



## A Note on Local Rigidity

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**Abstract.** The aim of this note is to give a geometric proof for classical local rigidity of lattices in semisimple Lie groups. We are reproving well known results in a more geometric (and hopefully clearer) way.

**Key words.** lattices, local rigidity, semisimple Lie groups.

### 1. Introduction

Let  $G$  be a semisimple Lie group, and let  $\Gamma \leq G$  be an irreducible lattice. We denote by  $\mathcal{R}(\Gamma, G)$  the space of deformations of  $\Gamma$  in  $G$ , i.e. the space of all homomorphisms  $\Gamma \rightarrow G$  with the topology of pointwise convergence. Recall that  $\Gamma$  is called *locally rigid* if there is a neighborhood  $\Omega$  of the inclusion map  $\rho_0: \Gamma \rightarrow G$  in  $\mathcal{R}(\Gamma, G)$  such that any  $\rho \in \Omega$  is conjugate to  $\rho_0$ . In other words, this means that if  $\Gamma$  is generated by the finite set  $\Sigma$ , then there is an identity neighborhood  $U \subset G$  such that if  $\rho: \Gamma \rightarrow G$  is a homomorphism such that  $\rho(\gamma) \in \gamma \cdot U$  for any  $\gamma \in \Sigma$ , then there is some  $g \in G$  for which  $\rho(\gamma) = g\gamma g^{-1}$  for any  $\gamma \in \Gamma$  (such  $\rho$  is called a trivial deformation).

Local rigidity was first proved by Selberg [17] for uniform lattices in the case  $G = \mathrm{SL}_n(\mathbb{R})$ ,  $n \geq 3$ , and by Calabi [3] for uniform lattices in the case  $G = \mathrm{PO}(n, 1) = \mathrm{Isom}(\mathbb{H}^n)$ ,  $n \geq 3$ . Then, Weil [22] generalized these results to any uniform irreducible lattice in any  $G$ , assuming that  $G$  is not locally isomorphic to  $\mathrm{SL}_2(\mathbb{R})$  (in which case lattices have many nontrivial deformations – the Teichmüller spaces). Later, Garland and Raghunathan [7] proved local rigidity for nonuniform lattices  $\Gamma \leq G$  when  $\mathrm{rank}(G) = 1^*$  under the necessary assumption that  $G$  is neither locally isomorphic to  $\mathrm{SL}_2(\mathbb{R})$  nor to  $\mathrm{SL}_2(\mathbb{C})$ . For nonuniform irreducible lattices in higher rank semisimple Lie groups, the local rigidity is a consequence of the much stronger property: super-rigidity, which was proved by Margulis [13].

Our aim here is to present a simple proof of the following theorem:

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\*Unless otherwise specified  $\mathrm{rank}(G)$  will always mean the real rank of  $G$ .

**THEOREM 1.1.** *Under the necessary conditions on  $G$  and  $\Gamma$ , if  $\rho$  is a deformation, close enough to the identity  $\rho_0$ , then  $\rho(\Gamma)$  is a lattice and  $\rho$  is an isomorphism.*

*Remark 1.2.* In the higher rank case, we need the lattice to be arithmetic, which is always true by Margulis' theorem. But this is of course cheating. Anyway our method provides, for example, a simple geometric proof of the local rigidity of  $\mathrm{SL}_n(\mathbb{Z})$  in  $\mathrm{SL}_n(\mathbb{R})$  for  $n \geq 3$ . Note also that Margulis' arithmeticity theorem is easier in the non compact case [14] (it does not rely on super-rigidity) which is the only case we need here.

*Remark 1.3.* In fact, we prove a 'stronger' result than the statement of Theorem 1.1. Namely, if  $X = G/K$  is the associated symmetric space, then for a small deformation  $\rho$ ,  $X/\rho(\Gamma)$  is homeomorphic to  $X/\Gamma$  and  $\rho: \Gamma \rightarrow \rho(\Gamma)$  is an isomorphism.

With Theorem 1.1, local rigidity follows from Mostow rigidity. Recall that Mostow's rigidity theorem [15] says that if  $G$  is a connected semisimple Lie group not locally isomorphic to  $\mathrm{SL}_2(\mathbb{R})$ , and  $\Gamma_1, \Gamma_2 \leq G$  are isomorphic irreducible lattices in  $G$ , then they are conjugate in  $G$ , i.e.  $\Gamma_1 = g\Gamma_2g^{-1}$  for some  $g \in G$ .

*Remark 1.4.* In Selberg's paper [17] a main difficulty was to prove a weaker version of 1.1 in his special case. Selberg also indicated that his proof (which is quite elementary and very elegant) would work for more general higher rank cases if the general version (1.1 above) of his Lemma 9 could be proved. This supports the general feeling that it should be possible to give a complete elementary proof of local rigidity which does not rely on Mostow's rigidity. However, the authors decided not to strain too much, in trying to avoid Mostow rigidity, since in the rank one case there are by now very elegant and straightforward proofs of Mostow rigidity (such as the proof of Gromov and Thurston (see [18]) for compact hyperbolic manifolds, and of Besson, Courtois and Gallot [2] for the general rank one case, note also that Mostow's original proof is much simpler in the rank one case) while for higher rank spaces the much stronger Margulis' super-rigidity holds.

Our proof of Theorem 1.1 relies on an old principle implicit in Ehresmann's work and to our knowledge first made explicit in Thurston's notes. This principle implies in particular that any sufficiently small deformation of a cocompact lattice in a Lie group stays a lattice. It seems to have remained unknown for many years, as both Selberg and Weil spent some effort in proving partial cases of it. It is by now fairly well known, but the authors could not find a complete written proof of it in the literature although everyone knows that it is implicit in Ehresmann's work. In the second section of this paper, we thus decided to write a complete proof, using Ehresmann's beautiful viewpoint, of a slight generalization of this principle to the non-compact case needed for our purpose. In the third section we prove Theorem

1.1 in the rank one case. In the last section we prove Theorem 1.1 for arithmetic lattices of higher rank.

Nonuniform lattices in the group  $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$  are not locally rigid, although Mostow rigidity is valid for  $\mathrm{PSL}_2(\mathbb{C})$ . This shows that local rigidity is not a straightforward consequence of Mostow rigidity. Moreover, it was shown by Thurston that if  $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$  is a nonuniform lattice then there is an infinite family of small deformations  $\rho_n$  of  $\Gamma$  such that  $\rho_n(\Gamma)$  is again a lattice, but not isomorphic to  $\Gamma$ , and in fact many of the  $\rho_n(\Gamma)$ 's may be constructed to be uniform. The proof which we present in this note is in a sense a product of our attempt to understand and formulate why similar phenomenon cannot happen in higher dimensions.

In fact, we shall show that some weak version of local rigidity is valid also for nonuniform lattices in  $\mathrm{PSL}_2(\mathbb{C})$ . This would allow us to obtain a direct proof of the fact that for any (nonuniform) lattice in  $G = \mathrm{SL}_2(\mathbb{C})$  (as well as in any connected  $G$  which is defined over  $\mathbb{Q}$  as an algebraic group and is locally isomorphic to  $\mathrm{SL}_2(\mathbb{C})$ ), there is an algebraic number field  $K$ , and an element  $g \in G$ , such that  $g\Gamma g^{-1} \leq G(K)$ . This fact (which is well known in the general case for any lattice in any  $G$  which is not locally isomorphic to  $\mathrm{SL}_2(\mathbb{R})$ ) can be proved by a few line argument when  $\Gamma$  is locally rigid in  $G$ . When  $G$  is locally isomorphic to  $\mathrm{SL}_2(\mathbb{C})$ , and  $\Gamma \leq G$  is nonuniform (and, hence, not locally rigid), this was proved in [7], Section 8, by a special and longer argument. Our method allows us to unify this special case to the general case.

In Section 2 we shall formulate and prove the Ehresmann–Thurston principle and use it in order to prove local rigidity for uniform lattices (Weil's theorem). In Section 3, which is in a sense the main contribution of this note, we shall present a complete and elementary proof for local rigidity of nonuniform lattices in groups of rank one (originally due to Garland and Raghunathan). Then, in Section 4, we shall apply our method to higher-rank nonuniform lattices. In some cases, our proof remains completely elementary. However, for the general case we shall rely on some deep results.

## 2. The Ehresmann–Thurston Principle

In this Section  $X$  is a model manifold and  $G$  a transitive group of real analytic homeomorphisms of  $X$ . Given such data one can construct all possible manifolds by choosing open subsets of  $X$  and pasting them together using restrictions of homeomorphisms from the group  $G$ . Following Ehresmann and Thurston we will call such a manifold a  $(G, X)$ -manifold. More precisely, a  $(G, X)$ -manifold is a manifold  $M$  together with an atlas  $\{\kappa_i: U_i \rightarrow X\}$  such that the changes of charts are restrictions of elements of  $G$ .

Such a structure enables to develop the manifold  $M$  along its paths (by pasting the open subsets of  $X$ ) into the model manifold  $X$ . We can do this in particular for closed paths representing generators of the fundamental group  $\pi_1(M)$ . Starting from one of the open sets  $U \subset M$  of the given atlas, the development produces a covering space  $M'$  of  $M$ , a representation  $\pi_1(M) \xrightarrow{\rho} G$  called the holonomy and a structure

preserving immersion, the developing map  $M' \xrightarrow{d} X$ , which is equivariant with respect to the action of  $\pi_1(M)$  on  $M'$  and of  $\rho(\pi_1(M))$  on  $X$ .

There is an ambiguity in the choice of one chart  $\kappa: U \rightarrow X$  about a base point. However, different charts around a point are always identified on a common sub-neighborhood modulo the action of  $G$ .

Ehresmann's way of doing mathematics was to find neat definitions for his concepts so that theorems would then follow easily from definitions. We shall try to follow his way. Recall first his fiber bundle picture of a  $(G, X)$ -structure and his neat definition of the developing map.

To make it simple, let us describe first the picture for one chart  $\kappa: U \rightarrow X$  which is a topological embedding. We can associate a trivial fiber bundle  $E_U = U \times X \rightarrow U$  by assigning to any  $m \in U$  the model space  $X_m = X$ . The manifold  $U$  is embedded as the diagonal cross-section  $s(U) = \{(m, \kappa(m)) : m \in U\} \subset U \times X$ . Its points are the points of tangency of fibers and base manifolds.

For the global picture of a  $(G, X)$ -structure on a manifold  $M$ , we will restrict ourselves to compatible charts that are topological embeddings  $\kappa: U \rightarrow X$  for open sets  $U \subset M$ . A point of the fiber bundle space  $E$  over  $M$  is, by definition, a triple

$$\{m, \kappa, x\},$$

where  $m \in U \subset M$ ,  $\kappa: U \rightarrow X$  is a compatible chart and  $x \in X$ , modulo the equivalence relation given by the action of  $G$ , i.e.  $\{m, \kappa, x\} \sim \{m, \kappa', x'\}$  iff there is  $g \in G$  such that  $x' = g \cdot x$  and  $\kappa' = g \circ \kappa$  on some sub-neighborhood  $V \subset U \cap U'$ . In  $E$ , the manifold  $M$  is embedded as the diagonal cross-section  $s(M)$ , whose points are represented by triples  $\{m, \kappa, \kappa(m)\}$ . The horizontal leaves represented by the triples  $\{U, \kappa, x\}$  give a foliation  $\mathcal{F}$  of the total space  $E$ , which induces an  $n$ -plane field  $\zeta$  in  $E$  transversal to the fibers and transversal to the cross-section  $s(M)$ . Given such data, Ehresmann has defined the classical notion of holonomy. The holonomy is obtained by lifting a closed curve starting and ending at  $m_0 \in M$ , into all curves tangent to  $\zeta$ , and by this getting an element of  $G$  acting on  $X_{m_0}$ . For contractible closed curves in the base space  $M$ , the holonomy is of course the identify map of the fiber  $X_{m_0}$ .

This leads us, following Ehresmann, to consider the general notion of *flat Cartan connections*. In fact  $(G, X)$ -manifolds are the flat cases of manifolds  $M$  with a general  $(G, X)$ -connection. A  $(G, X)$ -connection is defined in [6] as follows:

- (1) a fiber bundle  $X \rightarrow E \rightarrow M$  with fiber  $X$  over  $M$ ,
- (2) a fixed cross section  $s(M)$ ,
- (3) an  $n$ -plane field  $\zeta$  in  $E$  transversal to the fibers and transversal to the fixed cross-section, such that
- (4) the holonomy of each closed curve starting and ending at  $m_0 \in M$  is obtained by lifting it to all curves tangent to  $\zeta$ , and is a homeomorphism of  $X_{m_0}$  which is induced by an element of  $G$ .

A  $(G, X)$ -connection is said to be *flat* if contractible closed curves have trivial holonomy.

We have just seen that a  $(G, X)$ -structure on a manifold  $M$  yields a flat  $(G, X)$ -connection on  $M$ . Let us describe how we can recover the holonomy and the developing map of a given  $(G, X)$ -structure from the induced flat  $(G, X)$ -connection. Let  $M$  be a manifold with a flat  $(G, X)$ -connection. For general closed curves, starting and ending at  $m_0$ , the holonomy gives a representation of  $\pi_1(M)$  into the group  $G$  acting on  $X_{m_0}$ . When the  $(G, X)$ -connection is induced by a  $(G, X)$ -structure on  $M$ , this map is the usual holonomy map of the  $(G, X)$ -structure.  $E$  can be described as the quotient

$$\pi_1(M) \backslash (M' \times X)$$

where the action is the diagonal one, with  $\pi_1(M)$  acting on  $M'$  by covering transformations and on  $X$  via the holonomy map through the action of  $G$  on  $X$ . The development of a curve ending at  $m_0 \in M$ , is obtained by dragging along  $\xi$  the corresponding points until they arrive in the fiber  $X_{m_0}$ . As soon as the connection  $\xi$  is flat, homotopic curves with common initial and end points give the same image of the initial point in the end fiber and the development map  $M' \rightarrow X_{m_0}$  is well defined. We thus conclude that any flat  $(G, X)$ -connection on a manifold  $M$  yields a  $(G, X)$ -structure.

It is then an easy matter to prove the following useful result, which to our knowledge first appeared in a slightly different setup in Thurston's notes [18], but which, as we will see, and seems to be a common knowledge, is elementary using Ehresmann's viewpoint of  $(G, X)$ -structures. For a slightly different proof, from which we have borrowed some ideas, see [4] which is a very nice reference for all basic material on  $(G, X)$ -structures.

**THEOREM 2.1** (The Ehresmann–Thurston principle). *Let  $N$  be a smooth compact manifold possibly with boundary. Let  $N_{\text{Th}}$  be the reunion of  $N$  with a small collar  $\partial N \times [0, 1)$  of its boundary. Assume  $N_{\text{Th}}$  is equipped with a  $(G, X)$ -structure  $M$  whose holonomy is  $\rho_0: \pi_1(N) \rightarrow G$ . Then, for any sufficiently small deformation  $\rho$  of  $\rho_0$  in  $\mathcal{R}(\pi_1(N), G)$ , there is a  $(G, X)$ -structure on the interior of  $N$  whose holonomy is  $\rho$ .*

*Proof.* Let  $\rho \in \mathcal{R}(\pi_1(N), G)$  be a deformation of  $\rho_0$ . We define the following two fiber bundles over  $N_{\text{Th}}$  with fibers  $X$

$$E_{\rho_0} = \pi_1(N) \backslash (N'_{\text{Th}} \times X)$$

and

$$E_{\rho} = \pi_1(N) \backslash (N'_{\text{Th}} \times X)$$

where the action of  $\pi_1(N)$  on  $X$  is respectively induced via  $\rho_0$  and  $\rho$  by the natural action of  $G$  on  $X$ . We denote by  $(1, \rho_0)$  and  $(1, \rho)$  the two diagonal actions of  $\pi_1(N)$  on  $N_{\text{Th}} \times X$  considered above.

Note that since  $N$  is compact,  $\pi_1(N)$  is finitely generated and hence  $\mathcal{R}(\pi_1(N), G)$  has the structure of an analytic manifold and, in particular, it is locally arcwise connected. Theorem 2.1 is a consequence of the following easy claim.

CLAIM 2.2. If  $\rho$  is a deformation of  $\rho_0$  in the same connected component of  $\mathcal{R}(\pi_1(N), G)$  then there exists a fiber bundle map  $F: E_{\rho_0} \rightarrow E_\rho$  such that

- (1)  $F$  restricts to a diffeomorphism  $\Phi$  above  $N$ ,
- (2) as  $\rho$  gets closer to  $\rho_0$ , the lifted diffeomorphism  $\tilde{\Phi}: N' \times X \rightarrow N' \times X$  gets closer to the identity in the compact-open topology.

We temporarily postpone the proof of the claim and conclude the proof of Theorem 2.1.

When  $\rho$  is in the same connected component as  $\rho_0$ , we have, by Claim 2.2, a diffeomorphism  $\Phi$  between the compact fiber bundles  $E_{\rho_0}^c$  and  $E_\rho^c$  which are defined over  $N$ . Thus, if  $\mathcal{F}$  denotes the horizontal foliation in  $E_{\rho_0}$ , the restriction of  $\mathcal{F}$  to  $E_{\rho_0}^c$  induces via  $\Phi$  an  $n$ -plane field  $\xi$  on  $E_\rho^c$ . When  $\rho$  is close enough to  $\rho_0$ , the lifted diffeomorphism  $\tilde{\Phi}$  is close to the identity so that the  $n$ -plane field  $\xi$  is transversal to both the fibers and the image by  $\Phi$  of the diagonal cross section in  $E_{\rho_0}^c$ . Moreover, the holonomy which is obtained by lifting a closed curve starting and ending at  $m_0 \in N$ , to all curves tangent to  $\xi$ , is equal to the image by  $\rho$  in  $G$  of the homotopy class of the curve. Thus the restriction of the bundle  $E_\rho^c$  to a bundle over the interior of  $N$  yields a  $(G, X)$ -connection. This connection is obviously flat and, hence, gives a  $(G, X)$ -structure on the interior of  $N$  whose holonomy is  $\rho$ . This concludes the proof of Theorem 2.1 modulo Claim 2.2.

Let us now prove Claim 2.2. Let  $\{U_i\}_{0 \leq i \leq k}$  be a finite open covering of  $N_{\text{Th}}$  such that

- $U_0 = N_{\text{Th}} - N$ ,
- for each  $1 \leq i \leq k$ ,  $U_i$  is simply connected and the fiber bundles  $E_\rho$  and  $E_{\rho_0}$  are both trivial above  $U_i$ .

For each integer  $1 \leq i \leq k$ , we fix a trivialization  $(E_\rho)_{|U_i} \cong U_i \times X$  (resp.  $(E_{\rho_0})_{|U_i} \cong U_i \times X$ ) of  $E_\rho$  (resp.  $E_{\rho_0}$ ) above  $U_i$ . For each pair of integers  $1 \leq i \neq j \leq k$ , we then denote by  $g_\rho^{i,j}: (U_i \cap U_j) \times X \rightarrow (U_i \cap U_j) \times X$  (resp.  $g_{\rho_0}^{i,j}: (U_i \cap U_j) \times X \rightarrow (U_i \cap U_j) \times X$ ) the diffeomorphism, which is the product of the identity on the first factor with the corresponding change of charts on the second factor, between the trivializations  $(E_\rho)_{|U_i}$  and  $(E_\rho)_{|U_j}$  (resp.  $(E_{\rho_0})_{|U_i}$  and  $(E_{\rho_0})_{|U_j}$ ).

Let  $U_i^1 = U_i$  ( $1 \leq i \leq k$ ) and for each integer  $r > 0$ , let  $\{U_i^{r+1}\}_{1 \leq i \leq k}$  be a shrinking of  $\{U_i^r\}_{1 \leq i \leq k}$  (i.e. for each integer  $i$ ,  $U_i^{r+1}$  is an open set whose closure is included in  $U_i^r$ ) such that  $\{U_0\} \cup \{U_i^{r+1}\}_{1 \leq i \leq k}$  is still a covering of  $N_{\text{Th}}$ .

As both  $E_\rho$  and  $E_{\rho_0}$  are trivial above  $U_i$ , the identity map  $U_i \times X \rightarrow U_i \times X$  induces a diffeomorphism  $F_1: (E_{\rho_0})_{|U_i^1} \rightarrow (E_\rho)_{|U_i^1}$ .

We will define by induction on  $s$ , a diffeomorphism  $F_s$  between  $E_{\rho_0}$  and  $E_\rho$  above  $U_1^s \cup \dots \cup U_s^s$  which is equal to  $F_{s-1}$  above  $U_1^s \cup \dots \cup U_{s-1}^s$ . But for the sake of clarity let us first define the diffeomorphism  $F_2$ .

Let's first describe the  $s = 2$  step. First note that, as  $\rho$  is a deformation of  $\rho_0$  in the same connected component, the diffeomorphism  $g_\rho^{1,2} \circ (g_{\rho_0}^{1,2})^{-1}$  of  $(U_1 \cap U_2) \times X$  is

isotopic to the identity, by an isotopy which preserves each of the fiber  $\{*\} \times X$ . We denote by  $(m, x) \mapsto (m, \varphi_t(x))$ ,  $t \in [0, 1]$ , this isotopy.

We want to define  $F_2$  above  $U_2^2$ , recall it will be equal to  $F_1$  above  $U_2^2 \cap U_1^2$ . Let  $W$  be an open set such that

$$\overline{U_2^2} \subset W \subset \overline{W} \subset U_2^1.$$

We define  $F_2$  above  $U_2^2$  as follows:

- Above  $U_2^1 - \overline{W} \subset U_2$ , both  $E_{\rho_0}$  and  $E_\rho$  are trivial and the identity map  $(U_2^1 - \overline{W}) \times X \rightarrow (U_2^1 - \overline{W}) \times X$  induces a diffeomorphism  $f: (E_{\rho_0})_{(U_2^1 - \overline{W})} \rightarrow (E_\rho)_{(U_2^1 - \overline{W})}$ .
- We define  $f = F_1$  above  $U_1^2 \cap U_2^2$ . (In the coordinates  $(U_1^2 \cap U_2^2) \times X$  induced by the trivializations above  $U_2$ , the diffeomorphism  $f$  is equal to the restriction of  $g_\rho^{1,2} \circ (g_{\rho_0}^{1,2})^{-1}$ ).
- We extend  $f$  above a small collar neighborhood  $B \times [0, \varepsilon]$  of the boundary  $B$  of  $U_1^2 \cap U_2^2$  in  $W - (U_1^2 \cap U_2^2)$  by the map from  $(B \times [0, \varepsilon]) \times X$  to itself, given by  $((b, t), x) \mapsto ((b, t), \varphi_{g(t)}(x))$  where  $g: [0, \varepsilon] \rightarrow [0, 1]$  is the classical smooth bump function.
- Finally we can extend  $f$  (by the identity) above all  $U_2^1$  to get a smooth diffeomorphism.
- We then define  $F_2$  above  $U_2^2$  to be the restriction of  $f$  to that set.

If we let  $F_2$  to be equal to  $F_1$  above  $U_1^2$ , we get a well defined diffeomorphism  $F_2$  from  $E_{\rho_0}$  to  $E_\rho$  above  $U_1^2 \cap U_2^2$ .

To follow the induction we need to define  $F_{s+1}$  above  $U_{s+1}^{s+1}$ . First note that above  $U_s^{s+1}$ , the fiber bundles  $E_{\rho_0}$  and  $E_\rho$  are both trivial.

Let  $W$  be an open set such that

$$\overline{U_{s+1}^{s+1}} \subset W \subset \overline{W} \subset U_{s+1}^s.$$

We define  $F_{s+1}$  above  $U_{s+1}^{s+1}$  as follows:

- Above  $U_{s+1}^s - \overline{W}$ , both  $E_{\rho_0}$  and  $E_\rho$  are trivial and we define a function  $f$  to be the canonical diffeomorphism between their trivializations.
- Above  $U_{s+1}^{s+1} \cap (U_1^{s+1} \cup \dots \cup U_s^{s+1})$  we define  $f$  to be equal to  $F_s$ .
- We extend  $f$  as in the first inductive step, by using an isotopy between changes of charts composed with a smooth bump function, to get a smooth diffeomorphism between  $E_{\rho_0}$  and  $E_\rho$  above all  $U_{s+1}^s$ .
- We define  $F_{s+1}$  above  $U_{s+1}^{s+1}$  to be equal to the restriction of  $f$  above  $U_{s+1}^{s+1}$ .

If we let  $F_{s+1}$  to be equal to  $F_s$  above  $U_1^{s+1} \cup \dots \cup U_s^{s+1}$ , we get a well defined diffeomorphism  $F_{s+1}$  from  $E_{\rho_0}$  to  $E_\rho$  above  $U_1^{s+1} \cup \dots \cup U_{s+1}^{s+1}$ .

The local trivialization of the fiber bundle  $E_\rho$  depends continuously (in the  $C^\infty$ -topology) on  $\rho$ , and hence  $F_{s+1}$  depends continuously on  $F_s$  and  $\rho$ .

We continue the induction until  $s = k$  and get a diffeomorphism between  $E_{\rho_0}$  and  $E_\rho$  above  $U_1^k \cup \dots \cup U_k^k$  which restricts to a fiber bundle diffeomorphism between the

corresponding bundles above  $N$ . Moreover we have seen that the restriction of  $F$  above  $N$  depends continuously (in the  $C^\infty$ -topology) on  $\rho$ . This concludes the proof of the claim.  $\square$

Recall now Ehresmann's definition [6] of a complete  $(G, X)$ -structure on a manifold  $M$ . Let  $M$  be a  $(G, X)$ -manifold. Any curve  $c$  starting from a point  $m_0$  in  $M$  can be developed to a curve  $\tilde{c}$  in  $X$ . The  $(G, X)$ -manifold is said to be *complete* if, conversely, any curve  $\tilde{C}$  extending  $\tilde{c}$  in  $X$  is a development of a curve in  $M$  extending  $c$ . When  $X$  is a Riemannian homogeneous  $G$ -space, for a Lie group  $G$  of isometries of  $X$ , it is a classical theorem of Hopf and Rinow that this notion of completeness coincides with the usual notion of metric completeness.

The Ehresmann–Thurston principle above is especially useful when combined with the following.

**THEOREM 2.3** (Ehresmann [5]). *If  $M$  is a complete  $(G, X)$ -manifold, then the developing map induces an isomorphism between its universal cover and the universal cover of  $X$ .*

As a corollary of Theorems 2.1 and 2.3 we get the following classical result of Weil [22].

**COROLLARY 2.4.** *Let  $\Gamma$  be a cocompact lattice in a connected center free semi-simple Lie group without compact factors  $G$ . If  $\rho$  is a deformation of  $\Gamma$  in  $G$ , close enough to the identity, then  $\rho(\Gamma)$  is still a cocompact lattice and is isomorphic to  $\Gamma$ .*

*Proof of Corollary 2.4.* The group  $\Gamma$  is finitely generated and by Selberg's lemma it has a torsion-free finite index subgroup. It is easy to see that if 2.4 holds for a finite index subgroup of  $\Gamma$  then it also holds for  $\Gamma$ . Hence, we shall assume that  $\Gamma$  itself is torsion-free. Let  $X$  be the symmetric space associated to  $G$ . It is simply connected. Let  $M$  be the  $(G, X)$ -manifold  $\Gamma \backslash X$ . It is compact and its fundamental group is  $\Gamma$ . According to Theorem 2.1, if  $\rho$  is a sufficiently small deformation of the inclusion  $\Gamma \subset G$ , then there exists a new  $(G, X)$ -structure  $M'$  on the manifold  $M$ , whose holonomy is  $\rho: \Gamma \rightarrow G$ . But as  $M$  is compact and  $X$  is Riemannian, this new  $(G, X)$ -structure is complete and, by Theorem 2.3, the developing map is a  $\Gamma$ -equivariant isomorphism between its universal cover and  $X$ . This implies that  $\rho(\Gamma)$  acts properly discontinuously on  $X$  and that  $M' = \rho(\Gamma) \backslash X$ . As  $M'$  is homeomorphic to  $M$  this concludes the proof.  $\square$

We shall conclude this section by stating the following natural generalization of Weil's theorem, for which we could not find an available reference in the literature:

**PROPOSITION 2.5.** *Let  $G$  be a connected Lie group  $\Gamma \leq G$  a uniform lattice. Then for any sufficiently small deformation  $\rho$  of  $\Gamma$  in  $G$ ,  $\rho(\Gamma)$  is again a uniform lattice and  $\rho: \Gamma \rightarrow \rho(\Gamma)$  is an isomorphism.*

*Proof.* Assume first that  $G$  is simply connected. Then we can choose a left invariant Riemannian metric  $m$  on  $G$ , and argue verbatim as in the proof Corollary 2.4 letting  $(G, m)$  to stand for  $X$ , and  $G$  to act isometrically by left multiplication.

Now consider the general case. Let  $\tilde{G}$  be the universal covering of  $G$ . Let  $\Gamma \leq G$  be a uniform lattice and let  $\tilde{\Gamma}$  be its pre-image in  $\tilde{G}$ . Then  $\tilde{\Gamma}$  is a uniform lattice in  $\tilde{G}$ , and  $G/\Gamma$  is homeomorphic to  $\tilde{G}/\tilde{\Gamma}$ . Since  $\Gamma$  is finitely generated, the deformation space  $R(\Gamma, G)$  has the structure of a manifold and in particular it is locally arcwise connected. Therefore if  $\rho$  is sufficiently small deformation of  $\Gamma$  in  $G$  then there is a curve of deformations  $\rho_t \in R(\Gamma, G)$  with  $\rho_0 =$  the identity, and  $\rho_1 = \rho$ . Now for any  $\gamma \in \Gamma$  and any  $\tilde{\gamma} \in \tilde{\Gamma}$  above  $\gamma$ , the curve  $\rho_t(\gamma)$  lifts uniquely to a curve  $\tilde{\rho}_t(\tilde{\gamma})$ . The uniqueness guarantees that  $\tilde{\rho}_t$  is again a homomorphism for any  $t$ . Hence  $\rho_t$  lifts to a curve  $\tilde{\rho}_t \in R(\tilde{\Gamma}, \tilde{G})$ . In particular  $\rho = \rho_1$  lifts to a small deformation  $\tilde{\rho} = \tilde{\rho}_1$  of  $\tilde{\Gamma}$  in  $\tilde{G}$ . It is also easy to see that the lifting  $\rho \mapsto \tilde{\rho}$  on a neighborhood of the inclusion  $\rho_0$  is continuous as a map from an open set in  $R(\Gamma, G)$  to  $R(\tilde{\Gamma}, \tilde{G})$ . Thus, the general case follows from the simply connected one.  $\square$

The proof of Theorem 1.1 will follow the same lines, we just need to understand the noncompact parts, namely the ends.

### 3. The Rank One Case

Let  $G$  be a connected center free simple Lie group of rank one, with associated symmetric space  $X = G/K$ . Let  $\Gamma$  be a nonuniform lattice in  $G$ . We are going to investigate the possible small deformations of  $\Gamma$ , and to prove local rigidity when  $n = \dim(X) \geq 4$ , and some weak version of it for dimensions 2 and 3.

By Selberg's lemma  $\Gamma$  is almost torsion free. Note that a small deformation which stabilizes a finite index subgroup must be trivial. In the sequel we shall assume that  $\Gamma$  is torsion free.

We keep the notation  $\rho_0: \Gamma \rightarrow G$  for the inclusion, and  $\rho$  for a small deformation of  $\rho_0$ . We denote by  $M = \Gamma \backslash X$  the corresponding locally symmetric manifold. We shall show that under some conditions on  $\rho$ ,  $\rho(\Gamma)$  is again a lattice in  $G$  and  $M' = \rho(\Gamma) \backslash X$  is homeomorphic to  $M$ , and that these conditions are fulfilled when  $\dim X \geq 4$ .

Our strategy is to use Ehresmann–Thurston's principle. As  $M$  is not compact, we shall decompose it to a compact part  $M_0$  which 'exhausts most of  $M$ ' and to cusps. More precisely, let  $M_0 \subset M$  be a fixed compact submanifold with boundary, which is obtained by cutting all the cusps of  $M$  along horospheres. Each such cusp is contained in a connected component of the thin part of the thick-thin decomposition of  $M$  corresponding to the constant  $\epsilon_n$  of the Margulis lemma (see [19] or [1]), and is homeomorphic to  $T \times \mathbb{R}^{\geq 0}$  for some  $(n-1)$ -dimensional compact manifold  $T$ . We shall call such cusps canonical. More precisely, a *canonical cusp* is a quotient of a horoball by a discrete group of parabolic isometries which preserves the horoball and acts freely and cocompactly on its boundary horosphere.

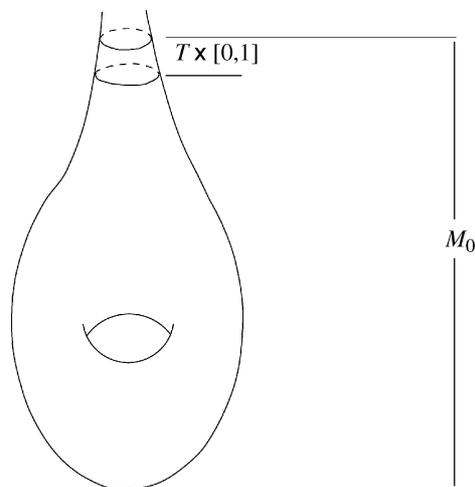


Figure 1. The manifold  $M_0$ .

Notice that a canonical cusp has finite volume. To see this, one can look at the Iwasawa decomposition  $G = KAN$  which corresponds to a lifting of the given cusp, express the Haar measure of  $G$  in terms of the Haar measures of  $K$ ,  $A$  and  $N$ , and estimate the volume of the cusp in the same way as one shows that a Siegel set has a finite volume.

In the argument below we shall assume, for the sake of clearness, that  $M$  has only one cusp, instead of finitely many. It is easy to see that the proof works equally well in the general case. We choose another horosphere which is contained in  $M_0$  and parallel to the boundary horosphere, and denote the collar between them by  $T \times [0, 1]$  (see Figure 1).

By the Ehresmann–Thurston principle, there is a  $(G, X)$ -structure  $M'_0$  on  $M_0$  whose holonomy is  $\rho$ .

We will show that under some conditions, the  $(G, X)$ -structure on  $T \times [0, 1]$  which is induced from  $M'_0$  coincides with a  $(G, X)$ -structure which is induced from some canonical cusp  $C$  (whose holonomy coincides with  $\rho(\pi_1(T))$ ) to a subset of it, which is homeomorphic to  $T \times [0, 1]$ . In such case we can glue  $M_0$  with  $C$  along  $T \times [0, 1]$  and obtain a complete  $(G, X)$ -manifold  $M'$  homeomorphic to  $M$  whose fundamental group is  $\rho(\Gamma)$ . Additionally,  $M'$  has a finite volume, being a union of the compact set  $M'_0$  with  $C$  which is a canonical cusp and hence has a finite volume.

Recall the following basic facts about isometries of rank one symmetric spaces:

- Maximal unipotent groups correspond to points in the ideal boundary  $X(\infty)$ . A maximal unipotent group acts simply transitively on each horosphere around its fixed point at  $\infty$ , its dimension is  $\dim(X) - 1$  and it is the unipotent radical of the parabolic group which is the stabilizer of this point. The maximal unipotent groups are metabelian.

- There are only three types of isometries: elliptic, hyperbolic and parabolic. An elliptic isometry fixes a point in  $X$  and is contained in some compact subgroup of  $G$ , it is semisimple and all its eigenvalues have absolute value 1.

A hyperbolic isometry  $\gamma$  has exactly two fixed points at the ideal boundary  $X(\infty)$  and it acts by translations on the geodesic connecting them, called the axis of  $\gamma$ , which is also characterized as the set of points where the displacement function  $d(x, \gamma \cdot x)$  attains its minimum. A hyperbolic element is semisimple and has an eigenvalue of absolute value  $> 1$ .

A parabolic element is one which has no fixed points in  $X$  and has a unique fixed point in the ideal boundary  $X(\infty)$ . If  $\gamma$  is parabolic then  $\inf\{d(x, \gamma \cdot x) : x \in X\} = 0$  and the infimum is not attained. An element is parabolic iff it is not semisimple. In particular, unipotents are parabolics. In general, all the eigenvalues of a parabolic element have absolute value 1.

- The quotient map  $G \rightarrow G/P = X(\infty)$  is open, and the action map  $G \times X(\infty) \rightarrow X(\infty)$  is continuous.

Let  $\Gamma_T = \pi_1(T \times [0, \infty)) \leq \Gamma$  be a subgroup of  $\Gamma$  which corresponds to the fundamental group of the cusp of  $M$ . Then  $\gamma_T$  is group of parabolic elements, and by [16] Corollary 8.25, it contains a subgroup  $\Gamma_T^0$  of finite index which is unipotent, i.e. a lattice in the unipotent radical of the corresponding parabolic group. We want to show that  $\rho(\Gamma_T)$  is the fundamental group of some canonical cusp. So, in particular, we need to show that  $\rho(\Gamma_T)$  is again a group of parabolic elements. This fact is true only when  $n = \dim X \geq 4$  (but the rest of the argument applies also for dimensions 2 and 3). We shall prove it by showing that otherwise,  $\rho(\Gamma_T)$  is contained in the stabilizer group of some geodesic or some point, and then derive a contradiction since  $\rho(\Gamma_T)$  is ‘too large’ and there is ‘no room’ for it in such a subgroup. We shall divide this argument into few claims.

LEMMA 3.1. *There exists an integer  $i = i(n)$  such that any nilpotent subgroup  $A$  of  $\mathbb{R} \times \mathrm{SO}_{n-1}(\mathbb{R})$  contains a subgroup  $B \leq A$  such that*

- $|A/B| \leq i$ ,
- $B$  is contained in a connected Abelian group.

*Proof.* Dividing  $\mathbb{R} \times \mathrm{SO}_{n-1}(\mathbb{R})$  by the factor  $\mathbb{R}$ , we can prove the analogous statement for  $\mathrm{SO}_{n-1}(\mathbb{R})$ . Let  $\Omega \subset \mathrm{SO}_{n-1}(\mathbb{R})$  be a small symmetric identity neighborhood with the following property: a subset  $\Sigma \subset \Omega^2$  generates a nilpotent group if and only if  $\log \Sigma \subset \mathfrak{so}_{n-1}(\mathbb{R}) = \mathrm{Lie}(\mathrm{SO}_{n-1}(\mathbb{R}))$  generates a nilpotent Lie sub-algebra (cf. [16], lemma 8.20).

Let  $\mu$  be a Haar measure on  $\mathrm{SO}_{n-1}(\mathbb{R})$ , and take

$$i = \frac{\mu(\mathrm{SO}_{n-1}(\mathbb{R}))}{\mu(\Omega)}.$$

Let now  $A$  be a nilpotent subgroup of  $\mathrm{SO}_{n-1}(\mathbb{R})$ . Take  $B = \langle A \cap \Omega^2 \rangle$ . Then  $B$  is contained in the connected nilpotent group which corresponds to the nilpotent Lie

subalgebra generated by  $\log A \cap \Omega^2$ . The closure of this connected nilpotent group is compact, connected and nilpotent, hence it is Abelian, and in particular so is  $B$ .

Now let  $F = \{a_j\}$  be a set of coset representatives for  $A/B$ . The sets  $a_j \cdot \Omega$  must be disjoint, for otherwise we will have  $a_k^{-1}a_j \in \Omega^2 \cap A \subset B$  for some  $i \neq j$ . Therefore

$$|F| \leq \frac{\mu(\mathrm{SO}_{n-1}(\mathbb{R}))}{\mu(\Omega)} = i. \quad \square$$

*Remark 3.2.* The proof of Lemma 3.1 shows that the analogous statement holds for the group  $\mathrm{SO}_n(\mathbb{R})$  as well. Below, we shall assume that the index  $i$  is good for both groups  $\mathbb{R} \times \mathrm{SO}_{n-1}(\mathbb{R})$  and  $\mathrm{SO}_n(\mathbb{R})$ .

The maximal dimension of a connected abelian subgroup of  $\mathrm{SO}_{n-1}(\mathbb{R})$  is  $[(n-1)/2]$  – the absolute rank of  $\mathrm{SO}_{n-1}$ , and hence the maximal dimension of a connected Abelian subgroup of  $\mathbb{R} \times \mathrm{SO}_{n-1}(\mathbb{R})$  is  $[(n-1)/2] + 1$ . Moreover, the stabilizer group of a geodesic  $c \subset X$  embeds naturally in  $\mathbb{R} \times \mathrm{SO}_{n-1}(\mathbb{R})$ . This implies:

**PROPOSITION 3.3.** *Assume  $n \geq 4$ . Let  $B \leq G$  be a connected Abelian subgroup. If  $B$  is not unipotent, then  $\dim B < n - 1$ .*

*Proof.* By Lie's theorem,  $B$  is triangulable over  $\mathbb{C}$ . Thus, since  $B$  is not a unipotent group, there is a semisimple element  $a \in B$ . The isometry  $a$  is either elliptic or hyperbolic. If  $a$  is hyperbolic, then as  $B$  is Abelian, any  $b \in B$  preserves the axis of  $a$ . By the above paragraph we get

$$\dim B \leq \left\lceil \frac{n-1}{2} \right\rceil + 1 < n - 1$$

as  $n \geq 4$ .

If  $a$  is elliptic then its set of fixed points  $\mathrm{Fix}(a)$  is a totally geodesic sub-symmetric space of dimension  $d = \dim(a)$  which is  $B$ -invariant. Hence  $B$  embeds in  $\mathrm{SO}_{n-d} \times \mathrm{Isom}(\mathrm{Fix}(a))$  and the result follows by an inductive argument on the dimension, keeping in mind that any connected unipotent subgroup of  $\mathrm{Fix}(a)$  has dimension  $\leq d - 1$ .  $\square$

Let  $p \in X(\infty)$  be the fixed point of  $\Gamma_T$ , let  $P \leq G$  be the parabolic group which corresponds to  $p$  and let  $P = MAN$  be the Langland's decomposition of  $P$ . Then  $N$  is the maximal unipotent group which corresponds to  $p$ , and  $MN$  can be characterized as the subgroup of  $G$  which preserve the horospheres around  $p$ , and we have  $\Gamma_T = \Gamma \cap MN$ , and  $\Gamma_T^0 = \Gamma \cap N$ . Additionally,  $\Gamma_T$  (resp.  $\Gamma_T^0$ ) is a cocompact lattice in  $MN$  (resp. in  $N$ ) because  $\Gamma_T \backslash N \cdot x$  is homeomorphic to  $T$  (where  $x \in X$  is an arbitrary point and  $N \cdot x$  the corresponding horosphere). In particular any subgroup of finite index  $B \leq \Gamma_T^0$  is Zariski-dense in the unipotent group  $N$ , and thus, such subgroup  $B$  is not contained in a connected proper subgroup of  $N$ . As  $N$  acts simply

transitively on each horosphere, its dimension is  $n - 1$ . So we see that  $\text{span } \log(B)$  is  $n - 1$ -dimensional and conclude:

**CLAIM 3.4.** Assume  $n \geq 4$ . No subgroup of finite index of  $\Gamma_T$  is contained in a connected Abelian subgroup of  $G$  of dimension  $< n - 1 = \dim(N)$ .

The following claim is obvious.

**CLAIM 3.5.** For a finitely generated group  $\Delta$ , and a convergent sequence of homomorphisms  $f_n: \Delta \rightarrow G$ , the property: ' $f_n(\Delta)$  is contained in a connected Abelian group of dimension  $\leq m$ ' is preserved by taking the limit.

We can now prove the following main proposition:

**PROPOSITION 3.6.** *Assume  $n \geq 4$ . If  $\rho$  is a sufficiently small deformation of  $\rho_0$ , then the group  $\rho(\Gamma_T)$  is contained in a parabolic group. Moreover,  $\rho(\Gamma_T)$  contains no hyperbolic elements.*

*Proof.* Let  $z_1 \in \Gamma_T^0$  be a central element. If  $\rho(z_1)$  is parabolic then  $\rho(\Gamma_T^0)$  is contained in the parabolic group corresponding to the unique fixed point of  $\rho(z_1)$  at  $X(\infty)$ .

Assume that  $\rho(z_1)$  is not parabolic. Then it is either elliptic or hyperbolic. But if  $\rho(z_1)$  is hyperbolic, then, as it is central,  $\rho(\Gamma_T^0)$  is contained in the stabilizer group of the axis of  $\rho(z_1)$ . However, by Lemma 3.1 we obtain that  $\rho(\Gamma_T^0)$  contains a subgroup of index  $\leq i$  which is contained in a connected Abelian group, which by Proposition 3.3 has dimension  $< n - 1$ . Since  $\Gamma_T^0$  is finitely generated (being a lattice in  $N$ ) it has only finitely many subgroups of index  $\leq i$ . Therefore the property 'to have a subgroup of index  $\leq i$  which is contained in a connected Abelian group of dimension  $\leq n - 2$ ' is preserved by taking a limit. This, however, contradicts Claim 3.4.

Suppose therefore that  $\rho(z_1)$  is elliptic. Then its set of fixed points is a subsymmetric space  $X_1$  of smaller dimension, on which  $\rho(\Gamma_T^0)$  acts nilpotently, and we continue by taking  $z_2 \in \Gamma_T^0$  which is central with respect to this action. If  $\rho(z_2)$  is parabolic we conclude that  $\rho(\Gamma_T^0)$  is contained in the parabolic group corresponding to the unique fixed point of  $\rho(z_2)$  at  $X_1(\infty) \subset X(\infty)$ . If  $\rho(z_2)$  is not parabolic, then it cannot act parabolically on  $X_1$ . As above  $\rho(z_2)$  can not act hyperbolically. If  $z_2$  acts elliptically, we continue by induction, defining  $X_2 = \text{Fix}(\rho(z_1)) \cap \text{Fix}(\rho(z_2))$ . By this way we conclude that  $\rho(\Gamma_T^0)$  is contained in a parabolic group.

Now if all the  $z_i$ 's constructed above act elliptically, then  $\rho(\Gamma_T^0)$  is contained in a compact group, and (letting  $\rho \rightarrow \rho_0$ ) we derive a contradiction to Remark 3.2, Proposition 3.3 and Claim 3.4 as above. Therefore, one of the  $z_i$ 's, say  $z_{i_0}$ , must act as a parabolic. This implies that  $\rho(\Gamma_T^0)$  has a unique fixed point at infinity, and since  $\Gamma_T^0$  is normal in  $\Gamma_T$  it follows that  $\rho(\Gamma_T)$  fix this point and hence contained in the corresponding parabolic group.

Now if  $\rho(\Gamma_T)$  would contain a hyperbolic element then so would  $\rho(\Gamma_T^0)$  since a power of hyperbolic is hyperbolic. However, hyperbolic element cannot commute with parabolic, while the action of any element on the corresponding subsymmetric space  $X_{i_0-1}$  commute with the action of the parabolic element  $\rho(z_{i_0})$ . Therefore  $\rho(\Gamma_T)$  does not contain hyperbolic elements.  $\square$

Since the action of  $G$  at  $X(\infty)$  is continuous, the unique fixed point  $x$  of  $\rho(\Gamma_T)$  is close to the fixed point  $x_0$  of  $\Gamma_T$ , and since the orbit map  $G \rightarrow G/P = X(\infty)$  is open, there is some  $g \in G$  near 1 which takes  $x_0$  to  $x$ . Conjugating  $\rho$  by  $g$  we get  $\rho^g$  which is again a small deformation. Moreover,  $\rho^g|_{\Gamma_T}$  is a deformation inside the parabolic subgroup  $P$  which corresponds to the cusp.

CLAIM 3.7. The group  $\rho^g(\Gamma_T)$  is contained in  $MN$ .

*Proof.* This follows immediately from the previous proposition since  $MN$  is exactly the set of elements in  $P$  which are not hyperbolic.

We shall now use Ehresmann–Thurston’s principle (Proposition 2.5) to deduce:

PROPOSITION 3.8. *The group  $\rho^g(\Gamma_T)$  is a cocompact lattice in  $MN$ , and  $\rho^g: \Gamma_T \rightarrow \rho^g(\Gamma_T)$  is an isomorphism.*

Since  $\rho^g(\Gamma_T)$  (and  $\rho(\Gamma_T)$ ) is discrete and torsion free, it acts properly and freely on each horosphere  $N \cdot x$ , and hence  $\rho^g(\Gamma_T) \backslash N \cdot x$  (and, hence, also  $\rho(\Gamma_T) \backslash N^{g^{-1}} \cdot x$ ) is homeomorphic to  $\Gamma_T \backslash N \cdot x$  – the boundary of our cusp. This implies that  $\rho(\Gamma_T)$  is the fundamental group of some canonical cusp homeomorphic to  $T \times [0, \infty)$ . This completes the proof of the following theorem and thus completes the rank one case:

THEOREM 3.9. *Let  $G$  be a connected center free simple Lie group of rank one neither isomorphic to  $\mathrm{PSL}_2(\mathbb{R})$  nor to  $\mathrm{PSL}_2(\mathbb{C})$ , and let  $\Gamma \leq G$  be a nonuniform lattice. Then there is a neighborhood  $\Omega$  of the inclusion  $\rho_0: \Gamma \rightarrow G$  in the deformation space  $\mathcal{R}(\Gamma, G)$  such that for any  $\rho \in \Omega$ ,  $\rho(\Gamma)$  is a lattice in  $G$ ,  $\rho: \Gamma \rightarrow \rho(\Gamma)$  is an isomorphism, and  $X/\rho(\Gamma)$  is homeomorphic to  $X/\Gamma$ .*

From Theorem 3.9 and Mostow rigidity, we conclude local rigidity in the rank one case.

It is well known that if  $\Gamma$  is a locally rigid lattice in the group of real points  $G = \mathbb{G}(\mathbb{R})$  of some  $\mathbb{Q}$ -algebraic group  $\mathbb{G}(\mathbb{C}) \leq \mathrm{GL}_n(\mathbb{C})$ , then it can be conjugated by an element  $g \in G$  into  $\mathbb{G}(K)$  for some number field  $K$ . This follows from the fact that  $\Gamma$  is finitely generated and, hence, the deformation space  $\mathcal{R}(\Gamma, G)$  has the structure of a real algebraic variety  $V$  which is defined over  $\mathbb{Q}$ . If we let  $\overline{\mathbb{Q}}$  denote the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ , then this implies that  $V(\overline{\mathbb{Q}} \cap \mathbb{R})$  is dense in  $V(\mathbb{R})$  in the real topology. Thus we can find a point  $\rho \in V(\overline{\mathbb{Q}} \cap \mathbb{R})$  arbitrarily close to

the inclusion  $\rho_0$ . As  $\Gamma$  is locally rigid,  $\rho$  is given by conjugation, and as  $\Gamma$  is finitely generated,  $\rho(\Gamma)$  lies in some algebraic number field.

Let now  $G \cong \mathrm{SL}_2(\mathbb{C})$ , and let  $\Gamma \leq G$  be a (torsion free) non-uniform lattice. Then  $\Gamma$  is not locally rigid. For this reason Garland and Raghunathan had to deal with this case separately in [7], Section 8. Using the above presented proof, one can treat the  $\mathrm{SL}_2(\mathbb{C})$  case in the same manner as the general case, since, as explained above, if  $\rho$  is a small deformation of  $\Gamma$  which sends unipotents to unipotents, then  $\rho(\Gamma)$  is a lattice and  $\rho: \Gamma \rightarrow \rho(\Gamma)$  is an isomorphism, and hence by Mostow rigidity, it is given by conjugation. In fact, the fundamental group of each cusp is finitely generated, and it is enough to choose such finite generating set for each cusp, and to require that  $\rho$  sends all these finitely many elements to unipotents. Then any unipotent in  $\Gamma$ , being conjugate to some element of the chosen fundamental group of some cusp, must also be sent to a unipotent.

We can choose a large finite generating set  $\Sigma$  for  $\Gamma$  which contains a generating set for a chosen fundamental group for each of the finitely many cusps. Then, we add to the set of relations defining  $\mathcal{R}(\Gamma, G)$  the conditions that every unipotent in  $\Sigma$  must be sent to unipotent, and call this subspace  $\mathcal{R}_U(\Gamma, G) \subset \mathcal{R}(\Gamma, G)$ . Since, the unipotents are the zeros of the  $\mathbb{Q}$ -polynomial  $(\mathrm{Ad}(g) - 1)^n$  we see that the space  $\mathcal{R}_U(\Gamma, G)$  has the structure of a real algebraic variety defined over  $\mathbb{Q}$ . We conclude that:

**COROLLARY 3.10** (Garland–Raghunathan). *Let  $\mathbb{G}$  be a  $\mathbb{Q}$ -algebraic linear group with  $G = \mathbb{G}(\mathbb{R})$  isomorphic to  $\mathrm{PSL}_2(\mathbb{C})$ , and let  $\Gamma$  be a nonuniform lattice in  $G$ . Then  $\Gamma$  is conjugate to a subgroup of  $\mathbb{G}(K)$  for some number field  $K \leq \mathbb{R}$ .*

In the next section, we shall proceed by induction on rank ( $G$ ). The rank one case, thus, would be the base step in this induction argument. We shall need the following weak version of local rigidity, which holds in the general case (also for  $G = \mathrm{SL}_2(\mathbb{R})$ ), and is proved by the same argument as above.

**LEMMA 3.11.** *Let  $G$  be a connected rank one simple Lie group with associated symmetric space  $X$ , and let  $\Gamma \leq G$  be a torsion-free lattice. Let  $\rho$  be a small deformation of  $\Gamma$  which takes all the unipotents in  $\Gamma$  to unipotents. Then  $\rho(\Gamma)$  is a lattice,  $\rho(\Gamma) \backslash X$  is homeomorphic to  $\Gamma \backslash X$ , and  $\rho: \Gamma \rightarrow \rho(\Gamma)$  is an isomorphism.*

#### 4. The Higher Rank Case

We shall now consider the case in which  $G$  is a connected center-free semisimple Lie group of rank  $\geq 2$  without compact factors, with associated symmetric space  $X = G/K$ , and  $\Gamma \leq G$  a nonuniform (torsion-free) irreducible lattice, corresponding to the locally symmetric manifold  $M = \Gamma \backslash X$ .

As in the rank one case, we are going to apply Ehresmann–Thurston’s principle. So we need to work with a compact submanifold with boundary  $M_0 \subset M$  which exhausts most of  $M$ . We want to obtain  $M_0$  from  $M$  by cutting out its ends, and

to have enough information on the structure of the ends, so that we can argue as we did in the rank one case. Of course, it would be nicer to give a simple proof of local rigidity for general lattices, without assuming arithmeticity. However, we shall use the description of Leuzinger [9] for the structure of  $M \setminus M_0$ , and hence assume, as in [9], that  $\Gamma$  is arithmetic. We shall prove the following result, which originally is a consequence of Margulis super-rigidity theorem.

**THEOREM 4.1.** *Let  $\Gamma \leq G$  be a non-uniform arithmetic irreducible lattice, and let  $\rho$  be a small deformation of the inclusion  $\rho_0: \Gamma \rightarrow G$ . Then  $\rho(\Gamma)$  is a lattice,  $\rho(\Gamma) \backslash X$  is homeomorphic to  $M$ , and  $\rho: \Gamma \rightarrow \rho(\Gamma)$  is an isomorphism.*

The case of  $\mathbb{Q}$ -rank( $\Gamma$ ) = 1 is very similar to the  $\mathbb{R}$ -rank one case. However, the picture is much more sophisticated when  $\mathbb{Q}$ -rank( $\Gamma$ )  $\geq 2$ . Then  $M$  has only one end, i.e. if  $M_0$  is defined appropriately, then  $M \setminus M_0$  is connected. Moreover, ‘from infinity’  $M$  ‘looks like’ a simplicial complex of dimension  $\mathbb{Q}$ -rank( $\Gamma$ ). Our argument will use an induction on the  $\mathbb{R}$ -rank.

Let  $B$  be a parabolic subgroup of  $G$  corresponding to a cusp of  $\Gamma$ , and  $B = MAN$  be its Langland’s decomposition over  $\mathbb{Q}$ . Let  $\Gamma_0 := \Gamma \cap B$ ,  $\Lambda = \Gamma \cap N$ , and  $C$  be the center of  $\Lambda$ . By arithmeticity,  $\Lambda$  is a lattice in the unipotent group  $N$ , and in particular it is finitely generated.

As in the rank one case, a crucial point in the proof is the following proposition:

**PROPOSITION 4.2.** *The group  $\rho(\Lambda)$  is unipotent.*

In order to prove 4.2 we shall use the notion of  $U$ -elements from [8] and [10].

**DEFINITION 4.3.** Let  $\Delta$  be a finitely generated group. An element  $\gamma \in \Delta$  is called a  $U$ -element if the norm of  $\gamma^n$ , with respect to the word metric on  $\Delta$  corresponding to a fixed finite generating set, grows like  $\log(n)$ .

*Remark 4.4.* The notion of  $U$ -elements is well defined and independent of the choice of the finite generating set.

**LEMMA 4.5.** *If  $\Delta$  is a linear group and  $\gamma \in \Delta$  is a  $U$ -element, then all the eigenvalues of  $\gamma$  are roots of unity.*

*Proof.* If  $\gamma \in \Delta$  has an eigenvalue which is not a root of unity, then, as shown in the proof of Tits’ alternative, [20] (or more simply using Kronecker’s lemma), there is a local field  $k$  and a representation  $r: \Delta \rightarrow \mathrm{GL}_m(k)$ , such that  $r(\gamma)$  has an eigenvalue with absolute value  $> 1$ . This implies that the norm of  $r(\gamma)^n$  in  $\mathrm{GL}_m(k)$  grows exponentially. If  $\Sigma$  is a finite generating set, then the norms of the elements in  $r(\Sigma)$  are uniformly bounded. As the norm is a sub-multiplicative function, this implies that the length of  $\gamma^n$  in the word metric grows linearly with  $n$ .  $\square$

**THEOREM 4.6** (Lubotzky–Mozes–Raghunathan [10]). *If  $\Delta \leq G$  is an irreducible lattice in a connected higher rank semisimple Lie group  $G$ , then any unipotent element  $\gamma \in \Delta$  is a  $U$ -element.*

Returning to our case, we conclude that:

**COROLLARY 4.7.** *If  $\gamma \in \Gamma$  is unipotent, and  $\rho$  is sufficiently small, then  $\rho(\gamma)$  is unipotent.*

*Proof.* We know that  $\gamma$  is a  $U$ -element in  $\Gamma$  by Theorem 4.6 and, hence,  $\rho(\gamma)$  is a  $U$ -element in  $\rho(\Gamma)$ . By Lemma 4.5 all the eigenvalues of  $\rho(\gamma)$  are roots of unity.

Recall that  $\mathcal{R}(\Gamma, G)$  is locally arcwise connected, i.e. if  $\rho$  is close enough to  $\rho_0$  then they are connected by an arc of deformations  $\rho_t$ . Since the eigenvalues depend continuously on the matrix, and since the set of roots of unity is totally disconnected, we conclude that for any  $t$ , all the eigenvalues of  $\rho_t(\gamma)$  must be 1. Hence  $\rho_t(\gamma)$ , and in particular  $\rho(\gamma)$  is unipotent.  $\square$

**COROLLARY 4.8.** *Given a finite generating set  $\Sigma$  for  $\Lambda$ , there is a neighborhood  $\Omega$  of the inclusion  $\rho_0: \Gamma \rightarrow G$ , such that if  $\rho \in \Omega$  and  $\gamma \in \Sigma$ , then  $\rho(\gamma)$  is unipotent.*

*Proof of Proposition 4.2.* Fix a generating set  $\Sigma$  for  $\Lambda$  and assume that  $\rho \in \Omega$  as in Corollary 4.8. As  $\rho(\Lambda)$  is nilpotent, its Zariski closure is nilpotent, and hence, by Lie's theorem, its identity component  $A = \overline{(\rho(\Lambda))^Z}^0$  is triangulable over  $\mathbb{C}$ .

Assume that  $\overline{(\rho(\Lambda))^Z}$  has  $i$  connected components. Let  $\rho(\Lambda)^0 = \langle \rho(\gamma)^n : \gamma \in \Sigma \rangle$ . Being a subgroup of  $A$ ,  $\rho(\Lambda)^0$  is triangulable over  $\mathbb{C}$ . Since  $\rho(\Sigma)$  is made of unipotents,  $\rho(\Lambda)^0$  is generated by unipotents, and hence it is a unipotent group. It follows that  $\overline{\rho(\Lambda)^0}^Z$  is a unipotent algebraic group, and as such it is also connected.

Let  $\gamma$  be an element of  $\Sigma$ . Since  $\rho(\gamma)$  is unipotent  $\overline{\langle \rho(\gamma) \rangle}^Z = \overline{\langle \rho(\gamma)^n \rangle}^Z$ . This implies that  $\rho(\gamma)$  is actually contained in  $\overline{\rho(\Lambda)^0}^Z$ , and the proposition follows.  $\square$

*Remark 4.9.* It might be possible to avoid the notion of  $U$ -elements and the use of Theorem 4.6, and to find a more elementary proof for Proposition 4.2. This is not so hard in the special case where  $G = \mathrm{SL}_3(\mathbb{R})$  and  $\Gamma \leq \mathrm{SL}_3(\mathbb{R})$  is any lattice. Moreover, for the special case of  $G = \mathrm{SL}_n(\mathbb{R})$  and  $\Gamma = \mathrm{SL}_n(\mathbb{Z})$  Theorem 4.6 also has an elementary proof (see [11]).

**CLAIM 4.10.** The group  $\rho(\Gamma_0)$  fixes a point at infinity (i.e. in  $X(\infty)$ ).

*Proof.* According to [1], Appendix 3, since  $\rho(\Lambda)$  is unipotent, the set of common fixed points at  $X(\infty)$

$$\mathcal{B} = \bigcap_{\gamma \in \Lambda} \mathrm{Fix}(\rho(\gamma))$$

is non-empty, and so is the set

$$\mathcal{B}_0 = \left\{ z \in \mathcal{B} : \mathrm{Td}(z, y) \leq \frac{\pi}{2}, \text{ for any } y \in \mathcal{B} \right\},$$

(here  $Td$  denotes the Tits distance on  $X(\infty)$ ) which clearly has diameter  $\leq \pi/2$  and thus has a unique Chebyshev center  $O$ . Since  $\Gamma_0$  normalizes  $\Lambda$ , the point  $O$  must be invariant under the action of  $\rho(\Gamma_0)$ . Therefore,  $\rho(\Gamma_0)$  is contained in the parabolic subgroup which corresponds to  $O$ .  $\square$

Let  $P$  be a parabolic subgroup which contains  $\rho(\Gamma_0)$ , and let  $P = M_P \Lambda_P N_P$  be a Langland's decomposition of  $P$ . Note that the group  $N_P$  is well defined, the group  $M_P$  is determined up to conjugation by elements of  $N_P$ , and the group  $M_P N_P$  is well defined. The group  $M_P N_P$  can be characterized as the set of all elements in  $P$  which acts unimodularly on the Lie algebra  $\text{Lie}(N_P)$  through the adjoint action, or as the set of all elements in  $P$  which preserve the Busemann functions and their level sets – the horospheres around the corresponding fixed points at  $X(\infty)$ . In fact we can choose  $P$  so that

- (1)  $\rho(\Lambda) \subset N_P$ , and
- (2)  $N_P$  is the Zariski closure  $N_P = \overline{\rho(\Lambda)}^Z$ .

We need the following analogue of Claim 3.5:

CLAIM 4.11. For a finitely generated group  $\Delta$ , and a convergent sequence of homomorphisms  $f_n: \Delta \rightarrow G$ , the property ' $f_n(\Delta)$  is contained in a connected unipotent group of dimension  $\leq m$ ' is preserved by taking a limit.

This claim can be easily proved using the fact that for connected unipotent groups the exponential map is a diffeomorphism.

It follows that if  $\rho$  is sufficiently small:

- (1)  $\dim(N_P) \geq \dim(N)$ ,
- (2)  $P$  must be conjugate to a parabolic which contains  $B$ .

The second statement follows from the fact that  $\Gamma_0$  is an arithmetic subgroup and hence a lattice in  $MN$  and in particular its Zariski closure contains  $M'N$ , where  $M'$  is the product of all noncompact factors of  $M$ . Hence, when  $\rho$  is small enough,  $P$  must contain a conjugate of  $M'N$ , but this implies that it contains a conjugate of  $B$ .

Since the inclusion relation is opposite for parabolic subgroups and for their unipotent radicals, these two facts together imply:

COROLLARY 4.12. *If  $\rho$  is sufficiently small,  $P$  is conjugate to  $B$ .*

Since the canonical map  $G \rightarrow G/B$  is continuous and open, we have  $P = g^{-1}Bg$  for some  $g \in G$  close to 1, and by replacing  $\rho$  by its conjugation with  $g$ ,  $\rho^g$  (which is still a small deformation of  $\rho_0$ ), we may assume that  $P = B$ .

We conclude that

CLAIM 4.13. The group  $\rho^g(\Lambda)$  is a (cocompact) lattice in  $N$ .

This is a subsequent of the fact that  $\rho^g$  induces a small deformation of  $\Lambda$  (a cocompact lattice) in  $N$  together with either some classical theorems on lattices of nilpotent Lie groups (Macev theory [12]) or, more simply, Ehresmann–Thurston’s principle.

The next step is:

CLAIM 4.14. The group  $\rho(\Gamma_0)$  is contained in  $M_P N_P$ .

This is because the group  $\rho(\Gamma_0)$  normalizes the cocompact lattice  $\rho(\Lambda)$  of  $N_P$ , and hence  $\text{Ad}(\rho(\Gamma_0))$  acts unimodularly on  $\text{Lie}(N_P)$ .

As noted above  $\Gamma_0$  is a lattice in  $MN$ . Moreover, the projection of  $\Gamma_0$  in  $M$  is a lattice in  $M$ , let’s denote this lattice by  $L$ . The morphism  $\rho^g$ , composed with the projection  $MN \rightarrow M$ , induces a small deformation  $\tilde{\rho}^g(L)$  of  $L$  in  $M$ . Moreover, by Theorem 4.6 and Corollary 4.7,  $\rho^g$  and hence  $\tilde{\rho}^g$  takes unipotents to unipotents. Hence, by an induction procedure on the real rank, starting with rank one, of which we already took care in the previous section, we conclude that:

CLAIM 4.15. The group  $\tilde{\rho}^g(L)$  is a lattice in  $M$ .

Since the Haar measure of  $MN$  is the product of the Haar measures of  $M$  and of  $N$ , and since  $\Lambda$  is a cocompact lattice in  $N$ , we get:

COROLLARY 4.16. *The group  $\rho(\Gamma_0)$  is a lattice in  $MN$ .*

To conclude the proof, we now need, as in the rank one case, to introduce a notion of cusp, but this time it will depend on the group  $\Gamma$ . By [9], Theorem 4.2, on the locally symmetric manifold  $M = \Gamma \backslash X$ , there exists a continuous and piecewise real analytic exhaustion function  $h: M \rightarrow [0, \infty)$  such that, for any  $s \geq 0$ , the sub-level set  $V(s) := \{h \leq s\}$  is a compact submanifold with corners of  $V$ . Moreover, the boundary of  $V(s)$ , which is a level set of  $h$ , consists of projections of subsets of horospheres in  $X$ .

We can choose a sufficiently large  $s_0$  in order to ensure that for any  $s > s_0$ ,  $V(s) - V(s_0)$  is a collar neighborhood of the boundary of  $V(s_0)$  inside  $V(s)$ . In the following we let  $M_0 = V(s)$  for some real  $s > s_0$ .

By the Ehresmann–Thurston principle, there is a  $(G, X)$ -structure  $M'_0$  on  $M_0$  whose holonomy is  $\rho$  and which induces a new  $(G, X)$ -structure on  $V(s) - V(s_0)$ . The fundamental group of  $V(s) - V(s_0)$  is a finite complex of groups  $\mathcal{C}$  where each simplex group is the intersection  $\Gamma \cap M_P N_P$  for some  $\mathbb{Q}$ -parabolic subgroup  $P$  of  $G$  and the inclusion order for simplices is the reversed inclusion order for the corresponding parabolic groups. In particular the vertex groups correspond to the  $\Gamma$ -conjugacy classes of maximal  $\mathbb{Q}$ -parabolics.

The proof of Proposition 3.3 in [9] implies that any finite complex of groups as above leaves invariant a countable union of horospheres of  $X$  (level sets of Busemann functions) centered at the points of the ideal boundary of  $X$  which corresponds to the vertices of the complex. In fact, Leuzinger proves that the developing map of  $M_0$  embeds  $\tilde{M}_0$  in  $X$  as the complement  $X(s)$  of countable union of open horoballs. These horoballs are disjoint if and only if  $\Gamma$  is an arithmetic subgroup of a  $\mathbb{Q}$ -rank one group. The projection  $\pi: X \rightarrow M$  maps  $X(s)$  to  $V(s)$  which is a compact manifold with corners whose fundamental group is isomorphic to  $\Gamma$ . The image of the developing map of  $V(s) - V(s_0)$  with the  $(G, X)$ -structure induced from  $M_0$  is  $X(s) - X(s_0)$ . It is a  $\pi_1(V(s) - V(s_0))$ -invariant subset of  $X$ .

The image  $Y$  of the developing map of  $V(s) - V(s_0)$  with the  $(G, X)$ -structure induced from  $M'_0$  is a subset of  $X$  invariant under  $\rho(\pi_1(V(s) - V(s_0)))$  which converges toward  $X(s) - X(s_0)$  on every compact subset as  $\rho$  tends to  $\rho_0$ . According to the above claims, when  $\rho$  is sufficiently small, the group  $\rho(\pi_1(V(s) - V(s_0)))$  is still a complex of groups isomorphic to  $\mathcal{C}$ , and each simplex of the new complex is associated with a parabolic group  $B'$  of the same type as the parabolic  $B$  associated to the corresponding simplex of  $\mathcal{C}$ . Moreover,  $\rho: (\Gamma \cap B) \rightarrow (\rho(\Gamma) \cap B')$  is an isomorphism, its image  $\rho(\Gamma) \cap B'$  is contained in  $M_{B'}N_{B'}$  and it is a lattice there. The group  $\rho(\pi_1(V(s) - V(s_0)))$  thus leaves invariant a countable family of horospheres as above. More precisely, we get that for  $\rho$  sufficiently close to  $\rho_0$ , the image  $Y$  of the developing map of  $V(s) - V(s_0)$  with the  $(G, X)$ -structure induced from  $M'_0$  contains the subset of  $X$  which is bounded between two families of horospheres as above. By adding the corresponding countably many horoballs,  $Y$  can be completed by a  $\rho(\pi_1(V(s) - V(s_0)))$ -invariant subset of  $X$ , whose boundary is one part of the boundary of  $Y$  with the reversed orientation and whose quotient  $C$  by  $\rho(\pi_1(V(s) - V(s_0)))$  has a finite volume. As the restriction of the  $(G, X)$ -structures from  $M'_0$  and from  $C$  coincide on  $V(s) - V(s_0)$ , these two sets can be glued along  $V(s) - V(s_0)$  in order to yield a complete  $(G, X)$ -manifold  $M'$  whose fundamental group is  $\rho(\Gamma)$ . Note that  $M'$  has finite volume, being a union of a compact set and a finite volume set. The finiteness of the volume of each of the (finitely many) parts composing the ‘cusp’  $M' \setminus M'_0$  can be proved in the same way as one proves the finiteness of the volume of an ordinary Siegel set. Namely, by expressing the Riemannian measure of  $X$  in terms of the Haar measures of  $M_P, N_P$  and  $A_P$ , and using the Fubiny theorem, integrating the function 1 first along the horosphere corresponding to  $P$  in  $V(s_0)$  (here we get a finite number since  $\rho(\Gamma) \cap M_P N_P$  is a lattice in  $M_P N_P$ ), and then in the directions of the end – the simplex of the Tits’ building which corresponds to  $P$ . Note that the Riemannian measure of  $X$  is obtained from the product measure of the corresponding horosphere and the Haar measure of  $A_P$  by multiplying by a factor which tends to zero exponentially fast as one moves toward the end.

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