H\lambda_n(L) \xrightarrow{i_n} \overline{T}(H\lambda(L))_n = \pi\lambda_{n+1}(\Sigma L).$$

Theorem 4.6. *Let (L, ∂) be a n-connected differential graded Leibniz algebra. The Freudenthal suspension morphism is an isomorphism for $k \leq 2n$ and an epimorphism for $k = 2n+1$.*

Proof. Theorem 4.3 asserts that the Hurewicz morphism $\phi\lambda_k$ is an isomorphism for $k \leq 2n$ and an epimorphism for $k = 2n+1$. Moreover $H\lambda_k(L) = 0$ for $k \leq n$, so $\overline{T}(H\lambda(L, \partial))_k = H\lambda_k(L, \partial),\ \forall k \leq 2n+1$. Hence i_k is an isomorphism for $k \leq 2n+1$. $\square$

5. Leibniz spheres and classical spheres

In rational homotopy theory, it is well known that a rational space, which has the cohomology of a sphere, has in fact the same homotopy type. The aim of this section is to construct a n-Leibniz sphere: a differential graded Leibniz algebra L such that $H\lambda^n(L) = K$ and is zero elsewhere. We prove that such an object is unique up to homotopy type. We compute its homotopy, which turns out to be periodic.

Definition 5.1. *Two reduced differential graded Leibniz algebras are said to have the* **same homotopy type** *if their minimal model are isomorphic.*

Definition 5.2. *Let $n \geq 2$. The n-***Leibniz sphere***, denoted by $\mathbb{S}\lambda^n$, is the differential free graded Leibniz algebra generated by one generator in degree $n-1$, equipped with the zero differential.*

Theorem 5.3. *For $n \geq 2$, the cohomology of the n-Leibniz sphere is K in degree n and is 0 elsewhere. Its homotopy is periodic of period $n-1$. Explicitly $\pi\lambda_j(\mathbb{S}\lambda^n) = K$ if $j = k(n-1)+1,\ k \geq 1$ and is 0 otherwise. Besides, any differential graded Leibniz algebra which has the same cohomology as the n-Leibniz sphere has the same homotopy type.*

Proof. The homotopy calculation is immediate. Since the Leibniz sphere $\mathbb{S}\lambda^n$ is a minimal Leibniz algebra, its homology is K in degree n and is 0 elsewhere (see theorem 3.10). Hence, the cohomology is the same. Let (L, ∂) be a reduced differential graded Leibniz algebra whose cohomology is the same as the cohomology of $\mathbb{S}\lambda^n$. Let $(\overline{T}(V), \partial')$ be its minimal model. By theorem 3.10, we deduce that V is necessarily concentrated in degree $n-1$ and that $V_{n-1} \simeq K$. Let z be a generator of V_{n-1}. Since ∂' is decomposable of degree -1, we deduce immediatly that $\partial'(z) = 0$. Thus, the minimal model of (L, ∂) is $(\overline{T}(K_{n-1}), 0)$ which is isomorphic to $\mathbb{S}\lambda^n$. □

We would like now to compare Leibniz spheres and classical spheres. The Quillen model of the classical n-sphere will be denoted by $\mathbb{S}^n$. It is the free graded Lie algebra generated by one generator in degree $n-1$ together with the zero differential. To any graded Leibniz algebra L, we can associate a graded Lie algebra, taking the quotient of L by the relations $[x, y] + (-1)^{|x||y|}[y, x],\ \forall x, y \in L$ (see [Lo1]). Theorem 5.4 asserts that $\mathbb{S}^n$ is the Lie algebra associated to $\mathbb{S}\lambda^n$.

Since a Lie algebra is obviously a Leibniz algebra, the Leibniz homotopy and Leibniz homology of $\mathbb{S}^n$ can be computed. Clearly we have $\pi\lambda(\mathbb{S}^n) = \pi(\mathbb{S}^n)$. Theorem 5.5 computes the homology of $\mathbb{S}^n$.

Theorem 5.4. *The graded Lie algebra associated to the n-Leibniz sphere is the Quillen model of the classical n-sphere.*

Theorem 5.5. *For $n \geq 2$, the Leibniz homology of the classical n-sphere is periodic of period n if n is odd and of period $3n-1$ if n is even. More precisely*

a) If n is odd $H\lambda_i(\mathbb{S}^n) \simeq K$, if $i = kn,\ k \geq 1$, and $H\lambda_i(\mathbb{S}^n) = 0$ otherwise.
b) If n is even $H\lambda_i(\mathbb{S}^n) \simeq K$ for $i = n + k(3n-1), k \geq 0$, or for $i = k(3n-1), k \geq 1$, and $H\lambda_i(\mathbb{S}^n) = 0$ otherwise.

Proof. Recall that $\mathbb{S}^n$ is the free graded Lie algebra generated by one generator y_n in degree $n-1$ together with the zero differential. Hence the Leibniz homology of $\mathbb{S}^n$ is the homology of $(\mathcal{L}^{!}(\mathbb{L}(Ky_n)), d_2) = (\overline{T}(s\mathbb{L}(Ky_n)), d_2)$, where d_2 is given by

$$d_2(sx_1 \otimes \cdots \otimes sx_l) = \sum_{1 \le i < j \le l} (-1)^{t_{i,j}} sx_1 \otimes \cdots \otimes s[x_i, x_j] \otimes \cdots \otimes sx_l.$$

a) Since n is odd, $[y_n, y_n] = 0$; thus $d_2 = 0$ and the result follows.

b) In case n is even, $\mathbb{L}(Ky_n)$ is a graded vector space generated by y_n in degree $n-1$ and by $[y_n, y_n]$ in degree $2(n-1)$. Hence $(\overline{T}(s\mathbb{L}(Ky_n), d_2) = (\overline{T}(Ky \oplus Kz), d)$, where $|y| = n$, $|z| = 2n-1$. We call *elementary element* an element $x = x_1 \otimes \cdots \otimes x_l$ such that x_i is either y or z. The differential d is given on elementary elements by

$$d(x) = \sum_{\substack{(i,j), x_i = y, x_j = y \\ 1 \le i < j \le n}} (-1)^{t_{i,j}} x_1 \otimes \cdots \otimes x_{i-1} \otimes z \otimes x_{i+1} \otimes \cdots \otimes \hat{x_j} \otimes \cdots \otimes x_l,$$

where $t_{i,j}$ denotes the number of z in the decomposition of x lying before x_i.

Let $x = x_1 \otimes \cdots \otimes x_l$ be an elementary element of $\overline{T}(Ky \oplus Kz)$. Define $a(x)$ (resp. $b(x)$) to be the number of occurences of y (resp. of z) in x. The sub-vector space of $\overline{T}(Ky \oplus Kz)$ generated by the set of x such that $a(x) + 2b(x) = k$ is denoted by A_k. Since A_k is stable under d, it is in fact a subcomplex of $(\overline{T}(Ky \oplus Kz), d)$ and we have the identity $(\overline{T}(Ky \oplus Kz), d) = \oplus_{k \ge 1}(A_k, d)$. Thus, it is sufficient to compute the homology of A_k, $k \ge 1$. $\square$

Lemma 5.6. *The homology of the complexes* A_{3k}, A_{3k+1} *and* A_{3k+2} *is periodic of period* $3n-1$. *More precisely,*

$$\begin{aligned}
&H_i(A_{3k}) \simeq K && \textit{for } i = k(3n-1),\ k \ge 1, && \textit{and is 0 otherwise,}\\
&H_i(A_{3k+1}) \simeq K && \textit{for } i = n + k(3n-1),\ k \ge 0, && \textit{and is 0 otherwise,}\\
&H_i(A_{3k+2}) = 0,\ \forall\, i,\ \forall k.
\end{aligned}$$

Proof. This lemma is proved by induction on k. The lemma is true for A_1 and A_2: A_1 is concentrated in degree n and its differential is zero; the complex (A_2, d) is $0 \to K(y \otimes y) \to Kz \to 0$ with $d(y \otimes y) = z$, then A_2 is acyclic. Assume that the lemma is true for $A_{3(k-1)+1}$ and $A_{3(k-1)+2}$ and prove it for A_{3k}, A_{3k+1} and A_{3k+2}.

We have the following natural short exact sequence

$$0 \to A_{k-2} \xrightarrow{\phi} A_k \xrightarrow{\psi} A_{k-1} \to 0,$$

where

$$\phi(x_1 \otimes \cdots \otimes x_l) = z \otimes x_1 \otimes \cdots \otimes x_l,\ \ \psi(y \otimes x_1 \otimes \cdots \otimes x_l) = x_1 \otimes \cdots \otimes x_l$$

and $\psi(z \otimes x_1 \otimes \cdots \otimes x_l) = 0$. It is easy to check that $d \circ \phi = -\phi \circ d$, $d \circ \psi = \psi \circ d$, ϕ is injective, ψ is surjective and $\operatorname{Im} \phi = \operatorname{Ker} \psi$. It yields a long exact sequence in homology

$$\cdots \to H_p(A_{k-2}) \xrightarrow{\phi} H_{p+2n-1}(A_k) \xrightarrow{\psi} H_{p+n-1}(A_{k-1}) \xrightarrow{\gamma} H_{p-1}(A_{k-2}) \to \cdots$$

where γ is the connecting morphism. Actually, the connecting morphism is induced by the morphism

$$\begin{array}{rccc} \gamma: & A_k & \to & A_{k-1} \\ & x = x_1 \otimes \cdots \otimes x_l & \mapsto & \displaystyle\sum_{i \mid x_i = y} x_1 \otimes \cdots \otimes \hat{x_i} \otimes \cdots \otimes x_l. \end{array}$$

Let us prove that A_{3k} satisfies the lemma. The result is obtained by combining the induction hypothesis for $A_{3(k-1)+1} = A_{3k-2}$ and $A_{3(k-1)+2} = A_{3k-1}$ with the long exact sequence in homology. Moreover ϕ induces an isomorphism

$$H_{n+(k-1)(3n-1)}(A_{3(k-1)+1}) \xrightarrow{\phi} H_{k(3n-1)}(A_{3k}).$$

Let $t_{3(k-1)+1}$ be a cycle whose homology class generates $H_{n+(k-1)(3n-1)}(A_{3(k-1)+1})$. Then, the homology class of $\phi(t_{3(k-1)+1}) = z \otimes t_{3(k-1)+1} =: t_{3k}$ generates $H_{k(3n-1)}(A_{3k})$.

We prove, as above, that A_{3k+1} satisfies the lemma. Moreover ψ induces the following isomorphism

$$H_{n+k(3n-1)}(A_{3k+1}) \xrightarrow{\psi} H_{k(3n-1)}(A_{3k}).$$

Let t_{3k+1} be a cycle whose homology class generates $H_{n+k(3n-1)}(A_{3k+1})$, and such that $\psi(t_{3k+1}) = t_{3k}$. Necessarily $t_{3k+1} = y \otimes t_{3k} + z \otimes \beta$ and $d(t_{3k+1}) = z \otimes \gamma(t_{3k}) - z \otimes d(\beta) = 0$. For instance, $\beta = y \otimes t_{3(k-1)+1}$ is suitable.

Remark. We have constructed, step by step, generators of $H_{i(3n-1)}(A_{3i})$, $1 \leq i \leq k$ (resp. $H_{n+i(3n-1)}(A_{3i+1}),\ 0 \leq i \leq k$), with one representative in A_{3i} (resp. in A_{3i+1}) denoted by t_{3i} (resp. t_{3i+1}) satisfying
$t_1 = y$,
$t_{3i} = z \otimes t_{3(i-1)+1}$,
$t_{3i+1} = y \otimes t_{3i} + z \otimes y \otimes t_{3(i-1)+1}$.

Finally, let us prove that A_{3k+2} satisfies the lemma. Applying last results and the long exact sequence in homology, we deduce that $H_i(A_{3k+2}) = 0$ for $i \neq k(3n-1) + 2n$ and $i \neq k(3n-1) + 2n - 1$. The following exact sequence remains (set $l = k(3n-1)$)

$$0 \to H_{l+2n}(A_{3k+2}) \xrightarrow{\psi} H_{l+n}(A_{3k+1}) \xrightarrow{\gamma} H_l(A_{3k}) \xrightarrow{\phi} H_{l+2n-1}(A_{3k+2}) \to 0.$$

It suffices to prove that γ is an isomorphism. But $\gamma(t_{3k+1}) = t_{3k} + y \otimes \gamma(t_{3k}) + z \otimes \gamma(\beta)$. Observing that $d(y \otimes \beta) = z \otimes \gamma(\beta) + y \otimes d(\beta) = z \otimes \gamma(\beta) + y \otimes \gamma(t_{3k})$, we deduce $[\gamma(t_{3k+1})] = [t_{3k}]$. Hence γ maps generator to generator, so γ is an isomorphism. □

Corollary 5.7. *a) If $n > 1$ is odd, then the Leibniz cohomology of the classical n-sphere is the free graded Leibniz-dual algebra generated by one generator in degree n.*

b) If n is even, then the Leibniz cohomology of the classical n-sphere is the graded Leibniz-dual algebra generated by two elements, denoted by a and b, such that $|a| = 3n - 1$, $|b| = n$, and satisfying the relations $b \cdot b = a \cdot b = 0$. Let a^k be the product $a \cdot a^{k-1}$, with $a^1 = a$. Then, a^k is a generator of $H\lambda^{k(3n-1)}(\mathbb{S}^n)$ and $b \cdot a^k$ is a generator of $H\lambda^{k(3n-1)+n}(\mathbb{S}^n)$.

Proof. The case n is even is immediate. We know that the homology of a graded Leibniz algebra is a graded Leibniz-dual coalgebra, so its cohomology is a graded Leibniz-dual algebra. Let Δ be the comultiplication defining the structure of graded Leibniz-dual coalgebra on $\mathcal{L}^!(\mathbb{S}^n)$. From an induction equality for Δ (see [O]) and the definition 1.5 of graded Leibniz-dual coalgebra we deduce the graded Leibniz-dual coalgebra structure $\tilde{\Delta}$ on the homology of the n-sphere. More explicitly, $\tilde{\Delta}$ is defined on the generators t_{3k} and t_{3k+1} by $\tilde{\Delta}(t_1) = 0$, $\tilde{\Delta}(t_3) = 0$, and

$$\begin{aligned}
\tilde{\Delta}(t_{3k+1}) &= \sum_{i=0}^{k-1} \gamma_i^k t_{3i+1} \otimes t_{3(k-i)}, \ \forall k \geq 1, \\
\tilde{\Delta}(t_{3k}) &= \sum_{i=1}^{k-1} \gamma_{i-1}^{k-1} t_{3i} \otimes t_{3(k-i)}, \ \forall k \geq 2,
\end{aligned}$$

where γ_i^k is given by induction:

$$\begin{aligned}
\gamma_0^k &= 1, \ \forall k \geq 1, \\
\gamma_i^k &= \gamma_{i-1}^{k-1} + (-1)^{i(k-i)} \gamma_{k-i-1}^{k-1}, \ \forall\, k \geq 1, \ \forall\, 1 \leq i < k.
\end{aligned}$$

Denote by a_k (resp. b_k) the linear dual of t_{3k} (resp. t_{3k+1}). Hence a_k is a generator of $H^{k(3n-1)}(\mathbb{S}^n)$ for $k \geq 1$ and b_k is a generator of $H^{n+k(3n-1)}(\mathbb{S}^n)$ for $k \geq 0$. Then, we have the relations

$$\begin{aligned}
b_i \cdot a_{k-i} &= \gamma_i^k b_k, \ \ \forall\, 0 \leq i \leq k-1, \ a_{k-i} \cdot b_i = 0, \ \forall\, 0 \leq i \leq k-1, \\
a_i \cdot a_{k-i} &= \gamma_{i-1}^{k-1} a_k, \ \forall\, 1 \leq i \leq k-1, \ \ b_i \cdot b_j = 0 \qquad \forall i, j.
\end{aligned}$$

By setting $a = a_1$ and $b = b_0$, it is easy to check that these conditions are equivalent to the conditions $b \cdot b = a \cdot b = 0$. □

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