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A BOUNDARY CRITERION FOR CUBULATION

By NICOLAS BERGERON and DANIEL T. WISE

Abstract. We give a criterion in terms of the boundary for the existence of a proper cocompact action of a word-hyperbolic group on a CAT(0) cube complex. We describe applications towards lattices and hyperbolic 3-manifold groups. In particular, by combining the theory of special cube complexes, the surface subgroup result of Kahn-Markovic, and Agol’s criterion, we find that every subgroup separable closed hyperbolic 3-manifold is virtually fibered.

1. Introduction. Let \( G \) be a finitely generated group with Cayley graph \( \Gamma \). A subgroup \( H \subset G \) is codimension-1 if it has a finite neighborhood \( N_r(H) \) such that \( \Gamma - N_r(H) \) contains at least two \( H \)-orbits of components that are deep in the sense that they do not lie in any \( N_s(H) \). For instance any \( \mathbb{Z}^n \) subgroup of \( \mathbb{Z}^{n+1} \) is codimension-1, and any infinite cyclic subgroup of a closed surface subgroup is as well.

Given a finite collection of codimension-1 subgroups \( H_1, \ldots, H_k \) of \( G \), Michah Sageev introduced a simple but powerful construction that yields an action of \( G \) on a CAT(0) cube complex \( C \) that is dual to a system of walls associated to these subgroups [16].

For each \( i \), let \( N_i = N_{r_i}(H_i) \) be an \( H_i \)-invariant neighborhood of \( H_i \) that separates \( \Gamma \) into at least two orbits of deep components. A wall associated to \( N_i \) is a partition \( \{ \hat{N}_i, \hat{N}_i \} \) consisting of \( H_i \)-invariant subsets called halfspaces, where \( N_i \subset \hat{N}_i \) and each component of \( \Gamma - N_i \) lies in one of the halfspaces. We shall moreover assume that both \( \hat{N}_i \) and \( \hat{N}_i \) are deep. More generally, the translated wall associated to \( gN_i \) is the partition \( \{ g\hat{N}_i, g\hat{N}_i \} \). We will implicitly associate each codimension-1 subgroup \( H_i \) with a chosen wall as above.

We presume a certain degree of familiarity with the details of Sageev’s construction here, but hope the interested reader will mostly be able to follow the arguments. We shall not describe the entire dual cube complex \( C \) here but will describe its 1-skeleton. A 0-cube of \( C \) is a choice of one halfspace from each wall such that each pair of chosen halfspaces have nonempty intersection and each element of \( G \) lies in all but finitely many of these chosen halfspaces. We regard each wall as facing the points in its chosen halfspace. Two 0-cubes are joined by a 1-cube precisely when their halfspace choices differ on exactly one wall.
The walls in \( \Gamma \) are in one-to-one correspondence with the hyperplanes of the \( \text{CAT}(0) \) cube complex \( C \) given by Sageev’s construction, and the stabilizer of each such hyperplane equals the codimension-1 subgroup that stabilizes the associated translated wall: The stabilizer of the hyperplane corresponding to a translated wall associated to \( gN_i \) is commensurable with \( gH_i g^{-1} \).

Cocompactness properties of the resulting action were analyzed in [17] where Sageev proved that:

**Proposition 1.1.** Let \( G \) be a word-hyperbolic group, and \( H_1, \ldots, H_k \) be a collection of quasiconvex codimension-1 subgroups. Then the action of \( G \) on the dual cube complex is cocompact.

We refer to [11] for a more elaborate discussion of the finiteness properties of the action obtained from Sageev’s construction, as well as for background and an account of the literature.

In parallel to Proposition 1.1, is a properness criterion which we state as follows (see for instance [11]). We use the notation \(#(p,q)\) for the number of walls separating \( p,q \).

**Proposition 1.2.** If \(#(1,g) \to \infty \) as \( d_\Gamma(1,g) \to \infty \) then \( G \) acts properly on \( C \).

An alternative to Proposition 1.2 is the following:

**Proposition 1.3.** Let \( H_1, \ldots, H_k \) be a collection of quasiconvex codimension-1 subgroups of the word-hyperbolic group \( G \).

Suppose that for each infinite order element \( g \) of \( G \), there is a translate \( fN_i \) such that \( g^{-n} fN_i \) and \( g^n fN_i \) are separated by \( fN_i \) for some \( n \) in the sense that the corresponding partitions are nested:

\[
\text{f}^{N_i} \subset g^{\pm n} \text{f}^{N_i} \quad \text{and} \quad \text{f}^{N_i} \subset g^{\mp n} \text{f}^{N_i}.
\]

Then \( G \) acts properly on the dual \( \text{CAT}(0) \) cube complex.

**Sketch.** By Proposition 1.1, \( G \) acts cocompactly, so it suffices to show that the stabilizer of each 0-cube of \( C \) is finite. If an infinite order element \( g \) fixes a 0-cube \( v \) of \( C \) then \( g^n \) would fix \( v \) for each \( n \). But then the sequence \( \{g^{nr} fN_i : r \in \mathbb{Z}\} \) is shifted by \( g^n \) and so the walls in this sequence would all face in the same direction. By traveling in one or the other direction of this infinite sequence, we see that there are infinitely many walls that do not face \( \{1 \in G \), which contradicts that \( v \) is a 0-cube. \( \square \)

We have found that in many cases it is difficult and sometimes quite messy to directly verify Proposition 1.2 or 1.3. Moreover, in many cases, when there is a profusion of available codimension-1 subgroups it is desirable to choose them in a
flexible enough way so that there are sufficiently many to satisfy the properness cri-
teron of Proposition 1.3, while maintaining a finite number so that Proposition 1.1
for cocompactness is satisfied.

We propose the following criterion which is our main result:

**Theorem 1.4.** Let $G$ be word-hyperbolic. Suppose that for each pair of dis-
tinct points $(u,v) \in (\partial G)^2$ there exists a quasiconvex codimension-1 subgroup $H$
such that $u$ and $v$ lie in $H$-distinct components of $\partial G - \partial H$.

Then there is a finite collection $H_1, \ldots, H_k$ of quasiconvex codimension-1 sub-
groups such that $G$ acts properly and cocompactly on the resulting dual $\text{CAT}(0)$
cube complex.

It is clear how the hypothesis of Theorem 1.4 relates to Proposition 1.3. Indeed,
let $g$ be an infinite order element in $G$. According to Theorem 1.4 there exists
a quasiconvex codimension-1 subgroup $H$ such that the attracting and repelling
limit points $g^{\pm \infty}$ lie in distinct components of $\partial G - \partial H$. The iterates $g^n \partial H$ form a
nested sequence in $\partial G$, and hence so do the corresponding walls for large multiples
of $n$.

We prove Theorem 1.4 in Section 2. The notion of virtual specialness is briefly
recalled in Section 3.

As an application we build upon a fundamental new result of Kahn-Markovic
to get the following theorem, see Section 4.

**Theorem 1.5.** Let $M$ be a closed hyperbolic 3-manifold. Then $\pi_1 M$ acts
freely and cocompactly on a $\text{CAT}(0)$ cube complex.

It follows from Theorem 1.5 that if $\pi_1 M$ is subgroup separable—i.e., every
f.g. subgroup is closed in the profinite topology—then $\pi_1 M$ is virtually special and
hence, $M$ is virtually fibered by Agol’s criterion, see Section 4 for definitions. With
some more work this holds under the milder assumption that the quasi-fuchsian
(quasiconvex) surface subgroups are separable. A much more elaborate proof that
$\pi_1 M$ is subgroup separable and virtually special is given in [18] when $M$ has a
geometrically finite incompressible surface.

In Section 5 we extend Theorem 1.4 to the relatively hyperbolic setting. As an
application we revisit our earlier work towards the cubulation of arithmetic lattices
in Section 6 and extend it to nonuniform lattices.

Over the last years, lectures on special cube complexes have included the ex-
planation that the existence of sufficiently many separable surface subgroups im-
plies the virtual specialness of hyperbolic 3-manifold groups. It is satisfying to
record this in writing in view of the results of Kahn-Markovic.

Since a first version of this paper was circulated, Guillaume Dufour has in-
formed us that, motivated by the cubulation of compact hyperbolic 3-manifolds, he
has obtained a criterion similar to Theorem 1.4 in the case of cocompact lattices of
$\mathbb{H}^n$. 
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2. Soft separation. Let $X$ be a compact metrizable space. A group $G$ acts by homeomorphisms on $X$ as a convergence group if it acts properly discontinuously on the space of pairwise distinct triples in $X$ (see below). Equivalently, $G$ acts as a convergence group if for every sequence $(g_n)_n$ in $G$, there exists a subsequence $(g_{n_k})_k$ and $a, b$ in $X$ such that the sequence $(g_{n_k})_k$ converges uniformly on compact subsets of $X - \{a\}$ to the point $b$. The group $G$ is a uniform convergence group if the action on the space of of pairwise distinct triples in $X$ is also cocompact. In that case, $X$ is unique up to equivariant homeomorphism. If $G$ acts properly discontinuously on some locally compact Hausdorff space $Y$, then there is a unique connected compactification $Y \cup X$ of $Y$ that the action of $G$ extends to.

Bowditch proved that any properly discontinuous group action on a locally compact $\delta$-hyperbolic metric space extends to a convergence group action on the boundary [6].

Let $G$ be a word-hyperbolic group. Then $G$ acts properly discontinuously and cocompactly on its Cayley graph $\Gamma$ and we may identify the Gromov boundary $\partial \Gamma$ with $\partial G$ (see [8] for more details). Note that $X = \Gamma \cup \partial \Gamma$ has a natural compact topology and is metrizable but has no preferred metric. The group $G$ acts as a convergence group on $X$ and as a uniform convergence group on $\partial G$, see [6, Prop 1.12 and 1.13]. That this last property characterizes word-hyperbolic groups was a longstanding open problem until it was resolved by Bowditch.

Triple space is the subspace $T \subset (\partial G)^3$ consisting of pairwise distinct triples of points $\{(x, y, z) \in (\partial G)^3 : x \neq y \neq z \neq x\}$. Note that $T = \emptyset$ precisely when $G$ is elementary. Since $G$ acts as a uniform convergence group on $\partial G$, it acts properly and cocompactly on $T$. Here the action is the restriction of the diagonal action $g(x, y, z) = (gx, gy, gz)$ induced by the action of $G$ on $\partial G$. It easily follows that:

**Lemma 2.1.** For each neighborhood $M'$ of $(u, v, w) \in T$ there is a subneighborhood $M \subset M'$ containing $(u, v, w)$ and a finite subgroup $K$ such that $gM \cap M = M$ for $g \in K$ and $gM \cap M = \emptyset$ for $g \notin K$.

**Proof.** By properness we can choose $M'' \subset M'$ to be a neighborhood of $(u, v, w)$ such that the set of elements $K = \{g \in G : gM'' \cap M'' \neq \emptyset\}$ is finite.

We can further assume that $K$ is the stabilizer of $(u, v, w)$. Indeed, if $g(u, v, w) \neq (u, v, w)$ then we can make $M''$ a bit smaller to discard $g$ as follows: by Hausdorffness, let $U, V$ be disjoint neighborhoods of $(u, v, w)$ and $g(u, v, w)$, and let $S = U \cap g^{-1}V$. We then have $gS \cap S = \emptyset$, and replace $M''$ by $M'' \cap S$.

Finally, we let $M = \cap_{g \in K} gM''$ to ensure that $gM = M$ for all $g \in K$. \qed

**Proof of Theorem 1.4.** When $T = \emptyset$, the group $G$ is elementary and hence acts properly on a CAT(0) cube complex consisting of a single point or a line. For the remainder of the proof we will assume that $T$ is nonempty.
For each pair of distinct points $u, v \in \partial G$, by hypothesis there is a quasiconvex codimension-1 subgroup $H$ such that $u$ and $v$ are separated by $\partial H \subset \partial G$. Let $U_H, V_H$ be the $u, v$ components of $\partial G - \partial H$ so $\partial H$ separates $U_H, V_H$. We let $N$ be a finite neighborhood of $H$ such that $\Gamma - N$ contains components $Q_u$ and $Q_v$ with $\partial Q_u = U_H$ and $\partial Q_v = V_H$. Note that $\partial N = \partial H$. Let $\overleftarrow{N} = HQ_u$, and let $\overrightarrow{N} = \Gamma - HQ_u$. For each $j \in G$ we define the translated wall associated to $jH$ to be the partition $\{ j\overleftarrow{N}, j\overrightarrow{N} \}$.

Let $w \in \partial G - \{ u, v \}$. By Hausdorffness of $\partial G$, there exists an open neighborhood $W'$ of $w$ that is disjoint from open neighborhoods of $u$ and $v$. We let $U' = U_H - W'$ and $V' = V_H - W'$, so that $M = U' \times V' \times W' \subset T$. Then $M' = U' \times V' \times W'$ is an open neighborhood of $(u, v, w)$ such that $U', V'$ are separated by $\partial H$. By Lemma 2.1 we can refine this to an open neighborhood $M = U \times V \times W$ of $(u, v, w)$ such that:

1. $U$ and $V$ are separated by $\partial H$
2. $gM \cap M = M$ for those $g$ in some finite subgroup of $G$, and otherwise $gM \cap M = \emptyset$.

Observe that $GM$ is saturated with respect to the quotient $T \to \overline{T} = G \setminus T$, in the sense that $GM$ is the union of fibers of $T \to \overline{T}$.

The above construction applies to each point $(u, v, w) \in T$ and we shall use the explicit notation $M_{(u, v, w)}$ for the neighborhood $M$ chosen above.

Consider the following collection of open saturated neighborhoods:

\[
\{ \cup_{j \in G} jM_{(u, v, w)} : (u, v, w) \in T \}.
\]

The collection forms an open covering of $T$.

Since $G$ is word-hyperbolic, it acts as a uniform convergence group on $\partial G$, and so the quotient $\overline{T}$ is compact. Since each element of collection (1) is saturated with respect to $T \to \overline{T}$, the compactness of $\overline{T}$ assures that there is a finite subcollection that also covers $T$.

For each infinite order $g \in G$ consider a point $(g^{+\infty}, g^{-\infty}, w') \in T$. We have shown that $(g^{+\infty}, g^{-\infty}, w')$ lies in one of the sets of our finite subcollection, so in particular, $(g^{+\infty}, g^{-\infty}, w')$ must lie in some $jM_{(u, v, w)}$ associated to some point $(u, v, w) \in T$ as above, where $M_{(u, v, w)} = U_{(u, v, w)} \times V_{(u, v, w)} \times W_{(u, v, w)}$ and where the associated quasiconvex codimension-1 subgroup $H_{(u, v, w)}$ has the property that $\partial H_{(u, v, w)}$ separates $U_{(u, v, w)}$ from $V_{(u, v, w)}$. Thus $g^{+\infty} \in jU_{(u, v, w)}$ and $g^{-\infty} \in jV_{(u, v, w)}$ and these are separated by $j\partial H_{(u, v, w)}$.

Now $G$ acts as a convergence group on $X$ and $N_{(u, v, w)} \cup \partial H_{(u, v, w)}$ is a compact subset of $X - \{ jg^{-\infty} \}$. This implies that $j^{-1}g^{\pm n}j$ converges to $j^{-1}g^{\pm \infty}$ uniformly on $N_{(u, v, w)} \cup \partial H_{(u, v, w)}$. Thus the translates $j^{-1}g^{n}jN_{(u, v, w)}$ and $j^{-1}g^{n}jN_{(u, v, w)}$ are separated by $N_{(u, v, w)}$ for some $n$. Equivalently $g^{-n}jN_{(u, v, w)}$ and $g^{n}jN_{(u, v, w)}$ are separated by $jN_{(u, v, w)}$. We note that $\overrightarrow{N}_{(u, v, w)}$ and $\overleftarrow{N}_{(u, v, w)}$ are both deep since
they respectively contain $g^+\infty$ and $g^-\infty$ in their boundaries. Proposition 1.3 thus implies that $G$ acts properly on the CAT(0) cube complex dual to our finite collection of $H_{(u,v,w)}$. Theorem 1.4 finally follows from Proposition 1.1.

Remark 2.2. When $H$ is a finite codimension-1 subgroup of $G$, then $\partial H = \emptyset$, and $\partial G$ is already disconnected. We can regard $\partial G$ as being “separated” by $\partial H$ by letting the two parts consist of $\partial \overrightarrow{N}$ and $\partial \overleftarrow{N}$, where $\Gamma = \overrightarrow{N} \sqcup \overleftarrow{N}$ is a partition into a pair of deep $H$-invariant subspaces. We will revisit this point in Section 5 where connectivity of $\partial G$ arises in the nonuniform generalization of Theorem 1.4.

3. Virtual specialness. A nonpositively curved cube complex is special if it admits a local isometry to the cube complex associated with a right-angled Artin group. Special cube complexes were introduced in [10] where they were initially defined in terms of illegal configurations of immersed hyperplanes. Recall that a hyperplane in a CAT(0) cube complex is a connected separating subspace consisting of a maximally extending sequence of “midcubes” each of which cuts its ambient cube in half. For each hyperplane $\overrightarrow{D} \subset \overrightarrow{C}$ in a CAT(0) cube complex $\overrightarrow{C}$, one obtains an immersed hyperplane $D \to C$ where $D = \text{Stabilizer}_{\pi_1 C}(\overrightarrow{D}) \setminus \overrightarrow{D}$. Recall that a subset $S$ of $\pi_1 C$ is separable if $S$ is closed in the profinite topology of $\pi_1 C$, which means that $S$ is the intersection of cosets of finite index subgroups of $\pi_1 C$.

Some of the most important properties of a special cube complex are [10]:

**Proposition 3.1.** Let $C$ be a special cube complex. Then
(1) $\pi_1 C$ embeds in a right-angled Artin group.
(2) Thus $\pi_1 C$ is residually torsion-free nilpotent.
(3) If $C$ is compact and $\pi_1 C$ is word-hyperbolic then every quasiconvex subgroup $H$ of $\pi_1 C$ is separable.

**Remark 3.2.** The proof of Proposition 3.1(3) produces a finite index subgroup $V$ of $\pi_1 C$ that retracts onto $H$, see [10, Thm 7.3]. This means that $H$ is contained in $V$ and there is a homomorphism $\rho : V \to H$ whose restriction to $H$ is the identity. Equivalently, $V$ is a semidirect product of $H$ with a normal subgroup $N$.

A cube complex is virtually special if it has a special finite cover. Likewise a group $G$ is virtually [compact] special if $G$ has a finite index subgroup that acts freely [and cocompactly] on a CAT(0) cube complex $C$ with special quotient. (We shall more generally say that a group $G$ virtually has a property $P$ if there is a finite index subgroup of $G$ having property $P$.) Not every nonpositively curved cube complex is virtually special, but the following criteria hold:

**Proposition 3.3.** (1) A nonpositively curved cube complex with finitely many immersed hyperplanes is virtually special if and only if $\pi_1 D \pi_1 E$ is separable for each pair $D, E$ of crossing immersed hyperplanes in $C$. 
(2) A compact nonpositively curved cube complex with word-hyperbolic \( \pi_1 \) is virtually special if each quasiconvex subgroup is separable.

(3) A compact nonpositively curved cube complex with word-hyperbolic \( \pi_1 \) is virtually special if \( \pi_1 D \) is separable for each immersed hyperplane.

(4) A compact nonpositively curved cube complex with word-hyperbolic \( \pi_1 \) is virtually special if each immersed hyperplane is embedded.

Note that Criteria (1) and (2) were obtained in [10], Criterion (3) is a more difficult criterion established in [9], and Criterion (4) is a considerably deeper result (depending also on the proof of Criterion (3)) that is established in [18].

4. Cubulating Hyperbolic 3-manifolds. An immersed surface \( f : F \to M \) in a hyperbolic 3-manifold \( M \) is quasi-fuchsian if \( \partial \tilde{f}(\tilde{F}_i) \subset \partial \mathbb{H}^3 \) is a topological circle. Jeremy Kahn and Vladimir Markovic obtained the outstanding result that every hyperbolic 3-manifold contains a closed quasi-fuchsian immersed surface. In fact, they proved the following [12, Thm. 1.1]:

**Proposition 4.1.** Let \( M \) be a closed hyperbolic 3-manifold, and regard \( \pi_1 M \) as acting on \( \mathbb{H}^3 \cong \tilde{M}^3 \). For each great circle \( C \subset \partial \mathbb{H}^3 \) there is a sequence of immersions \( (f_i : F_i \to M)_i \), where each \( F_i \) is a surface and \( f_*(\pi_1(F_i)) \) is a quasi-fuchsian subgroup in \( \pi_1 M \), such that \( \partial \tilde{f}_i(\tilde{F}_i) \subset \partial \mathbb{H}^3 \) pointwise converges to \( C \).

An immediate consequence is that:

**Corollary 4.2.** Let \( M \) be a closed hyperbolic 3-manifold. For each pair of distinct points \( p, q \in \partial \tilde{M} \) there is an immersed quasi-fuchsian surface \( F \to M \) and a choice of lift of its universal cover \( \tilde{F} \subset \tilde{M} \) such that \( \partial \tilde{F} \) separates \( p, q \) in \( \partial \tilde{M} \).

**Proof.** Let \( C \) be a great circle that separates \( p, q \) (e.g. choose a perpendicular bisector of the geodesic from \( p \) to \( q \) in the sphere \( \partial M \)). By Proposition 4.1, let \( (F_i \to M)_i \) be a sequence of surfaces whose universal covers have boundaries that converge to \( C \).

For sufficiently large \( i \), the \( \epsilon \)-neighborhood of \( C \) in \( \partial \tilde{M} \) has the property that \( \partial \tilde{F}_i \) separates its two bounding circles (just like \( C \)). Thus \( \partial \tilde{F}_i \) separates \( p, q \) in \( \partial \tilde{M} \).

By combining Corollary 4.2 with Theorem 1.4 and Proposition 1.1, we obtain Theorem 1.5.

**Remark 4.3.** The proof of Theorem 1.5 depends crucially on Proposition 4.1. We note that Marc Lackenby proved that any arithmetic 3-manifold contains a surface group [14]. It follows from his proof that when the 3-manifold is closed the surface subgroup he constructs is quasiconvex. Theorem 1.5 for closed arithmetic 3-manifolds thus follows from the combination of Lackenby’s theorem and Theorem 6.1.
Let $M$ be a closed hyperbolic 3-manifold $M$. The celebrated “virtual Haken conjecture” for $M$ claims that some finite cover of $M$ contains a $\pi_1$-injective embedded surface. The virtual Haken conjecture would follow by combining the result of Kahn-Markovic with the well-known conjecture that:

\[(2) \quad \text{quasi-fuchsian surface subgroups of } \pi_1 M \text{ are separable,}\]

which means that quasi-fuchsian surface subgroups are closed in the profinite topology of $\pi_1 M$. In fact it follows from Proposition 3.3(1) and the word-hyperbolicity of $\pi_1 M$ that if moreover all quasiconvex subgroups are closed in the profinite topology then $\pi_1 M$ is virtually special.

Agol shows even more: if $\pi_1 M$ is virtually special then $M$ is virtually fibered [1] in the sense that some finite cover of $M$ fibers over the circle. Specifically, he introduced the condition residually finite rational solvable (RFRS) and proves the following:

**Proposition 4.4.** Let $M$ be a closed 3-manifold. If $\pi_1 M$ is RFRS then $M$ is virtually fibered.

The RFRS condition on $G$ says that there is a decreasing sequence of finite index subgroups $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ such that $\cap G_i = \{1_G\}$ and such that $N_i \subset G_{i+1}$ for each $i$, where $G_i/N_i$ is torsion-free abelian. The condition essentially states that nontrivial elements can be detected in finite quotients by employing the free-abelianization. It obviously holds for free groups and free-abelian groups and in fact, any right-angled Artin groups satisfy the RFRS condition. Agol verified that right-angled Coxeter groups are RFRS, and this virtually obtains RFRS for right-angled Artin groups [1, Thm 2.2].

Using Proposition 3.3(3)—shown in [9]—we obtain the following result:

**Theorem 4.5.** If quasi-fuchsian surface subgroups of a hyperbolic 3-manifold $M$ are separable, then $M$ is virtually fibered.

**Proof.** By Theorem 1.5, $\pi_1 M = \pi_1 C$ where $C$ is a compact nonpositively curved cube complex. As surface subgroups are separable, we apply Proposition 3.3(3) to obtain a finite special cover of $C$ with $\hat{C}$. As recalled above this implies that $\pi_1 C$ virtually satisfies the (RFRS) condition of Agol. Consequently, the corresponding finite cover $\hat{M}$ satisfies Agol’s criterion, and so $\hat{M}$ has a finite cover that fibers.

According to [18] it even suffices for $\pi_1 M$ to have a single separable quasi-fuchsian subgroup to obtain virtual specialness and hence virtual fibering.

5. **Relatively hyperbolic extension.** In this section we generalize Theorem 1.4 to a relatively hyperbolic situation. The notion of a relatively hyperbolic group was introduced by Gromov and has been developed by various authors.
We mainly follow Brian Bowditch’s and Asli Yaman’s treatments as developed in [5, 19]. Let $G$ be a finitely generated group. A peripheral structure on $G$ consists of a set $\mathcal{G}$ of infinite subgroups of $G$ such that each $P \in \mathcal{G}$ is equal to its normalizer in $G$, and each $G$-conjugate of $P$ lies in $\mathcal{G}$. We refer to an element of $\mathcal{G}$ as a maximal parabolic subgroup.

The group $G$ acts on $\mathcal{G}$ by conjugation $P \mapsto P^g$. Let us assume that $\mathcal{G}$ contains only finitely many distinct $G$-conjugates, and let $P_1, \ldots, P_s$ be a choice of representatives so that $\mathcal{G} = \{P_i^g : i = 1, \ldots, s, g \in G\}$.

The group $G$ is hyperbolic relative to $\mathcal{G}$ or relative to $P_1, \ldots, P_s$ if $G$ admits an action on a connected graph $K$ with the following properties:

1. $K$ is hyperbolic and each edge of $K$ is contained in only finitely many length $n$ circuits for each $n$,
2. there are finitely many $G$-orbits of edges, and each edge stabilizer is finite,
3. the elements of $\mathcal{G}$ are precisely the infinite vertex stabilizers of $K$.

We now define the boundary of $G$ relative to $P_1, \ldots, P_s$. Let $\partial K$ denote the Gromov boundary of $K$ and let $V_\infty(K)$ be the subset of $K^0$ consisting of infinite valence vertices. The set $V_\infty(K) \cup \partial K$ has a natural topology as a compact Hausdorff space (see [19, p.37]). This topological space, which we denote by $\partial G$, only depends on $G$ and $P_1, \ldots, P_s$.

The group $G$ acts as a convergence group on $\partial G$. It thus follows from the dynamical definition of a convergence group that each infinite order element $g \in G$ has either one or two fixpoints in $\partial G$. The element $g$ is loxodromic if it has two fixpoints. The conjugates of $P_1, \ldots, P_s$, are precisely the maximal infinite subgroups that fix a point $p \in \partial G$ (maximal parabolic subgroups). Such a subgroup acts properly-discontinuously and cocompactly on $\partial G - \{p\}$. A parabolic point is the fixpoint of some $P \in \mathcal{G}$.

Let $T$ denote the triple space of $\partial G$. The quotient $\bar{T} = G \setminus T$ is the union of a compact set and finitely many quotients of “cusp neighborhoods” of parabolic points.

The triple space $T$ can be compactified by adding a copy of $\partial G$, see [6]. We thus topologize $T \cup \partial G$ so that a sequence $(u_i, v_i, w_i)$ in $T$ converges to $x$ in $\partial G$ if and only if at least two of the sequences $(u_i), (v_i)$ and $(w_i)$ converge to $x$ in $\partial G$.

A cusp neighborhood of a parabolic point $p \in \partial G$ is defined as follows: Let $C$ be a compact subset of $\partial G - \{p\}$ such that $\text{Stab}_G(p)C = \partial G - \{p\}$ and let $K$ be a compact subset containing an open neighborhood of $C$ in $T \cup \partial G$. A cusp neighborhood of $p$ in $T$ is an open subset of the form $T - \text{Stab}_G(p)K$.

Let $p_1, \ldots, p_s$ be representatives from the $G$-classes of parabolic points in $\partial G$ such that $P_i = \text{Stab}_G(p_i)$. The quotient $\bar{T}$ becomes a compact Hausdorff space upon the addition of the points $Gp_1, \ldots, Gp_s$, so that a neighborhood basis of $Gp_i$ is given by $\{Gp_i\} \cup G \setminus B$ where $B$ ranges over cusp neighborhoods of $p_i$. In particular we may choose an open cusp neighborhood $B_i$ of $p_i$, $i = 1, \ldots, s$, such that the $gB_i$, $g \in G/P_i$, $i = 1, \ldots, n$, are pairwise disjoint and the following space $T_{\text{thick}}$
projects onto a compact subspace of $\tilde{T}$:

\[
T_{\text{thick}} := T - \bigcup_{i=1}^{n} \bigcup_{g \in G} gB_i.
\]

Given a subgroup $H$ of $G$ we let $\partial H$ be the set of all accumulation points of $H$-orbits in $\partial G$. Note that if $N$ is a finite neighborhood of $H$ in the Cayley graph $\Gamma$ of $G$, we define $\partial N \subset \partial G$ as the intersection of $\partial G$ with the closure in $K \cup \partial K$ of the set $\{gk : g \in N \cap G\}$ where $k$ is some point of $K$. Observe that $\partial N$ does not depend upon the choice of $k \in K$ and that $\partial N$ equals $\partial H$.

Theorem 1.4 now generalizes as follows:

**Theorem 5.1.** Let $G$ be hyperbolic relative to a collection of maximal parabolic subgroups $P_1, \ldots, P_s$. Suppose the following hold:

1. For each parabolic point $p$, there exist finitely many quasi-isometrically embedded f.g. codimension-1 subgroups of $G$ whose intersections with $\text{Stab}_G(p)$ yield a proper action of $\text{Stab}_G(p)$ on the corresponding dual cube complex.
2. For each pair of distinct points $u, v \in \partial G$ there is a quasi-isometrically embedded f.g. codimension-1 subgroup $H$ such that $u, v$ lie in $H$-distinct components of $\partial G - \partial H$.

Then there exist finitely many quasi-isometrically embedded f.g. codimension-1 subgroups of $G$ such that the action of $G$ on the dual cube complex is proper.

An important ingredient in the proof of Theorem 5.1 is a relatively hyperbolic generalization of Proposition 1.3. This generalization is Lemma 5.4 which is the goal of the following subsection.

### 5.1. Relatively hyperbolic axis separation

Let $G$ be hyperbolic relative to $P_1, \ldots, P_s$. Let $H_1, \ldots, H_k$ be quasi-isometrically embedded f.g. codimension-1 subgroups of $G$. As in the introduction, let $N_i$ be a neighborhood of $H_i$ that separates the Cayley graph, let $W_i$ denote the wall $\{\overline{N_i^+}, \overline{N_i^-}\}$, and let $C$ be the dual cube complex.

While Proposition 1.3 immediately generalizes under the assumption that $C$ is locally finite, this is itself not always easily deduced. In fact, it is not always the case that $C$ is locally finite—even in the relatively hyperbolic case. For instance, let $G$ be the free product $P_1 \ast P_2$ where each $P_i$ is the fundamental group of a locally-infinite cube complex, and note that $G$ is hyperbolic relative to $P_1, P_2$.

The main result in [11] can be stated as follows:

**Proposition 5.2.** Let $G$ be a f.g. group that is hyperbolic relative to a collection of parabolic subgroups $P_1, \ldots, P_s$. Let $H_1, \ldots, H_k$ be a collection of quasi-isometrically embedded f.g. codimension-1 subgroups of $G$. Let $C$ denote the CAT(0) cube complex dual to the $G$-translates of $W_1, \ldots, W_k$. For each $i$, let $C_i$ denote the CAT(0) cube complex dual to the walls in $P_i$ corresponding to the
nontrivial walls obtained from intersections with translates of the $W_i$, and note that $C_i$ embeds in $C$ as a convex subcomplex. Then:

(1) there exists a compact subcomplex $K$ such that $C = GK \cup_{i=1}^{s} GC_i$, and

(2) $g_iC_i \cap g_jC_j \subset GK$ unless $i = j$ and $g_j^{-1}g_i \in \text{Stabilizer}(C_i)$.

**Remark 5.3.** We may further assume that $GK$ is connected. Indeed, since $G$ is finitely generated, and $C$ is connected, one can add a collection of paths $S_i$ to $K$ such that each $S_i$ starts at the basepoint in $K$ and ends at the translate of this basepoint by the $i$th generator of $G$.

We shall now state a properness criterion. Under the hypothesis of Proposition 5.2 we say that an infinite order element $g$ of $G$ satisfies the axis separation condition if there is a translate $fW_i$ such that the walls $g^{-n}fW_i$ and $g^n fW_i$ are separated by $fW_i$ for some $n$, in the sense that the partitions are nested.

**Lemma 5.4.** The group $G$ acts properly on the dual cube complex $C$ provided the following two conditions both hold:

(1) Each $P_i$ acts properly on $C$. More explicitly, $P_i$ stabilizes and acts properly on a convex subcomplex $C_i \subset C$. Moreover, $C_i$ is isomorphic to the cubulation of $P_i$ induced by intersections with the relevant $G$-translates of the walls.

(2) The axis separation condition holds for loxodromic elements.

Before proceeding to the main part of the proof of Lemma 5.4 we prove the following partial result. We follow the notation of Proposition 5.2 and assume that $GK$ is connected by Remark 5.3.

**Lemma 5.5.** The group $G$ acts properly on $GK$.

**Proof.** Since $G$ acts cocompactly on $GK$, it suffices to show that no vertex of $GK$ has infinite stabilizer.

Let $v$ be a vertex of $GK$. We first prove that $	ext{Stabilizer}(v)$ cannot contain a loxodromic element. Assume to the contrary that some loxodromic element $g$ stabilizes $v$. Since the axis separation condition holds for $g$, there is a translate $fW_i$ and an integer $n$ such that the walls $g^{-n} fW_i$ and $g^n fW_i$ are separated by $fW_i$. The sequence $\{g^{nr} fW_i : r \in \mathbb{Z}\}$ of disjoint separating walls is then shifted by $g^n$. As $g$ fixes $v$ we conclude that the walls $g^{nr}W_i$ have to face the same direction in $v$. This violates the property of a vertex (or rather 0-cube) that all but finitely many walls face $1 \in G$.

Since $	ext{Stabilizer}(v)$ contains no loxodromic element, it is either elliptic and hence finite, or it is contained in some parabolic subgroup $P = hP_i h^{-1}$. By hypothesis, $P$ acts properly on its stabilized subcomplex $hC_i$. As both $v$ and the convex subcomplex $hC_i$ are stabilized, the point $v'$ in $hC_i$ that is closest to $v$ is also stabilized by $	ext{Stabilizer}(v)$. But $	ext{Stabilizer}(v')$ is finite, so $	ext{Stabilizer}(v)$ is finite as well.
Proof of Lemma 5.4. Let $B$ be a finite ball in $C$. Then $B \cap GK$ lies in a finite ball $B'$ of $GK$. Since $K$ is compact, $G$ obviously acts cocompactly on $GK$, and by Lemma 5.5, $G$ acts properly on $GK$. We thus see that $B'$ intersects finitely many sets of the form $P_i^0K$. Thus $B$ lies in the union of $B'$ and finitely many $hC_i$ that intersect $B'$. (We note that the radius of $B'$ in $GK$ could be significantly larger than the radius of $B$ in $C$, as the map $GK \hookrightarrow C$ might not be a quasi-isometric embedding.) For each $i \in \{1, \ldots, s\}$ we let $J_i$ denote the finite subset of $G$ consisting of elements $j$ with $jC_i \cap B' \neq \emptyset$.

We may now conclude the proof of the Proposition. Suppose to the contrary that the set $S = \{g \in G : gB \cap B \neq \emptyset\}$ is infinite. There are finitely many values of $g$ such that $gB' \cap B' \neq \emptyset$, thus finitely many values where $g(B \cap GK) \cap B = g(B \cap GK) \cap (B \cap GK)$ is nonempty.

It hence follows from Proposition 5.2(2) that for some $i$ and $j_1, j_2 \in J_i$, there are infinitely many $g \in S$ such that $g(j_1C_i - GK) \cap (j_2C_i - GK) \neq \emptyset$. There are finitely many $jC_i$ that intersect $B'$, so again Proposition 5.2(2) implies that there exists $i$ and $j \in J_i$ and infinitely many $g \in S$ such that $g(jC_i - GK) \cap (jC_i - GK) \neq \emptyset$. So $j^{-1}g \in \text{Stabilizer}_{C_i}$ for infinitely many values of $g \in S$. Note that $P_i$—being a maximal parabolic subgroup—is not commensurated by any larger subgroup so Stabilizer$_{C_i} = P_i$. We thus have proved that $S \cap jP_i j^{-1}$ is infinite. Furthermore, if $g \in S$ we have $gB \cap B \neq \emptyset$ and the translation length of $g$ in $C$ is bounded by $2r$ where $r$ is the radius of the ball $B$. We conclude that there are infinitely many elements $g \in jP_i j^{-1}$ of translation length bounded by $2r$. This contradicts (the hypothesis) that the parabolic subgroup $jP_i j^{-1}$ acts properly on its cubulation $jC_i$. \hfill \Box

5.2. Proof of Theorem 5.1. While $\partial G$ is always connected when $G$ is an $\text{SO}(n, 1)$ lattice, as was addressed in Remark 2.2 in the hyperbolic case, in general it is possible that $\partial G$ is not connected. This creates some technical difficulties in the relatively hyperbolic case which we address through a fundamental result of Bowditch that we now describe. The reader may wish to gloss over this technicality as it is not essential to the main idea of the proof.

Bowditch proved the first part of the following result in [5, Prop. 10.2] and the second part in [5, Prop. 10.1].

**Proposition 5.6.** (Accessability Splitting) Let $G$ be hyperbolic relative to $P_1, \ldots, P_s$.

1. $G$ has a splitting as a graph of groups where each edge group $G_e$ is finite and each vertex group $G_v$ does not split nontrivially over a finite subgroup relative to peripheral subgroups.

2. For each vertex group $G_v$ of the splitting, $\partial G_v$ is a connected component of $\partial G$. 
Remark 5.7. As each edge group $G_e$ is finite, we have $\partial G_e = \emptyset$. Using the Bass-Serre tree of the accessibility splitting as a guide, we regard the translates of $\partial G_e = \emptyset$ as “separating” $\partial G$ according to the way a corresponding edge of the Bass-Serre tree separates its boundary (see Remark 2.2).

A more important difference between the proofs of Theorems 5.1 and 1.4, is that the quotient $\overline{T} = G \setminus T$ is no longer compact. Instead we use the decomposition in Equation (3) and the following:

**Lemma 5.8.** Let $u, v$ be distinct non-parabolic points in the same component of $\partial G$. There exists $w \in \partial G$ such that

$$ (u, v, w) \in T_{\text{thick}}. $$

**Proof.** Let $L$ be the component of $\partial G$ containing $u$ and $v$. Let $\phi(L)$ denote the image of $L$ in $T \cup \partial G$ under the continuous map $\phi(w) = (u, v, w)$. Since $\phi(L)$ is connected and Hausdorff and contains $\phi(u)$ and $\phi(v)$, it must contain a third point $\phi(w)$ that lies in $T$.

If $\phi(w)$ doesn’t lie in any cusp neighborhood then we are done, so let $B$ denote a cusp neighborhood containing $\phi(w)$. Observe that the closure of $B$ contains a single boundary point and thus contains neither $\phi(u)$ nor $\phi(v)$.

Since $T \cup \partial G$ is a compact Hausdorff space we may choose $U, V$ to be open neighborhoods of $\phi(u), \phi(v)$ that are also disjoint from the closure of $B$. Now, suppose that $\phi(L)$ is covered by $U, V$ and a collection of cusp neighborhoods, and observe that this collection must include $B$. Let $A$ denote the union of $U, V$ and all cusp neighborhoods besides $B$. Then $A, B$ provides a separation of $\phi(L)$—which is impossible. \hfill \Box

**Proof of Theorem 5.1.** The strategy of the proof is to choose a finite collection of quasi-isometrically embedded subgroups to which Lemma 5.4 applies.

Hypothesis (1) of Theorem 5.1 allows us to choose a finite collection of quasi-isometrically embedded subgroups for each $P_i$ so that Hypothesis (1) of Lemma 5.4 is satisfied.

Hypothesis (2) of Lemma 5.4 on a loxodromic element $g$ will be dealt with in two separate ways according to whether $g^{\pm \infty}$ lie in distinct components of $\partial G$, or $g^{\pm \infty}$ lie in a single component of $\partial G$.

Our element $g$ is either hyperbolic or elliptic with respect to the Bass-Serre tree. In the elliptic case, $g$ lies in a conjugate of some $G_v$, and hence $g^{\pm \infty}$ lie in the same component of $\partial G$ by Proposition 5.6(2). This is the main case we focus on below. When $g^{\pm \infty}$ lie in distinct components, we are in the hyperbolic case and so $g$ stabilizes an axis in the Bass-Serre tree. As in Remark 5.7, $g^{\pm \infty}$ are separated by some $j \partial G_e$ for some $j \in G$ and $G_e$. Therefore by including each edge group $G_e$ of the accessibility splitting in our collection of quasi-isometrically embedded
codimension-1 subgroups, we see that Hypothesis (2) of Lemma 5.4 is satisfied whenever \( g^{\pm\infty} \) lie in distinct components of \( \partial G \).

We now focus on the main part of the argument where we choose finitely many quasi-isometrically embedded codimension-1 subgroups so that the axis separation condition applies to a loxodromic \( g \) when \( g^{\pm\infty} \) lie in the same component of \( \partial G \). We will proceed as in the proof of Theorem 1.4. For each pair of distinct points \( u, v \in \partial G \), by hypothesis there is a quasi-isometrically embedded codimension-1 subgroup \( H \) such that \( u \) and \( v \) are separated by \( \partial H \subset \partial G \). Let \( U_H, V_H \) be the open neighborhoods of \( u, v \) corresponding to components of \( \partial G - \partial H \). For \( j \in G \) we define the translated wall associated to \( jH \) as in the proof of Theorem 1.4.

Let \( w \neq u, v \). As in the proof of Theorem 1.4, we can apply Lemma 2.1 to obtain an open neighborhood \( M(u,v,w) = U(u,v,w) \times V(u,v,w) \) such that:

1. \( U(u,v,w) \subset U_H \) and \( V(u,v,w) \subset V_H \).
2. \( gM(u,v,w) \cap M(u,v,w) = M(u,v,w) \) for those \( g \) in some finite subgroup of \( G \), and otherwise \( gM(u,v,w) \cap M(u,v,w) = \emptyset \).

We later use the notation \( H(u,v,w) \) for \( H \), but note that \( H(u,v,w) \) does not depend on \( w \).

Consider the following collection of open saturated neighborhoods:

\[
\{ \cup_{j \in G} jM(u,v,w) : (u,v,w) \in T_{\text{thick}} \}
\]

This collection forms an open covering of \( T_{\text{thick}} \). As the sets in collection (4) are saturated relative to \( T \to \bar{T} \), the compactness of the projection of \( T_{\text{thick}} \) to \( \bar{T} \) assures that there is a finite subcollection that also covers \( T_{\text{thick}} \).

Consider a loxodromic element \( g \in G \) whose distinct fixpoints \( g^{\pm\infty} \) lie in the same component of \( \partial G \). Since \( g^{\pm\infty} \) are not parabolic points, Lemma 5.8 provides a point \( w' \in \partial G \) such that:

\[
(g^{+\infty}, g^{-\infty}, w') \in T - \bigcup_{i=1}^{n} \bigcup_{g \in G} gB_i.
\]

We have shown that \( (g^{+\infty}, g^{-\infty}, w') \) lies in one of the sets of our finite subcollection, so in particular, \( (g^{+\infty}, g^{-\infty}, w') \) must lie in some \( jM(u,v,w) \) associated to a point \( (u,v,w) \in T \) as above, where the associated quasi-isometrically embedded codimension-1 subgroup \( H(u,v,w) \) has the property that \( \partial(H(u,v,w)) \) separates \( U_{(u,v,w)} \) from \( V_{(u,v,w)} \). Thus \( g^{+\infty} \in jU_{(u,v,w)} \) and \( g^{-\infty} \in jV_{(u,v,w)} \) and these are separated by \( j\partial H(u,v,w) \). Now note that \( G \) acts as a convergence group on \( T \cup \partial G \) and that considering the \( G \)-orbit of a point in \( T_{\text{thick}} \) yields a map from \( \Gamma \) to \( T \) that is proper when restricted to the axis of \( g \). As in the proof of Theorem 1.4 we thus conclude that the element \( g \) is “separated” by the wall associated to \( jN(u,v,w) \). □

6. Cubulating arithmetic hyperbolic lattices. In [3] and [9] we cubulated certain uniform hyperbolic lattices \( G \) by showing that there is a finite family of
regular hyperplanes $H_1, \ldots, H_k$ such that the complement of their translates $\mathbb{H}^d - GH_i$ consists of uniformly bounded pieces. The following shows that an analogous result holds in the nonuniform case:

**THEOREM 6.1.** Let $G$ be an arithmetic hyperbolic lattice with a quasi-isometrically embedded codimension-1 e.g. subgroup $H$. If $G$ is nonuniform assume moreover that $H$ intersects a parabolic subgroup of $G$ along an infinite subgroup. Then $G$ acts properly on a CAT(0) cube complex and the quotient has finitely many immersed hyperplanes.

**Proof.** Let $U, V$ be the components of $\partial G$ separated by $\partial H$. As it is arithmetic, the group $G$ has a commensurator group that is dense in the group of isometries of hyperbolic space. In particular: For any two distinct points $p, q \in \partial G$ there is an element $c$ of the commensurator of $G$ such that $cp \in U$ and $cq \in V$. Note that $H_c = c^{-1} H \cap G$ is of finite index in $c^{-1} H c$, and is thus itself a quasi-isometrically embedded codimension-1 subgroup. Observe that $\partial H_c = c^{-1} \partial H$ separates $p, q$ since $p \in c^{-1} U$ and $q \in c^{-1} V$. In the uniform case we have thus satisfied the criterion of Theorem 1.4.

In the nonuniform case we proceed similarly to get finitely many $H_c$ whose intersection with each parabolic subgroup $P$ yields a proper action of $P$ on the corresponding cube complex. □

Let $F$ be a totally real number field, and let $V$ be an $F$-vector space with a quadratic form $q$ of signature $(n, 1)$ at one place and definite at all other places. An arithmetic lattice $G \subset \text{SO}(V, F)$ is said to be of **simple type**. Such a lattice is uniform if and only if $(V, q)$ is anisotropic over $F$. All nonuniform arithmetic lattices in $\text{SO}(n, 1)$ are of simple type, and moreover, every arithmetic lattice in $\text{SO}(n, 1)$ is of simple type when $n$ is even.

Arithmetic lattices of simple type contain a quasi-isometrically embedded codimension-1 subgroup $H$ to which Theorem 6.1 applies. Indeed, $H$ arises as the stabilizer of a codimension-1 subspace of $V$ to which $q$ restricts to a form of signature $(n - 1, 1)$. Moreover, since these hyperplanes are regular—corresponding to algebraic subgroups of $\text{SO}(V, F)$—it is possible to deduce virtual specialness using the double separability criterion. This leads to the following extension of the main theorem of [3] to nonuniform lattices.

**THEOREM 6.2.** Let $G$ be an arithmetic hyperbolic lattice of simple type. Then $G$ is virtually special.

**Proof.** Theorem 6.1 applies to the codimension-1 subgroup associated to a regular hyperplane. The result is a proper action on a CAT(0) cube complex $\tilde{C}$ such that the quotient $\tilde{G} = G \backslash \tilde{C}$ has only finitely many immersed hyperplanes. The fundamental groups of immersed hyperplanes of $\tilde{C}$ correspond to the stabilizers of the chosen regular hyperplanes. And the double coset separability criterion for
virtual specialness stated in Proposition 3.3(1) holds by [2, Prop. 10 and proof of Thm. 6]. □

**Remark 6.3.** (Slight generalizations) If $G$ is an arithmetic hyperbolic lattice with a separable quasiconvex codimension-1 subgroup then the analogous proof shows that $G$ is virtually special, but Proposition 3.3(3) must be used. Likewise, if $G$ is an arithmetic hyperbolic lattice with a quasiconvex codimension-1 subgroup that does not self-cross its translates (on the boundary) then we obtain a quasiconvex hierarchy for $G$ which is thus virtually special using Proposition 3.3(4).

**Remark 6.4.** (Other lattices) Thomas Delzant and Misha Gromov proved that complex hyperbolic lattices cannot have codimension-1 quasiconvex subgroups [7]. Nevertheless, complex hyperbolic arithmetic lattices of simple type have positive virtual first Betti number (see [4]). This yields (non-quasiconvex) codimension-1 subgroups.

Excluding some exceptional classes arising in dimensions 3 and 7, each (simple or non simple type) real hyperbolic arithmetic lattice $\Lambda$ embeds as a quasiconvex subgroup of a simple type complex hyperbolic arithmetic lattice $\Gamma$, see e.g. [15]. It is thus of considerable interest to know whether a codimension-1 subgroup in $\Gamma$ intersects $\Lambda$ along a quasiconvex codimension-1 subgroup.

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