TORSION HOMOLOGY GROWTH AND CYCLE COMPLEXITY OF ARITHMETIC MANIFOLDS

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Abstract

Let $M$ be an arithmetic hyperbolic $3$-manifold, such as a Bianchi manifold. We conjecture that there is a basis for the second homology of $M$, where each basis element is represented by a surface of “low” genus, and we give evidence for this. We explain the relationship between this conjecture and the study of torsion homology growth.

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1. Introduction

In this article we formulate and discuss a conjecture about the topological complexity of arithmetic manifolds, that is, locally symmetric spaces associated to arithmetic groups. This conjecture is closely related to studying the growth of torsion in homology. Roughly speaking, the conjecture is that homology classes on arithmetic manifolds are represented by cycles of low complexity. From a strictly arithmetic perspective, what may be most interesting is that our proofs suggest that the topological complexity of these cycles reflects the arithmetic complexity of the...
(Langlands-)associated varieties (i.e., the height of equations needed to define the varieties).

We will study this in detail in a simple interesting case, namely, that of arithmetic hyperbolic 3-manifolds. To simplify matters as much as possible, we study only sequences that are coverings of a fixed base manifold $M_0$.

**Conjecture 1.1**

There is a constant $C = C(M_0)$ such that, for any arithmetic congruence hyperbolic 3-manifold $M \to M_0$ of volume $V$, there exist immersed surfaces $S_i$ of genus less than or equal to $V^C$ such that the $[S_i]$'s span $H_2(M, \mathbb{R})$.

Thus, the conjecture is related to understanding the Gromov–Thurston norm on $H_2$; it can also be phrased in terms of a harmonic norm on $H_2$ whose definition uses the hyperbolic metric (see Section 4). It follows from Gabai’s [31, p. 3] generalization of Dehn’s lemma to higher genus that we may as well ask the $S_i$’s to be embedded in Conjecture 1.1.

It is plausible, although we are not sure, that this conjecture is really a special feature of arithmetic manifolds. For the purpose of this article, “arithmetic manifold” means more properly arithmetic congruence manifold. First, our proofs certainly use number theory heavily. Second, it seems that any naive analysis yields only an exponential bound on $[S_i]$ in terms of $V$ or the topological complexity of $M$—indeed, work of Brock and Dunfield [14] strongly suggests that this exponential bound cannot be improved. Finally, numerical data (see, e.g., [14] or [61]), although far from conclusive, also appear to differ between nonarithmetic and arithmetic cases. (See Section 1.4 for a little further discussion.)

This conjecture is motivated by the study of torsion classes and, indeed, in trying to understand the obstruction to extending previous results (see [6]) on so-called strongly acyclic coefficient systems to the case of the trivial local system. We will prove the following.

**Theorem 1.2**

Let $(M_i \to M_0)_{i \in \mathbb{N}}$ be a sequence of arithmetic congruence hyperbolic 3-manifolds such that $M_0$ is compact and $V_i = \text{vol}(M_i)$ goes to infinity. Assume that the following two conditions are satisfied.

(i) Few small eigenvalues: For every $\varepsilon > 0$ there exists some positive real number $c$ such that

$$
\limsup_{i \to \infty} \frac{1}{V_i} \sum_{0 < \lambda \leq c} |\log \lambda| \leq \varepsilon.
$$
Here $\lambda$ ranges over eigenvalues of the 1-form Laplacian $\Delta$ on $M_i$. Indeed, we may even replace the condition by the condition that

$$\lim_{i \to \infty} \frac{1}{V_i} \sum_{0 < \lambda \leq V_i^{-\delta}} |\log \lambda| = 0$$

(1.2)

for every $\delta > 0$.

(ii) Small Betti numbers: $b_1(M_i, \mathbb{Q}) = o\left( \frac{V_i}{\log V_i} \right)$.

If Conjecture 1.1 holds, then as $i \to \infty$, we have

$$\frac{\log \#H_1(M_i, \mathbb{Z})_{\text{tors}}}{V_i} \to \frac{1}{6\pi}.$$ (1.3)

For the proof see Section 2. (It also uses results from Sections 3 and 4.) The main tool is the Cheeger–Müller theorem (the former Ray–Singer conjecture). It relates the determinants of the Laplacians on $M_i$ to the product of $\#H_1(M_i, \mathbb{Z})_{\text{tors}}$ by regulators. The story is similar to that of the behavior of the $L$-function of an elliptic curve at the central point in terms of the rank and the size of the Tate–Shafarevic group. In fact, one could express the various quantities involving the determinants of Laplacians in terms of the central values of corresponding Selberg zeta functions, making the analogy quite clear. Conjecture 1.1 is then used to de-correlate the size of $H_1(M_i, \mathbb{Z})_{\text{tors}}$ from the regulators.

Heuristically, we expect Theorem 1.2(i) to be valid with very few exceptions. Indeed, the basic models for the distribution of the $\lambda$’s are the following (see, e.g., [46] where the spectrum of hyperbolic surfaces is considered).

- The $\lambda$’s are random—that is, $\lambda_{j+1} - \lambda_j$ has a Poisson distribution.
- The $\lambda$’s are the eigenvalues of a random symmetric matrix (Gaussian orthogonal ensemble or GOE).
- The $\lambda$’s are the eigenvalues of a random Hermitian matrix (Gaussian unitary ensemble or GUE).

In each of these models, Theorem 1.2(i) is easy to check. Unfortunately, one has little in the way of techniques to attack this question.

As for Theorem 1.2(ii), we expect it to be always valid (see [5], [18] for evidence, and also [47] in a somewhat different direction). As explained in the proof of Lemma 2.7, one can show that $b_1(M_i, \mathbb{Q}) = O\left( \frac{V_i}{\log V_i} \right)$. But in general, going from big $O$ to small $o$ is quite challenging.

The proof of Theorem 1.2 also gives a partial converse. For instance, if we suppose (1.3) and a strengthening of Theorem 1.2(ii)—that the Betti numbers $b_1$ actually remain bounded—then Theorem 1.2(ii) must be true, and also a weak form of Conjecture 1.1, with “polynomial” replaced by “subexponential,” must hold.

Now the central result of our article is the following.
THEOREM 1.3

Conjecture 1.1 is true in the two following cases:

(i) when $M_0$ arises from a division algebra $D \otimes F$ where $D$ is a quaternion algebra over $\mathbb{Q}$, $F$ is an imaginary quadratic field, $M$ is defined by a principal congruence subgroup,\(^\dagger\) and all the cohomology of $M$ is of base-change type (see Section 6);

(ii) when $M_0$ is a Bianchi manifold (for us, an adelic manifold whose components are of the form $\mathbb{H}^3_\mathbb{A}$) and the cuspidal cohomology of $M$ is 1-dimensional, associated to a non-CM elliptic curve of conductor $n$, for which we assume the equivariant Birch–Swinnerton-Dyer (BSD) conjecture (see (8.14)) and the Frey–Szpiro conjecture (see [36, Conjecture F.3.2]).

What the proof of (ii) really gives is a relationship between the complexity of $H_2$-cycles and the height of the elliptic curve (i.e., the minimal size of $A, B$, so it can be expressed as $y^2 = x^3 + Ax + B$). Thus, the topological complexity of cycles in $H_2$ reflects the arithmetic height of $E$. This may be a general phenomenon. (It was also suggested in [19].)

The Langlands program predicts that when the cuspidal cohomology of $M$ has dimension 1—or, indeed, when one is given a Hecke eigenclass with rational eigenvalues—there should be an associated rank 2 motive over $F$ with Hodge numbers $(0,1), (1,0)$ and coefficient field equal to $\mathbb{Q}$ (see [22]). Such a motive arises either from an elliptic curve over $F$, which does not have CM by $F$, or from an abelian variety $A/F$ whose algebra of $F$-endomorphisms is a quaternion division algebra (see [64]). While we do not know of an infinite family of Bianchi manifolds for which the existence of associated motives is known, there is an overwhelming amount of data, going back to [34] and [23] (see Section 9.2), that leaves little doubt to the truth of this prediction.

The Frey–Szpiro conjecture is a conjecture in Diophantine analysis which follows from the abc conjecture (and thus is very strongly expected from a heuristic viewpoint). It asserts that the height of an elliptic curve cannot be too large relative to its conductor. Moreover, for the purposes of establishing the growth of torsion, as in Theorem 1.2, we do not need the full strength of Conjecture 1.1; a weaker version with subexponential bounds would suffice, and correspondingly a very weak subexponential version of the Frey–Szpiro conjecture would do.

We note that both case (i) and case (ii) are quite common over imaginary quadratic fields. For (i), we present data in Section 9.1; for example, for the first 40 rational primes $p$ that are inert in $\mathbb{Q}(\sqrt{-7})$, the cohomology of $\Gamma_0(p)$, where $p = (p)$, is

\(^\dagger\)This is not an onerous restriction; it is easy to reduce the conjecture for other standard subgroup structures, such as $\Gamma_0$-structure, to this case.
entirely base change in all but six cases. For (ii) we refer to [60, p. 17]; in the data there, at prime level, situation (ii) occurs in the majority of cases where $b_{1,!} > 0$ (see also Section 9.2).

Also, (i) and (ii) illustrate two different extremes of the theorem. For (i), it is easy to think of candidate surfaces in $H_2$—the challenge is, rather, that the dimension of $H_2$ is increasing rapidly and it is not clear that the candidate surfaces span enough homology. In fact, our result applies to all $M$, but bounds only the regulator of the base-change part of cohomology. One can see (i) as an effectivization of a result of Harder, Langlands, and Rapoport [35], although they work with Hilbert modular surfaces rather than hyperbolic 3-manifolds. The main global ingredient is a (polynomially strong) quantitative form of the multiplicity 1 theorem in the theory of automorphic forms but there is also (surprisingly) a nontrivial local ingredient: one needs good control on, for example, the support of matrix coefficients of supercuspidal representations. In fact, one motivation to study example (i) is that our result shows that the regulator $R_2$ (see Section 2) grows subexponentially, whereas this was not at all clear by looking at numerical evidence (see Section 9.3). (There is actually another setting where $H_2$ grows quickly for easily comprehensible reasons—the setting of oldforms, whereby one pulls back forms from a manifold of lower level. In that case, it is not difficult to see that the complexity of the cycles remains controlled.)

For (ii) the challenge is instead that there are no obvious cycles in $H_2$; we work with $H_1$ and modular symbols, and dualize. The main point is to replace a modular symbol with the sum of two well-chosen others to avoid unpleasant dominators. The equivariant BSD conjecture enters to compute cycle integrals over modular symbols. The Szpiro conjecture enters to give a lower bound on the period of an elliptic curve. We note that this result is closely related to prior work of Goldfeld [32], although the techniques of proof are necessarily different owing to the lack of an algebraic structure.

1.4. The role of arithmeticity

As we have mentioned, it seems plausible that Conjecture 1.1 is really specific to arithmetic. It would be desirable to have a specific counterexample in this direction, that is to say, exhibiting the behavior that Conjecture 1.1 disallows in the arithmetic case.

From the point of view of mirroring the situation of this article, it would be ideal to have an answer to the following.

**Question**

Can one produce a sequence of hyperbolic manifolds $M_i$ with the following properties:
• the volumes of $M_i$ go to infinity (or, even better, the sequence $(M_i)$ Benjamini–Schramm (BS) converges toward $\mathbb{H}^3$; see Section 2.3),
• the injectivity radii of $M_i$ remain bounded below, and
• in any basis for $H_2(M_i, \mathbb{Z})$, at least one basis element cannot be represented by a surface of genus less than or equal to $(\text{vol } M_i)^i$?

Brock and Dunfield [14] have made progress in constructing such a sequence.

Here is some intuition as to why arithmeticity might play a role. In general, generators for $H_2(M, \mathbb{Z})$ might be of exponential complexity. This comes down to analyzing the kernel of a matrix $M$ that expresses adjacency between 1-cells and 2-cells in a triangulation. Now, even given a matrix $A \in M_n(\mathbb{Z})$ of zeroes and ones, generators for the kernel of $A$ on $\mathbb{Z}^n$ could have exponentially large (in $n$) entries. However, in the arithmetic case, the existence of Hecke operators means that the adjacency matrix $A$ is (heuristically speaking) forced to commute with many other symmetries. One might expect this to reduce its effective size—a phenomenon that is perhaps parallel to the observed difference between eigenvalue statistics in the arithmetic and nonarithmetic cases (see [11], [58] for discussions).

Finally, the experiments of Brock and Dunfield (see [14, Figures 4.4 and 4.5]) seem to show that, in the case when $H_2(M, \mathbb{Q})$ vanishes, the torsion convergence (1.3) is even faster in the nonarithmetic case than in the arithmetic case. This may be related to the different eigenvalue statistics: the repulsion of eigenvalues in the random matrix setting would perhaps suggest that the few small eigenvalues condition (1.1) should hold even more strongly in the nonarithmetic setting.

2. Relationship to torsion and the proof of Theorem 1.2
In this section and the next, we will give the proof of Theorem 1.2. We first recall the definition of regulators from a prior article by the first and third authors [6].

2.1. Regulators
Let $M$ be a compact Riemannian manifold of dimension $n$. We define the $H_j$-regulator of $M$ as the volume of $H_j(M, \mathbb{Z})$ with respect to the metric on $H_j(M, \mathbb{R})$ defined by harmonic forms—the harmonic metric. That is,

$$R_j(M) = \frac{\det(\int \gamma_k \omega_\ell)}{\sqrt{\det(\omega_k, \omega_\ell)}},$$

where the $\gamma_k \in H_j(M, \mathbb{Z})$ project to a basis for $H_j(M, \mathbb{Z})/H_j(M, \mathbb{Z})_{\text{tors}}$ and the $\omega_\ell$’s are a basis for the space of $L^2$-harmonic forms on $M$. Note that $R_0(M) = 1/\sqrt{\text{vol}(M)}$, $R_n(M) = \sqrt{\text{vol}(M)}$, and by Poincaré duality, we have

$$R_j(M) \cdot R_{n-j}(M) = 1.$$
2.2
A celebrated theorem of Cheeger and Müller [21, Theorem 8.22], [51, Theorem 10.22] relates the torsion homology groups and the regulators to the analytic torsion of $M$.

In the special case $n = 3$, the theorem of Cheeger and Müller implies that

$$|H_1(M, \mathbb{Z})_{\text{tors}}| \cdot \frac{R_0 R_2}{R_1 R_3} = T_{\text{an}}(M)^{-1},$$

where $T_{\text{an}}(M)$ is the analytic torsion of the manifold $M$. We furthermore note that

$$\frac{R_0 R_2}{R_1 R_3} = \frac{R_0^2}{\text{vol}(M)} = \frac{1}{R_1^2 \text{vol}(M)}.$$
2.6. Third ingredient

Finally, the few small eigenvalues condition from Theorem 1.2 implies that

\[
\frac{\log T_{an}(M_i)}{V_i} \to \tau(2)_{\text{H}^3} = \frac{1}{6\pi}.
\]

It follows from the definition of analytic torsion and well-known properties of the spectrum of the Laplace operators on Riemannian 3-manifolds (see, e.g., [6]) that it is enough to prove that

\[
\frac{d}{ds} \bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{1}{V_i} \left( \int_{M_i} (\text{tr} e^{-t\Delta_i} (\bar{x}, \bar{x}) - \text{tr} e^{-t\Delta_i} (x, x)) dx + b_1(M_i) \right) dt
\]

\[\to 0.\]

(2.5)

Here \(\Delta_i\) (resp., \(\Delta^{(2)}\)) is the Laplace operator on square-integrable 1-forms on \(M_i\) (resp., \(\mathbb{H}^3\)), and \(\bar{x}\) is an arbitrary lift of \(x\) to \(\mathbb{H}^3\).

Since \(b(M_i)\) grows as \(o(V_i/(\log V_i))\), the proof of the limit (2.5) follows along the same lines as [6, Theorem 4.5] under the assumptions that

1. the injectivity radius of \(M_i\) goes to infinity, and
2. there exists some positive \(c\) such that, for all \(M_i\), the lowest eigenvalue of \(\Delta_i\) is bigger than \(c\).

The first assumption is used to handle the small \(t\) contribution to the limit (2.5). In fact, the proof only uses the fact that the local injectivity radius is almost everywhere going to infinity; the condition is precisely that the sequence \((M_i)_{i \in \mathbb{N}}\) BS-converges to \(\mathbb{H}^3\). We refer to [1, Section 8 and 9] for more details, in particular, on how to bound the size of the heat kernel at the bad points.

The second assumption is used to handle the large \(t\) contribution; it more precisely implies that for sufficiently large \(t\) each individual term of the difference in (2.5) can be made arbitrarily small. However, this spectral-gap assumption never holds for the trivial coefficient system; we replace that instead by Theorem 1.2(i).

Let \(\varepsilon\) and \(c\) be as in Theorem 1.2(i). Without loss of generality, \(c < 1\). Spectral expansion on the compact manifold \(M_i\) and classical Sobolev estimates yield that, for any \(t \geq 1\), we have

\[
\int_{M_i} \text{tr} e^{-t\Delta_i} (x, x) dx = \sum_{0 < \lambda} e^{-\lambda t}
\]

\[\ll \sum_{0 < \lambda \leq c} e^{-\lambda t} + e^{-c(t-1)} \sum_{\lambda > c} e^{-\lambda}
\]

\[\ll \sum_{0 < \lambda \leq c} e^{-\lambda t} + e^{-c(t-1)} V_i,
\]
where we have denoted by $\Delta'_i$ the restriction of $\Delta_i$ to the orthogonal complement of its kernel and the implicit constant does not depend on $i$; we used the fact that the trace of $e^{-\Delta'_i}$ on $M_i$ can be bounded by a multiple of $V_i$. To conclude the proof, we just have to remark that, for any $T \geq 1$ fixed,

$$\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_T^{+\infty} t^{s-1} \sum_{0<\lambda \leq c} e^{-t\lambda} \,dt$$

$$= \sum_{0<\lambda \leq c} \int_T^{\infty} \frac{e^{-t\lambda} \,dt}{t}$$

$$= \sum_{0<\lambda \leq c} \int_T^{\infty} \frac{e^{-t} \,dt}{t} + \sum_{0<\lambda \leq c} \int_T^{\infty} \frac{e^{-t\lambda} - e^{-t}}{t} \,dt$$

$$\ll \left( \text{number of eigenvalues in } (0,c] \right) e^{-T} + \left| \sum_{0<\lambda \leq c} \log \lambda \right|.$$

There is a constant $A$ so that the number of eigenvalues in $(0,c]$ is at most $AV_i$, and we may then choose $T$ sufficiently large so that $Ae^{-T} < \varepsilon$. Thus, the integral above contributes at most $2\varepsilon$ to the limit. By using (1.1), this holds for every $\varepsilon$, so the proof of (2.5) follows as in [6]. We have now completed the proof of the theorem, but assuming (i) in the stronger form (1.1). To see that (1.2) suffices, we prove the following lemma.

**Lemma 2.7**

Assume that (1.2) holds. Then, for every $\varepsilon > 0$ there exists some positive real number $c$ such that

$$\lim_{i \to \infty} \frac{1}{V_i} \sum_{0<\lambda \leq c} \left| \log \lambda \right| \leq \varepsilon.$$

Here $\lambda$ ranges over eigenvalues of the 1-form Laplacian $\Delta_i$ for $M_i$.

**Proof**

In [1, Theorem 1.12] a quantitative version of BS-convergence is proved; in particular, there exist positive constants $c$ and $\delta$ such that, for every $i$, one has

$$\text{vol}(M_i) < c \log V_i \leq V_i^{1-\delta}.$$

Fix $M = M_i$, and fix $V = V_i$. Employing the trace formula—as in [55] and [59]—with a test function supported in an interval of length $1/(c' \log V)$ for some positive constant $c'$ and using the estimates of [1, Lemma 7.23], we can show that, for every
$k \in \mathbb{N}$, the number of eigenvalues of $\Delta$ between $\frac{k}{c \log V}$ and $\frac{k+1}{c \log V}$ is bounded by some uniform constant times $V/ \log V$. It follows that

$$\prod_{\frac{k}{c \log V} < \lambda \leq \frac{k+1}{c \log V}} \lambda \gg \left( \frac{k}{\log V} \right)^{\frac{V}{\log V}}.$$  

Taking a further product for $k = 1, \ldots, \alpha \log V$ for some positive $\alpha$ (and using Stirling’s formula), we get that

$$\prod_{\frac{1}{c \log V} < \lambda \leq \alpha} \lambda \gg \left( \frac{(\alpha \log V)\log V}{(\log V) \log V} \right)^{\frac{V}{\log V}} \gg e^{-o(1)V}$$  

as $\alpha \to 0$.

Now given a positive real number $\delta$, we similarly have

$$\prod_{V^{-\delta} < \lambda \leq \frac{1}{c \log V}} \lambda \gg \left( \frac{1}{V^\delta} \right)^{\frac{V}{\log V}} = e^{-\delta V}.$$  

The lemma follows from (2.6), (2.7), and the few small eigenvalues assumption.

3. Bounding $R_1(M)$

Here we prove (2.2), which was used in the proof of Theorem 1.2. Let $M_0$ be a complete Riemannian $n$-dimensional manifold of pinched nonpositive sectional curvature. We more generally prove the following.

PROPOSITION 3.1

If $M$ varies through a sequence of finite coverings of a fixed compact manifold $M_0$, then we have

$$|R_1(M)| \ll \text{vol}(M)^{Cb(M)}.$$  

Here the implicit constants only depend on $M_0$. □

The following is a consequence of Sobolev estimates.

LEMMA 3.2

Let $M$ be as in Proposition 3.1, let $S \subset M$ be a $k$-submanifold of (Riemannian) volume $v$, and let $\omega$ be an $L^2$-normalized harmonic differential $k$-form on $M$. Then

$$\int_S \omega \ll v.$$  

We now explain how to prove Proposition 3.1 using Lemma 3.2.
3.3
Fix $M_0$, let $\Gamma_0$ be the fundamental group of $M_0$, let $S$ be a set of generators of $\Gamma_0$, and let $d_0$ be the cardinality of $S$. To any finite covering $M \to M_0$—corresponding to a finite-index subgroup $\Gamma < \Gamma_0$—we associate the Schreier graph $\mathcal{G}(\Gamma_0/\Gamma, S)$; it is a finite cover of degree $[\Gamma_0 : \Gamma]$ of the wedge product of $d_0$ circles. Computing the Euler characteristic, we conclude that $\mathcal{G}(\Gamma_0/\Gamma, S)$ has the homotopy type of the wedge product of $d$ circles where $$(d - 1) = [\Gamma_0 : \Gamma](d_0 - 1).$$

The group $\Gamma$ is therefore generated by at most $d$ elements; moreover, each of these elements has length at most $[\Gamma_0 : \Gamma]$ in the $S$-word metric of $\Gamma_0$. Since $\Gamma_0$ with the $S$-word metric is quasi-isometric to the universal cover $\bar{M}$ of $M$ with its induced Riemannian metric, we have the following.

**Lemma 3.4**
There exists a constant $c = c(M_0)$ such that $\Gamma$ is generated by at most $c[\Gamma_0 : \Gamma]$ elements which can be represented by closed geodesics of length at most $c[\Gamma_0 : \Gamma]$.

Note that up to a constant (depending only on $M_0$), $\text{vol}(M)$ equals $[\Gamma_0 : \Gamma]$. Hadamard’s inequality, Lemma 3.2, and Lemma 3.4 therefore imply Proposition 3.1. (Note that, in the definition (2.1) of the regulator, replacing the $\gamma_j$’s by elements $\gamma^j_i$’s that generate a finite-index sublattice of homology only increases the regulator.)

3.5
Assuming Conjecture 1.1, we can apply a similar scheme to bound $R_2(M)$, but we now need to compare two different norms on $H_2(M, \mathbb{R})$. This is the purpose of the next section.

4. Relationship of the harmonic norm and the Gromov–Thurston norm
In this section, $M$ will denote a compact hyperbolic 3-manifold. The second homology group $H_2(M, \mathbb{R})$ is equipped with two natural norms: the Gromov–Thurston norm, which measures the number of simplices needed to present a cycle, and the harmonic norm, which arises from the identification of $H_2(M, \mathbb{R}) \simeq H^1(M, \mathbb{R})$ with harmonic 1-forms on $M$. We will relate the two norms and use this relation to prove (2.3), which was used in the proof of Theorem 1.2.

More precisely, if $\delta \in H_2(M, \mathbb{R})$, then we set $\|\delta\|_{GT} = \inf \{\sum |n_k| \left| \sum n_k \sigma_k \right| = \delta \text{ where } \sum n_k \sigma_k \text{ is a singular chain}\}$. 
Note that Gabai [31, Corollary 6.18] shows that if \( \delta \in H_2(M, \mathbb{Z}) \), then
\[
\| \delta \|_{GT} = 2 \min \left\{ \sum_{i, \chi(S_i) < 0} |\chi(S_i)| \mid S = \bigcup_i S_i, \text{ where } S_i \text{ is a properly embedded connected surface in } M \text{ and } [S] = \delta \text{ in } H_2(M, \mathbb{Z}) \right\}. 
\] (4.1)

Note that, since \( M \) is compact hyperbolic, we may suppose that each \( S_i \) is actually a surface of genus at least 2, since if \( S \) is either a sphere or a torus, then the image of \( H_2(S, \mathbb{Z}) \) in \( H_2(M, \mathbb{Z}) \) will be trivial. In particular, to prove Theorem 1.2, it is enough to exhibit a set in \( H_2(M, \mathbb{Z}) \) of full rank and with polynomially bounded Gromov–Thurston norm.

We also define \( \| \delta \|_{L^2} = \| \omega \|_{L^2} \), where \( \omega \) is the \( L^2 \)-harmonic 1-form on \( M \) which is dual to \( \delta \), that is,
\[
\int_M \omega \wedge \alpha = \int_\delta \alpha, 
\]
for every closed 2-form \( \alpha \) on \( M \). Note, in particular, that
\[
\| \delta \|_{L^2}^2 = \left\| \int_\delta * \omega \right\|. 
\] (4.2)

In this section we compare \( \| \cdot \|_{L^2} \) and \( \| \cdot \|_{GT} \). In particular, we prove the following.

**Proposition 4.1**

If \( M \) varies through a sequence of finite coverings of a fixed manifold \( M_0 \), then we have
\[
\frac{1}{\text{vol}(M)} \| \cdot \|_{GT} \ll \| \cdot \|_{L^2} \ll \| \cdot \|_{GT}.
\]

**Proof**

The proof is found in Sections 4.2–4.5 below.

**4.2**

Given a cycle \( \delta \in H_2(M, \mathbb{R}) \) with \( \| \delta \|_{GT} \leq 1 \), we may write (see, e.g., [54, Theorem 11.4.2 and the remark following it])
\[
\delta = \sum_k n_k \sigma_k, 
\]
where each \( \sigma_k \) is a straight simplex (or triangle), that is, the image of the convex hull of 3 points in \( \mathbb{H}^3 \), and \( \sum_k |n_k| \leq 1 \). Now if \( \alpha \) is a harmonic 2-form, then
\[
\int_{\sigma_k} \alpha \ll \| \alpha \|_\infty \text{area}(\sigma_k) \leq \pi \| \alpha \|_\infty \ll \| \alpha \|_2.
\]
is uniformly bounded so that \( \int_\delta \alpha \ll \|\alpha\|_2 \). Since we can compute the harmonic norm of \( \delta \) as the operator norm of \( \alpha \mapsto \int_\delta \alpha \), this has shown the second inequality of Proposition 4.1; we pass now to the first inequality.

4.3
In the reverse direction, suppose that we are given an element \( \delta \in H_2(M, \mathbb{R}) \) of harmonic norm at most 1; equivalently, its image under \( H_2(M, \mathbb{R}) \cong H^1(M, \mathbb{R}) \) is represented by a harmonic 1-form \( \omega \) of \( L^2 \)-norm at most 1. Fix a triangulation \( K \) of \( M \) by lifting a triangulation \( K_0 \) of \( M_0 \). We can suppose that every edge has length at most 1 and every triangle has area at most 1. Let \( K'_0 \) and \( K' \) be the corresponding dual cell subdivisions. We denote by

\[
\langle \cdot, \cdot \rangle : C_i(K, \mathbb{Z}) \times C_{3-i}(K', \mathbb{Z}) \to \mathbb{Z}
\]

the (integer) intersection number; it canonically identifies \( C_{3-i}(K', \mathbb{Z}) \) with the dual \( C^i(K, \mathbb{Z}) = C_i(K, \mathbb{Z})^* \) of \( C_i(K, \mathbb{Z}) \). Furthermore, the boundary homomorphism

\[
\partial : C_{3-i}(K', \mathbb{Z}) \to C_{3-i-1}(K', \mathbb{Z})
\]

is (up to sign) dual to the corresponding boundary homomorphism \( C_{i+1}(K, \mathbb{Z}) \to C_i(K, \mathbb{Z}) \). In other words, \( \partial \) identifies (up to sign) with the coboundary homomorphism \( C^i(K, \mathbb{Z}) \to C^{i+1}(K, \mathbb{Z}) \). Both complexes \( C_\bullet(K, \mathbb{Z}) \) and \( C_\bullet(K', \mathbb{Z}) \) compute \( H_\bullet(M, \mathbb{Z}) \). Now the latter identifies with \( C^{3-\bullet}(K, \mathbb{Z}) \) and computes \( H^{3-\bullet}(M, \mathbb{Z}) \). This realizes the Poincaré duality.

4.4
Consider, then, the 2-cycle

\[
Z := \sum_e \left( \int_e \omega \right) e^* \in C^1(K, \mathbb{R}) = C_2(K', \mathbb{R}).
\]

Here we sum over edges \( e \) of \( K \), and \( e^* \in C^1(K, \mathbb{R}) = C_1(K, \mathbb{R})^* \) denotes the dual element. Since \( \omega \) is closed, it follows from the Stokes formula that

\[
\partial Z = \pm \sum_t \left( \int_{\partial t} \omega \right) t^* = 0.
\]

On the other hand, \( Z \) represents the image of the class of \( [\omega] \) under the Poincaré duality pairing

\[
H^1(M, \mathbb{R}) \cong H_2(M, \mathbb{R}).
\]
4.5 Subdivide $K_0$ to get a triangulation $T_0$ of $M_0$, and lift this triangulation to a triangulation $T$ of $M$. There exists a constant $c$ which only depends on $M_0$ (and $T_0$) such that the number of triangles of $T_0$ in all the cells of $K_0$ dual to an edge of $K_0$ is bounded by $c$. Then the number of triangles of $T$ in all the cells of $K$ is at most $c[M : M_0]$, where $[M : M_0]$ is the degree of the cover $M \to M_0$. By the definition of the Gromov–Thurston norm, we conclude that

$$\| [Z] \|_{GT} \ll \| \omega \|_{\infty} \text{vol}(M) \ll \text{vol}(M).$$

We obtain the last inequality by the Sobolev inequality. Proposition 4.1 now follows.

4.6 Relation with $R_2(M)$
Let us now assume Conjecture 1.1. Then each $[S_i]$ has Gromov–Thurston norm—and therefore, by Proposition 4.1, harmonic norm—which is bounded by a polynomial in $\text{vol}(M)$. Thus, Hadamard’s inequality shows that

$$R_2(M) \ll \text{vol}(M)^{Cb(M)},$$

where $b(M)$ is the Betti number. We have thus concluded the proof of Theorem 1.2.

5. Arithmetic manifolds
Let $F$ be an imaginary quadratic field. We consider arithmetic manifolds associated to an algebraic group $G$ over $\mathbb{Q}$ such that $G(\mathbb{R}) = \text{PGL}_2(\mathbb{C})$. In the remainder of the article we are interested in the two examples:

- $G_1 = \text{Res}_{F/\mathbb{Q}} \text{GL}_2$ mod center;
- $G_2 = \text{Res}_{F/\mathbb{Q}} \text{GL}_1(D')$ mod center, where $D'$ is a division algebra over $F$ of the form $D' = D \otimes F$, with $D$ a quaternion algebra over $\mathbb{Q}$.

Let $\mathbb{A}_F$ be the ring of adèles of $\mathbb{Q}$ and $F$, respectively. We denote by $\mathbb{A}_f$ and $\mathbb{A}_{F,f}$ the corresponding rings of finite adèles. We also write $F_\infty = F \otimes \mathbb{R} \simeq \mathbb{C}$.

In the remaining part of this article, $G$ stands for either $G_1$ or $G_2$.

In the second case, $G_2$ admits a $\mathbb{Q}$-subgroup which will be of importance to us. Let $H = \text{GL}_1(D)$ modulo center, considered as a subgroup of $G_2$. Thus, $H(\mathbb{R}) = \text{PGL}_2(\mathbb{R})$.

5.1 The arithmetic manifold $X(n)$
Let $n$ be an ideal of the ring of integers $\mathcal{O}$ of $F$. We associate to $n$ a compact open subgroup

$$K(n) = \prod_v K_v(n) \subset G(\mathbb{A}_f)$$
in the following way. If \( G = G_1 \) as usual, then we define \( K(n) = K_1(n) \) as the subgroup corresponding—after restriction of scalars and mod center—to

\[
\{ g \in \GL_2(\overline{\theta}) : g \equiv I_2 (n \overline{\theta}) \}. \tag{5.1}
\]

Here \( \overline{\theta} \) is the closure of \( \varnothing \) in \( \mathbb{A}_{F,f} \). In this case, we also define \( K_0(n) \) in the usual way:

\[
K_0(n) = \left\{ g \in \GL_2(\overline{\theta}) : g \equiv \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} (n \overline{\theta}) \right\}.
\]

Now if \( G = G_2 \), then we make a corresponding definition of \( K(n) \) as follows. Regard \( G_2(\mathbb{A}_f) \) as the \( \mathbb{A}_{F,f} \)-points of \( \GL_1(D') \) mod center. In the paragraph that follows, products over places \( v \) will be over places of \( F \). First, make an arbitrary choice \( K = \prod_v K_v \subset G_2(\mathbb{A}_f) \) of a compact open subgroup such that \( K_v \) is hyperspecial at each unramified place \( v \). At those places we may then fix isomorphisms \( \phi_v : G_2(\mathbb{Q}_v) \to \PGL_2(F_v) \) such that \( \phi_v(K_v) = \PGL_2(\mathcal{O}_v) \), where \( \mathcal{O}_v \subset F_v \) is the ring of integers. We finally define \( K(n) = \prod_v K_v(n) \) by setting \( K_v(n) = K_v \) at each ramified place and \( K_v(n) = \phi_v^{-1}(K_{1,v}(n)) \) at each unramified place, where \( K_{1,v}(n) \) is the local analogue of (5.1). We will also suppose that, for at least one ramified place \( v \), the subgroup \( K_v \) is sufficiently small so as to force any group \( G(\mathbb{Q}) \cap K(n) \) to be torsion-free.

Given any compact open subgroup \( K \subset G(\mathbb{A}_f) \), we define the arithmetic manifold \( X(K) \) by

\[
X(K) = G(\mathbb{Q}) \backslash (\mathbb{H}^3 \times G(\mathbb{A}_f))/K.
\]

We simply denote by \( X(n) \) the arithmetic manifold \( X(K(n)) \).

5.2. Connected components of \( X(n) \)

The connected components of \( X(n) \) can be described as follows. Write \( G(\mathbb{A}_f) = \bigsqcup_j G(\mathbb{Q}) g_j K(n) \). Then

\[
X(n) = \bigsqcup_j \Gamma_j \backslash \mathbb{H}^3,
\]

where \( \Gamma_j \) is the image in \( \PGL_2(\mathbb{C}) \) of \( \Gamma_j' = G(\mathbb{Q}) \cap g_j K(n) g_j^{-1} \). We let

\[
Y(n) = \Gamma \backslash \mathbb{H}^3
\]

denote the connected component of \( X(n) \) associated to the class \( g_j = e \) of the identity element so that \( \Gamma \) is the image in \( \PGL_2(\mathbb{C}) \) of \( G(\mathbb{Q}) \cap K(n) \).
Note that $X(n)$ is the quotient of 
$$G(\mathbb{Q}) \backslash (\text{PGL}_2(\mathbb{C}) \times G(\mathbb{A}_f))/K(n),$$
by
$$K_\infty = \text{image in PGL}_2(\mathbb{C}) \text{ of } \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\},$$
the maximal compact subgroup at infinity. Here we have chosen an identification of $G(\mathbb{R})$ with PGL$_2(\mathbb{C})$; in the $G_2$ case, we require that this identification carry $D^\times$ into PGL$_2(\mathbb{R})$, so that, in particular, $K_\infty$ intersects $H(\mathbb{R})$ in a maximal compact subgroup.

In the $G_2$ case, both $Y(n)$ and $X(n)$ are compact manifolds. In the $G_1$ case, both are noncompact of finite volume.

### 5.3. Hecke operators

Suppose that $g \in G(\mathbb{A}_f)$, suppose that $K'$ is another compact open subgroup of $G(\mathbb{A}_f)$, and suppose that $K' \subset gK(n)g^{-1}$. The map $G(\mathbb{A}) \to G(\mathbb{A})$ given by $h \mapsto hg$ defines a continuous mapping 
$$r(g) : X(K') \to X(n).$$
Taking $K' = K(n) \cap gK(n)g^{-1}$, we define the Hecke operator $T(g) : H^\bullet(X(n)) \to H^\bullet(X(n))$ to be the composition

$$H^\bullet(X(n)) \xrightarrow{r(g)^*} H^\bullet(X(K')) \xrightarrow{r(1)^*} H^\bullet(X(n)).$$

For a suitable choice of $g$ this gives rise to the usual Hecke operators $T_m$, which are attached to any ideal $m$ of $\mathcal{O}$ which is relatively prime to ramification, that is, no prime divisor of $m$ lies above any place $v$ of $\mathbb{Q}$ where $K_v$ is nonmaximal.

### 5.4. The truncation in the Bianchi case

Now assume that $G = G_1$, so that we are in the noncompact case. We denote by $X(n)_{tr}$ a truncation of $X(n)$, where we “chop off the cusps.” Thus, $X(n)_{tr}$ is a manifold with boundary, and up to homeomorphism it does not depend on the height at which the cusps were cut off.

Connected components of $X(n)_{tr}$ are homeomorphic to the compact quotient $\Gamma \backslash \mathbb{H}^3_*$ where

$$\mathbb{H}^3_* := \mathbb{H}^3 \setminus \bigcup_{\sigma \in P^1(F)} B(\sigma),$$

where the $B(\sigma)$’s are a disjoint collection of horospheres in $\mathbb{H}^3$ tangent to the rational boundary point $\sigma$. In particular, $\Gamma \backslash \mathbb{H}^3_*$ looks like a thickening of the 2-skeleton of $\Gamma \backslash \mathbb{H}^3$. 
6. Complexity of base-change cohomology classes

In this section, $G = G_2$ and $M = Y(n)$ is an associated congruence arithmetic manifold. We address Conjecture 1.1 for base-change cohomology classes of $M$. We recall below the definition of the base-change part $H^2_{bc}(M)$ of the cohomology $H^2(M)$. Note that in this section, when we write $H^*_{ETX}(M)$ and so on without coefficients, we always mean complex cohomology. Here we prove the following statement.

**Theorem 6.1**

There are an exponent $C$, depending only on the field $F$, and a constant $B$, depending only on the base manifold $Y$, such that, for any level $n$ and corresponding arithmetic hyperbolic manifold $M = Y(n)$ of volume $V$, there exist compact immersed surfaces $S_i$ of genus at most $BV^C$ such that the homology classes $[S_i] \in H_2(M)$ span $H^2_{bc}(M)$.

Note that Section 9 gives evidence that often we actually have $H^2(M) = H^2_{bc}(M)$.

It is enough to prove this theorem for the nonconnected $X(n)$ rather than for $M = Y(n)$. Since $X(n)$ is compact, we can compute $H^2(X(n))$ by means of $L^2$-cohomology, and indeed there is a Hecke-equivariant isomorphism (Matsushima’s theorem; see [12])

$$H^2(X(n)) = H^2(g, K_\infty; L^2([G]))^K,$$

where $L^2([G])$ is the Hilbert space of measurable functions $f$ on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ such that $|f|$ is square-integrable on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ and we abridge by $K = \prod_v K_v$ the compact open subgroup $K(n)$.

Since $G$ is anisotropic, the space $L^2([G])$ decomposes as a direct sum of irreducible unitary representations of $G(\mathbb{A})$ with finite multiplicities (in fact equal to 1). A representation $\sigma$ which occurs in this way is called an automorphic representation of $G$; it is factorizable as a restricted tensor product of admissible representations of $G(\mathbb{Q}_p)$ (or, more precisely, the unitary completions of these admissible representations). In particular,

$$\sigma = \sigma_\infty \otimes \sigma_f,$$

where $\sigma_\infty$ is a unitary representation of $G(\mathbb{R})$ and $\sigma_f$ is a representation of $G(\mathbb{A}_f)$. In the following we let $\mathcal{A}$ be the set of all irreducible automorphic representations $(\sigma, V_\sigma)$ of $G(\mathbb{A})$, realized on the subspace $V_\sigma \subset L^2([G])$.

\footnote{More explicitly, $B$ depends only the division algebra $D'$ together with the data $K_v$ that was fixed at ramified places in Section 5.1.}
6.2. Representations with cohomology

Let \( g = \mathfrak{sl}_2(\mathbb{C}) \) be the Lie algebra of the real Lie group \( G(\mathbb{R}) = \text{PGL}_2(\mathbb{C}) \). There exists a unique, nontrivial, and irreducible \((g, K_\infty)\)-module \((\pi, V_\pi)\) such that \( H^\bullet(g, K_\infty; V_\pi) \neq \{0\} \). Furthermore,

\[
H^q(g, K_\infty; V_\pi) = \begin{cases} 0 & \text{if } q \neq 1, 2, \\ \mathbb{C} & \text{if } q = 1, 2. \end{cases}
\]

If we let \( p = \mathfrak{sl}_2(\mathbb{C})/\mathfrak{su}_2 \), then the compact group \( K_\infty \) acts by conjugation on \( \bigwedge^q p \); this yields an irreducible representation of \( K_\infty \). There is a natural isomorphism

\[
H^q(g, K_\infty; V_\pi) \cong \text{Hom}_{K_\infty}(\bigwedge^q p, V_\pi).
\]

We denote by \( \mathcal{C} \) the subset of \( \mathcal{A} \) which consists of the automorphic representation \( \sigma = \sigma_\infty \otimes \sigma_f \) of \( G(\mathbb{A}) \) such that \( \sigma_\infty \cong \pi \) (where, by a slight use of notation, we use \( \pi \) also to denote the unitary completion of the \((g, K_\infty)\)-module described above).

6.3

Let \( \mathcal{H}_K \) be the Hecke algebra of finite \( \mathbb{Q} \)-linear combinations of \( K \)-double cosets in \( G(\mathbb{A}_f) \). It is generated by the Hecke operators \( T(g) := KgK \). If \( \sigma \) is a representation of \( G(\mathbb{A}) \), then \( \mathcal{H}_K \) acts on the space of \( K \)-fixed vectors of \( \sigma \). On the other hand, Section 5.3 gives a realization of \( \mathcal{H}_K \) on \( H^\bullet(X(K)) \).

To summarize the prior discussion, then, we have an \( \mathcal{H}_K \)-isomorphism

\[
H^q(X(K)) = \bigoplus_{\sigma \in \mathcal{C}} \text{Hom}_{K_\infty}(\bigwedge^q p, V_\sigma^K).
\]  

(6.1)

6.4. Base-change classes

Given an automorphic representation \( \sigma \) of \( G(\mathbb{A}) \), we let \( JL(\sigma) \) be the automorphic representation of \( \text{Res}_{F/\mathbb{Q}}(\text{GL}_2|_F) \) with trivial central character associated to \( \sigma \) by the Jacquet–Langlands correspondence. We say that an automorphic representation \( \sigma \) of \( G \) comes from base change if \( JL(\sigma) \) is isomorphic to \( \text{BC}(\sigma_0) \otimes \chi \), where \( \sigma_0 \) is a cuspidal automorphic representation of \( \text{GL}_2|_{\mathbb{Q}} \) and \( \chi \) is an idèle class character of \( F \), and \( \text{BC} \) denotes base change.

We denote by \( \mathcal{A}^{bc} \) the set of all such representations \((\sigma, V_\sigma)\) and we define

\[
H^2_{bc}(X(K)) \subset H^2(X(K))
\]

as the subspace corresponding, under (6.1), to those \( \sigma \in \mathcal{C} \) that actually belong to \( \mathcal{A}^{bc} \).

A priori, this defines only a complex subspace, but it is actually defined over \( \mathbb{Q} \), as one sees by consideration of Hecke operators. There is then a unique Hecke-invariant splitting.
\[ H^2(X(K), \mathbb{Q}) = H^2_{bc}(X(K), \mathbb{Q}) \oplus H^2_{\text{else}}(X(K), \mathbb{Q}), \]

and we define the base-change subspace \( H^2_{bc} \) of homology as the orthogonal complement of \( H^2_{\text{else}} \). As it turns out, \( H^2_{bc} \) is spanned by some special cycles that we now describe.

### 6.5. Special cycles

Let \( H \subset G_2 \) be as in Section 5; recall that \( H(\mathbb{R}) \simeq \text{PGL}_2(\mathbb{R}) \). Many notions we have defined for \( G \) make similar sense for \( H \). We will not recall definitions but just add \( H \) as a subscript to avoid confusion. For example, we write \( p_H \) for the image inside \( p \) of the Lie algebra of \( H(\mathbb{R}) \). (Recall that \( p \) is defined as a quotient of the Lie algebra of \( G_2(\mathbb{R}) \).)

Let \( L = K \cap H(\mathbb{A}_f) \). The quotient \( Z(L) = H(\mathbb{Q}) \backslash (\mathbb{C} - \mathbb{R}) \times H(\mathbb{A}_f) / L = H(\mathbb{Q}) \backslash H(\mathbb{A}) / L_{\infty} \) is a union of (compact) Shimura curves. Here \( L_{\infty} \) denotes the connected component of \( L_{\infty} \), and \( L_{\infty} / L_{\infty} \simeq \pm 1 \) acts on \( Z(L) \).

The inclusion \( H \hookrightarrow G \) defines a map \( Z(L) \to X(K) \). Note that, since we are supposing that \( K \) is sufficiently small (Section 5.1), both \( Z(L) \) and \( X(K) \) are genuine manifolds and not merely orbifolds. The submanifold \( Z(L) \) defines a class \([Z(L)]\) in \( H^2(X(K)) \).

More generally, for every \( g \in G(\mathbb{A}_f) \) we set \( L_g = gKg^{-1} \cap H(\mathbb{A}_f) \); then right multiplication by \( g \) gives a map \( Z(L_g) \to X(K) \), and by pushing forward the fundamental class from any component we obtain a class in \( H^2(X(K)) \). The components of \( Z(L_g) \) are indexed by \( \mathbb{Q}^\times \backslash \mathbb{A}^\times / (\det L_g)(\mathbb{A}^\times)^2 \), where \( \det \) denotes here the reduced norm. Accordingly, if \( \mu : \mathbb{Q}^\times \backslash \mathbb{A}^\times / (\det L_g)(\mathbb{A}^\times)^2 \to \mathbb{Z} \) is an integer-valued function, then we denote by \([Z(L)]_{g,\mu}\) the associated class in \( H^2(X(K)) \); in other words, \([Z(L)]_{g,\mu}\) is the image of

\[ \mu \in H^0(Z(L_g), \mathbb{Z}) \cong H^2(Z(L_g), \mathbb{Z}) \to H^2(X(K), \mathbb{Z}), \]

where the first map is Poincaré duality.

We let \( Z_K \) be the subspace of \( H^2(X(K)) \) spanned by all such \([Z(L)]_{g,\mu}\). Note that this subspace is spanned by classes of totally geodesic immersed surfaces that we call special cycles.

### 6.6

We will need a precise description of the dual pairing \( \langle \cdot, \cdot \rangle : H_2 \times H^2 \to \mathbb{C} \). Choose a Haar measure \( dh \) on \( H(\mathbb{Q}) \backslash H(\mathbb{A}) \), and fix a generator \( \nu_H \) of the line \( (\lambda^2 p_H) \). Now let
T \in \text{Hom}_{K_\infty}(\wedge^2 p, V_{\sigma}^K)$, for some $\sigma \in \mathcal{C}$. By (6.1) we can identify $T$ with an element of $H^2(X(K))$. We compute

$$\langle [Z(L)]_{g,\mu}, T \rangle = c \int_{\text{Hom}(\mathbb{Q})\setminus \text{Hom}(\mathbb{A})/L_{\infty}^0 L_g} T(v_H)(hg)\mu(\det(h)) \, dh,$$

where $c$ is a nonzero constant of proportionality, depending on $g$, the choice of measure $dh$, and the choice of $v_H$.

6.7. Distinguished representations
Let $(\sigma, V_\sigma) \in \mathcal{A}$. A function $\varphi \in V_\sigma$ can then be seen as a function in $L^2([G])$. Let $\chi$ be a quadratic idèle class character of $\mathbb{A}^\times / \mathbb{Q}^\times$. We define the period integral

$$P_\chi(\varphi) = \int_{\text{Hom}(\mathbb{Q})\setminus \text{Hom}(\mathbb{A})} \varphi(h)\chi(\det h) \, dh,$$

where $dh = \bigotimes_v dh_v$ is a Haar measure on $\text{Hom}(\mathbb{Q})\setminus \text{Hom}(\mathbb{A})$ as above. Let us say that $\sigma$ is $\chi$-distinguished if $P_\chi(\varphi) \neq 0$ for some $\varphi \in \sigma$. We say simply that $\sigma$ is distinguished if it is $\chi$-distinguished for some $\chi$. (For the following, see [3, Theorem 4.1]; see also the discussion above that theorem, [3, Section 3], and [2, Proposition 3.4].)

**Proposition 6.8**

An automorphic representation $\sigma \in \mathcal{A}$ is distinguished if and only if $\sigma$ comes from base change.

**Proposition 6.9**

Let $\sigma \in \mathcal{C}$ be such that $\sigma_f \neq \{0\}$. Then $\sigma$ is not distinguished if and only if the subspace

$$\text{Hom}_{K_\infty}\left(\bigwedge^2 p, V_{\sigma}^K\right) \subset H^2(X(K))$$

is orthogonal to the subspace $Z_K$ spanned by the cycles $[Z(L)]_{g,\mu}$. Equivalently, cycles $[Z(L)]_{g,\mu}$ span $H^2_{bc}$.

**Proof**

The direct implication “only if” follows immediately from (6.2), together with the fact that $\mu$ lies in the span of the quadratic idèle class characters $h \mapsto \chi \circ \det$.

In the converse direction, suppose that $P_\chi$ is not identically zero for some $\chi$, but $\text{Hom}_{K_\infty}(\wedge^2 p, V_{\sigma}^K)$ is orthogonal to $Z_K$. Then (6.2) says at least that $P_\chi(\varphi)$ vanishes for every $\varphi$ of the form $\varphi_\infty \otimes g \varphi_f \in \sigma_\infty \otimes \sigma_f$, where $\varphi_f$ is $K$-fixed, $g \in G(\mathbb{A}_f)$ is arbitrary, and $\varphi_\infty$ is the image of $v_H$ under a nontrivial element of $\text{Hom}_{K_\infty}(\wedge^2 p, \sigma_\infty)$. Since such vectors $g \varphi_f$ span all of $\sigma_f$, we see that $P_\chi$ vanishes on $\varphi_\infty \otimes \sigma_f$. Now
factor $P_X$ on $\sigma = \sigma_{\infty} \otimes \sigma_f$ as $P_{\infty} \otimes P_f$ (this can be done by multiplicity 1; see Section 6.13). It remains to show that $P_{\infty}(\varphi_{\infty}) \neq 0$.

However, $\chi_\infty$ is the nontrivial quadratic character of $\mathbb{R}^*$. This is because if $\sigma_{\infty} \simeq \pi$ were distinguished by the trivial character $\chi_\infty$, then $\sigma$—considered as a representation of $GL_2(\mathbb{C})$—would be distinguished by $GL_2(\mathbb{R})$. It is known from [29, Theorem 7] that such representations of $GL_2(\mathbb{C})$ are the unstable base changes of representations of $U(1, 1)$, but $\sigma$ is not such a representation; it is the stable base change of the weight 2 discrete series representation. Now, if $P_{\infty}(\varphi_{\infty}) = 0$, then the above argument shows that $[Z(L)]_{g, \mu}$ would be zero for every choice of $g, \mu$ as above, and this is not so, as follows, for example, from [49].

6.10. Outline of the proof of Theorem 6.1

Write $V = \text{vol}(Y(K))$. Fix an embedding of $\iota : G \hookrightarrow \text{SL}_N$ over $\mathbb{Q}$. For $g \in G(\mathbb{Q}_v)$, we denote by $\|g\|_v$ the largest $v$-adic valuation of any entry of $\iota(g)$. For $g = (g_v) \in G(\mathbb{A}_v)$, we put $\|g\| = \prod_v \|g_v\|_v$.

Let $\sigma_j$ (for $j$ in some index set $J$) be all the $\sigma \in \mathcal{C}$ such that $\sigma^K \neq 0$ and such that $\sigma$ comes from base change. Let $R$ be the set of finite ramified places, that is, the set of finite places at which $K_v \subset G(\mathbb{Q}_v)$ is not maximal or where $K_v \cap H(\mathbb{Q}_v) \subset H(\mathbb{Q}_v)$ is not maximal, and let $\mathbb{Q}_R = \prod_{v \in R} \mathbb{Q}_v$. Let $R^* = R \cup \{\infty\}$. We decompose accordingly each $\sigma = \sigma_j$ ($j \in J$) as

$$\sigma = \pi \otimes \sigma_{\infty} \otimes \sigma_R \otimes \sigma_f,$$

where $\sigma_R$ is a representation of $G(\mathbb{Q}_R)$ and where $\sigma_R$ is a representation of $G(\mathbb{A}_v)$, the group $G$ over the adèles omitting $R^*$.

The proof now proceeds in four steps. After giving the outline we discuss steps 1 and 3 in more detail (see Sections 6.11 and 6.13).

Fix $j_0 \in J$, and let $\sigma_0 = \sigma_{j_0}$. Let $\chi_0$ be so that $\sigma_0$ is $\chi_0$-distinguished. Factor $P_{\chi_0}$ on $\sigma_0 = \pi \otimes \sigma_{0,R} \otimes \sigma_{0,f}$:

$$P_{\chi_0} = P_{\infty} \otimes P_R \otimes P_f.$$

(1) We will show, first of all, that there exist ideals $p_1, \ldots, p_r$ of $F$ relatively prime to $R$ (that is, they do not lie above any place in $R$) whose norms $Np_i$ are all bounded by $aVb$ (with $b$ a constant depending only on $F$, and $a$ a constant that depends only on the base manifold $Y(1)$, as in the statement of Theorem 6.1), and constants $c_i \in \mathbb{C}$ such that the Hecke operator $\sum_i c_i Y^{p_i}$ is nonzero on $V_{\sigma_0}$ and trivial on $V_{\sigma_k}$ for every $k \in J, k \neq j_0$. In other words, if $\lambda_{q}(\sigma)$ is the eigenvalue by which $T_q$ acts on $\sigma$, then we have $\sum c_i \lambda_{p_i}(\sigma_0) \neq 0$, whereas $\sum c_i \lambda_{p_i}(\sigma_k) = 0$ for $k \neq j_0$. 


(2) If $v^{R}_{\text{sph}}$ denotes the nontrivial spherical vector in the space of $\sigma_0^{R}$, then we have
\[ P^{R}(v^{R}_{\text{sph}}) \neq 0. \]

We will omit the proof (for discussion of this type of result, see [56, Corollary 8.0.4]). (We have not verified that the auxiliary conditions of [56] apply here, although the method surely does. In any case, this can be verified here by direct computation. Because of multiplicity 1, it is enough for each $v \in R$ to show that there exists a function on $G(\mathbb{Q}_v)$ such that $f(hgk) = \chi(h)f(g)$ for $k \in K_v$ and $h \in H(\mathbb{Q}_v)$, such that $f(1) \neq 0$ and such that the Hecke eigenvalue of $f$ is the same as $\sigma_v$. Now the cosets $H(\mathbb{Q}_v) \backslash G(\mathbb{Q}_v)/K_v$ are parameterized by nonnegative integers and one constructs the required $f$ as a solution to a linear recurrence.)

(3) Now let $\varphi_1, \ldots, \varphi_s$ be a basis for $(\sigma_0, f)^R$. Write $\varphi_j = \varphi_{j,R} \otimes v^{R}_{\text{sph}}$. We will show that there exist $g_1, \ldots, g_s \in G(\mathbb{Q}_R)$ such that $\|g_i\| \leq cV^d$ for constants $c, d$ depending only on $F$, and the matrix $(P_R(g_k \cdot \varphi_{j,R}))_{1 \leq i, j \leq s}$ is nonsingular.

(4) Let $\varphi_\infty \in \pi$ be as in the proof of Proposition 6.9. From the two first steps we conclude that for every $j = 1, \ldots, s$ we have
\[ P_{X_0}(\varphi_\infty \otimes \sum_i c_i \mathbb{T}_{p_i} \varphi_j) = \left( \sum_i c_i \lambda_{p_i}(\sigma_0) \right) P_R(\varphi_{j,R}) P_\infty(\varphi_\infty) P^{R}(v^{R}_{\text{sph}}), \]
\[ P_{X_0}(\varphi_\infty \otimes \sum_i c_i \mathbb{T}_{p_i} \psi) = 0, \quad \psi \in \sigma_j \neq \sigma_0, \]

where—according to steps 1 and 2 and the discussion in the proof of Proposition 6.9—the scalars $\mu_1 := \sum_i c_i \lambda_{p_i}(\sigma_0)$, $\mu_2 := P^{R}(v^{R}_{\text{sph}})$, and $\mu_3 := P_\infty(\varphi_\infty)$ are all nonzero. Since the $g_k$’s belong to $G(\mathbb{Q}_R)$ and the ideals $p_i$ are relatively prime to $R$, step 3 finally implies that the matrix
\[ \left( P_{X_0}(\varphi_\infty \otimes g_k \cdot \sum_i c_i \mathbb{T}_{p_i} \cdot \varphi_j) \right)_{1 \leq k \leq s, 1 \leq j \leq s} = \mu_1 \cdot \mu_2 \cdot \mu_3 \cdot \left( P_R(g_k \cdot \varphi_{j,R}) \right)_{j,k} \]

is nonsingular.

Repeating the same reasoning for each $\sigma_j$ leads to the following refinement of Proposition 6.9: $H^b_2$ is spanned by cycles of the form $\mathbb{T}_p[Z(L)]_{g, \mu}$, where both $Np$ and $\|g\|$ are bounded by a polynomial in $V$. Using trivial estimates, we see that all the cycles appearing in this statement have volume bounded by a power of $V$. That will conclude the proof of Theorem 6.1. In the following sections we provide details for steps 1 and 3.
6.11. Step 1 of Section 6.10: A quantitative multiplicity 1 theorem

We first deal with automorphic representations of $GL_2|F$. Recall the definition of the analytic conductor of Iwaniec–Sarnak: Let $\pi = \bigotimes_v \pi_v$ be a cuspidal automorphic representation of $GL_2|F$. Each finite place $v$ we denote by $\text{Cond}_v(\pi) = q_v^{m_v}$, where $m_v$ is the smallest nonnegative integer such that $\pi_v$ possesses a fixed vector under the subgroup of $GL_2(\mathbb{O}_{F_v})$ consisting of matrices whose bottom row is congruent to $(0,1)$ modulo $\sigma_v^m$. Here $\sigma_v \in F_v$ is a uniformizer. For the infinite place $v$, let $\mu_{j,v} \in \mathbb{C}$ satisfy $L(s,\pi_v) = \prod (2\pi)^{-s-\mu_{j,v}} \Gamma(s+\mu_{j,v})$, and put $\text{Cond}_v(\pi) = \prod (2 + |\mu_{j,v}|)^2$.

We then put $\text{Cond}(\pi) = \prod_v \text{Cond}_v(\pi)$ (this is within a constant factor of the Iwaniec–Sarnak definition).

**Lemma 6.12 (Linear independence of Hecke eigenvalues)**

Given automorphic representations $\pi_1, \ldots, \pi_r$ of $GL_2|F$, all of which have analytic conductor at most $X$, let $\mathcal{Q}$ be a set of prime ideals of $F$ of cardinality at most $B \log X$ containing all ramified primes for the $\pi_i$’s, and let $\{n_j : j = 1, \ldots, s\}$ be the set of all ideals of $F$ of norm less than $Y$ that are relatively prime to $\mathcal{Q}$. Then the $r \times s$ matrix of Hecke eigenvalues

$$M_{ij} = (\lambda_{q_j}(\pi_i))_{i,j} \quad (j = 1, \ldots, s, i = 1, \ldots, r)$$

has rank $r$ so long as $Y \geq (rX)^A$, where $A$ is a constant depending only on $B$ and the field $F$.

Before the proof, we show how this gives step 1. The Jacquet–Langlands correspondence associates to any automorphic representation $\sigma_j$ as in Section 6.10 a cuspidal automorphic representation $\pi_j = JL(\sigma_j)$ of $GL_2|F$ with the same Hecke eigenvalues. Since $\sigma_j^K(n) \neq \{0\}$, $\text{Cond}(\pi_j) \ll N(n)^A$, where the implicit constant may depend on the ramified places of $D'$ and the choice of level structure $K_v$ at ramified places from Section 5.1. (We do not know a reference for this bound, but that such a polynomial bound exists can be readily derived by reducing to the supercuspidal case and using the relationship between depth and conductor (see [43] and references therein, especially [17]).) In particular, the conductor is bounded by a polynomial in $\text{vol}(Y(n))$.

First of all, we show that the norm of the level of $\pi_j$ can be bounded by a polynomial in $V$. To obtain step 1, then, we apply the lemma with $\mathcal{Q} = R$, the set of “bad” places—that is, the places that are ramified for $D$ together with primes dividing $n$. Note that the number of primes dividing $n$ is at most $\log_2(N(n))$: the desired result follows, since (in the setting of step 1) the integer $r$ is bounded by $\dim H^2(Y(n))$ and thus by a linear function in $\text{vol} Y(n)$.
Proof
This is a certain strengthening of multiplicity 1 and will be deduced from the quantitative multiplicity 1 estimate of Brumley [15] (see also [39], [50] for earlier results in the same vein). Consider, instead of the matrix $M$, the smoothed matrix $N$ wherein we multiply the matrix entry $M_{ij}$ by $h(\text{Norm}(q_j/Y))$, where $h$ is a smooth real-valued bump function on the positive reals such that $h(x) = 0$ when $x > 1$ and $h$ is positive for $x < 1$. Clearly the rank of $M$ and the rank of $N$ are the same.

It is enough to show that the square $(r \times r)$ Hermitian matrix

$$N \cdot \bar{N}$$

is of full rank $r$. Its $(i, j)$ entry is equal to

$$\sum N_{ik} \bar{N}_{jk} = \sum_q \lambda_q(\pi_i) \overline{\lambda_q(\pi_j)} h(q/Y)^2,$$

where the sum extends over the set of $q$ with norm less than $Y$ and prime to $Q$.

This is very close to [15, page 1471, equation (23)], with a minor wrinkle: [15] discusses the corresponding sum but with $\lambda_q(\pi_i) \overline{\lambda_q(\pi_j)}$ replaced by $\lambda_q(\pi_i \times \pi_j)$. But the proof of [15] applies word for word here, using the equality

$$\sum_q \lambda_q(\pi_i) \overline{\lambda_q(\pi_j)} \frac{L^Q(\pi_i \times \pi_j, s)}{L^Q(\omega_i \omega_j, 2s)} = \sum_q \frac{\lambda_q(\pi_i) \lambda_q(\pi_j)}{N(q)^s},$$

(6.4)

where $\omega_i$ is the central character of $\pi_i$, and the superscript $Q$ means we take the finite $L$-function and omit all factors at the set $Q$. It leads to the corresponding bound

$$\sum_q \lambda_q(\pi_i) \overline{\lambda_q(\pi_j)} h(q/Y)^2 = \delta_{ij} Y \cdot R_i + O(Y^{1-\theta} X^{B'}).$$

Here $R_i$ is a residue of the $L$-function on the right of (6.4), $\theta$ is a positive real number (one can take $\theta = 1/2$), and $B'$ is a constant that depends only on the constant $B$ and the field $F$. It moreover follows from [15, equation (21)] that $R_i$ is bounded below by $X^{-C}$ for some absolute (positive) constant $C$.

Now the proof follows from diagonal dominance: if we are given a square Hermitian matrix $S = (S_{ij})$ such that, for every $\alpha$,

$$S_{\alpha \alpha} > \sum_{j \neq \alpha} |S_{\alpha j}|,$$

(6.5)

then $S$ is nonsingular, by an elementary argument. Now one may choose $A$, depending only on $B$ and $F$, so that (6.5) holds as long as $Y \geq (rX)^A$. □
6.13. Step 3 of Section 6.10
Let \((\sigma, V^\sigma) \in \mathcal{C}\) and \(\chi\) be such that the functional \(P_\chi\) is not identically vanishing on \(\sigma\). For \(p\) a prime of \(\mathbb{Q}\), let \(H_p = H(\mathbb{Q}_p)\), and let \(G_p = G(\mathbb{Q}_p)\). The multiplicity 1 theorem shows that the functional \(P_\chi\) factorizes over places.

**Lemma 6.14**
For any irreducible \(G_p\)-module \(\sigma_p\) we have
\[
\dim \text{Hom}(H_p, \chi_p)(\sigma_p, \mathbb{C}) \leq 1.
\]

**Proof**
If \(p\) is split in \(F\), then the result is easy. If \(D_p\) is split, then this amounts to [28, Proposition 11] or [52, Theorem A]. (Note that by twisting, one reduces to the case of \(\chi_p = 1\), at the cost of allowing \(\sigma_p\) to have a central character, so one can indeed apply [28, Proposition 11].) If \(D_p\) is ramified, then there does not appear to be a convenient reference. The following proof was suggested to us by one anonymous referee. First, if \(\sigma_p\) is either a principal series or a Steinberg representation, then \(\sigma_p\) can be obtained as a quotient of a representation induced from the Borel subgroup \(B_p \subset G_p = \text{GL}_2(F_p)\). But \(D_p^\times\) acts transitively on lines in \(D_p\) and hence on \(\text{GL}_2(F_p)/B_p\). The lemma therefore holds for the full induced representation. Now if \(\sigma_p\) is supercuspidal, then the proof of [52, Theorem 5.1] implies the lemma. To be more precise, the comment “it therefore suffices to prove that the dimension of \(\text{GL}_2(k)\)-invariant linear forms on \(V\) is one more than the dimension of \(D_k^*\)-invariant linear forms in the finite dimension (virtual representation) \(V - P\) of \(D_k^*\)” at the end of [52, page 21] shows that the dimension of \(D_p^\times\)-invariant forms on the supercuspidal representation \(\sigma_p\) is equal to the dimension of \(\text{GL}_2(\mathbb{Q}_p)\)-invariant forms. The lemma reduces to the split case. □

For simplicity in what follows, we suppose that actually \(\chi_p\) is trivial; the general case is a twisted case of what follows. So let \(P_p\) be a nonzero \(H_p\)-invariant functional on \(\sigma_p\). Denote by \(V_p\) the index of \(K_p\) inside a maximal compact subgroup. We will now sketch a proof of the following result, which implies step 3:

If \(v_1, \ldots, v_r\) form a basis for \(\sigma_p^K_p\), then there exist \(g_i \in G_p\) with \(\|g_i^{-1}\| \leq c V_p^d\) —where \(c, d\) are constants depending only on the embedding \(i\) used in the definition of \(\|g\|\) —such that the matrix \((P_p(g_i^{-1} v_j))_{i,j}\) is nonsingular.

Consider the functions \(f_j\) on \(X = G_p/H_p\) defined by the rule \(g \mapsto P_p(g^{-1} v_j)\). We will show that, when restricted to the compact set
\[
\Omega = \{g H : \|g^{-1}\| \leq c \cdot V_p^d\},
\]
the functions \(f_j\) are linearly independent.
Suppose to the contrary that there exist \(a_1, \ldots, a_r\) not all 0 such that \(\sum a_j f_j\) is zero on \(\Omega\). However, the asymptotics of \(\sum a_j f_j\) can be computed by the theory of asymptotics on spherical varieties or even symmetric varieties (see [41], [42], or [57]); this theory of asymptotics shows that if \(\sum a_j f_j\) vanishes identically on a sufficiently large compact subset of \(X\), then it must in fact identically vanish everywhere on \(X\), which is impossible because associating to \(v\) the function \(P_p(g^{-1} v)\) is an embedding of the irreducible representation \(\sigma_p\) into functions on \(X\). More explicitly, the asymptotics of a function on \(X\) are described “near infinity” in terms of finite linear recurrences and, in particular, have the property that vanishing on a large enough compact set implies vanishing everywhere.

All that is needed is to give a sufficiently effective version of this asymptotic theory. We sketch this. This sketch actually departs slightly from the above outline, in that it uses asymptotics on \(G\) rather than \(X\).

First, the wavefront lemma (see [4, Proposition 3.2] or [57, Corollary 5.3.2]) shows that there is a set \(F \subset G\) such that \(FH = G\) and \(P_p(g^{-1} v_j)\) coincides for \(g \in F\) with a usual matrix coefficient \((g^{-1} v_j, u)\), where \(u\) is a vector obtained by smoothing \(P_p\). The desired asymptotics for \(\sum a_j f_j\) follow from known asymptotics of matrix coefficients (see, e.g., [20]). Now, by explicit construction (see the references given), \(F\) is a finite union of translates of the form \(g_i A^+ U\), where \(A^+\) is the positive cone in a maximal split torus, and \(U\) is compact; since the asymptotics of matrix coefficients on \(A^+\) satisfy “linear recurrences,” both in the interior and along the walls, we see that the vanishing of matrix coefficients on a sufficiently large compact subset of \(F\) implies vanishing on all of \(F\). Now what is needed is an explicit control on when matrix coefficients follow their asymptotic expansion. For supercuspidal representations of \(GL_n\), a sufficiently strong bound has been given by Finis, Lapid, and Müller [26, Corollary 2]. In our case of \(GL_2\) the remaining possibilities of principal series (and their subrepresentations) can be verified by direct computation. (An alternate approach that treats the two together is to compute in the Kirillov model, using the local functional equation to control support near 0.)

7. The noncompact case

7.1. The main result
In this section \(G = G_1\). If \(M\) is a noncompact manifold, then we define, as usual, \(H^i\) to be the image of the compactly supported cohomology \(H^i_c\) inside the cohomology \(H^i\), and we define \(H_{i,1}\) to be the image of the usual homology \(H_i\) inside the Borel–Moore homology \(H_{i,BM}\). All these definitions make sense with any coefficients, in particular, either integral or complex. If we do not specify the coefficients, then we will understand them to be \(\mathbb{C}\).
We now suppose the following.

(i) \( K = K_0(n) \), where \( n \) is a square-free ideal, that is, \( K = \prod K_v \) where

\[
K_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \PGL_2(\mathcal{O}_v) : c \in n\mathcal{O}_v \right\}.
\]

(ii) The corresponding symmetric space \( X_0(n) = X(K) \) satisfies \( \dim H^1_1(X_0(n), \mathbb{C}) = 1 \). Let \( \pi \) be the associated automorphic representation (i.e., the unique representation whose Hecke eigenvalues coincide with those of a class in this \( H^1_1 \)).

(iii) \( \pi \) is associated to an elliptic curve \( E \) of conductor \( n \) over \( F \), which we moreover assume to not have complex multiplication. (See after Theorem 1.3 for a discussion of this assumption, which is part of the “Langlands philosophy.”)

Assumption (ii) forces \( X_0(n) \) to have one connected component (equivalently, that the class number of \( F \) is odd), because otherwise one could produce other cohomology classes by twisting by everywhere-unramified quadratic characters. Therefore, \( X_0(n) \) coincides with its identity component \( Y_0(n) \). (We thank one of the referees for pointing this out.) This assumption is only to make the proof as clean as possible: if we restrict to the space of cohomology corresponding to elliptic curves as in (iii), a similar statement and proof apply even without assumption (ii). Under these assumptions our main result is as follows.

**Theorem 7.2**

There exists an \( L^2 \)-harmonic 1-form \( \omega \) representing a nonzero class in \( H^1_1(Y_0(n), \mathbb{C}) \), with integral periods (i.e., \( \int_{\gamma} \omega \in \mathbb{Z} \) for every \( \gamma \in H_1 \)), and moreover

\[
\langle \omega, \omega \rangle \leq A(\text{Norm} \, n)^B
\]

for some constants \( A \) and \( B \) depending only on \( F \).

By methods similar to Section 4, this proves Conjecture 1.1 in this case. Note, however, that \( Y_0(n) \) is now an orbifold rather than a manifold. We will therefore consider the (finite) orbifold cover \( Y(n) \), which is a manifold, and think of \( \omega \) as a \( \Gamma_0(n)/\Gamma(n) \)-invariant class in \( H^1_1(Y(n), \mathbb{C}) \). The analogue of Conjecture 1.1 for \( Y_0(n) \)—which will be deduced from Theorem 7.2—is then: if \( Y_0(n) \) is as above, then there exist compact surfaces \( S_i \) in \( Y(n) \) of genus much less than \( \text{vol}(Y_0(n))^C \) such that the images of the classes \( [S_i] \) in \( H_2(Y(n)) \) under the map \( H_2(Y(n)) \to H_2(Y_0(n)) \) span \( H_2(Y_0(n), \mathbb{R}) \).

Indeed, fix a \( \Gamma_0(1) \)-invariant triangulation of \( \mathbb{H}^3 \) such that the boundary is a full subcomplex. It then follows that the boundary is a \( \Gamma_0(1) \)-equivariant deformation retract of the subcomplex which consists of all simplices that intersect the bound-
ary. We will also assume that all the edges of the dual cell subdivision have length at most 1. The triangulation of $\mathbb{H}_3^n$ projects onto a triangulation $K$ of $Y(n)_{tr}$. As in Section 4 we denote by $K'$ the dual cell subdivision. Given a 1-simplex $e$ of $K$, we denote by $e^*$ the dual element in $C^1(K, \mathbb{R}) = C_1(K, \mathbb{R})^*$ and by $e'$ the dual cell in $K'$. The map $e^* \mapsto e'$ then induces an isomorphism $C^1(K, \mathbb{R}) \to C_2(K', \mathbb{R})$ that takes cocycles into cycles modulo the boundary of $Y(n)_{tr}$. Then

$$ Z := \sum_e \left( \int_e \omega \right) e' \in C_2(K', \mathbb{R}) \quad (7.2) $$

is a 2-cycle modulo the boundary, and its image under $H_2(BM(Y(n))) \to H_2(BM(Y_0(n)))$ represents the image of the class of $[\Gamma(n) : \Gamma(n)]\omega$ under the Poincaré–Lefschetz duality.

Now, as in Section 4, we have

$$ \inf \left\{ \sum |n_k| \left| \sum n_k \sigma_k \right| = [Z] \quad \text{where} \quad \sum n_k \sigma_k \text{ is a singular chain in } C_2(Y(n)_{tr}, \partial Y(n)_{tr}) \right\} \ll N(n)^A. \quad (7.3) $$

Gabai’s theorem—used in Section 4—holds for $H_2(Y(n)_{tr}, \partial Y(n)_{tr})$: the 2-cycle $Z$ is homologous into a (maybe disconnected) embedded surface

$$(S, \partial S) \subset (Y(n)_{tr}, \partial Y(n)_{tr})$$

such that the left-hand side of (7.3) is $\sum_{\chi(S_i) < 0} -2\chi(S_i)$, the sum being taken over components $S_i$ of $S$. Since $\partial Y(n)_{tr}$ is incompressible and $Y(n)_{tr}$ is atoroidal and aspherical, we may furthermore assume that all components $S_i$ have negative Euler characteristic.

Note that the surface $S$ could a priori have boundary, but since $[S] = [Z]$ belongs to the image of $H_2(Y(n)_{tr})$ in $H_2(Y(n)_{tr}, \partial Y(n)_{tr})$, the image of $[S]$ in $H_1(\partial Y(n)_{tr})$ by the boundary operator in the long exact sequence associated to the pair $(Y(n)_{tr}, \partial Y(n)_{tr})$ is trivial. We can close $S$ using discs or annuli on the boundary tori, because $\partial S$ intersects each boundary torus in a union of disjoint simple closed curves $\gamma_j$. One first closes each $\gamma_j$ which is null-homotopic by a disc. The remaining $\gamma_j$’s must be parallel and all define, up to sign, the same primitive class in homology; we can close them in pairs by annuli.

Let $f$ be the total number of discs adjoined when closing the boundary curves. The closing process has only increased the total Euler characteristic of $S$ by $f$, so we arrive now at a closed surface $S'$ with Euler characteristic

$$ \chi(S') = \chi(S) + f = \sum_{\chi(S_i) < 0} \chi(S_i) + f. $$
Finally, we may remove from $S'$ all components that are either tori or spheres, because both cases must have trivial class inside $H_2(Y(n)_{\mathbb{R}}, \partial Y(n)_{\mathbb{R}})$. Removing the tori components does not change the Euler characteristic, but removing the sphere components decreases it and therefore increases the complexity. This is the last issue we have to deal with.

Each component $S'_i$ of $S'$ that is a sphere meets $S$ along spheres with at least 3 boundary components. So each such $S'_i$ corresponds to a component $S'^*_i$ of $S$ with $\chi(S'^*_i) \leq -1$, and distinct $i$'s give rise to distinct components. So

$$\sum_{\chi(S'_i) = 2} \chi(S'_i) \leq \sum_{\chi(S_i) < 0} -2\chi(S_i).$$

Therefore, the total Euler characteristic of all sphere components of $S'$ is at most $\sum_{\chi(S_i) < 0} -2\chi(S_i)$, and removing these and tori gives a closed surface $S''$ with Euler characteristic

$$\chi(S'') \geq \sum_{\chi(S_i) < 0} 3\chi(S_i) + f \geq \sum_{\chi(S_i) < 0} 3\chi(S_i),$$

where $S''$ still represents $Z \in H_2(Y(n)_{\mathbb{R}}, \partial Y(n)_{\mathbb{R}})$. Its image in $H_2(Y_0(n)_{\mathbb{R}}, \partial Y_0(n)_{\mathbb{R}})$ is a generator for the (1-dimensional) image of $H_2(Y_0(n))$ in $H_{2,\text{BM}}(Y_0(n))$ whose complexity is bounded. Finally, since the homology classes of the cusps of $Y(n)$ are represented by images of surfaces of genus 1, the conjecture follows.

### 7.3. Modular symbols

We henceforth suppose that we are in the situation of Section 5 with $G = G_1 (= \text{Res}_{F/\mathbb{Q}} \text{PGL}_2)$. In what follows, we will usually think of $G$ as $\text{PGL}_2$ over $F$, rather than the scalar-restricted group to $\mathbb{Q}$.

Let $\alpha, \beta \in \mathbb{P}^1(F)$, and let $g_f \in G(\mathbb{A}_f)/K_f$. Then the geodesic from $\alpha$ to $\beta$ (considered as elements of $\mathbb{P}^1(\mathbb{C})$, the boundary of $\mathbb{H}^3$), translated by $g_f$, defines a class in $H_{1,\text{BM}}(X(K))$ that we denote by $\langle \alpha, \beta; g_f \rangle$. Evidently these satisfy the relation

$$\langle \alpha, \beta; g_f \rangle + \langle \beta, \gamma; g_f \rangle + \langle \gamma, \alpha; g_f \rangle = 0,$$

the left-hand side being the (translate by $g_f$ of the) boundary of the Borel–Moore chain defined by the ideal triangle with vertices at $\alpha$, $\beta$, $\gamma$. Note that $\langle \alpha, \beta; g_f \rangle = \langle \gamma \alpha, \gamma \beta; \gamma g_f \rangle$ for $\gamma \in \text{PGL}_2(F)$.

For a finite place $v$ of $F$, let $F_v$ denote the completion of $F$ at $v$, let $\mathcal{O}_v$ denote the ring of integers of $F_v$, and let $q_v$ denote the cardinality of the residue field of $F_v$. By the valuation at $v$ of the triple $\langle \alpha, \beta; g_f \rangle$ we mean the distance between

- the geodesic from $\alpha_v, \beta_v \in \mathbb{P}^1(F_v)$ inside the Bruhat–Tits tree of $G(F_v)$ and
- the point in that tree defined by $g_f \mathcal{O}_v^2$.
that is, the minimum distance between a vertex on this geodesic and the vertex whose stabilizer is \( \text{Ad}(g_f) \) PGL\(_2(\mathcal{O}_v)\).

Let \( n_v \) be the valuation of the symbol \( \langle \alpha, \beta; g_f \rangle \) at \( v \). We define the conductor of the symbol to be \( \mathfrak{f} = \prod_v q_v^{n_v} \), where \( q_v \) is the prime ideal associated to the place \( v \). The denominator of the symbol \( \langle \alpha, \beta; g_f \rangle \) is then defined as

\[
\text{denom}(\langle \alpha, \beta; g_f \rangle) = |(\mathcal{O}/\mathfrak{f})^\times| = \prod_{v: n_v \geq 1} (q_v^{n_v-1}(q_v - 1)). \tag{7.4}
\]

where \( q_v \) is the norm of \( q_v \). We sometimes write this as the Euler \( \varphi \)-function \( \varphi(f) \).

Let \( T \) be the stabilizer of \( \alpha, \beta \) in PGL\(_2\); it is isomorphic to the multiplicative group \( T \simeq \mathbb{G}_m \) and the isomorphism is unique up to sign. Then \( T(\mathcal{O}_v) \cap \text{Ad}(g_f) \times \text{PGL}_2(\mathcal{O}_v) \) corresponds to the subgroup \( 1 + q_v^{n_v} \subset \mathbb{G}_m(F_v) = F_v^\times \) if \( n_v \geq 1 \), and otherwise it corresponds to the maximal compact subgroup of \( F_v^\times \). (For example, to see the latter statement, note that \( T(\mathcal{O}_v) \) fixes exactly the geodesic from \( \alpha \) to \( \beta \) inside the building of \( \text{PGL}_2(F_v) \).)

In particular, any finite-order character \( \psi \) of \( T(\mathbb{A}_F)/T(F) \simeq \mathbb{A}_F^\times /F^\times \) that is trivial on \( T(\mathbb{A}_F) \cap \text{Ad}(g_f) \text{PGL}_2(\mathcal{O}) \) has conductor dividing \( \mathfrak{f} \) and order dividing \( h_F \varphi(f) \), where

\[
h_F = \text{order of narrow class group } C_F \text{ of } F.
\]

More generally, if \( \psi \) is trivial on \( T(\mathbb{A}_F) \cap \text{Ad}(g_f)K_0(n) \), with \( n \) a square-free ideal, then—by a similar argument—the conductor of \( \psi \) divides \( nf \) and its order divides \( h_F \varphi(nf) \); in particular, its order divides

\[
h_F \varphi(f) \cdot \text{Norm}(n) \cdot \varphi(n). \tag{7.5}
\]

Note that another way to present our arguments would be to use a stronger version of conductor designed so that it takes into account the level structure at \( n \). This leads to a more elaborate version of Section 7.4 but simplifies other parts of the argument, because the factors of \( n \) are no longer present in (7.5) (see Section 7.6 for comments on that).

### 7.4. Denominator avoidance and its proof

**Lemma 7.5**

Fix any integer \( M \). Let \( p \) be a prime number. If \( p > 5 \) (resp., \( p \leq 5 \)), then any class in \( H_{1,\text{BM}}(Y(K), \mathbb{Z}) \) is represented as a sum of symbols \( \langle \alpha, \beta, g_f \rangle \), each of which has conductor relatively prime to \( Mp \) and denominator indivisible by \( p \) (resp., divisible by at most \( p^A \), for an absolute constant \( A \)).
Proof

This is a slight sharpening of results in [19, Section 6.7.5]. In fact, there is a slight error in [19] which does not deal properly with the case when \( g_f \not\in \text{PGL}_2(\mathcal{O}_v) \); the argument below in any case fixes that error.

As in [19], the Borel–Moore homology is generated by \( \langle 0, \infty; g_f \rangle \) for varied \( g_f \).

(In the classical case, this goes back to Manin, and the proof is the same here.) Set \( A_p = 1 \) for \( p > 5 \), set \( A_p = 3 \) for \( 2 < p \leq 5 \), and set \( A_2 = 4 \). One writes

\[
\langle 0, \infty; g_f \rangle = \langle 0, x; g_f \rangle + \langle x, \infty; g_f \rangle
\]

for a suitable \( x \in \mathbb{P}^1(F) \).

First of all, if \( g_v \in \text{PGL}_2(\mathcal{O}_v) \), and the prime ideal \( q_v \) associated to \( v \) divides the conductor of either \( \langle 0, x; g_f \rangle \) or \( \langle x, \infty; g_f \rangle \), then \( v(x) \neq 0 \). Now suppose that \( v \) belongs to the finite set \( \mathcal{B} \) of finite places such that \( g_v \not\in \text{PGL}_2(\mathcal{O}_v) \). In the Bruhat–Tits tree of \( \text{G}(F_v) \), consider the subtree rooted at \( \langle g_v \mathcal{O}_v^2 \rangle \) which consists of the half-geodesics that intersect the geodesic from 0 to \( \infty \) at most in the vertex \( g_v \mathcal{O}_v^2 \). Its boundary at infinity defines an open subset \( S_v \subset \mathbb{P}^1(F_v) \), and the conductors of both \( \langle 0, x; g_f \rangle \) and \( \langle x, \infty; g_f \rangle \) are prime to \( q_v \) if \( x \) belongs to this subset.

Being open, \( S_v \) contains a subset \( S'_v \) of the form

\[
S'_v = \omega_v^{n_v} \beta_v (1 + \omega_v^{m_v} \mathcal{O}_v),
\]

where \( n_v \) is an integer, \( m_v \) is an integer that is at least 1, \( \omega_v \) is a uniformizer, and \( \beta_v \in \mathcal{O}_v^\times \). Write \( n_v^+ = \max(n_v, 0) \), write \( n_v^- = \max(-n_v, 0) \), and set

\[
n_0 = \prod_{v \in \mathcal{B}} q_v^{n_v}, \quad a_1 = \prod_{v \in \mathcal{B}} q_v^{n_v^+}, \quad a_2 = \prod_{v \in \mathcal{B}} q_v^{n_v^-}.
\]

We say a prime ideal \( p \) is good if it is prime to \( Mp \), its norm is not congruent to 1 modulo \( p^{A_p} \), and it does not lie in the set \( \mathcal{B} \).

Now we claim that we may always find \( x = \frac{a_1 b_1}{a_2 b_2} \) with the following properties.

(i) First, \( a_1, a_2 \) have the prime factorization

\[
(a_i) = a_i \cdot a_i^{'},
\]

where the \( a_i^{' \prime} \)’s are good prime ideals. In particular, \( v(\frac{a_1}{a_2}) = n_v \) for every \( v \in \mathcal{B} \).

(ii) Second,

\[
\frac{b_1}{b_2} \in \left( \frac{a_2}{a_1} \omega_v^{n_v} \right) \beta_v (1 + \omega_v^{m_v} \mathcal{O}_v)
\]

for every \( v \in \mathcal{B} \). (Note that this forces \( x \in S'_v \) for every \( v \in \mathcal{B} \).)

(iii) Third, \( b_1, b_2 \) generate principal good prime ideals \( b_1, b_2 \).
Given such $a_i, b_i$, we are done. Because of $(7.7)$ and $(7.6)$, the conductor of $\langle 0, x; g_f \rangle$ and $\langle x, \infty; g_f \rangle$ is not divisible by $q_v$ if $v \in \mathcal{B}$. Otherwise, if $v \notin \mathcal{B}$, then $g_v \in \text{PGL}_2(\mathcal{O}_v)$. In that case, $q_v$ divides the conductor of either symbol only when $v(x) \neq 0$. In other words, the only primes dividing the conductor will be primes in the set $\{a_1', a_2', b_1, b_2\}$. Any prime $q$ in this set is prime to $Mp$, so that the conductor is prime to $Mp$. Also, for any prime $q$ in this set, $Nq - 1$ is not divisible by $p^Aq$. Thus, the denominator of either symbol is divisible at most by $p^{2(Aq-1)}$. We first find $a_1, a_2$ to satisfy (i). We then find $b_1, b_2$ to satisfy (ii) and (iii).

For (i), we apply the Chebotarev density theorem to the homomorphism $\text{Gal}(\tilde{F}/F) \to C_F \times (\mathbb{Z}/p^Aq\mathbb{Z})^\times$ arising from the Hilbert class field (for the $C_F$ = class group factor) and from the extension $F(\mu_{p^Am}) \supset F$ (for the $(\mathbb{Z}/p^Aq\mathbb{Z})^\times$). Now the kernel of $\text{Gal} \to C_F$ does not project trivially to the second factor; considering inertia shows that the image has size at least $\frac{p^Aq-1}{p-1} > 1$. The Chebotarev density theorem now shows that there are infinitely many prime ideals $p$ whose image in $C_F$ is in the same class as $a_1^{-1}$ (or $a_2^{-1}$) and whose image in $(\mathbb{Z}/p^Aq\mathbb{Z})^\times$ is nontrivial. Now take $a_1$ to be a generator for the principal ideal $pa_1$, where the norm of $p$ is taken sufficiently large to guarantee that $p$ is prime to $Mp\mathcal{B}$. Take $a_2$ similarly.

Now, once we have found $a_1$ and $a_2$, condition (ii) amounts to the following. For a certain class $\lambda \in (\mathcal{O}_F/n_0)^\times$ defined by the right-hand side of $(7.7)$, we want to have

$$\frac{b_1}{b_2} \equiv \lambda \mod n_0. \quad (7.8)$$

To get $(7.8)$ and (iii) is another application of the Chebotarev density theorem. Write $n_0 = n_1n_2$, where $n_1$ is prime to $p$ and where $n_2$ is divisible only by primes above $p$. Choose $\tilde{b}_1, \tilde{b}_2 \in (\mathcal{O}_F/p^Aq_{n_2})^\times$ such that $\tilde{b}_1 \equiv \lambda\tilde{b}_2 \mod n_2$ and the norms of $\tilde{b}_1, \tilde{b}_2$ under the map

$$(\mathcal{O}_F/p^Aq_{n_2}) \to (\mathcal{O}_F/p^Aq)^\times \to \mathbb{Z}/p^Aq\mathbb{Z}$$

are not congruent to 1. This can be done, because the image of the norm map $(\mathcal{O}_F/p^Aq)^\times \to (\mathbb{Z}/p^Aq)^\times$ has size strictly larger than 2. Now take for $b_1$ a lift of

$$(\lambda \mod n_1) \times \tilde{b}_1 \in (\mathcal{O}_F/n_1)^\times \times (\mathcal{O}_F/p^Aq_{n_2})^\times \simeq (\mathcal{O}_F/n_1n_2p^Aq)^\times$$

to a generator of a principal prime ideal $\pi$, and take $b_2$ similarly to be a lift of $1 \times \tilde{b}_2$ to a generator of a prime ideal $\pi'$. These lifts can be done in infinitely many ways, so certainly the prime ideals can be taken prime to $Mp\mathcal{B}$. Moreover, the norm of $(b_1)$ equals the norm of $\pi$ (note this is automatically positive) and thus is not congruent to 1 modulo $p^Aq$. This holds similarly for $(b_2)$. \qed
7.6. *Digression*

This section is not necessary for the proof. It is rather a commentary on how parts of the proof could be simplified at the cost of expanding the preceding section.

A complication in the later proof arises at various points because of primes dividing \( n \). For example, we have to explicitly evaluate some local integrals in (7.17), we cannot assume that the conductors of \( E, \psi \) are relatively prime in Proposition 7.8, and so on. We outline here a refined version of Lemma 7.5 that would allow us to avoid these points.

Suppose for finitely many places \( V \) we specify a geodesic segment \( \ell_v \) (\( v \in V \)) of length 1 inside the Bruhat–Tits tree of \( \text{PGL}_2(F_v) \) containing \( \mathcal{O}_v^2 \) (i.e., \( \mathcal{O}_v^2 \) and one adjacent vertex). Now define the valuation at \( v \in V \) of a triple \( \langle \alpha; \beta; g_f \rangle \) to be the distance between the geodesic from \( \alpha_v \) to \( \beta_v \) and the set of vertices of \( g_v \ell_v \), that is,

\[
\text{valuation at } v = \max(\text{distance between } P \text{ and } [\alpha_v, \beta_v], \text{ for } P \in g_v \ell_v).
\]

Thus, the valuation is 0 if and only if the segment \( g_v \ell_v \) is contained in the geodesic from \( \alpha_v \) to \( \beta_v \). At places outside \( V \), the valuation is defined as before. Then with this refined notion, the same statement as in the lemma still holds.

The proof, however, is slightly more involved. In the proof above, take \( \mathcal{B} \) to consist of all places in \( V \) together with all places where \( g_v \notin \text{PGL}_2(\mathcal{O}_v) \). The problem is that the set of \( x \) such that \( \langle 0, x; g_f \rangle \) and \( \langle x, \infty; g_f \rangle \) both have conductor indivisible by \( v \), for \( v \in V \), need not contain an open subset of \( \mathbb{P}^1(F_v) \). The problem arises when \( g_v \ell_v \subset [0, \infty] \).

Call a modular symbol \( \langle \alpha, \beta; g_f \rangle \) *good* if, for every \( v \in V \), the segment \( g_v \ell_v \) is not contained in \( [\alpha, \beta] \) for every \( v \in V \). Thus, what the proof still gives is:

a good modular symbol is the sum of two modular symbols

with the desired properties, \( (7.9) \)

where “desired properties” refers to the relevant divisibility statements for conductor and denominator.

Now if a modular symbol—without loss of generality \( \langle 0, \infty; g_f \rangle \)—is not good, then, for every \( v \in V \), the set of \( x \in \mathbb{P}^1(F_v) \) such that

- \( \langle 0, x; g_f \rangle \) has \( v \)-valuation 0 and
- \( \langle x, \infty; g_f \rangle \) is good

is open and nonempty. The above argument then works to show that we can write

\[
\langle 0, \infty; g_f \rangle = \langle 0, x; g_f \rangle + \langle x, \infty; g_f \rangle,
\]

where \( \langle 0, x; g_f \rangle \) has the desired divisibility properties and \( \langle x, \infty; g_f \rangle \) is good. Then we are done by \( (7.9) \).
7.7. The integral of a differential form over a modular symbol
In this section we will normalize (see Section 7.7.2) a differential form representing a cohomology class in $H^1(Y_0(n), \mathbb{C})$. We will compute bounds (7.12) for its $L^2$-norm and finally compute its integral over a modular symbol (7.18).

Fix in what follows a symbol $\langle \alpha, \beta; g_f \rangle$ with conductor $f$ and denominator $D = \varphi(f)$. Without loss of generality we can suppose that $\alpha = 0$ and $\beta = \infty$. We write $N$ for the norm of $f$. Also we can factorize $D = D_v$ over places $v$ of $F$, that is, $D_v = 1$ for Archimedean $v$, and otherwise $D_v = \varphi(f_v)$, where $f_v$ is the $v$-component of $f$. Finally we write $N_E = \text{Norm}(n)$ for the absolute conductor of the elliptic curve $E$.

7.7.1. Normalizations
Fix an additive character $\chi$ of $A(F)$. For definiteness we take the composition of the standard character of $A(\mathbb{Q}) = \mathbb{Q}$ with the trace. Fix the measure on $A(F)$ that is self-dual with respect to $\chi$, and similarly on each $F_v$.

For a function $\varphi$ on $G(\mathbb{Q}) \setminus G(\mathbb{A}) \cong \text{PGL}_2(F) \setminus \text{PGL}_2(\mathbb{A}_F)$ we define the Whittaker function $W_\varphi$ by the rule

$$W_\varphi(g) = \int_{x \in A(F)} \theta(x) \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx.$$

Let $X = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ in the Lie algebra of the diagonal torus $A$ of $G$. We will also think of it as an element in the Lie algebra of $\text{pgl}_2$. For $y \in F$ or $F_v$, we set $a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$. Write $U_\infty = A(F_\infty) \cap K_\infty$. It is a maximal compact subgroup of $A(F_\infty)$.

On every $A(F_v)$, for $v$ finite, choose the measure $\mu_v$ which assigns the maximal compact subgroup mass 1. On the 1-dimensional Lie group $A(F_\infty) / U_\infty$, we put the measure that is dual to the vector field $X$ defined by $X \in \text{Lie}(A)$ (in other words, induced by a differential form dual to $X$). Finally, on $A(F_\infty)$ itself, take the Haar measure which projects to the measure just defined on $A(F_\infty) / U_\infty$.

The product measure $\mu = \prod_v \mu_v$ has been chosen to have the following property. If $\nu$ is a 1-form on the quotient $A(F) \setminus A(\mathbb{A}_F) / U_\infty U$ for some open compact $U \subset A(\mathbb{A}_{F,f})$, then we have

$$\int_{A(F) \setminus A(\mathbb{A}_F) / U_\infty U} \nu = \frac{1}{\text{vol}(U)} \int_{A(F) \setminus A(\mathbb{A}_F)} \langle X, \nu \rangle d\mu.$$

(7.10)

Here is how to interpret the right-hand side: $X$ defines a vector field $X$ on $A(F) \setminus A(\mathbb{A}_F) / U_\infty U$; pairing with $\nu$ gives a function, which we then pull back to $A(F) \setminus A(\mathbb{A}_F)$ and integrate against the measure we have just described. The volume $\text{vol}(U)$ is measured with respect to the measure $\prod_v \mu_v$ over finite $v$. Finally, the left-hand side requires an orientation to make sense; we orient so that $X$ is positive.
To prove (7.10), note that the $\nu$-integral is a sum of integrals over components. Each component is a quotient of $A(F) / U$. On each such component, the integral is (by definition) obtained by pushing forward the measure $\langle X, \nu \rangle \mu$ to this quotient and integrating. One also computes the right-hand side to induce the same measure on each component.

7.7.2. Normalization of $T(X)$
Let $T \in \text{Hom}_{K_g}(g, \pi)^K$. Here $\pi$ is the unique cohomological representation of level $n$ as in our assumptions (see Section 7.1), and as in the previous section, $g$ and $\mathfrak{k}$ are the Lie algebra of the groups $G(F)$ and its maximal compact subgroup, respectively. Now $T$ defines a differential form on $Y_0(n)$, which we call simply $T$. We normalize $T$ by requiring that the Whittaker function $W_T(X)$ of $T(X)$ is (7.11)

where $W_v$ is the new vector of $\pi$ (in particular, $W_v(e) = 1$ when $\theta_v$ is unramified) and at $\infty$ we normalize by the requirement

$$\int_{F_\infty} W_\infty \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy = 1,$$

where $dy$ is chosen to correspond to the measure on $A(F)$ fixed above. (A simple computation is necessary to check that this is possible, since the integral might, a priori, always equal 0.) By Rankin–Selberg and standard estimates, we check that

$$N(n)^{-e} \ll \frac{(\omega, \omega)_{L^2(Y_0(n))}}{\text{vol}(Y_0(n))} \ll N(n)^e.$$

Indeed, all we need is polynomial bounds of this form, with lower bound $N(n)^{-A}$ and upper bound $N(n)^A$ for a constant $A$ depending only on $F$. Such bounds are given in [15, equation (10) and Theorem 5]. For the case of $F = \mathbb{Q}$ the sharper lower bound is due to Hoffstein and Lockhart [37], and that also contains references for the sharper upper bound. The general statement is proved in [10, Section 2.9, Lemma 3].

7.7.3. Adelic torus orbits versus modular symbols
We want to express the integral of $\omega$ over a modular symbol $\langle 0, \infty ; g_f \rangle$ in terms of an adelic integral, similar to what was done in (6.2). We will assume that the conductor of $\langle 0, \infty ; g_f \rangle$ is relatively prime to $n$.

Let $U = A(\mathbb{A}_{F, f}) \cap g_f K g_f^{-1}$. Consider now the map $A(F) \backslash A(\mathbb{A}_F) / U \rightarrow Y(K)$
defined by $t \mapsto tg_f$. Its image can be regarded as a finite union of modular symbols
$(0, \infty; tg_f)$ where $t$ varies through representatives in $A(A_{F,f})$ for the group $Q = A(A_{F,f})/A(F)\mathcal{U}$. There is an exact sequence
\[ \mu_F / \mu_F \cap g_f \mathcal{U} \mathcal{F}_f^{-1} \rightarrow A(\Omega) / U \rightarrow Q \rightarrow \text{class group}, \]
where $\mu_F$ is the group of roots of unity and we regard it as a subgroup of $A(F) \simeq F^\times$ via $\mu_F \subseteq F^\times$. Call $w'_F$ the size of the group on the far left-hand side. So $|Q| w'_F = h_F \text{vol}(U)^{-1}$. Any character $\psi$ of $Q$ extends to a character of $[A] = A(F) \backslash A(A_{F})$, which is trivial at infinity, and it follows from (7.10) that
\[ \sum_{q \in Q} \psi(q) \int_{(0,\infty;tg_f)} \omega = \frac{1}{\text{vol}(U)} \cdot \int_{[A]} \psi(t)((g_f)^* \omega, X)(t) d\mu(t), \tag{7.13} \]
where the right-hand side is interpreted in the same way as in (7.10). Recall that $\omega$ has been defined in Section 7.7.2. Fourier analysis on the finite group $Q$ then gives
\[ \int_{(0,\infty;tg_f)} \omega = \frac{1}{|Q|} \sum_{\psi \in \hat{Q}} \frac{1}{\text{vol}(U)} \cdot \int_{[A]} \psi(t)((g_f)^* \omega, X)(t) d\mu(t) \]
\[ = \frac{w'_F}{h_F} \sum_{\psi \in \hat{Q}} \psi(t)((g_f)^* \omega, X)(t) d\mu(t) \]
\[ = \frac{w'_F}{h_F} \sum_{\psi \in \hat{Q}} T_X(tg_f) \psi(t) d\mu \tag{7.14} \]
\[ = \frac{w'_F}{h_F} \sum_{\psi} L\left(\frac{1}{2}, \pi \times \psi\right) \cdot \prod_{v \text{ finite}} \frac{I_v}{L_v\left(\frac{1}{2}, \pi \times \psi\right)}. \tag{7.15} \]
Here
\[ I_v := \int_{y \in F_v^\times} W_v(a(y)g_v) \psi_v(y) dy, \]
where $g_v$ is the component at $v$ of $g_f$, $W_v$ is in (7.11), and measures are as normalized earlier. For step (7.15), we have used unfolding as in the theory of Hecke integrals (see [16, Section 3.5]).

Let $S$ be the set of Archimedean places, together with all places where the conductor of the symbol $(0,\infty;g_f)$ is not 1. Let $S'$ be the set of finite places dividing $n$. Because of our assumption, $S$ and $S'$ are disjoint.

Note that if $v \notin S \cup S'$, then we have $g_v \in A(F_v) \cdot \text{PGL}_2(\mathcal{O}_v)$. So $\psi_v$ must be unramified for $I_v$ to be nonzero. By the choice of $W_v$ we have $I_v = u_v \cdot L_v\left(\frac{1}{2}, \pi_v \times \psi_v\right)$ whenever $v \notin S \cup S'$, where $u_v \in \mathbb{Z}_{\mathbb{Z}}$ is an algebraic unit.
For finite \( v \in S \), the values of \( W_v^{g_v} \) at least lie in \( \mathbb{Z}[\frac{1}{q_v}] \), as one verifies by explicit computation. On the other hand, the function \( y \mapsto W_v(a(y)g_v) \) is now constant on each coset of \( 1 + q_v^{n_v} \), where \( n_v \) is the local conductor (see the discussion just before Section 7.4). For the integral to be nonzero, \( \psi_v(y) \) must be identically 1 on \( 1 + q_v^{n_v} \) and constant on each of its cosets. Each of these cosets has measure \( D_v^{-1} \), where \( D_v \) is the local denominator.

Now
\[
J_v(s) := \frac{\int_{F_v} W_v(a(y)g_v)\psi_v(y)|y|^s \, dy}{L(s + 1/2, \pi_v \times \psi_v)}
\]
can be rewritten as \( \int_{F_v} f(y)\psi_v(y)|y|^s \, dy \) where \( f \) is a certain sum of translates of \( W_v \). But \( J_v(s) \) is a polynomial in \( q_v^{-s} \) (again by the theory of Hecke integrals) and so \( f_v(y) \) is compactly supported. Also, \( a_v, \beta_v \in \mathbb{Z}[1/q_v] \). So \( J_v(0) \) is actually a finite sum of elements, each lying in \( \mathbb{Z}[\frac{1}{q_v}] \cdot D_v^{-1} \). This shows that
\[
D_v \cdot I_v \in L_v\left(\frac{1}{2}, \pi_v \times \psi_v\right)\mathbb{Z}\left[\frac{1}{N}\right] \quad (v \in S),
\]
where \( \mathbb{Z} \) is the ring of algebraic integers, \( N \) is the norm of the conductor of \( \{0, \infty; g_f\} \), and \( D_v \) is the contribution of \( v \) to the denominator of \( \{0, \infty; g_f\} \).

Now consider the case in which \( v \in S' \). Although exactly the same reasoning that was just applied to \( v \in S \) also applies to \( v \in S' \), we will argue separately because we actually want a slightly more precise result for \( v \in S' \), that is, the set of primes dividing \( n \), with better denominator control. Because \( S \cap S' = \emptyset \), we have \( g_v \in A(F_v) \cdot \text{PGL}_2(\mathcal{O}_v) \) for each \( v \in S' \). In particular, we may suppose that \( g_v \in \text{PGL}_2(\mathcal{O}_v) \) while only modifying the value of \( I_v \) by an algebraic unit. By a direct computation with Steinberg representations, we find that, for \( k_v \in \text{PGL}_2(\mathcal{O}_v) \) and \( W_v \) the new vector for a Steinberg representation \( \pi_v \), we have
\[
\int W_v(a(y)k_v)\psi_v(y) \, dy = \frac{1}{q_v(q_v - 1)}L\left(\frac{1}{2}, \pi_v \times \psi_v\right) \cdot \mathbb{Z}.
\]
This is a matter of explicit computation, as we now detail.

(i) If \( k_v \) belongs to \( K_0(n) \), then this is clear.
(ii) Otherwise, we can write \( k_v = (\frac{1}{0} \ y) \cdot w \cdot k' \) where \( x \in \mathcal{O}_v \), \( k' \in K_0(n) \), and \( w = (\frac{0}{1} \ -1) \). In that case we can rewrite the integral as \( \int W_v^{wv}(a(y))\theta_v(xy) \times \psi_v(y) \, dy \). The function \( W_v^{wv}(a(y)) \) is supported on \( u(y) \geq -1 \) and its values are algebraic units when \( u(y) = -1 \) (see [66, (11.14)])]. Moreover, it is invariant on each coset \( \mathcal{O}_v^{x} \).

\[\text{Namely, write } L(s + 1/2, \pi_v \times \psi_v)^{-1} \text{ as } (1 - \alpha_v \psi_v(\sigma_v)q_v^{-s})(1 - \beta_v \psi_v(\sigma_v)q_v^{-s}), \text{ where } \sigma_v \text{ is a uniformizer and } \alpha_v, \beta_v \text{ could be } 0, \text{ and then take } f(y) = (1 - \alpha_v T)(1 - \beta_v T)W_v(a(y)g_v), \text{ where } T \text{ is the operation which translates a function by } \sigma_v.\]
If \( \psi \) is ramified, then the only contribution comes from \( \nu(y) = -1 \), since on any other coset \( y \mathcal{O}^\times_v \) with \( \nu(y) \geq 0 \) both \( W^w_v(a(y)) \) and \( \theta_v(x,y) \) remain constant on that coset. The integral over \( \nu(y) = -1 \) then amounts to a Gauss sum; it belongs to \( \frac{1}{q_v - 1} \mathbb{Z} \). On the other hand, if \( \psi \) is unramified, then the value of the integral by explicit computation is \( \pm q_v^{-1} \cdot L_v(1/2, \pi_v \times \psi_v) + \frac{u}{q_v - 1} \), where \( u \) is an algebraic unit. In fact, the term \( \frac{u}{q_v - 1} \) comes from \( \nu(y) = -1 \), and the remaining term \( \pm q_v^{-1} \cdot L_v(1/2, \pi_v \times \psi_v) \) comes from the contribution of \( \nu(y) \geq 0 \).

We deduce that if the conductor of \( (0, \infty; g_f) \) is relatively prime to \( n \), then

\[
\int_{(0,\infty; g_f)} \omega = \frac{1}{h_F D N(n) \psi(n)} \sum_{\psi} L\left( \frac{1}{2}, \pi \times \psi \right) \cdot a_{\psi}, \quad a_{\psi} \in \mathbb{Z}\left[ \frac{1}{N} \right],
\]

(7.18)

where \( D = \prod D_v \) is the denominator of \( (0, \infty; g_f) \), every character \( \psi \) that occurs on the right-hand side has conductor dividing \( n \) with \( f \) the conductor of \( (0, \infty; g_f) \), and \( N = \text{Norm}(f) \). Note also that the order of \( \psi \) is bounded, as in (7.5).

We will now apply equivariant BSD. We first normalize a period \( E \). We will deduce that if the conductor of \( E \) is bounded, as in (7.5). We regard it as a submodule of the \( F \)-vector space \( \Omega^1 \) of all differential 1-forms. Pick a \( \mathbb{Z} \)-basis \( \xi_1, \xi_2 \) for \( \Omega^1 \). (We will only use \( \xi_2 \) later.) Now put

\[
\Omega_E = \left[ \frac{1}{\Omega^1 \mathcal{O}_F: \mathcal{O}_E \xi_1} \right] \int_{E(C)} \xi_1 \wedge \xi_1.
\]

(7.19)

This is independent of the choice of \( \xi_1 \). For later usage, note the following: if \( a = \left[ \Omega^1 \mathcal{O}_F: \mathcal{O}_E \xi_1 \right] \), then we have \( \sqrt{-D^2} / a = \text{Im}(\xi_2 / \xi_1) \) by an area computation.

**PROPOSITION 7.8** (Equivariant BSD conjecture; see Section 8 for full discussion)

Assume the equivariant BSD conjecture in the formulation (8.14). Let \( E \) be a non-CM elliptic curve over the imaginary quadratic field \( F \) of conductor \( n_E \). Let \( \Omega_E \) be as in (7.19). Let \( \psi \) be a character of \( \mathbb{A}_F^\times / F^\times F_\infty^\times \) of finite order \( d \) and conductor \( n_\psi \). Suppose that \( E \) has semistable reduction at all primes dividing \( (n_E, n_\psi) \). Then we have

\[
(d N(n_\psi))^2 \cdot L\left( \frac{1}{2}, \pi \times \psi \right) \in \mathbb{Z}\left[ \frac{1}{N_\psi'} \right] \cdot \frac{\Omega_E}{|E(F_\psi)_{\text{tors}}|^2},
\]

(7.20)

where \( E(F_\psi)_{\text{tors}} \) denotes the torsion subgroup of the points of \( E \) over the abelian extension \( F_\psi \) corresponding to \( \psi \), and \( N_\psi' = \prod_p 2 \cdot n_\psi N(p) \).
Note that the elliptic curve $E$ in our context does have semistable reduction at every place, because its conductor is square-free, so we can freely apply this result.

### 7.9. Proof of Theorem 7.2

We now collect together what we have shown in order to complete the proof. As above, $\omega$ is a differential 1-form of level $n$ belonging to the automorphic representation $\pi$.

Fix a prime $l$ of $\mathbb{Z}$ above a prime $\ell$ of $\mathbb{Z}$. Let $\mathbb{Z}_l$ consist of algebraic numbers with valuation at least 0 at $l$. Also, let $F_\ell$ denote the largest abelian extension of $F$ that is unramified at all primes above $\ell$ if $\ell$ is relatively prime to $n$. Otherwise, let $F_\ell$ be the largest abelian extension of $F$ that is at worst tamely ramified at primes of $F$ above $\ell$.

Begin with $\int_\gamma \omega$ for arbitrary $\gamma \in H_{1, BM}$, and use Lemma 7.5 to write $\gamma$ as a sum of symbols $(0, \infty; g_f)$, where the conductor of each symbol is relatively prime to $N_E \ell$, and the denominator is prime to $\ell$ (or divisible by at most $\ell^A$ if $\ell \leq 5$).

Now (7.18) writes $\int_\gamma \omega$ as a sum of $L$-values $L(\frac{1}{2}, E \times \psi)$, where the $\psi$’s which occur have conductor dividing $n \cdot j$, where $j$ is prime to $N_E \ell$. In particular, for any prime $\lambda$ above $\ell$, the square $\lambda^2$ does not divide the conductor of $\psi$; that is, $\psi$ is tamely ramified at $l$, and it is actually unramified at $\lambda$ if $\ell$ is relatively prime to $n$. Therefore, $F_\ell \supset F_\psi$.

Combine (7.5), (7.20), and (7.18) to arrive at

$$\int_\gamma \omega \in \frac{1}{(30h_F N(n)\varphi(n))^B} \cdot \frac{\Omega_E}{|E(F_\ell)_{\text{tors}}|^2} \cdot \mathbb{Z}_l,$$

for some absolute constant $B$. Set

$$M = \prod_{\ell} \# E(F_\ell)[\ell^\infty] \quad \text{and} \quad M' = (30h_F N(n)\varphi(n))^B.$$

$M$ is finite because $E(F^{ab})_{\text{tors}}$ is finite—that is a simple consequence of Serre’s open image theorem (see, e.g., [63]), using the fact that $E$ does not have CM. Then

$$\int_\gamma \omega \in \frac{1}{M'} \cdot \frac{\Omega_E}{M^2} \cdot \mathbb{Z}_l,$$

$^1$Note that, by using Section 7.6, the situation can be simplified in the following way. Take $\ell_v$ of Section 7.6 to be the set of vertices fixed by $K_0(n)$. Then Section 7.6 allows us to write $\gamma$ as a sum of symbols $(0, \infty; g_f)$, where the “refined” conductor of each symbol is relatively prime to $N_E \ell$, and with controlled denominator as above. Now, (7.18) writes $\int_\gamma \omega$ as a sum of $L$-values $L(\frac{1}{2}, E \times \psi)$, where the $\psi$’s which occur have conductor relatively prime to $N_E \ell$ also. In particular, we can assume that $\psi$ and $E$ have relatively prime conductor, simplifying our later discussion. The reason is as follows. For any symbol $(0, \infty; g_f)$, refined valuation 0 actually means that $g_F K_0(n) v g_F^{-1}$ contains the maximal compact subgroup of $\text{Aut}(F_v)$. In particular—looking above (7.15)—if $v$ is any place of “refined” valuation 0, then the vector $W(\alpha(v) g_v)$ is actually invariant by $v \in \Theta_v^c$, and then $\psi_v$ must actually be unramified for the local integral $I_v$ to be nonzero. So, in the above reasoning, the only $\psi$’s that occur have conductor relatively prime to $N_E \ell$, because the modular symbols which occurred had “refined” conductor relatively prime to $N_E \ell$. 


and because this is true for all \( l \), we get
\[
\int_{\gamma} \omega \in \frac{1}{M'} \cdot \frac{\Omega_E}{M^2} \cdot \mathbb{Z}.
\]

Set
\[
\omega' = \Omega_E^{-1} \sqrt{|\Delta_F|} M' M^2 \cdot \omega,
\]
where \( \Delta_F \) is the discriminant of \( F \). Then the form \( \omega' \) has algebraic-integral periods, that is, \( \int_{\gamma} \omega' \in \mathbb{Z} \). In fact, it also has rational periods, and therefore \( \int_{\gamma} \omega' \in \mathbb{Z} \).

Because the subspace of compactly supported cohomology \( H^1_c \) which transforms with the same Hecke eigenvalues as \( \pi \) is 1-dimensional, it suffices to produce just one element \( \gamma \in H_{1, \text{BM}} \) with \( \int_{\gamma} \omega' \in \mathbb{Q} - \{0\} \). Now, one can apply (7.13) with \( g_f = 1 \) and \( \psi = 1 \) to get \( L(\frac{1}{2}, E) \) as a sum of integrals of \( \omega \) over modular symbols. (In more detail, one can proceed just as after (7.13), without even carrying out the Fourier inversion over \( \mathbb{Q} \), and all factors \( \frac{L_v}{L_v(1/2, \pi \times \psi_v)} \) are equal to 1.) This yields the classical expression (see, e.g., [24, Proposition 2.1]) of the \( L \)-series in terms of geodesic integrals. Moreover, \( L(\frac{1}{2}, E) \in \mathbb{Q} \cdot \sqrt{|\Delta_F|} \Omega_E \) by (usual) BSD. So if \( L(\frac{1}{2}, E) \neq 0 \), then this produces the desired class with \( \int_{\gamma} \omega' \in \mathbb{Q} - \{0\} \). Even if \( L(\frac{1}{2}, E) = 0 \), we can carry out the same argument with quadratic twists: in that case one deduces\(^\dagger\) from [30, Theorem B] the existence of a quadratic character \( \psi \) with \( L(\frac{1}{2}, E \times \psi) \neq 0 \) and one can reason similarly,\(^\ddagger\) now using (usual) BSD for the quadratic twist \( E_\psi \) of \( E \) by \( \psi \). The period \( \Omega_{E_\psi} \) differs from \( \Omega_E \) by a rational multiple of \( \sqrt{q} \), where \( q \) is the norm of the conductor of \( \psi \). Also, the corresponding integrals \( \frac{L_v}{L_v(1/2, \pi \times \psi_v)} \) are now rational multiples of Gauss sums at the ramified places for \( \psi \) (in the corresponding situation over \( \mathbb{Q} \), this is a well-known computation of Birch; see [7, page 399]); the Gauss sum is in the current setting always a rational multiple of \( \sqrt{q} \).

Our desired result (Theorem 7.2) follows from (7.12) together with the bounds
\[
\Omega_E^{-1} \text{ and } M \ll_F A(N(n))^B \tag{7.21}
\]
for absolute constants \( A, B \). We now explain how to check (7.21).

For \( \Omega_E \), one uses the relationship with the Faltings height, together with Szpiro’s conjecture and Frey’s conjecture (see [36, Conjecture F.3.2]). As mentioned in that reference, these conjectures are, up to the exact value of the constants involved, equivalent to the \( abc \) conjecture. Then, up to constant factors, \( \Omega_E^{-1/2} \) coincides with the

\(^\dagger\) There is a technicality here to be aware of: in order to check the assumption on root numbers of [30, Theorem B], one needs to suppose that the level \( n \) is not the trivial ideal, that is, \( E \) is not everywhere unramified; we can freely do this because the implicit constants of our theorem can depend on \( F \).

\(^\ddagger\) In more detail, one should apply (7.13) with \( g_f \) taken with local component \( 1 \) at places \( v \) at which \( \psi_v \) is unramified, and with local component \( \left( \begin{array}{cc} 1 & m_v^{-1} \\ 0 & 1 \end{array} \right) \) at every ramified place \( v \) for \( \psi \).
exponential of the Faltings height; [36, Conjecture F.3.2] now yields (7.21), using also the result stated in [36, Exercise F.5(c)].

Now let us examine $M$.

- For $\ell > 3$ and $\ell$ relatively prime to $n$, $E(F_{\ell})[\ell]$ must be either trivial or cyclic because if $E(F_{\ell})[\ell]$ were all of $E[\ell]$, then that means that inertia groups $I_l \subset \text{Gal} (\overline{F} / F)$ for any prime $l$ of $F$ above $\ell$ would act trivially on $E[\ell]$. But this is never true, because the determinant of this action is the mod $\ell$ cyclotomic character, which is nontrivial.

- For $\ell = 2, 3$ or $\ell$ dividing $n$, we see similarly that $E(F_{\ell})[\ell^3]$ cannot be all of $E[\ell^3]$.

Write $Q = (6 \text{Norm}(n))^2$.

So $Q \cdot E(F_{\ell})[\ell^\infty]$ is cyclic for every $\ell$. Write $K$ for the subgroup of $E(F_{\ell})[\ell^\infty]$ generated by all the $Q \cdot E(F_{\ell})[\ell^\infty]$. As we have just seen, $K$ is a cyclic subgroup of order at least $M / Q^2$, and it is stable by the Galois group of $\overline{F} / F$. Consider the isogeny $\varphi : E \to E' := E / K$. Masser–Wüstholz (see, e.g., the main theorem of [48]) give an isogeny $\varphi' : E' \to E$ in the reverse direction, whose degree is bounded by a polynomial in the Faltings height of $E$. The composite isogeny $\theta = \varphi' \circ \varphi : E \to E$ must be multiplication by an integer $r$, because $E$ does not have CM, and also $\#K$ divides $r$ because $\theta(K) = 0$ and $K$ is cyclic. From

$$(\#K) \cdot \text{deg} \varphi' = r^2$$

we get the desired bound: $M \leq Q^2 \text{deg} \varphi'$.

8. The equivariant conjecture of Birch and Swinnerton-Dyer

In the preceding section we used Proposition 7.8, which says (see that section for notation) the following.

Assume equivariant BSD. Let $E$ be an elliptic curve over the imaginary quadratic field $F$ of conductor $n_E$. Let $\psi$ be a character of $\mathbb{A}_F^\times / F^\times F_{\psi}^\times$ of finite order $d$ and conductor $n_{\psi}$. We assume that $E$ has semistable reduction at every prime dividing $(n_{\psi}, n_E)$. Put $N'_{\psi} = \prod_{p \mid n_{\psi}} N(p)$. We have

$$(d \text{Norm}(n_{\psi}))^2 \cdot L \left( \frac{1}{2}, \pi \times \psi \right) \in \mathbb{Z} \left[ \frac{1}{N'_{\psi}} \right] \cdot \frac{\Omega_E}{|E(F_{\psi})_{\text{tors}}|^2}, \quad (8.1)$$

where $N_E$ and $N_{\psi}$ are the respective norms of $n_E$ and $n_{\psi}$, respectively, $F_{\psi}$ is the abelian extension determined by $F$, and $\Omega_E$ is the period normalized as in (7.19).

This was quoted as a consequence of the equivariant BSD conjecture. Unfortunately, there is no standardized form of such a conjecture in the literature, to our
knowledge, in the generality we need. That is why we have written this section, to spell out exactly what we mean and how it gives rise to (8.1). We have chosen to directly formulate an equivariant BSD conjecture in (8.14) in a way that directly mirrors the formulation given by Gross [33] for CM elliptic curves. In principle, this should be routinely verifiable to be equivalent to the equivariant Tamagawa number conjecture of [27, Section 4], although we did not attempt to verify the details. In summary, when we say “equivariant BSD” in this article, we mean the conjecture that is formulated in (8.14) below. We anticipate, but have not verified, that this can be verified to be compatible with [27] in a routine fashion.

Here is the basic idea. To understand \( L(\frac{1}{2}, E \times \psi) \) as below one needs to understand the \( L \)-function of \( E \) over a certain abelian extension \( F_\psi \), that is, the \( L \)-function of abelian variety \( \text{Res}_{F_\psi/F} E \), but \textit{equivariantly} for the action of the Galois group \( G \) of \( F_\psi \) over \( F \). One difficulty encountered is that \( \mathbb{Z}[G] \) is not a Dedekind ring. This issue comes up in other work on the subject (see [9]). Of course, our goal is much less precise, since we may lose arbitrary denominators at \( N_\psi \) and also some denominator at \( d \). In any case we deal with this by instead passing to an abelian subvariety of \( A \) which admits an action of a Dedekind quotient of \( \mathbb{Z}[G] \). As a simple example, if \( \psi \) is a quadratic character, then one can analyze \( L(\frac{1}{2}, E \times \psi) \) as the \( L \)-function of a quadratic twist of \( E \), rather than working with \( E \) over the quadratic extension defined by \( \psi \).

In the actual derivation we will try to write formulas that are as explicit as possible. We write for short \( \mathbb{Z}' = \mathbb{Z}[\frac{1}{N_\psi}] \), and if \( M \) is a \( \mathbb{Z} \)-module, then we sometimes write \( M' \) for \( M \otimes \mathbb{Z}' \).

8.1. Basic setup
Choose a prime \( \ell \) that does not divide \( N_\psi \), and extend the valuation at \( \ell \) to a valuation of \( \mathbb{C} \). We will prove (8.1) at \( \ell \), that is, verify that the \( \ell \)-adic valuation of the ratio \( \text{LHS/RHS} \) behaves as predicted.

We regard \( F \) as a subfield of \( \mathbb{C} \), that is, we choose a fixed embedding \( \iota \) of \( F \) into \( \mathbb{C} \). When we write \( E(\mathbb{C}) \), we understand it as the complex points of \( E \) considered as a complex variety via \( \iota \).

Let \( F_\psi \) be the abelian field extension of \( F \) determined, according to class field theory, by the kernel of \( \psi \). Note that \( F_\psi \) is tamely ramified above \( F \) at all primes above \( \ell \), because any prime \( \ell \) above \( \ell \) divides \( n_\psi \) with multiplicity at most 1.

Let \( \mu \) be the cokernel of \( \psi \) (so that \( \psi \) gives an isomorphism of \( \mu \) with a cyclic subgroup of \( \mathbb{C}^\times \)); thus, \( \text{Gal}(F_\psi/F) \cong \mu \). We fix an extension of \( \iota \) to an embedding \( \sigma_1 : F_\psi \rightarrow \mathbb{C} \). For \( \alpha \in \mu \) we put \( \sigma_\alpha = \sigma_1 \circ \alpha \).
8.2. Background on cyclotomic rings

The size of \( \mu \) is \( d \), that is, the order of \( \psi \). Let \( R = \mathbb{Z}[\mu] \) be the group algebra of \( \mu \) so that \( \psi \) gives an algebra homomorphism \( \psi : R \to \mathbb{C} \). For \( a \in R \) we will sometimes write \( a^\psi \) instead of \( \psi(a) \); we will also use this notation for \( a \in R_\mathbb{R} := R \otimes \mathbb{R} \) (i.e., \( a^\psi \) is the value, in \( \mathbb{C} \), of the real-linear extension \( \psi : R_\mathbb{R} \to \mathbb{C} \)).

Choose a generator \( \zeta \) for \( \mu \). Let \( \phi_d \in \mathbb{Z}[x] \) be the \( d \)th cyclotomic polynomial, and let \( \theta_d = \frac{x^d - 1}{\phi_d} \in \mathbb{Z}[x] \). Let \( \Phi_d = \phi_d(\zeta) \) and \( \Theta_d = \theta_d(\zeta) \) be the elements of \( R \) obtained by evaluating these at \( \zeta \). Note that \( \Theta_d \Phi_d = 0 \in R \).

Set

\[
S = R/(\Phi_d).
\]

Then \( S \) is a Dedekind ring, isomorphic to the ring of integers in the \( d \)th cyclotomic field, and the homomorphism \( \psi : R \to \mathbb{C} \) then factors through \( S \). Note that, by differentiating,

\[
\text{the image in } S \text{ of } \left( \Theta_d \cdot \phi'_d(\zeta) \right) = \frac{d}{dx} (x^d - 1) \bigg|_{x=\zeta} = d \zeta^{-1},
\]

where this equality is in \( S \). This shows that \( d \) is divisible in \( S \) by the product of \( \Theta_d \) and \( \phi'_d(\zeta) \). Note that the image of \( \phi'_d(\zeta) \) in \( S \) is exactly the different of \( S \) over \( \mathbb{Z} \).

Consider the abelian category \( S\text{-modf} \) of finite \( S \)-modules: modules that are finite as abelian groups. Then the rule \( S/n \mapsto n \) gives an isomorphism

\[
K_0(\text{S-modf}) \simeq \{ \text{fractional ideals of } S \}.
\]

We use \([X]\) to denote the (fractional) ideal corresponding to a torsion \( S \)-module \( X \), and write \([X] \geq [Y]\) if the ideal for \( X \) is divisible by the ideal for \( Y \). We write \([X] \geq_\ell [Y]\) if this holds at \( \ell \), that is, the valuation at any prime \( \ell \) above \( \ell \) for \([X]\) is greater than or equal to that for \([Y]\).

If \([X]\) corresponds to a principal ideal, then we will say that \( X \) is virtually principal. By an abuse of notation, we may regard \([X]\) as an element of \( S_{\mathbb{Q}}^\times/S^\times \), namely, a generator for that principal ideal. Here we have written \( S_{\mathbb{Q}} \) as an abbreviation for \((S \otimes \mathbb{Q})\).

We denote by \( x \mapsto x^* \) the involution of \( R \) that is induced by inversion on \( \mu \). This descends to the canonical complex conjugation on the CM-field \( S \). Later we also consider the complex conjugation \( x \mapsto \overline{x} \) on \( R_{\mathbb{C}} := R \otimes_{\mathbb{Z}} \mathbb{C} \) arising from the conjugation of \( \mathbb{C}/\mathbb{R} \).

Given \( x \in F_\psi \), we define \([x] \in R_{\mathbb{C}}\) by

\[
[x] = \sum a_\alpha(x) \alpha^{-1}.
\]
The morphism \( x \mapsto [x] \) is actually equivariant for the action of \( \mathcal{O}_F[\mu] \), which acts on \( F_\psi \) by linear extension of the \( \mu \)-action and acts on \( R_C \) via the map \( \mathcal{O}_F[\mu] \to R_C \) induced from the natural embedding \( \mu \to R \) and the inclusion \( i : \mathcal{O}_F \hookrightarrow \mathbb{C} \).

### 8.3. The abelian varieties and their Néron models

We put

\[
A = \text{Res}_{F_\psi / \mathbb{Q}} E.
\]

Then \( A \) is a \( 2d \)-dimensional abelian variety which admits an action of \( \mu \simeq \text{Gal}(F_\psi / F) \) and, consequently, also of the algebra \( R \). Consider the \( 2\varphi(d) \)-dimensional abelian variety \( B \) which is given by the connected component of the kernel of \( \Phi_d \) acting on \( A \):

\[
B = \ker(\Phi_d : A \to A)^0.
\]

Then the action of \( R \) on \( B \) factors through \( S \). Also \( \Theta_d \) gives a surjection of abelian varieties \( A \to B \).

Denote by \( \mathcal{E} \) the Néron model of \( E \) over \( \mathcal{O}_F \), denote by \( \mathcal{B} \) the Néron model of \( B \) (now over \( \mathbb{Z} \)), and finally denote by \( \mathcal{A} \) that of \( A \) (also over \( \mathbb{Z} \)). Denote by \( \text{Lie}(\mathcal{E}) \) the tangent space to \( \mathcal{E} \) above the identity section, and denote by \( \Omega^1_{\mathcal{E}} \) its \( \mathcal{O}_F \)-linear dual; these are both locally free \( \mathcal{O}_F \)-modules of rank 1. We use similar notation for \( \mathcal{A} \) and \( \mathcal{B} \); in that case, they are free \( \mathbb{Z} \)-modules of rank \( 2d \) and \( 2\varphi(d) \), respectively. Thus, for example, \( \text{Lie}(\mathcal{E}) \) is the set of \( \text{Spec}(\mathcal{O}_F[\epsilon]/\epsilon^2) \)-valued points of \( \mathcal{E} \) that extend the identity section \( \text{Spec} \mathcal{O}_F \to \mathcal{E} \).

The connected Néron model of \( E \otimes F_\psi \) over \( \mathcal{O}_{F_\psi} \) coincides with the base change of the connected Néron model for \( \mathcal{E} \). Indeed, the universal property gives a map from \( \mathcal{E} \otimes_{\mathbb{Z}} \mathcal{O}_{F_\psi} \) to this Néron model, and this is an open immersion. We check this after localizing at each prime \( p \).

- If a prime \( p \) of \( F \) does not divide \( n_{\psi} \), then this is the commutation of Néron models with unramified base change (see [13, Chapter 7, Theorem 1]).
- If a prime \( p \) of \( F \) does divide \( n_{\psi} \), then by assumption \( E \) has semistable reduction at \( p \), and the result is known (see [13, Chapter 7, Proposition 3]).

Now it is known (see [25, Proposition 4.1]) that \( \mathcal{A} \) is the restriction of scalars \( \text{Res}_{\mathcal{O}_{F_\psi} / \mathbb{Z}} \) for the Néron model of \( E \) over \( \mathcal{O}_{F_\psi} \). Using the description of Lie algebra noted above and the defining property of restriction of scalars, we see that

\[
\text{Lie}(\mathcal{A}) = (\text{Lie}(\mathcal{E})) \otimes_{\mathcal{O}_F} \mathcal{O}_{F_\psi} \bigotimes \mathcal{O}_F \mathcal{O}_F^{-1},
\]

where \( \mathcal{O}_F \) denotes...
the different. The pairing \( x \in M, \; y \otimes \delta \in \text{Hom}_\mathcal{O}_\mathcal{F}(M, \mathcal{O}_\mathcal{F}) \otimes \mathcal{O}_\mathcal{F} \mathcal{d}_\mathcal{F}^{-1} \mapsto \text{trace}_{\mathcal{F}/\mathcal{Z}}((x, \; y)\delta) \) induces this isomorphism. Thus,

\[
\Omega^{1}_{\mathcal{A}/\mathcal{Z}} = \text{Hom}_\mathcal{Z}(\text{Lie}(\mathcal{A}), \mathcal{Z}) = \text{Hom}_\mathcal{Z}(\text{Lie}(\mathcal{E}) \otimes \mathcal{O}_\mathcal{F}, \mathcal{O}_\mathcal{F}/\mathcal{Z}) \\
= (\text{Hom}_{\mathcal{O}_\mathcal{F}}(\text{Lie}(\mathcal{E}) \otimes \mathcal{O}_\mathcal{F}, \mathcal{O}_\mathcal{F}/\mathcal{Z})) \\
= (\text{Hom}_{\mathcal{O}_\mathcal{F}}(\text{Lie}(\mathcal{E}), \mathcal{O}_\mathcal{F}/\mathcal{Z})) \\
= (\Omega^{1}_{\mathcal{E}} \otimes \mathcal{O}_\mathcal{F} \mathcal{d}_{\mathcal{F}^{-1}}) \\
= (\Omega^{1}_{\mathcal{E}} \otimes \mathcal{O}_\mathcal{F} \mathcal{d}_{\mathcal{F}^{-1}}/\mathcal{F}).
\]

To be more precise, there is a natural map \( \Omega^{1}_{\mathcal{E}} \otimes \mathcal{O}_\mathcal{F} \mathcal{d}_{\mathcal{F}^{-1}} \rightarrow \Omega^{1}_{\mathcal{A}}, \) and the assertion is that the image consists of 1-forms on \( \mathcal{A} \) which extend to the Néron model. In the last equations, \( \mathcal{d}_{\mathcal{F}^{-1}}/\mathcal{F} \) denotes the relative different and we used the transitivity of the different, which means that \( \mathcal{d}_{\mathcal{F}^{-1}} = \mathcal{d}_{\mathcal{F}^{-1}}/\mathcal{F} \).

Our next order of business is to get some understanding of \( \Omega^{1}_{\mathcal{B}}. \) There is a natural morphism \( \Omega^{1}_{\mathcal{A}} \rightarrow \Omega^{1}_{\mathcal{B}} \) induced by \( \mathcal{B} \rightarrow \mathcal{A}, \) and we want to put an upper bound on the size of the cokernel. To do so we examine the morphism \( \mathcal{B} \rightarrow \mathcal{A} \) given by multiplication by \( \Theta_d. \) The composite \( \mathcal{B} \rightarrow \mathcal{A} \rightarrow \mathcal{B} \) is given by multiplication by \( \Theta_d \) on \( \mathcal{B}; \) that shows us that

\[ \Theta_d \Omega^{1}_{\mathcal{B}} \subset \text{image of } \Omega^{1}_{\mathcal{A}}. \]

Note that \( \Omega^{1}_{\mathcal{B}} \) is a locally free \( S \)-module of rank 2. (In fact, it is a free \( \mathcal{Z} \)-module of rank 2\( \varphi(\mathcal{d}) \), and if we tensor \( \otimes_{\mathcal{Z}} \mathcal{C}, \) then we get a free \( S \otimes \mathcal{C} \) module of rank 2. From there we see that \( \Omega^{1}_{\mathcal{B}} \) is contained with finite index in a free \( S \)-module, which easily implies it is locally free.)

So in the exact sequence

\[ \Omega^{1}_{\mathcal{A}} \rightarrow \Omega^{1}_{\mathcal{B}} \rightarrow C, \]

the cokernel \( C \) has the property that \( [C] \leq 2[S/\Theta_d]. \) In particular, \( [C] \leq [S/d^2] \) (see (8.2)).

8.4. The homology of \( \mathcal{A}(\mathcal{C}) \)

Note that

\[ \mathcal{A}(\mathcal{C}) = E(\mathcal{F}_\psi \otimes \mathcal{C}) = \bigoplus_{\sigma: \mathcal{F}_\psi \rightarrow \mathcal{C}} E^\sigma(\mathcal{C}). \]

The set of \( \sigma \)'s which occur is the set \( \sigma_\alpha \) (\( \alpha \in \mu \)) defined earlier, together with their conjugates \( \overline{\sigma_\alpha} \) (\( \alpha \in \mu \)).
Choose generators \( \gamma_1, \gamma_2 \) for \( H_1(E^\sigma_1(\mathbb{C})) \), and let \( \overline{\gamma}_1, \overline{\gamma}_2 \) be their images under the antiholomorphic map \( E^\sigma_1 \to E^{\sigma_T} \). A free \( R \)-basis for \( H_1(A(\mathbb{C}), \mathbb{Z}) \) is given by \( \gamma_1, \gamma_2, \overline{\gamma}_1, \overline{\gamma}_2 \). The complex conjugation of \( \mathbb{C}/\mathbb{R} \) induces an antiholomorphic involution of \( A(\mathbb{C}) \); that involution switches \( \gamma_1 \) and \( \overline{\gamma}_1 \). Later we will also set \( \delta_i = \gamma_i + \overline{\gamma}_i \).

Now \( H_1(B(\mathbb{C}), \mathbb{Z}) \) is given by the kernel of \( \Phi_d \) acting on \( H_1(A(\mathbb{C}), \mathbb{Z}) \). Since the latter is free, as an \( R \)-module on \( \gamma_1, \gamma_2, \overline{\gamma}_1, \overline{\gamma}_2 \), it follows that \( H_1(B(\mathbb{C}), \mathbb{Z}) \) is free as an \( S \)-module on the same generators multiplied by \( \delta_d \).

8.5. Torsion subgroups

Later we will need to understand the torsion subgroups of both \( B(\mathbb{Q}) \) and \( \hat{B}(\mathbb{Q}) \) where \( \hat{B} \) is the dual abelian variety. Clearly, \( B(\mathbb{Q})_{\text{tors}} \subset A(\mathbb{Q})_{\text{tors}} = E(F_\psi)_{\text{tors}} \). To bound torsion in \( \hat{B} \) note that we have a map \( \Theta_d : A \to B \) and thus also a dual map \( \hat{\Theta}_d : \hat{B} \to \hat{A} \). We compute the kernel of \( \hat{\Theta}_d \) over \( \mathbb{C} \); it is dual, as an abelian group, to the cokernel of \( \Theta_d : H_1(A(\mathbb{C}), \mathbb{Z}) \to H_1(B(\mathbb{C}), \mathbb{Z}) \), but this is trivial, as we have seen. Thus, also \( \hat{B}(\mathbb{Q})_{\text{tors}} \) is isomorphic to a subgroup of \( \hat{A}(\mathbb{Q})_{\text{tors}} = E(F_\psi)_{\text{tors}} \). (We have that \( A \) carries a principal polarization and so is isomorphic to \( \hat{A} \).)

8.6. Integration

By integration we get a mapping

\[
\int : \Omega^1_B \otimes \mathbb{R} \to \left( H^1(B(\mathbb{C}), \mathbb{R})_+ \right),
\]

where on the right-hand side the subscript + denotes coinvariants of complex conjugation considered as an antiholomorphic involution of \( B(\mathbb{C}) \). We can regard the right-hand side as the \( \mathbb{R} \)-dual of \( H_1(B(\mathbb{C}), \mathbb{R})^+ \), the conjugation invariants on homology, and then the map is \( \omega \mapsto \int \omega \) for \( \gamma \in H_1(B(\mathbb{C}), \mathbb{R})^+ \). This is an isomorphism of free \( \mathbb{R} \)-modules, both of rank \( r = 2\varphi(d) \).

Both sides have integral structures: on the left-hand side \( \Omega^1_B \), and on the right-hand side we put the integral structure that is the image of \( H^1(B(\mathbb{C}), \mathbb{Z}) \). Thus, we can compute the period determinant of (8.8), well defined up to sign. That determinant is given by the volume

\[
\Lambda = \pm \int_{B(\mathbb{C})^0} |\omega_1 \wedge \cdots \wedge \omega_r|,
\]

where \( \omega_i \) is an integral basis for \( \Omega^1_B \).
The usual BSD conjecture in [8] for $B$ says

$$L\left(\frac{1}{2}, B\right) = \pm \frac{\prod_v c_v (B)}{B(\mathbb{Q}) \cdot \hat{B} (\mathbb{Q})} \cdot \Lambda,$$  

(8.10)

where we allow ourselves to write a finite group in place of its order, and if $B(\mathbb{Q})$ is infinite, then we understand the right-hand side as 0. Note that every term on the right-hand side is a finite $S$-module (e.g., $c_v (B)$ is the local component group of the Néron model, and $S$ acts on it too). Also note that (the way we have set things up) the Archimedean component group $c_\infty (B) \simeq B(\mathbb{R})/B(\mathbb{R})^\circ$ also counts.

As preparation for the equivariant version, we phrase this a little differently. Suppose that we give ourselves finite-index subgroups $H, W$ of the respective integral structures. We can form the period determinant $\hat{\theta}$ with respect to $H$ and $W$, that is, $\hat{\theta}(\det W) = \Lambda' \cdot \det \mathcal{H}$, where for example $\det \mathcal{H}$ denotes the element of the top exterior power of $H \otimes \mathbb{R}$ determined by the lattice $\mathcal{H}$. Since $[H^1_+: \mathcal{H}] \cdot \det (H^1_+) = \det (\mathcal{H})$ and similarly for $W$, we deduce the following variant form of BSD:

$$L\left(\frac{1}{2}, B\right) = \pm \frac{\prod_v c_v (B)}{B(\mathbb{Q}) \cdot \hat{B} (\mathbb{Q})} \cdot [H^1_+: \mathcal{H}] \cdot \Lambda'.$$  

(8.11)

8.7. Statement of the conjecture

In order to make the equivariant conjecture, we need to break up the right-hand side of (8.10) in a way that corresponds to the factorization $L\left(\frac{1}{2}, B\right) = \prod_{\chi} L\left(\frac{1}{2}, E \times \chi\right)$, where the product is taken over all powers $\chi = \psi^i$ with $i \in (\mathbb{Z}/d)\times$. First of all, choose integral elements $e_1, e_2 \subset H^1 (B(\mathbb{C}), \mathbb{R})_+$ so that the $S_\mathbb{R}$-module generated by $e_1, e_2$ is free, and similarly choose $\nu_1, \nu_2 \in \Omega^1_\mathbb{B}$. Then (8.11) says

$$L\left(\frac{1}{2}, B\right) = \left(\frac{\prod_v c_v (B)}{B(\mathbb{Q}) \cdot \hat{B} (\mathbb{Q})} \cdot \frac{H^1_+/S \nu_1 + S e_2}{\Omega^1_\mathbb{B} / S \nu_1 + S e_2}\right) \cdot \Lambda'.$$  

(8.12)

where $\Lambda'$ is the period determinant taken relative to the integral lattices $S \nu_1 + S e_2$ and $S \nu_1 + S e_2$. Note that all the finite groups inside the brackets on the right-hand side are actually $S$-modules. We will next examine how to refine each term on the right-hand side to an element of $S_\mathbb{R}^\times$, so that we recover (8.11) by taking norms.

First, let us examine $\Lambda'$. The map (8.8) is an isomorphism of free $S_\mathbb{R} = (S \otimes \mathbb{Q} \mathbb{R})$ modules of rank 2. We will obtain an element of $S_\mathbb{R}^\times$ by comparing generators for $\wedge^2 S_\mathbb{R}$ LHS and $\wedge^2 S_\mathbb{R}$ RHS. We have

$$\text{the image of } \nu_1 \wedge S_\mathbb{R} \nu_2 = \lambda (e_1 \wedge S_\mathbb{R} e_2) \text{ for some } \lambda \in S_\mathbb{R}$$  

(8.13)
and \( \lambda \in S_\mathbb{R} \) is the desired element. Its norm is equal to \( \Lambda' \). A more explicit way to think about this is the following. There are elements \( \alpha, \beta, \gamma, \delta \in S_\mathbb{R} \) so that the period map (8.8) is given by

\[
f \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.
\]

Then simply \( \lambda = \alpha \delta - \beta \gamma \in S_\mathbb{R} \); also the norm of \( \lambda \) is \( \Lambda' \), as before.

We can now state the equivariant BSD conjecture. The \( \mathbb{R} \)-linear extension of \( \psi \) gives \( \psi : S_\mathbb{R} \to \mathbb{C} \). We then allow ourselves to denote \( \psi(a) \) also by \( a^\psi \). Then

\[
L \left( \frac{1}{2}, E \times \psi \right) = \left( \left[ \prod B \cdot \prod \frac{c_v(B)}{B(\mathbb{Q})B(\mathbb{Q})} \cdot \frac{(H^1_1/Se_1 + Se_2)}{(\Omega^1_B/S_v1 + S_v2)} \right]^\lambda \right) \psi \pmod{\psi(S^\times)}.
\] (8.14)

Part of the conjecture is that the square-bracketed term is a virtually principal \( S \)-module (see Section 8.2) so that it gives an element of \( S_\mathbb{Q}^x/S^\times \) according to our conventions. As before, we regard the right-hand side as 0 if \( B(\mathbb{Q}) \) is infinite. Also, \( \pmod{\psi(S^\times)} \) means that the ratio of the two sides belongs to \( \psi(S^\times) \).

As noted previously, it is likely that one can derive this from the equivariant Tamagawa number conjecture [27, p. 6], although we did not verify the details of this process. For the purpose of this article, the phrase “equivariant BSD” refers to the formulation (8.14) above. Taking the corresponding conjecture with \( \psi \) replaced by \( \psi^i \) and taking the product over \( i \in (\mathbb{Z}/d)^\times \), we recover the original BSD conjecture for \( B \)—at least up to algebraic units.

8.8. Explication

Assume equivariant BSD. Now since we are interested only in proving Proposition 7.8 we may suppose that \( L \left( \frac{1}{2}, E \times \psi \right) \neq 0 \); then also (by (8.14)) we have that \( B(\mathbb{Q}) \) is finite and so \( L \left( \frac{1}{2}, B \right) \neq 0 \). We assume these in what follows.

Next let us explicitly choose \( v_1, v_2, e_1, e_2 \) as in the discussion above (8.12). The inclusion \( B \hookrightarrow A \) induces \( H^1(A(\mathbb{C})) \to H^1(B(\mathbb{C})) \). It is enough to produce forms \( v_1, v_2, e_1, e_2 \) on \( A \) so that the \( R \)-modules spanned by \( v_1, v_2, e_1, e_2 \) are free, and then we pull them back to \( B \). Then the \( S \)-modules spanned by \( v_1, v_2 \) and by \( e_1, e_2 \) are also free. In order to compute the number \( \lambda \) as above, it will be enough to do the corresponding computation on \( A \) and then pull back to \( B \).

(i) Choice of \( e_i \).

Recall that \( H^1(A(\mathbb{C}), \mathbb{Z}) \) is free as an \( R \)-module on basis \( \gamma_1, \gamma_2, \gamma_1, \gamma_2 \). We let \( x_1, x_2, y_1, y_2 \in H^1(A(\mathbb{C}), \mathbb{Z}) \) be the dual basis (to the basis for \( H^1(A(\mathbb{C}), \mathbb{Z}) \).
as the $\mathbb{Z}$-module obtained by applying $\mu$ to $\gamma_1, \gamma_2, \overline{\gamma_1}, \overline{\gamma_2}$). In other words, for any $\alpha \in \mu$,

$$
\{x_1, \alpha(\gamma_i)\} =\begin{cases} 
1 & i = 1, \alpha = \text{id}, \\
0 & \text{else},
\end{cases} \quad \{x_1, \alpha(\gamma_i)\} = 0,
$$

and $x_2$ is similarly dual to $\gamma_2, y_1$ to $\overline{\gamma_1}$, and $y_2$ to $\overline{\gamma_2}$. Then $H^1(A(\mathbb{C}), \mathbb{Z})$ is a free $R$-module on $x_1, x_2, y_1, y_2$. Also, the images of $x_1, y_1$ in $H^1_+$ coincide, as do $x_2, y_2$; and

$$
H^1(A(\mathbb{C}), \mathbb{Z})_+ \text{ is a free } R\text{-module on } x_1, x_2,
$$

where we abuse notation by writing $x_1$ also for its image in $H^1_+$.

(ii) Choice of $v_i$. We have an isomorphism, from (8.6),

$$
\Omega^1_\ell [\mathcal{O}_F \otimes F_{\psi} / F] \cong \Omega^1_{\mathcal{A}/\mathbb{Z}}.
$$

Now choose a $\mathbb{Z}$-basis $\xi_1, \xi_2$ for $\Omega^1_\ell [\mathcal{O}_F \otimes F_{\psi} / F]$, and take $v_i$ to be (the image of) $\xi_i \otimes x$, where $x \in \mathcal{O}_F \otimes F_{\psi}$ is chosen to have the property that

$$
\left[ F_{\psi} / F : \sum_{\alpha \in \mu} \mathcal{O}_F x^\alpha \right]
$$

is prime to $\ell$ (see the next paragraph for why this is possible). In particular,

$$
\Omega^1_{\mathcal{A}} / (Rv_1 + Rv_2) \text{ is prime to } \ell.
$$

As for why we can choose such an $x$, consider the following. We want to show that $\mathcal{O}_F \otimes \mathbb{Z}_\ell$ has a normal basis over $\mathcal{O}_F \otimes \mathbb{Z}_\ell$; that is, there is $x \in \mathcal{O}_F \otimes \mathbb{Z}_\ell$ so that $\alpha x (\alpha \in \mu)$ spans as an $\mathcal{O}_F \otimes \mathbb{Z}_\ell$-module. This comes down to the fact that $F_{\psi} / F$ is tamely ramified at primes above $\ell$, and Galois-stable ideals in tamely ramified extensions have (locally) normal bases. In other words, let $l_i$ be the primes of $F$ above $\ell$, and let $\lambda_{ij}$ be the primes of $F_{\psi}$ above $l_i$. Let $m_{ij}$ be the valuation of $\mathcal{O}_F \otimes F_{\psi} / F$ at $\lambda_{ij}$. We are asking that

$$
\prod_{i,j} \lambda_{ij}^{m_{ij}} \mathcal{O}_{F_{\psi}, \lambda_{ij}} \text{ have a normal basis over } \prod \mathcal{O}_{F,l_i}.
$$

It is enough that $\prod_j \lambda_{ij}^{m_{ij}} \mathcal{O}_{F_{\psi}, \lambda_{ij}}$ have a normal basis over $\mathcal{O}_{F,l_i}$ for each $i$ separately, say, $i = 1$. Next, because the Galois group permutes the various $\lambda_{ij}$’s and $m_{ij}$ does not depend on $j$, it is enough to show that $\lambda_{11}^{m_{11}} \mathcal{O}_{F_{\psi}, \lambda_{11}}$ has a normal basis over $\mathcal{O}_{F,l_1}$. But that is a theorem of Ullom [65, Theorem 1] because $F_{\psi} / F$ is tamely ramified.
Finally, we note for later use that in fact
\[
\text{Norm}(n_\psi)x \in \mathcal{O}_{F_\psi} \otimes \mathbb{Z}_\ell,
\]
that is, it is an algebraic integer above \(\ell\). Here \(\text{N}(n_\psi) \in \mathbb{Z}\) is the absolute ideal norm from ideals of \(F\). In fact, it is enough to see that \(n_\psi x \subset \mathcal{O}_{F_\psi} \otimes \mathbb{Z}_\ell\), that is, \(n_\psi \mathcal{O}_{F_\psi}^{-1}/F \subset \mathcal{O}_{F_\psi} \otimes \mathbb{Z}_\ell\). But, if \(L/K\) is a tamely ramified extension of global fields, then \(\prod_q q \cdot \mathcal{O}_{L/K}^{-1} \subset \mathcal{O}_L\), where the product is over ramified primes \(q\). (We want just the “version above \(\ell\)” of this.) One reduces immediately to the case of a tamely ramified extension of local fields, say, with ramification index \(e\) and residue field degree \(f\). In that case, we can check the inclusion by taking the norms of both sides. The valuation of the norm of \(q\) is \(ef\) and the valuation of the norm of \(\mathcal{O}_{L/K}^{-1}\) is \(e/NAK\). Clearly, \(ef \geq e - 1\).

Recall that we may choose \(\xi_1, \xi_2\) in such a way that
\[
\text{Im} \left( \frac{\xi_2}{\xi_1} \right) = \frac{\sqrt{-\Delta F/4}}{a},
\]
where \(a = [\Omega_\mathcal{F}^{-1}: \mathcal{O}_F \xi_1]\) is as in (7.19).

### 8.9. Exterior product computations

We compute \(\lambda \in S_\mathbb{R}\) as in (8.13)—with respect to the images of \(e_1, e_2, v, v_2\) under the natural maps induced by \(B \hookrightarrow A\)—by computing its analogue \(\tilde{\lambda} \in R_\mathbb{R}\) computed on \(A\). There exists \(\tilde{\lambda} \in R_\mathbb{R}\) such that
\[
\text{the image of } v_1 \wedge_{R_\mathbb{R}} v_2 = \tilde{\lambda}(e_1 \wedge_{R_\mathbb{R}} e_2) \quad \text{for some } \tilde{\lambda} \in R_\mathbb{R}
\]
(where everything is computed on the abelian variety \(A\)). Then the desired \(\lambda \in S_\mathbb{R}\) is simply the image of \(\tilde{\lambda} \in R_\mathbb{R}\) under the natural map. In what follows, we write simply \(v_1 \wedge v_2\) instead of \(v_1 \wedge_{R_\mathbb{R}} v_2\). Recall that we take \(e_1 = x_1, e_2 = x_2\) (see above).

Put \(\delta_i = \gamma_i + \bar{\gamma}_i\). Regard integration on \(\delta_i\) as being functionals \(H^1(A(\mathbb{C}), \mathbb{R}) \to \mathbb{R}\); they factor through \(H^1_+\). By averaging them over \(\mu\) we get \(R\)-linear functionals: we define \(\Delta_i : H^1(A(\mathbb{C}), \mathbb{R})_+ \to R_\mathbb{R}\) by the rule
\[
\Delta_i(\omega) = \sum \alpha \int_{\alpha(\delta_i)} \omega \quad (i = 1, 2; \omega \in H^1(A(\mathbb{C}), \mathbb{R}))
\]
which is now \(R\)-linear. We now pair both sides of (8.18) with \(\Delta_1 \wedge \Delta_2\) in order to compute \(\lambda\).

First,
\[
\langle x_1, \Delta_1 \rangle = \sum \alpha \int_{\alpha(\delta_1)} x_1 = 1 \in R,
\]
and similarly
\[ \langle x_2, \Delta_2 \rangle = 1, \quad \langle x_1, \Delta_2 \rangle = \langle x_2, \Delta_1 \rangle = 0. \]

Therefore,
\[ \langle x_1 \land x_2, \Delta_1 \land \Delta_2 \rangle = x_1(\Delta_1)x_2(\Delta_2) - x_2(\Delta_1)x_1(\Delta_2) = 1. \] (8.19)

Next, compute \( \langle v_1 \land v_2, \Delta_1 \land \Delta_2 \rangle \). It equals
\[
\begin{align*}
v_1(\Delta_1)v_2(\Delta_2) - v_2(\Delta_1)v_1(\Delta_2) \\
= \sum_{\alpha, \beta} \alpha^{-1} \beta^{-1} \left( \langle v_1, \alpha^{-1}(\delta_1) \rangle \langle v_2, \beta^{-1}(\delta_2) \rangle - \langle v_2, \alpha^{-1}(\delta_1) \rangle \langle v_1, \beta^{-1}(\delta_2) \rangle \right),
\end{align*}
\]
where \( \int_{\alpha^{-1}(\delta_1)} v_1 \) has been abbreviated \( \langle v_1, \alpha^{-1}(\delta_1) \rangle \) and so on. In turn,\(^\dagger\)

\[
\frac{1}{4} \left( \langle v_1, \alpha^{-1}(\delta_1) \rangle \langle v_2, \beta^{-1}(\delta_2) \rangle - \langle v_2, \alpha^{-1}(\delta_1) \rangle \langle v_1, \beta^{-1}(\delta_2) \rangle \right) \\
= \left( \text{Re} \left( \int_{y_1} \sigma_\alpha(x) \xi_1 \right) \right) \cdot \left( \text{Re} \left( \int_{y_2} \sigma_\beta(x) \xi_2 \right) \right) \\
- \left( \text{Re} \left( \int_{y_1} \sigma_\alpha(x) \xi_1 \right) \right) \cdot \left( \text{Re} \left( \int_{y_2} \sigma_\beta(x) \xi_2 \right) \right) \\
= \text{Im} \left( \frac{\sigma_\beta(x) \xi_2}{\sigma_\alpha(x) \xi_1} \right) \cdot \text{(area of lattice of } E \text{ with respect to 1-form } \sigma_\alpha(x) \xi_1) \quad (8.20) \\
= \frac{i}{2} \left( \int_{E(\mathbb{C})} \xi_1 \land \overline{\xi_1} \right) \cdot \text{Im} \left( \frac{\sigma_\beta(x) \sigma_\alpha(x) \xi_2}{\xi_1} \right) \quad (8.21)
\]

and \( \langle v_1 \land v_2, \Delta_1 \land \Delta_2 \rangle \) is obtained by summing this expression multiplied by \( 4\alpha^{-1} \times \beta^{-1} \) over \( \alpha \) and \( \beta \).

Note now that if we modify \( \xi_2 \) by a real multiple of \( \xi_1 \), the answer is unchanged. (The contribution of \( \alpha, \beta \) and of \( \beta, \alpha \) cancel in the summation.) Thus, we may take \( \xi_2 = \frac{1}{a} \sqrt{\Delta_F / 4} \cdot \xi_1 \), where \( a \) is as in (8.17), and then from the definition of \( \Omega_E \) we see that
\[
\langle v_1 \land v_2, \Delta_1 \land \Delta_2 \rangle = \sum_{\alpha, \beta} \sqrt{-\Delta_F} \Omega_E \cdot \text{Re}(\sigma_\beta(x) \sigma_\alpha(x)) \alpha^{-1} \beta^{-1} \\
= \sqrt{-\Delta_F} \cdot [\xi_1] \Omega_E \in \mathbb{R}.
\]

\(^\dagger\) Write \( \xi_2/\xi_1 \) for the element \( t \in F \) with \( t \xi_1 = \xi_2 \). We used the following simple fact in (8.21), with \( q = \frac{\sigma_\beta(x) \xi_2}{\sigma_\alpha(x) \xi_1} \):

\[ \Re(z_1)\Re(qz_2) - \Re(z_2)\Re(qz_1) = \text{Im}(q) \cdot \Re(z_1 z_2). \]
(Recall that \([x] = \sum_\alpha \sigma_\alpha(x)\alpha^{-1}\), and recall that \([\bar{x}] = \sum_\alpha \sigma_\alpha(\bar{x})\cdot\alpha^{-1}\). These belong to \(R_C\) but their product \([x][\bar{x}]\) belongs to \(R_R\).) Comparing with (8.19) and (8.18), we see that

\[
\tilde{\lambda} = \sqrt{-\Delta_F}[x][\bar{x}]\Omega_E \in R_R.
\]

We can now rewrite equivariant BSD from the form (8.14). We have seen that \(H^1_{\text{new}}\) is free on \(e_1, e_2\), so

\[
\frac{L(1/2, E \times \psi)}{\Omega_E \sqrt{-\Delta_F}} = \left( [x][\bar{x}] \cdot \left[ \prod_B \cdot \prod_v c_v(B) \right] \cdot \frac{1}{(\Omega_{\mathcal{B}}/Sv_1 + Sv_2)} \right)^\psi \mod \psi(S^\times).
\] (8.23)

### 8.10. Conclusion

We are almost finished. First note that for \(M\) a finite \(S\)-module, we always have \([M] \leq [S/#M]\); this is simply the fact that an ideal of \(S\) divides its norm. As we have mentioned, we can suppose that \(B(\mathcal{Q})\) and \(\hat{B}(\mathcal{Q})\) are finite. So, examining the denominator of (8.23), we have

\[
\left[ \hat{B}(\mathcal{Q})_{\text{tors}} \right] + \left[ B(\mathcal{Q})_{\text{tors}} \right] + [\Omega_{\mathcal{B}}/Sv_1 + Sv_2]
\leq 2\left[ E(\mathcal{F}_{\psi})_{\text{tors}} \right] + [\Omega_{\mathcal{A}}^1 / \text{image of } \Omega_{\mathcal{A}}^1] + [\text{image of } \Omega_{\mathcal{A}}^1 / Rv_1 + Rv_2]
\leq \ell [S/K].
\]

Here \(K = #E(\mathcal{F}_{\psi})_{\text{tors}} \cdot d^2 \cdot (\#\Omega_{\mathcal{A}}^1 / Rv_1 + Rv_2)\) and \(\leq \ell\) means that the equality holds at \(\ell\), as mentioned before. This last inequality follows from (8.7). Since by (8.16) we have that \(N(n_{\psi})^2 \cdot \psi([x][\bar{x}])\) is an algebraic integer, we have proved

\[
N(n_{\psi})^2 K \cdot \frac{L(1/2, E \times \psi)}{\Omega_E \sqrt{-\Delta_F}}\text{ has valuation at least } 0 \text{ at } \ell,
\]

which finishes the proof, because \(#\Omega_{\mathcal{A}}^1 / Rv_1 + Rv_2\) is prime to \(\ell\) by (8.15).

### 9. Numerical computations

9.1. How much of the cohomology is base change at higher levels?

Let \(\mathcal{F}\) be an imaginary quadratic field with its ring of integers \(\mathcal{O}_F\). Given a positive integer \(N\), consider the congruence subgroup \(\Gamma_0((N))\) of level \((N)\) inside the associated Bianchi group \(\text{SL}_3(\mathcal{O}_F)\). Let \(H_{bc}^2(\Gamma_0((N)), \mathbb{C})^\text{new}\) denote the subspace of \(H^2(\Gamma_0((N)), \mathbb{C})^\text{new}\) which corresponds to Bianchi modular forms that are base changes of classical elliptic newforms and their twists (see Section 6.4). We are interested in the following question.
Question
How much of $H^2_{bc}(\Gamma_0((N)), \mathbb{C})^{\text{new}}$ is exhausted by $H^2_{bc}(\Gamma_0((N)), \mathbb{C})^{\text{new}}$?

Note that this question was investigated in [53] for $N = 1$ and more general coefficient modules. Tsaknias and the second author in [62] computed the dimension of $H^2_{bc}(\Gamma_0((N)), \mathbb{C})^{\text{new}}$ for the following special case: $F$ is ramified at a unique prime $p > 2$ and $N$ is square-free and prime to $p$. Over the fields $F = \mathbb{Q}(\sqrt{-d})$ with $d = 3, 7, 11$, we have collected data to investigate the above question. For efficiency reasons, we computed $H^1_{bc}(\Gamma_0((N)), \mathbb{C})^{\text{new}}$ for six primes $\ell$ lying between 50 and 100 and took the minimum of the dimensions we got from these six mod $\ell$ computations. By the universal coefficient theorem, this minimum is an upper bound on the dimension of $H^1_{bc}(\Gamma_0((N)), \mathbb{C})^{\text{new}}$. However, in practice, this upper bound is very likely to give the actual dimension.

We focused on three classes of ideals $(N) \triangleleft \mathcal{O}_F$.

- $N = p$ with $p$ rational prime that is inert in $F$ (see Table 1). Here there are no oldforms. Thus, the base-change dimension formula, together with the number cusps, provides a lower bound for the dimension of $H^1_{bc}(\Gamma_0((N)), \mathbb{C})$. If this lower bound agrees with our upper bound coming from the mod $\ell$ computations, then we know for sure that the whole (co)homology is exhausted by base change. As a result, the zero entries in the “Non-BC” column of Table 1 are proven correct. We also directly computed the characteristic 0 dimensions. (The scope was smaller of course.) The nonzero entries in the Non-BC columns of Table 1 which are in bold are proven to be correct as a result of these characteristic 0 computations.

- $N = p$ with $p$ rational prime that is split in $F$ (see Table 2). Here there may be oldforms and we can compute size of the oldforms using the data computed in [60]. Now the base-change dimension formula, the dimension of the old part, and the number cusps provide a lower bound for the dimension of $H^1_{bc}(\Gamma_0((N)), \mathbb{C})$. As above, if this lower bound agrees with our upper bound coming from the mod $\ell$ computations, then we know for sure that the whole (co)homology is exhausted by base change. As a result, the zero entries in the Non-BC columns of Table 2 are proven correct. We also directly computed the characteristic 0 dimensions for Table 2. The nonzero entries in the Non-BC columns of Table 2 which are in bold are proven to be correct as a result of these characteristic 0 computations.

- $N = pq$ with $p, q$ rational primes that are inert in $F$ (Table 3). To compute the size of the oldforms, one can use the data computed for Table 1. (Note that we only have to refer to entries of Table 1 which are proven correct.) As before, the zero entries in the Non-BC columns of Table 3 are...
Table 1. Level is \( (p) \) with \( p \) a rational prime, *inert* in \( F \).

<table>
<thead>
<tr>
<th>( d = 3 )</th>
<th>( d = 7 )</th>
<th>( d = 11 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>New</td>
<td>Non-BC</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
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Proven correct. We also directly computed the characteristic 0 dimensions for Table 3. The nonzero entries in the Non-BC columns of Table 3 which are in bold are proven to be correct as a result of these characteristic 0 computations.

Of course, there can be non-base-change classes in the oldforms part, but this is not common. As we mentioned before, in case 1, there are no oldforms. In case 2, extensive computations in [60] show that most of the time the cuspidal cohomology of \( \Gamma_0(p) \) (with trivial coefficients) vanishes, where \( (p) = pp \). So usually, we do not have oldforms in case 2. However, when we do, they are completely non-base-change. For case 3, there will be lots of oldforms, however, with few non-base-change classes among them (which can be detected via Table 1).

In Tables 1, 2, and 3, the columns labeled “New” denote the dimension of the new subspace and the columns labeled “Non-BC” denote the dimension of the complement of the base-change subspace inside the new subspace.

9.2. Cases with 1-dimensional cuspidal cohomology

As mentioned in Section 1, experiments in [60] show that, for the five Euclidean imaginary quadratic fields \( F \), the cuspidal part of \( H_1(\Gamma_0(p), \mathbb{C}) \), for \( \Gamma_0(p) \leq \text{PSL}_2(\mathbb{O}_F) \) with residue degree 1 prime level \( p \) of norm at most 45000, vanishes roughly ninety percent of the time. In the remaining nonvanishing cases, the dimension is observed to be one in the majority of cases (see [60, Table 16] for details).

In a new experiment, we computed the dimension of the cuspidal part of \( H_1(\Gamma_0(n), \mathbb{C}) \) for \( \Gamma_0(n) \leq \text{PGL}_2(\mathbb{O}_F) \) with all levels \( n \) of norm at most 10000 for
Table 2. Here the level is \((p)\) with \(p\) a rational prime, \textit{split} in \(F\).

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</table>
the fields $F = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$. Again we see that a significant proportion of the nonvanishing cases have dimension exactly one. The distribution of the levels according to the dimension is given in Table 4.

(Recently, Warren Moore compiled a database of elliptic curves over the five Euclidean imaginary quadratic fields as part of his Warwick senior thesis written under the guidance of John Cremona. His data exhibited (not proven to be, but numerically seeming so) associated elliptic curves for approximately ninety percent of the weight 2 Bianchi newforms with rational Hecke eigenvalues whose level has norm at most 10000. The remaining ten percent of the missing cases were later filled in by Cremona (there are a small number of newforms which correspond to surfaces with quaternionic multiplication; see Section 1), and the complete database is now part of the beta version of the $L$-functions and modular forms database [44], which can be reached via http://beta.lmfdb.org/EllipticCurve/.)
9.3. Growth of regulators of hyperbolic tetrahedral groups

In this section, we report on our numerical experiments related to the growth of regulators in the case of hyperbolic tetrahedral groups. Here we deal with combinatorial regulators rather than analytic ones. One may, however, prove that they both have either subexponential or exponential growth with respect to the index (see [45]).

9.3.1. Tetrahedral groups

A hyperbolic tetrahedral group is the index-two subgroup consisting of orientation-preserving isometries in the discrete group generated by reflections in the faces of a hyperbolic tetrahedron whose dihedral angles are submultiples of \( \pi \). It is well known that there are 32 hyperbolic tetrahedral groups, and nine of them are cocompact. Among the nine cocompact ones, only one is nonarithmetic (see [61]).

Let \( \mathbb{H}^3 \) be one of the nine compact tetrahedra mentioned above that is sitting in hyperbolic 3-space \( \mathbb{H}^3 \), and let \( \Gamma \) be the associated hyperbolic tetrahedral group. Let \( \Sigma \) be a fundamental domain for \( \Gamma \), viewed as a 3-dimensional simplicial complex. We can take \( \Sigma \) to be the union \( \Delta \cup \Delta^* \) where \( \Delta^* \) is the copy of \( \Delta \) obtained by reflecting \( \Delta \) along one of its faces. Let \( T \) denote the triangulation of \( \mathbb{H}^3 \) obtained from \( \Delta \), viewed as an infinite, locally finite simplicial complex with a cocompact cellular action of \( \Gamma \) so that \( \Gamma \backslash T = \Sigma \).

Let \( H \) be a finite-index subgroup of \( \Gamma \), and consider the vector space \( M = \mathbb{R}[H \backslash \Gamma] \) with the natural \( \Gamma \)-action. It is well known that the \( \Gamma \)-equivariant cohomology of \( T \) is isomorphic to the usual cohomology of \( \Gamma \):

\[
H^*(\Gamma, M) \simeq H^*_T(T, M).
\]

Put an inner product on \( M \) by declaring the basis \( H_n \) to be orthonormal. We define the combinatorial Laplacians \( \{ \Delta_i \}_i \) on the \( \Gamma \)-equivariant cochain complex \( \{ C^i(T, M)^\Gamma \}_i \) (which computes the right-hand side) using the inner product

\[
\langle f, g \rangle_i^\Gamma := \sum_{\sigma \in \Sigma_i} \frac{1}{|\Gamma(\sigma)|} \langle f(\tilde{\sigma}), g(\tilde{\sigma}) \rangle,
\]

where \( \tilde{\sigma} \) is a lift of \( \sigma \) in \( T_i \) (see, e.g., [40, Section 2] for details).

For \( i = 1, 2 \), let \( r_i \) denote covolume of \( H^1(\Gamma, \mathbb{Z}[H \backslash \Gamma]) \) inside \( H^1(\Gamma, \mathbb{R}[H \backslash \Gamma]) \) with respect to the above inner product. According to Proposition 4.1, asymptotically \( r_i \) behaves like the regulator \( R_i \). For computational efficiency, we will compute another quantity, which, asymptotically speaking, gives us the desired information. Let \( \tilde{r}_i \) denote the covolume (with respect to the same inner product) of the subspace of harmonic \( i \)-cochains, that is, the kernel of \( \Delta_i \). Then it is not hard to see that

\[
\tilde{r}_i \geq r_i \geq \frac{1}{\tilde{r}_i}.
\]
Table 5. Data for projective subgroups $H$ of $T_6$.

<table>
<thead>
<tr>
<th>$[T_6 : H]$</th>
<th>Rank</th>
<th>$\log(\tilde{r}_1)$</th>
<th>$\log(\tilde{r}_1)/[T_6 : H]$</th>
<th>$\log(\tilde{r}_2)$</th>
<th>$\log(\tilde{r}_2)/[T_6 : H]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>122</td>
<td>1</td>
<td>5.4161004</td>
<td>0.04439426559</td>
<td>4.4755991</td>
<td>0.03668523930</td>
</tr>
<tr>
<td>170</td>
<td>5</td>
<td>28.1536040</td>
<td>0.1656094354</td>
<td>44.7684568</td>
<td>0.2633436863</td>
</tr>
<tr>
<td>290</td>
<td>5</td>
<td>36.2058878</td>
<td>0.1248478891</td>
<td>30.5155222</td>
<td>0.1052259389</td>
</tr>
<tr>
<td>362</td>
<td>7</td>
<td>45.4762539</td>
<td>0.1256250109</td>
<td>44.4415985</td>
<td>0.1227668467</td>
</tr>
<tr>
<td>458</td>
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<td>9.26712597</td>
<td>0.02023389951</td>
<td>6.4856782</td>
<td>0.01416086959</td>
</tr>
<tr>
<td>674</td>
<td>1</td>
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<td>0.01134831485</td>
<td>8.3297444</td>
<td>0.01235867125</td>
</tr>
<tr>
<td>962</td>
<td>11</td>
<td>78.0538394</td>
<td>0.08113704729</td>
<td>79.9289185</td>
<td>0.0830861934</td>
</tr>
<tr>
<td>1034</td>
<td>2</td>
<td>17.8191345</td>
<td>0.01723320555</td>
<td>21.3528238</td>
<td>0.02065070001</td>
</tr>
<tr>
<td>1370</td>
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<td>7.0476963</td>
<td>0.00514430392</td>
<td>7.52250931</td>
<td>0.005490882711</td>
</tr>
<tr>
<td>1682</td>
<td>15</td>
<td>105.2828487</td>
<td>0.0625938458</td>
<td>116.4427193</td>
<td>0.06922872726</td>
</tr>
<tr>
<td>1850</td>
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<td>0.01090259953</td>
<td>20.7570214</td>
<td>0.01122001160</td>
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<tr>
<td>2210</td>
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<td>109.5835840</td>
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<tr>
<td>2522</td>
<td>2</td>
<td>19.5918702</td>
<td>0.0077683863</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6. Data for projective subgroups $H$ of $T_8$.

<table>
<thead>
<tr>
<th>$[T_8 : H]$</th>
<th>Rank</th>
<th>$\log(\tilde{r}_1)$</th>
<th>$\log(\tilde{r}_1)/[T_8 : H]$</th>
<th>$\log(\tilde{r}_2)$</th>
<th>$\log(\tilde{r}_2)/[T_8 : H]$</th>
</tr>
</thead>
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<tr>
<td>42</td>
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<td>5.67652517</td>
<td>0.1351553613</td>
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<td>0.1966572686</td>
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<tr>
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<td>0.0828454854</td>
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<tr>
<td>1682</td>
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<td>90.93035031</td>
<td>0.05406085036</td>
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<tr>
<td>2402</td>
<td>4</td>
<td>26.74244163</td>
<td>0.01113340617</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We computed $\tilde{r}_i$ for prime-level $\Gamma_0$-type subgroups $H$ (see [61]) of two cocompact hyperbolic tetrahedral groups $T_6$ and $T_8$, which, in the notation of [61], can be identified as $T(4,3,2;4,3,2)$ and $T(5,3,2;4,3,2)$. While $T_6$ is arithmetic, $T_8$ is nonarithmetic. The data we collected are depicted in Tables 5 and 6, respectively. In the tables, the first column shows the index of the subgroup $H$ inside the tetrahedral group, and the second column shows the dimension of the cohomology $H_1(H;\mathbb{R})$. For the cases where this dimension is zero, the space of harmonic cochains is trivial and thus these cases are not included in the tables.

9.3.2. Arithmetic versus nonarithmetic

As discussed at the end of Section 1 (also see [6, Chapter 9]), the subexponential growth of the regulator with respect to the volume might be related to arithmeticity. Unfortunately, the scope of the data we collected here on the growth of the regulator is too limited to infer anything on this speculation. However, the experiments in [14] and [61], which inspect the growth of torsion, all suggest that if $M_0$ is nonarithmetic, then for a sequence $(M_i \to M_0)_{i \in \mathbb{N}}$ of finite covers of $M_0$ which is BS-converging to $\mathbb{H}^3$, the sequence

$$\frac{\log \# H_1(M_i, \mathbb{Z})_{\text{tors}}}{V_i}$$
does not necessarily converge to \(1/(6\pi^2)\). (The convergence is broken at covers with positive Betti numbers.) If we believe that analytic torsion converges in this general setting, then it must be that the regulator does not disappear in the limit, giving support to the above speculation.

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