SEMI-CLASSICAL ESTIMATES FOR NON-SELFADJOINT OPERATORS

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dedicated to Professor Salah Baouendi on his seventieth birthday

Abstract. This is a survey paper on the topic of proving or disproving a priori L^2 estimates for non-selfadjoint operators. Our framework will be limited to the case of scalar semi-classical pseudodifferential operators of principal type. We start with recalling the simple conditions following from the sign of the first bracket of the real and imaginary part of the principal symbol. Then we introduce the geometric condition $(\overline{\psi})$ and show the necessity of that condition for obtaining a weak L^2 estimate. Considering that condition satisfied, we investigate the finite-type case, where one iterated bracket of the real and imaginary part does not vanish, a model of subelliptic operators. The last section is devoted partly to rather recent results, although we begin with a version of the 1973 theorem of R.Beals and C.Fefferman on solvability with loss of one derivative under condition (P); next, we present a 1994 counterexample by N.L. establishing that $(\overline{\psi})$ does not ensure an estimate with loss of one derivative. Finally, we show that condition $(\overline{\psi})$ implies an estimate with loss of 3/2 derivatives, following the recent papers by N.Dencker and N.L. Our goal is to provide a general overview of the subject and of the methods; we do not enter in the details of the proofs, although we provide some key elements of the arguments, in particular in the last section.

1. Presentation of the problem

1.1. A few basic facts on pseudodifferential operators. We recall a few basic facts on pseudodifferential operators, focusing our attention on the semi-classical case.

Definition 1.1.1. Symbol classes. Let $n \ge 1$ be an integer and $m \in \mathbb{R}$. We shall say that a function $a : \mathbb{R}^n \times \mathbb{R}^n \times (0,1] \longrightarrow \mathbb{C}$ is in S^m_{scl} if the functions $(x,\xi) \mapsto a(x,\xi,h)$ are C^∞ for all $h \in (0,1]$ and are such that for all multi-indices α, β,

$$\sup_{\mathbb{R}^n \times \mathbb{R}^n \times (0,1]} |(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi,h)| h^{-|\beta|+m} = \gamma_{\alpha\beta}(a) < \infty. \tag{1.1.1}$$

A typical example of such a symbol is a function $p(x, h\xi)h^{-m}$ where $p \in C_b^\infty(\mathbb{R}^{2n})$ (C^∞ functions bounded as well as all their derivatives). We define $S^{-\infty}_{scl} = \cap_{m\in\mathbb{R}} S^m_{scl}$.

If Ω is some open subset of $\mathbb{R}^{2n}$, the set $S^m_{scl}(\Omega)$ is defined as the set of functions $c : \mathbb{R}^n \times \mathbb{R}^n \times (0,1] \longrightarrow \mathbb{C}$ such that $c \in S^N_{scl}$ for some N and $\sup_{\Omega\times(0,1]} |(\partial_x^\alpha \partial_\xi^\beta c)(x,\xi,h)| h^{-|\beta|+m} < \infty$. Accordingly the set $S^{-\infty}_{scl}(\Omega)$ stands for $\cap_{m\in\mathbb{R}} S^m_{scl}(\Omega)$.

We recall the definition of the Weyl quantization: to $a \in S^m_{scl}$ we associate an operator a^w, bounded on $L^2(\mathbb{R}^n)$ and given by the formula[1]

$$\big(a^w u\big)(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\big(\frac{x+y}{2}, \xi, h\big) u(y) dy d\xi. \tag{1.1.2}$$

The standard quantization formula is

$$(a(x,D,h)u)(x) = \int e^{2i\pi x\cdot\xi} a(x,\xi,h)\hat u(\xi) d\xi, \tag{1.1.3}$$

where the Fourier transform $\hat u$ is given by $\hat u(\xi) = \int e^{-2i\pi x\xi} u(x) dx$. One pleasant fact about the Weyl quantization is that real-valued symbols are quantized in formally selfadjoint operators and more generally that the adjoint of a^w is simply $(\bar a)^w$. The composition formula is $a^w b^w = (a\sharp b)^w$ with

$$(a\sharp b)(\underbrace{X}_{(x,\xi)}) = 2^{2n} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-4i\pi[X-Y,X-Z]} a(Y) b(Z) dY dZ, \tag{1.1.4}$$

where the symplectic form $[,]$ is given for $X = (x,\xi), Y = (y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, by

$$[X,Y] = [(x,\xi),(y,\eta)] = \xi\cdot y - \eta\cdot x. \tag{1.1.5}$$

[1] The formula (1.1.2) does not obviously make sense, and one should introduce the Wigner function $\mathcal{H}(u,v)(x,\xi) = \int e^{-2i\pi z\xi} u(x+\frac{z}{2})\bar v(x-\frac{z}{2}) dz$ which belongs to $\mathcal{S}(\mathbb{R}^{2n})$ for $u,v \in \mathcal{S}(\mathbb{R}^n)$ (and to $L^2(\mathbb{R}^{2n})$ for $u,v \in L^2(\mathbb{R}^n)$) to define

$$\langle a^w u, v\rangle_{L^2(\mathbb{R}^n)} = \langle a, \mathcal{H}(u,v)\rangle_{\mathcal{S}'(\mathbb{R}^{2n}),\mathcal{S}(\mathbb{R}^{2n})}.$$

It is then possible (but not trivial) to prove that if a is a smooth function bounded as well as all its derivatives, then a^w is indeed bounded on $L^2(\mathbb{R}^n)$.

It is convenient to give an asymptotic version of these compositions formulae: one has for $a \in S_{\rm scl}^{m_1}$ and $b \in S_{\rm scl}^{m_2}$, the expansion

$$(a\sharp b)(x,\xi,h) = \sum_{0\le k<N} 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\beta!} D_\xi^\alpha \partial_x^\beta a \ D_\xi^\beta \partial_x^\alpha b + r_N(a,b), \tag{1.1.6}$$

with $r_N(a,b) \in S_{\rm scl}^{m_1+m_2-N}$, $D = \partial/2i\pi$. The beginning of this expansion is thus

$$ab + \frac{1}{4i\pi}\{a,b\},$$

where $\{a,b\} = \sum_{1\le j\le n} \partial_{\xi_j} a \partial_{x_j} b - \partial_{x_j} a \partial_{\xi_j} b$ is the Poisson bracket. The sums inside (1.1.6) with k even are symmetric in a, b and skew-symmetric for k odd. We note in particular that, for a, b as above

$$a\sharp b + b\sharp a \equiv 2ab \quad \text{mod } S_{scl}^{m_1+m_2-2}, \qquad a\sharp b - b\sharp a \equiv \frac{1}{2i\pi}\{a,b\} \quad \text{mod } S_{scl}^{m_1+m_2-3}. \tag{1.1.7}$$

Remark 1.1.2. We note that if Ω is an open set of $\mathbb{R}^{2n}$, $c \in S_{scl}^{m_1}(\Omega)$, $\psi \in S_{scl}^{m_2}$ with $\operatorname{supp}\psi \subset \Omega$, we have $c\sharp\psi$ and $\psi\sharp c \in S_{scl}^{m_1+m_2}$ since c belongs to some S_{scl}^N so that $c\sharp\psi \in S_{scl}^{N+m_2}$ and from (1.1.6), for all nonnegative integers M,

$$c\sharp\psi = \sum_{0\le k<M} \omega_k(c,\psi) + r_M, \quad \omega_k(c,\psi) \in S_{scl}^{m_1+m_2-k}, \quad r_M \in S^{N+m_2-M},$$

so that choosing $M \ge N - m_1$ gives the answer. As a consequence, if $c \in S_{scl}^{-\infty}(\Omega)$, $\psi \in S_{scl}^m$ with $\operatorname{supp}\psi \subset \Omega$, we have $c\sharp\psi$ and $\psi\sharp c \in S_{scl}^{-\infty}$: from the previous argument we know that $c\sharp\psi$ and $\psi\sharp c$ belong to $S_{scl}^{m_1+m}$ for any real m_1, i.e. to $S_{scl}^{-\infty}$.

When p belongs to $C_b^\infty(\mathbb{R}^{2n})$, one may use the formalism developed in [Ro] and introduce the semi-classical Weyl quantization

$$p^{w_h} = p(x,h\xi)^w, \quad \text{so that } (p^{w_h}u)(x) = h^{-n} \iint e^{2i\pi h^{-1}(x-y)\xi} p(\frac{x+y}{2},\xi) u(y) dy. \tag{1.1.8}$$

For $p, q \in C_b^\infty(\mathbb{R}^{2n})$, we note that (1.1.7-8) give quite conveniently

$$p^{w_h} q^{w_h} + q^{w_h} p^{w_h} = 2(pq)^{w_h} + h^2 r^w, \quad r \in S_{scl}^0, \tag{1.1.9}$$

$$[p^{w_h}, q^{w_h}] = \frac{\hbar}{i}\{p,q\}^{w_h} + h^3 s^w, \quad s \in S_{scl}^0, \quad \hbar = h/(2\pi). \tag{1.1.10}$$

Since the mapping $a \ni S_{scl}^0 \mapsto a^w \in \mathcal{L}(L^2(\mathbb{R}^n))$ is continuous, a consequence of the previous formulae is that

$$p^{w_h} q^{w_h} + q^{w_h} p^{w_h} = 2(pq)^{w_h} + O(h^2), \tag{1.1.11}$$

$$[p^{w_h}, q^{w_h}] = \frac{\hbar}{i}\{p,q\}^{w_h} + O(h^3), \tag{1.1.12}$$

where the $O(h^\kappa)$ is to be understood in $\mathcal{L}(L^2(\mathbb{R}^n))$. In the sequel of this paper, we shall stick to the original Weyl formula (1.1.2) without parameter and use the notation (1.1.8) p^{w_h} occasionally as an abbreviation.

We shall also need some asymptotic version of our classes of symbols with the following definition.

Definition 1.1.3. Let $n \geq 1$ be an integer and $m \in \mathbb{R}$. We shall say that a function $a : \mathbb{R}^n \times \mathbb{R}^n \times (0, 1] \longrightarrow \mathbb{C}$ is in S^m_{psc} if there exists a sequence $p_j \in C^\infty_b(\mathbb{R}^{2n})$ such that

$$h^m a(x, \xi, h) \sim \sum_{j\geq 0} h^j p_j(x, h\xi), \quad p_j \in C^\infty_b(\mathbb{R}^{2n}), \tag{1.1.13}$$

$$\text{i.e. for all } N \in \mathbb{N}, \quad h^m a(x, \xi, h) - \sum_{0\leq j<N} h^j p_j(x, h\xi) \in S^{-N}_{scl}.$$

Note that $S^m_{psc} \subset S^m_{scl}$ and also that, given a family $(p_j)_{j\in\mathbb{N}}$ of functions of $C^\infty_b(\mathbb{R}^{2n})$, there exists $a \in S^m_{psc}$ such that (1.1.13) is satisfied; the function p_0 above is called the principal symbol of the operator a^w.

Note that the sequence (p_j) is uniquely determined by the equality (1.1.13) since the identity $0 \sim \sum_{j\geq 0} h^j p_j(x, h\xi)$ implies $p_0(x, h\xi) \in S^{-1}_{scl}$ and in particular we obtain the inequality $\sup_{(x,\xi,h)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,1]} |p_0(x, h\xi)|h^{-1} < \infty$ so that $p_0 = 0$.

For future reference, we also need to recall the definition of the semi-classical wave-front set, similar to the usual wave-front set and well adapted to the semi-classical framework that we follow here.

Definition 1.1.4. Let $h_0 \in (0, 1]$ and $(u_h)_{0<h\leq h_0}$ be a family of functions with L^2 norms smaller than 1. The semi-classical wave-front set of the family (u_h), denoted by $WF_{sc}(u_h)$, is a subset of $\mathbb{R}^{2n}$ whose complement is the set of points $(x_0, \xi_0) \in \mathbb{R}^{2n}$ such that there exists a function $\chi_0 \in C^\infty_b(\mathbb{R}^{2n})$ with

$$\chi_0(x_0, \xi_0) = 1, \qquad \|\chi_0(x, h\xi)^w u_h\|_{L^2(\mathbb{R}^n)} = O(h^\infty). \tag{1.1.14}$$

1.2. A priori estimates. We want to check under which classical conditions on the function $p \in C^\infty_b(\mathbb{R}^{2n})$ we can prove or disprove an estimate of the following type: there exists $\mu \geq 0, C > 0, h_0 > 0$, for all $h \in (0, h_0]$ and all $u \in L^2(\mathbb{R}^n)$,

$$h^\mu \|u\|_{L^2(\mathbb{R}^n)} \leq C \|p^{w_h} u\|_{L^2(\mathbb{R}^n)} . \tag{1.2.1}$$

In the sequel, we limit our attention to microlocal estimates: we know the behaviour of the symbol p near some point (x_0, ξ_0), and we want to prove the following modification of (1.2.1): there exists a neighborhood V_0 of (x_0, ξ_0), there exists $\mu \geq 0$ such that for all $\chi_0 \in C^\infty_b(\mathbb{R}^{2n})$ with $\operatorname{supp} \chi_0 \subset V_0$, there exists $C > 0$ and $r \in S^{-\infty}_{scl}(V^h_0)$, such that for all $u \in L^2(\mathbb{R}^n)$,

$$h^\mu \|\chi^{w_h}_0 u\|_{L^2(\mathbb{R}^n)} \leq C \|p^{w_h} u\|_{L^2(\mathbb{R}^n)} + \|r^w u\|_{L^2(\mathbb{R}^n)} , \tag{1.2.2}$$

with

$$V^h_0 = \{(x, \xi) \in \mathbb{R}^{2n}, (x, h\xi) \in V_0\}. \tag{1.2.3}$$

We shall say that the above estimate is loosing μ derivatives.

The elliptic case. As a matter of fact, if p is elliptic at (x_0, ξ_0), i.e. $p(x_0, \xi_0) \neq 0$, the semiclassical symbol $a(x, \xi, h) = p(x, h\xi)$ belongs to S^0_{scl} and $|a| \geq \epsilon_0 > 0$ on V_0^h (see (1.2.3)) where V_0 is some neighborhood of (x_0, ξ_0). Standard arguments of symbolic calculus ensure that there exists $b \in S^0_{scl}$ such that

$$b \sharp a = 1 + r, \quad r \in S^{-\infty}_{scl}(V_0^h) \cap S^0_{scl}.$$

As a consequence, for a function $\chi_0 \in C_b^\infty(\mathbb{R}^{2n})$ supported in V_0, we have

$$\chi_0^{w_h} + \chi_0^{w_h} r^w = \chi_0^{w_h} b^w a^w,$$

entailing from the remark 1.1.2, $\chi_0^{w_h} = c^w p^{w_h} + r_1^w, c \in S^0_{scl}, r_1 \in S^{-\infty}_{scl}$ so that with L^2 norms, $\|\chi_0^{w_h} u\| \leq C \|p^{w_h} u\| + \|r_1^w u\|$, which is indeed (1.2.2) with $\mu = 0$.

Remark 1.2.1. Note that a consequence of (1.2.2) is $h^\mu \|\chi_0^{w_h} u\| \leq C \|p^{w_h} u\| + \gamma_N h^N \|u\|$ and if we assume furthermore that the function u is somehow "concentrated" near (x_0, ξ_0), e.g. $\|u\| \leq C \|\chi_0^{w_h} u\|$, we get for h small enough the estimate $h^\mu \|u\| \leq C \|p^{w_h} u\|$, which is indeed (1.2.1) for this class of u.

What happens at the characteristic points, $p(x_0, \xi_0) = 0$? We shall always assume that p is principal type, i.e.

$$dp \neq 0 \text{ at } p = 0. \tag{1.2.4}$$

When p is real-valued, it is easy to get (1.2.2) with $\mu = 1$. As a matter of fact, thanks to (1.2.4), one can solve with $a \in C_c^\infty(\mathbb{R}^{2n})$, real-valued, supported on some neighborhood V_0 of $\gamma_0 = (x_0, \xi_0)$

$$H_p(a) = \{p, a\} = \chi_0^2 - \chi_1^2, \tag{1.2.5}$$

where $\chi_0 \in C_c^\infty(\mathbb{R}^{2n})$ is supported in a neighborhood $W_0 \subset V_0$ of γ_0 and $\chi_1 \in C_c^\infty(\mathbb{R}^{2n})$ is supported in a neighborhood $W_1 \subset V_0$ of $\gamma_1 = (x_1, \xi_1)$, with $(x_0, \xi_0) \neq (x_1, \xi_1)$ on the same integral curve of H_p (bicharacteristic of p). Quantifying with the h-Weyl formula (1.1.8) the equality (1.2.5), one can even get some propagation estimates.

Remark 1.2.2. Even for D_{x_1}, no better estimate than with $\mu = 1$ is true: in fact with L^2-norms, it is not possible to do better than

$$T \|hD_1 u\| \geq \hbar \|u\|, \quad \operatorname{supp} u \subset \{(x_1, x'), 0 \leq x_1 \leq T\}. \tag{1.2.6}$$

Although semi-global or global estimates may be difficult to obtain for real-principal type operators, it is not the case of the microlocal estimates, which can essentially be reduced to (1.2.6).

Some motivations and examples. What happens to these microlocal estimates for a nonelliptic principal-type complex-valued symbol ? There are several reasons to get interested in this problem. First of all, it is certainly a natural question to look at complex-valued symbols. Next, complex non-singular (i.e. nonvanishing) vector fields are basic geometrical objects and for instance the Hans Lewy operator

$$L_0 = \partial_{x_1} + i\partial_{x_2} + i(x_1 + ix_2)\partial_{x_3} \text{ is not locally solvable} \tag{1.2.7}$$

nor is $\partial_{x_1} + ix_1\partial_{x_2}$ whereas $\partial_{x_1} + ix_1^2\partial_{x_2}$ is indeed locally solvable (the latter models were studied by S.Mizohata in [Mi] and by L.Nirenberg and F.Treves in [NT1]). What is the geometric explanation? Nonelliptic boundary value problems, such as the oblique-derivative problem (see e.g. [L2] and the references therein) are also natural problems of interest: take Ω a smooth open subset of $\mathbb{R}^n$ and try to find u such that $\Delta u = 0$ in Ω and, with T tangent vector field to the boundary, and $\partial/\partial\nu$ the exterior normal,

$$Xu = Tu + \alpha\frac{\partial u}{\partial \nu} = f \quad \text{on } \partial\Omega.$$

When $\alpha \neq 0$ it is an elliptic problem but when α vanishes and $T \neq 0$ it is a nonselfadjoint principal type problem: if G is the Green kernel for the Dirichlet problem, we get $XGw = f$, where XG is a (nonlocal) pseudodifferential operator on the boundary. A more recent topic of interest in nonselfadjoint problems comes from the study of the pseudospectrum: the spectrum of nonnormal operators is generically very unstable, since the resolvent may be large far away from the spectrum and the spectrum of a perturbation $A + \epsilon R$ may be far from the spectrum of the nonnormal A. Some recent work by L.Boulton [Bo], E.Davies [Da1], [Da2], N.Dencker-J.Sjöstrand-M.Zworski [DSZ], M.Hager [Ha], K.Pravda-Starov [P1], [P2], L.Trefethen [TE] established a clear link between the pseudospectrum and geometric properties of locally solvable pseudodifferential operators. A remarkable point in these studies is that the pseudospectrum was introduced by numerical analysts a long time ago to tackle large nonselfadjoint matrices, and it is only recently that the key rôle of semi-classical estimates was identified for the determination of the pseudospectrum.

1.3. The first bracket. We consider now and in the sequel of this section a complex-valued function $p \in C_b^\infty(\mathbb{R}^{2n})$ and a point $(x_0, \xi_0) \in \mathbb{R}^{2n}$ such that $p(x_0, \xi_0) = 0, dp(x_0, \xi_0) \neq 0$. It turns out that a very simple study of the first bracket is leading us at once to rather sharp results: the sign of the first bracket

$$\{\operatorname{Re} p, \operatorname{Im} p\} = \frac{1}{2}\operatorname{Im}\{\bar p, p\} = \frac{1}{2i}\{\bar p, p\}$$

will play a determinant rôle. If we assume now that this first bracket is negative at a characteristic point (x_0, ξ_0):

$$p(x_0, \xi_0) = 0, \quad \{\operatorname{Re} p, \operatorname{Im} p\}(x_0, \xi_0) < 0, \tag{1.3.1}$$

then there exists a family of functions (u_h) with L^2 norm 1, and $WF_{sc}(u_h) = \{(x_0, \xi_0)\}$ such that

$$\|p^{w_h} u_h\| = O(h^\infty),$$

i.e. we can construct a quasi-mode. The proof is quite simple, once the apparatus of microlocal analysis is in place. After multiplication by an elliptic factor, the microlocal model for p is indeed $\xi_1 - ix_1$, quantized by $hD_{x_1} - ix_1$: we have

$$hD_{x_1} - ix_1 = \frac{h}{2i\pi}\big(\partial_{x_1} + 2\pi h^{-1}x_1\big), \quad \big(\partial_{x_1}+2\pi h^{-1}x_1\big)(2^{1/4}h^{-1/4}e^{-\pi\frac{x_1^2}{h}}) = 0,$$

and it is enough to multiply by a fixed cutoff function in the x'-variables to get the quasi-mode

$$u_h(x_1, x') = 2^{1/4}h^{-1/4}e^{-\pi\frac{x_1^2}{h}}w(x'), \quad \|w\|_{L^2(\mathbb{R}^{n-1})} = 1.$$

Let us consider now the case in which the first bracket is positive, i.e.

$$p(x_0, \xi_0) = 0, \quad \{\operatorname{Re} p, \operatorname{Im} p\}(x_0, \xi_0) > 0. \tag{1.3.2}$$

After multiplication by an elliptic factor, the microlocal model for p is $\xi_1 + ix_1$, quantized by $hD_{x_1} + ix_1$: we have by a simple integration by parts, say for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\|hD_{x_1}u + ix_1u\|^2 = \|hD_{x_1}u\|^2 + \|x_1u\|^2 + \underbrace{2\operatorname{Re}\langle hD_{x_1}u, ix_1u\rangle}_{=\hbar\|u\|^2} \geq \hbar\,\|u\|^2$$

an estimate which is easily seen as optimal in terms of power of h since

$$\|hD_{x_1}u + ix_1u\|^2 = \|hD_{x_1}u - ix_1u\|^2 + 4\operatorname{Re}\langle hD_{x_1}u, ix_1u\rangle \geq \frac{h}{\pi}\,\|u\|^2 \tag{1.3.3}$$

and h/π is the best constant since

$$\|(hD_{x_1} + ix_1)(e^{-\pi x_1^2h^{-1}}) = e^{-\pi x_1^2h^{-1}}(2ix_1)\|^2 = \int 4x_1^2e^{-2\pi x_1^2h^{-1}}dx_1 = 2^{-1/2}\pi^{-1}h^{3/2},$$
$$2\hbar\|e^{-\pi x_1^2h^{-1}}\|^2 = \hbar h^{1/2}2^{1/2} = h^{3/2}2^{-1/2}\pi^{-1}.$$

The estimate (1.3.3) is (1.2.1) with $\mu = 1/2$ (for u such that $x_1u, \partial_{x_1}u \in L^2$) that is an estimate with loss of 1/2 derivatives. The situation with a nonvanishing first bracket at a characteristic point is thus very simple: either it is negative and there exists a quasi-mode, or it is positive and one can prove an estimate with loss of 1/2 derivative.

Remark 1.3.1. The geometric explanation of the Hans Lewy counterexample ([Lw]) (1.2.7) given by L.Hörmander ([H2], [H3]) showed that whenever the first bracket is positive at some point (x_0, ξ_0) of the cotangent bundle where the homogeneous principal symbol q_0 is vanishing (we leave the semi-classical symbols in this remark to deal with smooth homogeneous symbols on the cotangent bundle of a manifold: see the appendix 5.1), i.e.

$$\{\operatorname{Re} q_0, \operatorname{Im} q_0\}(x_0, \xi_0) > 0,$$

then the operator $q(x, D) \in \operatorname{Op}(S^m_{phg})$ is not locally solvable at x_0: there exists some C^∞ right-hand-side f such that the equation $q(x, D)u = f$ has no distribution solution in any neighborhood of x_0. On the other hand if the first bracket $\{\operatorname{Re} q, \operatorname{Im} q\}$ is negative at every characteristic point, the transposed operator $q(x, D)^*$ is subelliptic with loss of $1/2$ derivative and thus the equation $q(x, D)u = f$ is locally solvable with a loss of $1/2$ derivatives: for $f \in H^s_{loc}$, there exists $u \in H^{s+m-\frac12}_{loc}$ (m is the order of the operator) solving the equation in a neighborhood of x_0. After a microlocalization and a Littlewood-Paley decomposition, this results boils down to dealing with semi-classical estimates as above.

1.4. The geometry: condition (Ψ). The elliptic points are well understood as well as the characteristic points with a non vanishing first bracket. What happens if somewhere $p = \{\operatorname{Re} p, \operatorname{Im} p\} = 0$?

Examples. Before dealing with a more geometrical setting, let us first check a couple of examples. For $k \in \mathbb{N}$, there exists C_k such that for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$C_k \|hD_{x_1} u + i x_1^{2k+1} u\| \geq h^{\frac{2k+1}{2k+2}} \|u\|, \tag{1.4.1}$$

that is an estimate with loss of $\frac{2k+1}{2k+2}$ derivatives. We have proven this when $k = 0$ (and did comput the best constant in that case with (1.3.3)) by expanding the square. Expanding the square is not enough if $k \geq 1$, but it makes sense since the first bracket is $(2k+1)x_1^{2k}$, which is nonnegative. Note that for the operator

$$hD_{x_1} - i x_1^{2k+1}, \quad \text{one can construct a quasi-mode,} \tag{1.4.2}$$

with a function $e^{-\alpha h^{-1} x_1^{2k+2}}$ with a positive α, a construction similar to the one following (1.3.1). On the other hand, for $k \in \mathbb{N}$, there exists C'_k such that for all $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$C'_k \left\|hD_{x_1} u \pm i x_1^{2k} u\right\| \geq h^{\frac{2k}{2k+1}} \|u\|, \tag{1.4.3}$$

an estimate with loss of $\frac{2k}{2k+1}$ derivative which holds whatever is the sign $\pm$. Note that expanding the square does not make sense, even for $k = 1$: the first bracket is $2k x_1^{2k-1}$, which may be negative. It is nevertheless a trivial matter to solve the ODE

$$\partial_{x_1} u \mp 2\pi h^{-1} x_1^{2k} u = f 2i\pi h^{-1}$$
$$u(x_1) = \int_{\pm\infty}^{x_1} e^{\pm \frac{2\pi}{h(2k+1)}(x_1^{2k+1} - y_1^{2k+1})} f(y_1) dy_1 2i\pi h^{-1}$$

so that the proof of (1.4.3) is reduced to the L^2-boundedness of the operator with kernel

$$H(t-s)e^{-\frac{2\pi}{h(2k+1)}(t^{2k+1}-s^{2k+1})}h^{-1+\frac{2k}{2k+1}},$$

($H = \mathbf{1}_{\mathbb{R}^+}$ is the Heaviside function), which is unitarily equivalent to the operator with kernel $H(t-s)e^{-\frac{2\pi}{2k+1}(t^{2k+1}-s^{2k+1})}$, the latter being trivially L^2-bounded by the Schur criterion. These examples are ODE and their simplicity could be misleading. Let us check a more involved example: for $\alpha \in C_b^\infty(\mathbb{R}^2)$, we have

$$C\|hD_{x_1}u + ix_1\big(x_1 - \alpha(x_2, hD_{x_2})\big)^2 u\| \ge h^{3/4}\,\|u\|\,. \tag{1.4.4}$$

That example is a mixture of (1.4.1) and (1.4.3) ($k = 0$ if α is far from zero, $k = 1$ otherwise). Since α can be arbitrarily close to zero, one has to find a good multiplier and for instance compute $\operatorname{Re}\langle hD_{x_1}u + ix_1\big(x_1 - \alpha(x_2, hD_{x_2})\big)^2 u, i\operatorname{sign}(x_1)u\rangle$, but the complete proof is not so easy. Increasing the complexity without leaving operators with a simple expression, we can prove

$$C\|hD_{x_1}u + ix_1^2\big(hD_{x_2} + x_1x_2^2\big)u\| \ge h^{9/10}\,\|u\|\,, \tag{1.4.5}$$

where the operator involved is definitely not an ODE and the 9/10 is (very) hard to get. To conclude with that short list of examples, it is possible to prove that

$$C\|hD_{x_1}u + ia(x_1,x',hD')b(x_1,x',hD')u\| \ge h^{3/2}\,\|u\|\,,\quad \text{whenever } a\ge 0, \partial_1 b \ge 0, \tag{1.4.6}$$

with $a, b \in C_b^\infty(\mathbb{R}^{2n-1})$, but the proof is rather tricky. Also it is not known if the 3/2 is optimal, i.e. smallest possible. Moreover, the estimate with loss of one derivative is not true in general for these models.

Condition (ψ). L.Nirenberg and F.Treves proposed in 1971 ([NT2],[NT3],[NT4]) a geometric condition, the so-called condition (ψ), as an iff geometric condition for local solvability.

Definition 1.4.1. Let p be a smooth complex-valued function on a symplectic manifold. The function p is said to satisfy condition (ψ) if for all non-vanishing complex-valued functions e, the imaginary part of ep does not change sign from $-$ to $+$ along the oriented bicharacteristic flow of the real part of ep: let $\dot\gamma(t) = H_{\operatorname{Re}(ep)}(\gamma(t))$ be a (null) bicharacteristic curve of $\operatorname{Re}(ep)$ (integral curve of the Hamiltonian vector field of $\operatorname{Re}(ep)$ along which p vanishes), then

$$\operatorname{Im}(ep)(\gamma(t)) < 0, s > t \Longrightarrow \operatorname{Im}(ep)(\gamma(s)) \le 0.$$

We shall say that p satisfies condition $(\overline{\psi})$ whenever $\bar p$ satisfies condition (ψ).

Remark 1.4.2. Assuming (1.2.4), it is possible to prove that condition (ψ) is satisfied in a neighborhood of a point whenever the above condition holds for $e = 1$ and $e = i$.

Remark 1.4.3. Assuming $H_{\operatorname{Re} p} \neq 0$ at a point, the complex-valued symbol p satisfies condition $(\overline{\psi})$ in a neighborhood of this point means that for a null bicharacteristic curve of the real part, $\dot\gamma(t) = H_{\operatorname{Re} p}(\gamma(t))$,

$$\operatorname{Im}\big(p(\gamma(t)\big) > 0 \text{ and } s > t \quad \Longrightarrow \operatorname{Im}\big(p(\gamma(s)\big) \geq 0. \tag{1.4.7}$$

Remark 1.4.4. That condition is consistent with the propagation-of-singularities result since the set $\{\operatorname{Im} p \geq 0\}$ (resp. $\{\operatorname{Im} p \leq 0\}$) is a forward (resp.backward) region in the sense that, if P is a classical pseudodifferential operator of order m, $Pu \in H^s$, I interval of $\mathbb{R}$,

$$\gamma(I) \subset \operatorname{Int}\{\operatorname{Im} p \geq 0\}, \quad \gamma(t) \in WF_{s+m-1}u, \quad s \geq t, \quad s,t \in I \Longrightarrow \gamma(s) \in WF_{s+m-1}u,$$
$$\gamma(I) \subset \operatorname{Int}\{\operatorname{Im} p \leq 0\}, \quad \gamma(t) \in WF_{s+m-1}u, \quad s \leq t, \quad s,t \in I \Longrightarrow \gamma(s) \in WF_{s+m-1}u.$$

When $H_{\operatorname{Re} p} \neq 0$, one may give a more accurate definition of the forward $\mathcal{R}_+$ (resp. backward $\mathcal{R}_-$) region with

$$\mathcal{R}_+ = \cup_{t\geq 0}\Phi^t(\{\operatorname{Im} p > 0\}), \qquad \mathcal{R}_- = \cup_{t\leq 0}\Phi^t(\{\operatorname{Im} p < 0\}), \tag{1.4.8}$$

where Φ stands for the bicharacteristic flow of $\operatorname{Re} p$. Note that $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) is stable by the forward (resp. backward) flow and the condition (1.4.7) expresses the fact that

$$\mathcal{R}_+ \cap \mathcal{R}_- = \emptyset. \tag{1.4.9}$$

In fact, if $X \in \mathcal{R}_+ \cap \mathcal{R}_-$, with $t \geq 0 \geq s$, we have $\Phi^s(Z) = X = \Phi^t(Y), \operatorname{Im} p(Y) > 0 > \operatorname{Im} p(Z)$, and thus $\operatorname{Im} p(Y) > 0 > \operatorname{Im} p(\Phi^{t-s}(Y))$, which contradicts (1.4.7) since $t - s \geq 0$. Conversely, if (1.4.7) is violated, it means that with $s > t$,

$$\operatorname{Im} p(\Phi^t(X)) > 0 > \operatorname{Im} p(\Phi^s(X)) = \operatorname{Im} p(\Phi^{s-t}(\Phi^t(X)))$$

so that $\Phi^t(X) \in \mathcal{R}_+ \cap \mathcal{R}_-$. The violation of condition $(\overline{\psi})$ at a point X_0 means that in any neighborhood of X_0, one can find a bicharacteristic of the real part such that, with $a < b$

$$\operatorname{Im}\big(p(\gamma(a))\big) > 0 > \operatorname{Im}\big(p(\gamma(b))\big).$$

In that case, we have with some $c_1 \leq c_2$, $\operatorname{Im}\big(p(\gamma([a,c_1)))\big) > 0 > \operatorname{Im}\big(p(\gamma((c_2,b])\big)$ so that the singularities are travelling from $\gamma(a)$ to $\gamma(c_1)$ and from $\gamma(b)$ to $\gamma(c_2)$ and are somehow trapped there, e.g. at $\gamma(c_1)$ if $c_1 = c_2$.

Checking the examples (1.4.1), (1.4.3), (1.4.4), (1.4.5), (1.4.6), we see that they are all of the form $\xi_1 + ia(x_1,x',\xi')b(x_1,x',\xi')$ with $a \geq 0$ and $\partial_{x_1} b \geq 0$. The condition $(\overline{\psi})$ is thus satisfied since $H_{\xi_1} = \partial_{x_1}$ and

$$\operatorname{Im} p(x_1,0,x',\xi') = a(x_1,x',\xi')b(x_1,x',\xi') > 0, y_1 > x_1 \Longrightarrow b(x_1,x',\xi') > 0$$
$$\Longrightarrow b(y_1,x',\xi') > 0 \Longrightarrow a(y_1,x',\xi')b(y_1,x',\xi') = \operatorname{Im} p(y_1,0,x',\xi') \geq 0.$$

Moreover, with with $q = ab$ on the null bicharacteristic of q, we have $-\dot\xi_1 = \partial_{x_1} q(x_1,x',\xi')$ and we already know that, at $q = 0$, we have $\partial_{x_1} q(x_1,x',\xi') \geq 0$ so that $\dot\xi_1 \leq 0$ and thus $-\xi_1$ cannot change sign from $+$ to $-$. On the other hand the example (1.4.2) clearly violates condition $(\overline{\psi})$ since the function $x_1 \mapsto -x_1^{2k+1}$ actually changes sign from $+$ to $-$ for increasing x_1.

2. The necessity of condition $(\overline{\psi})$ for an priori estimate

2.1. The Moyer-Hörmander result. The example (1.4.2) is providing a very simple situation in which a quasi-mode can be constructed in a way which is not so different of the operator $hD_1 - ix_1$. Obviously for these examples (1.4.2), the condition $(\overline{\psi})$ is violated, but in a very particular way. We are willing here to formulate a semi-classical result which implies that even the weakest form of estimate of type (1.2.1) will imply that condition $(\overline{\psi})$ is satisfied, that is the necessity of condition $(\overline{\psi})$ for an a priori estimate of type (1.2.1) to hold.

We consider a function $p_0 \in C_b^\infty(\mathbb{R}^{2n})$. We assume that there exists a nonvanishing function $q_0 \in C_b^\infty(\mathbb{R}^{2n})$ and a null bicharacteristic curve γ of $\operatorname{Re}(q_0 p_0)$ such that for some $a < b$ we have

$$\operatorname{Im}\big((q_0p_0)(\gamma(a))\big) > 0 > \operatorname{Im}\big((q_0p_0)(\gamma(b))\big). \tag{2.1.1}$$

We define

$$L_0 = \inf\{(t-s), a \le s < t \le b, \operatorname{Im}(q_0p_0)(\gamma(s)) > 0 > \operatorname{Im}(q_0p_0)(\gamma(t))\}. \tag{2.1.2}$$

When $L_0 > 0$, one can find a_0, b_0 such that $L_0 = b_0 - a_0$ and $a \le a_0 < b_0 \le b$; moreover, for any neighborhood V_{a_0}, V_{b_0} of a_0, b_0, with $V_{a_0}^- = V_{a_0} \cap (-\infty, a_0[,\ V_{b_0}^+ = V_{a_0} \cap]b_0, +\infty)$, we have from (2.1.2),

$$\begin{aligned} V_{a_0}^- \cap \{t \in [a,b], \operatorname{Im}(q_0p_0)(\gamma(t)) > 0\} &\ne \emptyset, \\ V_{b_0}^+ \cap \{t \in [a,b], \operatorname{Im}(q_0p_0)(\gamma(t)) < 0\} &\ne \emptyset, \\ \forall t \in [a_0, b_0], \quad \operatorname{Im}(q_0p_0)(\gamma(t)) &= 0. \end{aligned}$$

When $L_0 = 0$, one has $a_0 = b_0$ and, using (2.1.2), we see that for any neigborhood V_{a_0} of a_0, there exists $a_1 < b_1$ in V_{a_0} such that

$$\operatorname{Im}(q_0p_0)(\gamma(a_1)) > 0 > \operatorname{Im}(q_0p_0)(\gamma(b_1)).$$

Note that when the change of sign occurs at a finite order, we have $L_0 = 0$ and $\operatorname{Im}(q_0p_0) > 0$ (resp.< 0) on some $V_{a_0}^-$ (resp. $V_{a_0}^+$). When the change of sign occurs at an infinite order, we may possibly have some oscillation of the function $\operatorname{Im}(q_0p_0)$.

The following result is a semi-classical version of a theorem for homogeneous pseudo-differential operators due in two dimensions to R.D.Moyer ([Mo]) and to L.Hörmander in general ([H6], Theorem 26.4.7 in [H7]).

Theorem 2.1.1. *Let $a \in S^0_{psc}$ with a principal symbol p_0 (cf. definition 1.1.3). Assume that there exists a nonvanishing function $q_0 \in C_b^\infty(\mathbb{R}^{2n})$ and a null bicharacteristic curve γ of $\operatorname{Re}(q_0p_0)$ such that (2.1.1) holds.*

Then, using the above notations, for any open neighborhood V of the compact set $\gamma([a_0,b_0])$ in $\mathbb{R}^{2n}$, there exists $h_0>0$ and a family of functions $(u_h)_{0<h\le h_0}$ of $\mathcal{S}(\mathbb{R}^n)$ with $L^2(\mathbb{R}^n)$ norms equal to 1, such that

$$\|a^w u_h\|_{L^2(\mathbb{R}^n)} = O(h^\infty).$$

Moreover the semi-classical wave-front set (see def. 1.1.4) of $(u_h)_{0<h\le h_0}$ is included in $\overline{V}$ and $\overline{V}$ is a confinement-set for the family (u_h) (see def. 5.2.1 in the appendix).

Comments 2.1.2: the necessity of condition $(\overline{\psi})$ for an priori estimate. If $p_0(x_0,\xi_0)=0$ and if the condition $(\overline{\psi})$ is violated in any neighborhood V_0 of (x_0,ξ_0), that is if there exists a nonvanishing $q_0\in C_b^\infty(\mathbb{R}^{2n})$ and a null bicharacteristic curve γ of $\operatorname{Re}(q_0p_0)$ such that for some $a<b$ we have with $\gamma([a,b])\subset V_0$,

$$\operatorname{Im}(q_0p_0)(\gamma(a)) > 0 > \operatorname{Im}(q_0p_0)(\gamma(b)),$$

then we can apply the theorem above and disprove the estimate (1.2.1), no matter how large μ is chosen. On the other hand the estimate (1.2.2) cannot be true either since it would imply, for all $\chi_0\in C_b^\infty(\mathbb{R}^{2n})$ supported in some neighborhood V_0 of (x_0,ξ_0), for all $u\in L^2(\mathbb{R}^n)$,

$$h^\mu \left\|\chi_0^{w_h} u\right\|_{L^2(\mathbb{R}^n)} \le C\left\|p_0^{w_h}u\right\| + \left\|r^w u\right\|, \qquad r\in S^{-\infty}_{scl}(V_0^h).$$

But we can choose a neighborhood V of (x_0,ξ_0) such that $V\Subset V_0$ in the theorem 2.1.1, since condition $(\overline{\psi})$ is violated in any neighborhood of (x_0,ξ_0): we have then a family of functions (u_h), with $L^2(\mathbb{R}^n)$ norms 1 such that $\|p_0^{w_h}u_h\| = O(h^\infty)$. On the other hand, $\overline{V}$ is a confinement-set for (u_h), so that choosing $\chi_0\in C_c^\infty(V_0)$ with $\chi_0=1$ near $\overline{V}$, we get from the definition 5.2.1 that $\|(1-\chi_0)^{w_h}u_h\| = O(h^\infty)$. Since $r\in S^{-\infty}_{scl}(V_0^h)$, we get

$$\begin{aligned} h^\mu \left\|\chi_0^{w_h}u_h\right\| \le O(h^\infty) + \|r^w u_h\| \le O(h^\infty) + \Big\|\Big(\overbrace{\chi_0(x,h\xi)r(x,\xi,h)}^{\in S^{-\infty}_{scl}}\Big)^w u_h\Big\| \\ + \Big\|\big((1-\chi_0(x,h\xi))r(x,\xi,h)\big)^w u_h\Big\|. \end{aligned}$$

The symbol $(1-\chi_0(x,h\xi))r(x,\xi,h) = \omega(x,h\xi,h)$ with $(x,\xi)\mapsto\omega(x,\xi,h)\in C_b^\infty(\mathbb{R}^{2n})$ uniformly in h and ω is supported in the complement of a neighborhood of $\overline{V}$, so we get from the previous inequalities and the definition 5.2.1 that $\|\chi_0^{w_h}u_h\| = O(h^\infty)$. As a consequence, we obtain

$$1 = \|u_h\| \le \|\chi_0^{w_h}u_h\| + \|(1-\chi_0)^{w_h}u_h\| = O(h^\infty)$$

which is impossible.

Comments 2.1.3. A direct proof of the semi-classical statement of the theorem 2.1.1 can be done by following the lines of the proof of Theorem 26.4.7′ in [H7], and it was the path followed by K.Pravda-Starov in [P2]. The arguments needed to deal with an homogeneous situation (like in [H7]) are slightly different at some points than the required arguments to tackle a semi-classical setting. However, one can actually deduce a semi-classical theorem in dimension n from the same statement in an homogeneous framework in $n+1$ dimensions: in $n+1$ dimensions, considering a positively homogeneous symbol of degree m (i.e. in S^m satisfying (5.1.2)), $p(x, x_{n+1}; \xi, \xi_{n+1})$, $x \in \mathbb{R}^n, x_{n+1} \in \mathbb{R}, \xi \in \mathbb{R}^n, \xi_{n+1} \in \mathbb{R}$, one may assume that p is supported in a conic neighborhood of $(0,0;0,1)$ where $\xi_{n+1} \geq |\xi|$, and by a Littlewood-Paley decomposition, we can check the symbol

$$p(x, x_{n+1}; \xi, \xi_{n+1})\varphi(\xi_{n+1}2^{-\nu})$$

where φ is supported in $[1/2, 2]$. Moreover, p can be assumed to be independent of x_{n+1} and this symbol can be considered as a semi-classical symbol with $h = 2^{-\nu}$, acting on functions $u(x) \otimes \theta(x_{n+1})$, where θ is a fixed function (say a Gaussian function), so that we have indeed to deal with a semi-classical setting in n dimensions.

2.2. Notes on the proof. We shall not give here the details of the proof of the theorem 2.1.1, but we explain some lines of the demonstration.

The simplest model. It is of course the already mentioned (1.4.2) which corresponds to the semi-classical quantization of the symbol given in a neighborhood of the origin in the symplectic $\mathbb{R}^2$ by

$$p_0(t, \tau) = \tau - it^{2k+1}, \quad k \in \mathbb{N}. \tag{2.2.1}$$

There is in that case an explicit construction of a quasi-mode, which is not difficult to perform and is outlined in section 1.3 when $k = 0$. There is indeed a geometric description of the models (2.2.1): take a function $p_0 \in C_b^\infty(\mathbb{R}^{2n})$ such that at a point $(x_0, \xi_0) \in \mathbb{R}^{2n}$,

$$p_0(x_0, \xi_0) = 0, \; d \operatorname{Re} p_0(x_0, \xi_0) \neq 0 \tag{2.2.2}$$

and such that there exists a neighborhood V_0 of (x_0, ξ_0), such that for all $(x, \xi) \in V_0 \cap \{\operatorname{Re} p_0 = 0\}$, if γ is the bicharacteristic of $\operatorname{Re} p_0$ starting at (x, ξ), the function $t \mapsto \operatorname{Im} p_0(\gamma(t))$ has a zero at 0 of order $2k+1$ with a change of sign from $+$ to $-$. The theorem 21.3.5 in [H7] provides a choice of symplectic coordinates and an elliptic factor e such that $ep_0 = \xi_1 - ix_1^{2k+1}$.

The finite type case. We can consider the more general case where (2.2.2) is satisfied and, if γ_0 is the bicharacteristic of $\operatorname{Re} p_0$ starting at (x_0, ξ_0), the function $t \mapsto \operatorname{Im} p_0(\gamma_0(t))$ has a zero at 0 of order $2k+1$ with a change of sign from $+$ to $-$. This situation cannot be reduced in general to the previous one, except if $k = 0$, since it is the case (with $k = 1$) of the two-dimensional

$$\xi_1 - ix_1(x_1 - x_2)^2.$$

When $x_2 = 0$, $\operatorname{Im} p$ has a triple zero with a change of sign from $+$ to $-$, but when $x_2 \neq 0$, the zero where the change of sign occurs is simple. However this finite-type assumption simplifies a great deal the construction and the obtention of a quasi-mode is in fact quite close to the previous model.

The general case. In that case, we may assume (2.2.2), and that in any neighborhood V of (x_0, ξ_0), we can find a null bicharacteristic curve γ of $\operatorname{Re} p_0$ such that for some $a < b$ we have with $\gamma([a, b]) \subset V$, $\operatorname{Im} p_0(\gamma(a)) > 0 > \operatorname{Im} p_0(\gamma(b))$. Using the Malgrange-Weierstrass theorem, it is possible to assume that $p_0 = \xi_1 + iq(x_1, x', \xi')$. The idea is grounded on the use of a complex WKB method, which amounts to find a quasi-mode of type

$$u_h(x) = e^{ih^{-1}w(x)} \sum_{0 \leq j \leq M} h^j \phi_j(x),$$

where the phase w is complex-valued with $\operatorname{Im} w \geq 0$ and satisfies approximately the eiconal equation

$$\partial_{x_1} w + iq(x_1, x', \partial_{x'} w) = 0.$$

It is a quite technical matter whose details are explained in the section 26.4 of [H7](pp.104–110), and since the function q is not analytic and w is complex-valued, the meaning of the plain eiconal equation as written above has to be formulated in a suitable approximate way. The determination of the amplitudes ϕ_j is following a more classical course, analogous to the standard WKB method.

3. Finite type geometry

We have seen in section 2 that condition $(\overline{\psi})$ is necessary for obtaining any type of a priori estimate such as (1.2.2), so we assume in the sequel of the paper that condition $(\overline{\psi})$ is satisfied and we describe what type of estimate we can get out of this assumption, possibly reinforced by some other conditions. The first class of cases that we want to investigate is linked to some sort of finite-type assumption, related to subellipticity. First we assume that one of the iterated bracket of the real and imaginary part is not vanishing: we shall say that the geometry is finite type.

3.1. Classical assumptions for subellipticity. We consider $p = p_1 + ip_2$ a complex-valued function in C_b^∞. We shall assume that $(\overline{\psi})$ holds for p in a neighborhood V_0 of a point $(x_0, \xi_0) \in \mathbb{R}^{2n}$. Moreover, we define

$$p_{12} = \{p_1, p_2\} = -p_{21}, \quad H_j = H_{p_j}, \tag{3.1.1}$$

$$p_{112} = H_1^2(p_2), p_{212} = -p_{221}, p_{221} = H_2^2(p_1), p_{121} = -p_{112},$$

$$\text{for } j_k \in \{1,2\}, \quad p_{j_1,\dots,j_{l+1}} = H_{j_1} \dots H_{j_l}(p_{j_{l+1}}), \quad |(j_1, \dots, j_l)| = l. \tag{3.1.2}$$

We assume that one of these iterated brackets is nonzero and we define the integer k so that

$$\begin{aligned} &\text{for all } |J| \le k, \quad p_J(x_0, \xi_0) = 0, \\ &\text{there exists } |J| \text{ with } |J| = k+1 \text{ such that} \quad p_J(x_0, \xi_0) \ne 0. \end{aligned} \tag{3.1.3}$$

Examples. Note that

$$\begin{aligned} &|p| \ne 0 \text{ means } k = 0, \text{ that is ellipticity}, \\ &p = 0, \{p_1, p_2\} > 0, \text{ means } k = 1, \\ &p = \{p_1, p_2\} = 0, H_{p_1}^2(p_2) \ne 0 \text{ or } \ H_{p_2}^2(p_1) \ne 0, \text{ means } k = 2. \end{aligned}$$

We have already seen some of the simplest ODE-like models in section 1.4 such as

$$\xi_1 + ix_1^k, \ k \in 2\mathbb{N}+1, \qquad \xi_1 \pm ix_1^k, \ k \in 2\mathbb{N}.$$

Also the example (1.4.5) above is a special case of

$$\xi_1 + ix_1^s\big(\xi_2 + V(x_1, x_2)\big), \quad s \in 2\mathbb{N}, \ \partial_{x_1} V \ge 0,$$

where V is a nonzero polynomial. Let us check for instance that for (1.4.5), we have indeed $k = 9$ at $(0,0)$: for $p_1 + ip_2 = \xi_1 + ix_1^2(\xi_2 + x_1 x_2^2)$, we have

$$\frac{1}{6} H_1^3(p_2) = x_2^2, \quad \frac{1}{2} H_1^2(p_2) = \xi_2 + 3x_1 x_2^2 \qquad \text{so that}$$

$\frac{1}{24}\big\{H_1^2(p_2), H_1^3(p_2)\big\} = x_2, \big\{H_1^2(p_2), \big\{H_1^2(p_2), H_1^3(p_2)\big\}\big\} \neq 0$ i.e.

$$0 \neq H^2_{H_1^2(p_2)}\big(H_1^3(p_2)\big) = H^2_{\{p_1,H_1(p_2)\}}\big(H_1^3(p_2)\big)$$

and since $H_{\{p_1,H_1(p_2)\}} = [H_1,[H_1,H_2]]$ we get

$$[H_1,[H_1,H_2]]^2 H_1^3(p_2) \neq 0$$

which forces $H^I(p_2) \neq 0$ with some $|I| = 9$. We leave for the reader to check that $p_J = 0$ at $(0,0)$ for $|J| \leq 8$.

3.2. Subellipticity under condition (P): a coherent states method. The general theory of subelliptic operators, as exposed in the chapter 27 of Hörmander's treatise [H7] is quite involved. However when a strengthened version of $(\overline{\psi})$ is satisfied, there is a great deal of simplifications: we shall essentially assume in the present section that our symbol p satisfies condition (ψ) and condition $(\overline{\psi})$, which means that when $d\operatorname{Re} p \neq 0$,

NO change of sign occurs for $\operatorname{Im} p$ along the bicharacteristic flow of $\operatorname{Re} p$: condition (P).

In general that does not imply that $\operatorname{Im} p$ should have a constant sign as shown by the aforementioned degenerate CR operators. For $D_1 + ix_1^2 D_2, x_1 \mapsto x_1^2\xi_2$ doesn't change sign for ξ_2 fixed, although the function $x_1^2\xi_2$ does not have a constant sign. However, when the geometry is finite-type, one can get a microlocal reduction to a model

$$hD_t + iq(t,x,\xi)^{w_h}, \quad q \geq 0, \partial_t^k q \neq 0, \text{ with } k \text{ even.}$$

With this reduction at hand, we can show that a coherent states method, based upon a nonnegative quantization formula, provides a rather simple proof. This was observed first by F.Treves in 1971 ([Tv]) and section 27.3 in [H7] is devoted to that case, much simpler to handle. The plan of the proof is quite clear: first we prove an estimate for an ODE with parameters, such as $\frac{d}{dt} - q(t,x,\xi)$, essentially with q as above, then using a nonnegative quantization for the symbol q (a coherent states method) we show that the ODE estimates can be transferred to the semi-classical level.

We begin with the introduction of our nonnegative quantization formula, which will turn out to be useful also in some different contexts.

Definition 3.2.1. Let $Y = (y,\eta)$ be a point in $\mathbb{R}^{2n}$. The operator Σ_Y is defined as $\big[2^n e^{-2\pi|\cdot - Y|^2}\big]^w$. This is a rank-one orthogonal projection: $\Sigma_Y u = (Wu)(Y)\tau_Y\varphi$ with

$$(Wu)(Y) = \langle u, \tau_Y\varphi\rangle_{L^2(\mathbb{R}^d)},$$

where $\varphi(x) = 2^{n/4}e^{-\pi|x|^2}$ and $(\tau_{y,\eta}\varphi)(x) = \varphi(x-y)e^{2i\pi\langle x-\frac{y}{2},\eta\rangle}$. Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick quantization of a is defined as

$$a^{\mathrm{Wick}} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY. \tag{3.2.1}$$

Proposition 3.2.2. *Let a be in $L^\infty(\mathbb{R}^{2n})$. Then $a^{\mathrm{Wick}} = W^* a^\mu W$ and $1^{\mathrm{Wick}} = Id_{L^2(\mathbb{R}^n)}$ where a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_H = WW^*$ is the orthogonal projection on a closed proper subspace H of $L^2(\mathbb{R}^{2n})$. Moreover, we have*

$$\left\| a^{\mathrm{Wick}} \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \|a\|_{L^\infty(\mathbb{R}^{2n})},$$

$$a(X) \ge 0 \text{ for all } X \text{ implies } a^{\mathrm{Wick}} \ge 0,$$

$$a^{\mathrm{Wick}} = a^w + r^w, \quad r(X) = \iint_0^1 (1-\theta) a''(X+\theta Y) Y^2 2^n e^{-2\pi|Y|^2} dY d\theta.$$

The proof is standard and can be found e.g. in [L5].

The main result of this section is the following

Theorem 3.2.3. *Let n be an integer and $q(t,x,\xi)$ be a real-valued symbol in $C_b^\infty(\mathbb{R}^{2n})$: q is defined on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, smooth with respect to t,x,ξ and such that, for all multi-indices α,β, $\sup|(\partial_x^\alpha\partial_\xi^\beta q)(t,x,\xi)| < +\infty$. Assume moreover that $\tau + iq$ satisfies condition $(\overline{\psi})$ and*

$$q(t,x,\xi) = 0 \implies d_{x,\xi} q(t,x,\xi) = 0, \tag{3.2.2}$$

$$\text{for some } k \in \mathbb{N}, \quad \inf|\partial_t^k q(t,x,\xi)| > 0. \tag{3.2.3}$$

Then there exists some positive constants C, h_0, such that, for $h \in (0, h_0]$, for any $u(t,x) \in C_c^1(\mathbb{R}, L^2(\mathbb{R}^n))$,

$$C\left\|hD_t u + iq(t,x,h\xi)^w u\right\|_{L^2(\mathbb{R}^{n+1})} \ge h^{\frac{k}{k+1}} \|u\|_{L^2(\mathbb{R}^{n+1})}. \tag{3.2.4}$$

The above estimate will be called a subelliptic estimate with loss of $k/(k+1)$ derivative, consistently with (1.2.2).

Let us note right now that the condition (3.2.2) is satisfied by nonnegative (and nonpositive) functions. However that condition may be satisfied by some functions which may change sign such as

$$q = ta(t,x,\xi), \quad a \ge 0.$$

In fact, if $a = 0$ we have $da = 0$, so that $dq = 0$; at $t = 0$, we have $d_{x,\xi} q = 0$.

Going back to our operator $hD_t + iq(t,x,h\xi)^w$, we shall first replace it by the unitary equivalent $hD_t + iq(t, h^{1/2}x, h^{1/2}\xi)^w$ acting on $C_c^1(\mathbb{R}_t; L^2(\mathbb{R}^n))$. Next, one can check, using the proposition 3.2.2, that

$$q(t, h^{1/2}x, h^{1/2}\xi)^w = q(t, h^{1/2}x, h^{1/2}\xi)^{\mathrm{Wick}} + O(h), \quad \text{in } \mathcal{L}(L^2(\mathbb{R}^n)).$$

Also, defining $\Phi(t) = Wu(t) \in L^2(\mathbb{R}^{2n})$ (see def.3.2.1), we are reduced to proving an estimate for

$$P = hD_t + i\pi_H q(t, h^{1/2}X)\pi_H,$$

where the Toeplitz orthogonal projection $\pi_H = WW^*$. Now we see that for $\Phi \in C_c^1(\mathbb{R}_t; H)$, with $K = H^\perp$, $L^2(\mathbb{R}^{2n})$ norms,

$$\begin{aligned}\|(hD_t + iq(t, h^{1/2}X))\Phi\|^2 = \|P\Phi\|^2 + \|\pi_K q(t, h^{1/2}X)\Phi\|^2 \\ = \|P\Phi\|^2 + \|[\pi_H, q(t, h^{1/2}\cdot)]\Phi\|^2.\end{aligned}$$

Handling the linear ODE $\quad \mathcal{P} = hD_t + iq(t, h^{1/2}X)$ is a simple matter and we obtain

$$C\,\|\mathcal{P}\Phi\|\,\|\Phi\| \ge \langle |q(t, h^{1/2}X)|\Phi, \Phi\rangle + h^{k/k+1}\,\|\Phi\|^2\,.$$

The nasty term $\|[\pi_H, q(t, h^{1/2}\cdot)]\Phi\|^2$ can be estimated by $h\|\nabla q(t, h^{1/2}\cdot)\Phi\|^2$, and thanks to (3.2.2), we can prove

$$\langle |\nabla q(t, h^{1/2}X)|^2\Phi, \Phi\rangle \le 2\langle |q(t, h^{1/2}X)|\Phi, \Phi\rangle \sup|\nabla^2 q(t, \cdot)|,$$

yielding

$$\begin{aligned}\langle |q(t, h^{1/2}\cdot)|\Phi, \Phi\rangle + h^{k/k+1}\,\|\Phi\|^2 &\le C_1\,\|P\Phi\|\,\|\Phi\| + C_1\|[\pi_H, q(t, h^{1/2}\cdot)]\Phi\|\,\|\Phi\| \\ &\le C_1\,\|P\Phi\|\,\|\Phi\| + \underbrace{C_2\langle |q(t, h^{1/2}\cdot)|\Phi, \Phi\rangle^{1/2} h^{1/2}\,\|\Phi\|}_{\text{can be absorbed in the lhs}}.\end{aligned}$$

3.3. Subellipticity under condition $(\overline{\psi})$. The previous argument can be generalized beyond condition (P) and also in many non-finite type situations. We should keep in mind that the most important point in the previous argument is a way to handle the commutator $[\pi_H, q]$. If we can control this commutator, the coherent states method outlined above reduces the problem to a linear ODE with parameters, a much more elementary problem. The conclusion of the theorem 3.2.3 still holds when (3.2.2-3) are replaced by the existence of J with length k (see section 3.1) such that $\inf|p_J| > 0, p = \tau + iq$. A semi-classical version is given in proposition 27.6.1 of [H7] but is much more difficult to reach than in the previous section.

Let us quote an excerpt of L.Hörmander's contribution [H10] to the book *Fields medallists' lectures*: *"For the scalar case, Egorov* [Eg] *found necessary and sufficient conditions for subellipticity with loss of δ derivatives; the proof of sufficiency was completed in* [H5]*. A slight modification of the presentation is given in* [H7]*, but it is still very complicated technically. Another approach which also covers systems operating on scalars has been given by Nourrigat* [No] *(see also the book* [HN] *by Helffer and Nourrigat), but it is also far from simple so the study of subelliptic operators may not yet be in a final form."*

4. Non-finite type geometry

First of all, one should not think that it concerns complicated operators. A simple analytic example is the degenerate Cauchy-Riemann symbol $\xi_1 + ix_1^2\xi_2$. We have, using the notations of section 3.1,

$$\begin{aligned} p_{12} &= \{\xi_1, x_1^2\xi_2\} = 2x_1\xi_2, \\ p_{112} &= \{\xi_1, 2x_1\xi_2\} = 2\xi_2, \quad p_{212} = \{x_1^2\xi_2, 2x_1\xi_2\} = 0. \end{aligned}$$

Assume that $p_J = \xi_2\alpha(x,\xi)$ which is true for $|J| = 2, 3$. Then

$$\{p_1, p_J\} = \{\xi_1, \alpha\xi_2\} = \xi_2\alpha'_{x_1}, \quad \text{and} \quad \{x_1^2\xi_2, \alpha\xi_2\} = \{x_1^2\xi_2, \alpha\}\xi_2,$$

so that all the iterated brackets are of the form $\xi_2\alpha$ and thus vanish at $\xi_1 = 0 = \xi_2$ as well as p.

4.1. Condition (P), two dimensions. Let p a principal type complex-valued semi-classical symbol such that p and $\bar p$ satisfy $(\overline{\psi})$, i.e. p satisfies condition (P), i.e. no change of sign occurs for $\operatorname{Im} p$ along the bicharacteristic flow of $\operatorname{Re} p$. Note that in the homogeneous case, that property is indeed a consequence of $(\overline{\psi})$ for *differential operators* since they have a symbol such that

$$p(x, -\xi) = (-1)^m p(x,\xi),$$

so that prohibit a change of sign from + to − is equivalent to forbid any change of sign. Under this assumption, R.Beals and C.Fefferman ([BF]) and also under condition $(\overline{\psi})$ in the homogeneous 2D case (N.L. [L1]) and for the classical oblique derivative problem (N.L., [L2]) , it is possible to prove an estimate with loss of one derivative, the same estimate as for ∂_{x_1}.

Theorem 4.1.1. *Let n be an integer and $q(t,x,\xi)$ be a real-valued symbol in $C_b^\infty(\mathbb{R}^{2n})$: q is defined on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, smooth with respect to t,x,ξ and such that, for all multi-indices α,β, $\sup|(\partial_x^\alpha\partial_\xi^\beta q)(t,x,\xi)| < +\infty$. Assume moreover that $\tau + iq$ satisfies condition $(\overline{\psi})$ and condition (ψ):*

$$q(t,x,\xi)q(s,x,\xi) \geq 0.$$

Then there exist some positive constants C, h_0 and δ, such that, for $h \in (0, h_0]$, for any $u(t,x) \in C_c^1((-\delta,\delta), L^2(\mathbb{R}^n))$,

$$C\left\|hD_t u + iq(t,x,h\xi)^w u\right\|_{L^2(\mathbb{R}^{n+1})} \geq h\left\|u\right\|_{L^2(\mathbb{R}^{n+1})}. \tag{4.1.1}$$

The proof is not simple, but the steps are neatly identified.

(1) Using the Malgrange-Weierstrass preparation theorem on normal forms and the Egorov result on conjugation by Fourier integral operators, we get indeed a reduction to

$$hD_t + iq(t, xh^{1/2}, \xi h^{1/2})^w. \tag{4.1.2}$$

(2) Using a Calderón-Zygmund decomposition of the family of functions

$$(q(t, h^{1/2}x, h^{1/2}\xi))_{|t|\le 1}$$

and we can cut the $\mathbb{R}^{2n}_{x,\xi}$ phase space via a new calculus of pseudodifferential. One defines $H(x,\xi)$ so that

$$1 \le H^{-1}(x,\xi) = \max\bigl(\sup_{|t|\le 1} h^{-1}|q(t, h^{\frac12}x, h^{\frac12}\xi)|, \sup_{|t|\le 1} h^{-1}|(\nabla_{x,\xi}q)(t, h^{\frac12}x, h^{\frac12}\xi)|^2, 1\bigr) \le Ch^{-1}.$$

This implies with $a(t,x,\xi) = q(t, h^{\frac12}x, h^{\frac12}\xi)$,

$$\begin{aligned} |a(t,x,\xi)| &\le hH^{-1},\\ \max\bigl(|\partial_x a(t,x,\xi)|, |\partial_\xi a(t,x,\xi)|\bigr) &\le hH^{-1/2} = hH^{-1}H^{1/2},\\ \text{for } |\alpha|+|\beta|\ge 2, \quad |\partial_x^\alpha\partial_\xi^\beta a(t,x,\xi)| &\le C_{\alpha\beta}h^{\frac12(|\alpha|+|\beta|)} \le C_{\alpha\beta}hH^{-1}H^{\frac12(|\alpha|+|\beta|)} \end{aligned}$$

since the two first lines are obvious whereas in the last one $|\alpha|+|\beta|-2\ge 0$ and $h \le CH$. The weight $H(x,\xi)$ is shown to be slowly varying, i.e. there exists $r>0$ such that for all $(x,\xi),(y,\eta)\in\mathbb{R}^{2n}$

$$|x-y| + |\xi-\eta| \le rH(x,\xi)^{-1/2} \Longrightarrow r \le H(x,\xi)H(y,\eta)^{-1} \le r^{-1}.$$

As a result, the symbol $h^{-1}q(t, h^{1/2}x, h^{1/2}\xi)$ behaves like a symbol in S^1_{scl} with a new small parameter H, so that the operator (4.1.2) is of type

$$hH^{-1}\bigl(HD_t + iQ(t, H^{1/2}x, H^{1/2}\xi)^w\bigr)$$

(3) Reduction to three models: the first two are $HD_t + iQ(t, xH^{1/2}, \xi H^{1/2})$, $\pm Q \ge 0$, and quite simple to handle, and the last model is

$$HD_t + ia(t, xH^{1/2}, \xi H^{1/2})b(xH^{1/2}, \xi H^{1/2}), \quad a \ge 0. \tag{4.1.3}$$

Although a factorization (4.1.3) can be obtained for differential operators with analytic regularity satisfying condition $(\overline{\psi})$, as shown in [NT3], such a factorization is not true in the C^∞ case, even microlocally in the standard sense, according to the remark 4.1.3 below.

The energy method introduced by Nirenberg and Treves in [NT3] can be applied to this last model. The method of proof in [L1], [L2] is also based upon a factorization analogous to (4.1.3) but where $b(x,\xi)$ is replaced by $\beta(t,x)|\xi|$ and β is a smooth function such that $t \mapsto \beta(t,x)$ does not change sign from $+$ to $-$ when t increases. Then a properly defined sign of $\beta(t,x)$ appears as a non-decreasing operator and the Nirenberg-Treves energy method can be adapted to this situation.

Remark 4.1.2. The Beals-Fefferman result mentioned above proved the local existence of H^{s+m-1}_{loc} solutions u to the equation $Lu = f$ with a source f in H^s_{loc}, whenever L is an operator of order m satisfying condition (P); since the size of the neigbourhood where the equation is satisfied may depend on the index s, this is not enough to get C^∞ solutions whenever f is smooth. The existence of C^∞ solutions for C^∞ sources was proved by L.Hörmander in [H4] for pseudodifferential equations satisfying condition (P). We refer the reader to the papers [H9], [L6], for a more detailed historical overview of this problem. On the other hand, it is clear that our interest is focused on solvability in the C^∞ category. Let us nevertheless recall that the sufficiency of condition (ψ) in the analytic category (for microdifferential operators acting on microfunctions) was proved by J.-M.Trépreau [Tr] (see also [H8], chapter VII).

Remark 4.1.3. Consider the C^∞ function q defined on $\mathbb{R}^3$ by

$$q(t,x,\xi) = \begin{cases} (\xi - te^{-1/x})^2 & \text{if } x > 0, \\ \xi(\xi - e^{1/x}) & \text{if } x < 0. \end{cases}$$

For every fixed (x,ξ), the function $t \mapsto q(t,x,\xi)$ does not change sign since we have $q(t,x,\xi)q(s,x,\xi) \geq 0$. Nevertheless one can show that it is not possible to find some C^∞ functions a, b such that a is nonnegative and b independent of t such that $q = ab$.

4.2. The estimate with loss 1 does not follow from $(\overline{\psi})$. For many years, repeated claims were made that condition $(\overline{\psi})$ implies (1.2.2) with $\mu = 1$, entailing that solvability with loss of 1 derivative is a consequence of condition (ψ). It turned out that these claims were wrong, as shown in [L3] by the following result (see also section 6 in the survey [H9]).

Theorem 4.2.1. *There exists a homogeneous principal type first-order pseudo-differential operator L in three dimensions, satisfying condition (ψ), a sequence $(u_k)_{k\geq 1}$ of C^∞_c functions with* $\operatorname{supp} u_k \subset \{x \in \mathbb{R}^3, |x| \leq 1/k\}$ *such that*

$$\|u_k\|_{L^2(\mathbb{R}^3)} = 1, \quad \lim_{k\to+\infty} \|L^* u_k\|_{L^2(\mathbb{R}^3)} = 0. \tag{4.2.1}$$

As a consequence, for this L, there exists $f \in L^2$ such that the equation $Lu = f$ has no local solution u in L^2. We shall now briefly examine some of the main features of this counterexample, leaving aside the technicalities which can be found in the papers quoted above. Let us try, with $(t,x,y)\in\mathbb{R}^3$,

$$L = D_t - ia(t)\bigl(D_x + H(t)V(x)|D_y|\bigr), \tag{4.2.2}$$

with $H=\mathbf{1}_{\mathbb{R}_+}$, $C^\infty(\mathbb{R})\ni V\ge 0$, $C^\infty(\mathbb{R})\ni a\ge 0$ flat at 0. Since the function

$$q(t,x,y,\xi,\eta) = -a(t)\bigl(\xi + H(t)V(x)|\eta|\bigr)$$

is the product of the non-positive function $-a(t)$ by the non-decreasing function

$$t\mapsto \xi + H(t)V(x)|\eta|,$$

the operator L satisfies condition (ψ) (and thus L^* satisfies condition $(\overline{\psi})$).

To simplify the exposition, let us assume that $a\equiv 1$, which introduces a rather unimportant singularity in the t-variable, let us replace $|D_y|$ by a positive (large) parameter h^{-1}, which allows us to work now only with the two real variables t,x and let us set $W=h^{-1}V$. We are looking for a non-trivial solution $u(t,x)$ of $L^*u=0$, which means then

$$\partial_t u = \begin{cases} D_x u, & \text{for } t<0,\\ \bigl(D_x+W(x)\bigr)u, & \text{for } t>0.\end{cases}$$

The operator D_x+W is unitarily equivalent to D_x: with $A'(x)=W(x)$, we have

$$D_x + W(x) = e^{-iA(x)}D_x e^{iA(x)},$$

so that the negative eigenspace of the operator $D_x+W(x)$ is $\{v\in L^2(\mathbb{R}), \operatorname{supp}\widehat{e^{iA}v}\subset\mathbb{R}_-\}$. Since we want u to decay when $t\to\pm\infty$, we need to choose $v_1,v_2\in L^2(\mathbb{R})$, such that

$$u(t,x) = \begin{cases} e^{tD_x}v_1, \quad \operatorname{supp}\widehat{v_1}\subset\mathbb{R}_+ & \text{for } t<0,\\ e^{t(D_x+W)}v_2, \quad \operatorname{supp}\widehat{e^{iA}v_2}\subset\mathbb{R}_- & \text{for } t>0.\end{cases} \tag{4.2.3}$$

We shall not be able to choose $v_1=v_2$ in (4.2.3), so we could only hope for L^*u to be small if $\|v_2-v_1\|_{L^2(\mathbb{R})}$ is small. Thus this counterexample is likely to work if the unit spheres of the vector spaces

$$E_1^+ = \{v\in L^2(\mathbb{R}), \operatorname{supp}\widehat{v}\subset\mathbb{R}_+\} \quad\text{and}\quad E_2^- = \{v\in L^2(\mathbb{R}), \operatorname{supp}\widehat{e^{iA}v}\subset\mathbb{R}_-\}$$

are close. Note that since $W\ge 0$, we get $E_1^+\cap E_2^-=\{0\}$: in fact, with $L^2(\mathbb{R})$ scalar products, we have

$$v\in E_1^+\cap E_2^- \Longrightarrow 0 \overset{v\in E_1^+}{\le} \langle Dv,v\rangle \overset{0\le W}{\le} \langle (D+W)v,v\rangle \overset{v\in E_2^-}{\le} 0 \Longrightarrow \langle Dv,v\rangle = 0,$$

which gives $v = 0$ since $v \in E_1^+$. Nevertheless, the "angle" between E_1^+ and E_2^- could be small for a careful choice of a positive W. It turns out that $W_0(x) = \pi\delta_0(x)$ is such a choice. Of course, several problems remain such as regularize W_0 in such a way that it becomes a first-order semi-classical symbol, redo the same construction with a smooth function a flat at 0 and various other things.

Anyhow, these difficulties eventually turn out to be only technical, and *in fine*, the actual reason for which the theorem 4.2.1 is true is simply that the positive eigenspace of D_x (i.e. $L^2(\mathbb{R})$ functions whose Fourier transform is supported in $\mathbb{R}_+$) could be arbitrarily close to the negative eigenspace of $D_x + W(x)$ for some non-negative W, triggering nonsolvability in L^2 for the three-dimensional model operator $D_t - ia(t)\big(D_x + \mathbf{1}_{\mathbb{R}_+}(t)W(x)|D_y|\big)$, and the existence of a quasi-mode for the adjoint operator

$$D_t + ia(t)\big(D_x + \mathbf{1}_{\mathbb{R}_+}(t)W(x)|D_y|\big), \tag{4.2.4}$$

where a is some non-negative function, flat at 0. This phenomenon is called the "drift" in [L3] and could not occur for differential operators or for pseudodifferential operators in two dimensions.

A more geometric point of view is that for a principal type symbol p, satisfying condition $(\overline{\psi})$, one may have bicharacteristics of $\operatorname{Re} p$ which stay in the set $\{\operatorname{Im} p = 0\}$. This can even occur for operators satisfying condition (P). However condition (P) ensures that the nearby bicharacteristics of $\operatorname{Re} p$ stay either in $\{\operatorname{Im} p \geq 0\}$ or in $\{\operatorname{Im} p \leq 0\}$. This is no longer the case when condition $(\overline{\psi})$ holds, although the bicharacteristics are not allowed to pass from $\{\operatorname{Im} p > 0\}$ to $\{\operatorname{Im} p < 0\}$. The situation of having a bicharacteristic of $\operatorname{Re} p$ staying in $\{\operatorname{Im} p = 0\}$ will generically trigger the drift phenomenon mentioned above when condition (P) does not hold. So the counterexamples to solvability with loss of one derivative are in fact very close to operators satisfying condition (P).

4.3. An estimate with loss of 3/2 derivatives follows from condition $(\overline{\psi})$. In the preprint [D2] and in the paper [D3], N.Dencker has proven that $(\overline{\psi})$ implies (1.2.2) with $\mu = 2$, establishing as a consequence the Nirenberg-Treves conjecture on local solvability of pseudodifferential operators, i.e. $(\psi) \Longleftrightarrow$ local solvability. Later on, in [D4], he proved that $(\overline{\psi})$ implies (1.2.2) with $\mu = \epsilon + 3/2$ for any $\epsilon > 0$ and N.L. ([L7]) obtained that condition $(\overline{\psi})$ implies (1.2.2) with $\mu = 3/2$, erasing the ϵ.

4.3.1. Preliminary comments. One of the difficulty related to the handling of (1.2.1) when the loss μ is > 1 is the following: condition $(\overline{\psi})$ is only concerned with the principal symbol, so that the estimate (1.2.1) should be preserved when the principal-type P is perturbed by a pseudodifferential operator of order -1. However, the estimate (1.2.1) is too weak to absorb directly a perturbation of order -1 and there is no way to avoid this situation under the sole condition $(\overline{\psi})$ since an estimate with loss of one derivative is not a consequence of condition

$(\overline{\psi})$(section 4.2) (it could be possible that the analyticity of the symbol and (ψ) imply a factorization of type (4.1.3)). The method of proof used by N.Dencker is based upon an energy method, rather classical in its principles, which was introduced by L.Nirenberg and F.Treves and developed by R.Beals and C.Fefferman. But although these authors were able to separate sharply the forward and backward regions of propagation for operators satisfying condition (P), N.Dencker defines these regions in the more general case of condition $(\overline{\psi})$ and construct a multiplier smoother than a sign function. Although that smoothness forces a loss of derivatives larger than one, he can take advantage of it to handle some calculus of pseudodifferential operators. A version of one of his most striking arguments appears below as Lemma 4.3.10 and shows that the rigidity of condition $(\overline{\psi})$ entails strong regularity properties for the set where the key change of sign occurs.

4.3.2. The geometry of condition $(\overline{\psi})$. Here we shall consider that the phase space is equipped with a *symplectic quadratic form* Γ (Γ is a positive definite quadratic form such that $\Gamma = \Gamma^\sigma$, see the definition 5.3.1(ii) in the appendix). It is possible to find some linear symplectic coordinates (x, ξ) in $\mathbb{R}^{2n}$ such that $\Gamma(x, \xi) = |(x, \xi)|^2 = \sum_{1 \le j \le n} x_j^2 + \xi_j^2$. The running point of our Euclidean symplectic $\mathbb{R}^{2n}$ will be usually denoted by X or by an uppercase letter such as Y, Z. The open Γ-ball with center X and radius r will be denoted by $B(X, r)$. Let $q(t, X, \Lambda)$ be a smooth real-valued function[2] defined on $\Xi = \mathbb{R} \times \mathbb{R}^{2n} \times [1, +\infty)$, vanishing for $|t| \ge 1$ and satisfying

$$\forall k \in \mathbb{N}, \sup_{\Xi} \|\partial_X^k q\|_\Gamma \Lambda^{-1+\frac{k}{2}} = \gamma_k < +\infty, \text{ i.e. } q(t, \cdot) \in S(\Lambda, \Lambda^{-1}\Gamma), \tag{4.3.1}$$

$$s > t \quad \text{and } q(t, X, \Lambda) > 0 \Longrightarrow q(s, X, \Lambda) \ge 0. \tag{4.3.2}$$

Notation. The Euclidean norm $\Gamma(X)^{1/2}$ is fixed and the norms of the vectors and of the multilinear forms are taken with respect to that norm. We shall write everywhere $|\cdot|$ instead of $\|\cdot\|_\Gamma$. Furthermore, we shall say that C is a "fixed" constant if it depends only on a finite number of γ_k above and on the dimension n.

We shall always omit the dependence of q with respect to the large parameter Λ and write $q(t, X)$ instead of $q(t, X, \Lambda)$. The operator $Q(t) = q(t)^w$ will stand for the operator with Weyl symbol $q(t, X)$. We introduce now for $t \in \mathbb{R}$, following [H11],

$$\mathbb{X}_+(t) = \cup_{s \le t} \{X \in \mathbb{R}^{2n}, q(s, X) > 0\},\ \mathbb{X}_-(t) = \cup_{s \ge t} \{X \in \mathbb{R}^{2n}, q(s, X) < 0\}, \tag{4.3.3}$$

$$\mathbb{X}_0(t) = \mathbb{X}_-(t)^c \cap \mathbb{X}_+(t)^c. \tag{4.3.4}$$

Thanks to (4.3.2), $\mathbb{X}_+(t), \mathbb{X}_-(t)$ are disjoint open subsets of $\mathbb{R}^{2n}$; moreover $\mathbb{X}_0(t), \mathbb{X}_0(t) \cup \mathbb{X}_\pm(t)$ are closed since their complements are open. The three sets $\mathbb{X}_0(t), \mathbb{X}_\pm(t)$ are two by

[2] Since our semi-classical symbol q in this section is in fact of type $h^{-1}F(t, h^{1/2}x, h^{1/2}\xi)$ where F belongs to $C_b^\infty(\mathbb{R}^{2n+1})$, we have preferred, to avoid confusion, using a large parameter $\Lambda \ge 1$ and symbols satisfying (4.3.1). A standard semi-classical result is given at the end of this section with Theorem 4.3.22.

two disjoint with union $\mathbb{R}^{2n}$ (note also that $\overline{\mathbb{X}_\pm(t)} \subset \mathbb{X}_0(t) \cup \mathbb{X}_\pm(t)$ since $\mathbb{X}_0(t) \cup \mathbb{X}_\pm(t)$ are closed). When t increases, $\mathbb{X}_+(t)$ increases and $\mathbb{X}_-(t)$ decreases. The following three lemmas are easy and can be found as Lemmas 2.1.1-2-3 in [L7].

Lemma 4.3.1. *Let (E, d) be a metric space, $A \subset E$ and $\kappa > 0$ be given. We define $\Psi_{A,\kappa}(x) = \kappa$ if $A = \emptyset$ and if $A \neq \emptyset$, we define*

$$\Psi_{A,\kappa}(x) = \min\bigl(d(x, A), \kappa\bigr).$$

The function $\Psi_{A,\kappa}$ is valued in $[0, \kappa]$, Lipschitz continuous with a Lipschitz constant ≤ 1. Moreover, the following implication holds: $A_1 \subset A_2 \subset E \Longrightarrow \Psi_{A_1,\kappa} \geq \Psi_{A_2,\kappa}$.

Lemma 4.3.2. *For each $X \in \mathbb{R}^{2n}$, the function $t \mapsto \Psi_{\mathbb{X}_+(t),\kappa}(X)$ is decreasing and for each $t \in \mathbb{R}$, the function $X \mapsto \Psi_{\mathbb{X}_+(t),\kappa}(X)$ is supported in $\mathbb{X}_+(t)^c = \mathbb{X}_-(t) \cup \mathbb{X}_0(t)$. For each $X \in \mathbb{R}^{2n}$, the function $t \mapsto \Psi_{\mathbb{X}_-(t),\kappa}(X)$ is increasing and for each $t \in \mathbb{R}$, the function $X \mapsto \Psi_{\mathbb{X}_-(t),\kappa}(X)$ is supported in $\mathbb{X}_-(t)^c = \mathbb{X}_+(t) \cup \mathbb{X}_0(t)$. As a consequence the function $X \mapsto \Psi_{\mathbb{X}_+(t),\kappa}(X)\Psi_{\mathbb{X}_-(t),\kappa}(X)$ is supported in $\mathbb{X}_0(t)$.*

Lemma 4.3.3. *For $\kappa > 0, t \in \mathbb{R}, X \in \mathbb{R}^{2n}$, we define*[3]

$$\sigma(t, X, \kappa) = \Psi_{\mathbb{X}_-(t),\kappa}(X) - \Psi_{\mathbb{X}_+(t),\kappa}(X). \tag{4.3.5}$$

The function $t \mapsto \sigma(t, X, \kappa)$ is increasing and valued in $[-\kappa, \kappa]$, the function $X \mapsto \sigma(t, X, \kappa)$ is Lipschitz continuous with Lipschitz constant less than 2; we have

$$\sigma(t, X, \kappa) = \begin{cases} \min(|X - \mathbb{X}_-(t)|, \kappa) & \text{if } X \in \mathbb{X}_+(t), \\ -\min(|X - \mathbb{X}_+(t)|, \kappa) & \text{if } X \in \mathbb{X}_-(t). \end{cases}$$

We have $\{X \in \mathbb{R}^{2n}, \sigma(t, X, \kappa) = 0\} \subset \mathbb{X}_0(t) \subset \{X \in \mathbb{R}^{2n}, q(t, X) = 0\}$, and

$$\begin{aligned} \{X \in \mathbb{R}^{2n}, \pm q(t, X) > 0\} \subset \mathbb{X}_\pm(t) \subset \{X \in \mathbb{R}^{2n}, \pm\sigma(t, X, \kappa) > 0\} \\ \subset \{X \in \mathbb{R}^{2n}, \pm\sigma(t, X, \kappa) \geq 0\} \subset \{X \in \mathbb{R}^{2n}, \pm q(t, X) \geq 0\}. \end{aligned} \tag{4.3.6}$$

Definition 4.3.4. Let $q(t, X)$ be as above. We define

$$\delta_0(t, X) = \sigma(t, X, \Lambda^{1/2}) \tag{4.3.7}$$

and we notice that from the previous lemmas, $t \mapsto \delta_0(t, X)$ is increasing, taking its values in $[-\Lambda^{1/2}, \Lambda^{1/2}]$, satisfying

$$|\delta_0(t, X) - \delta_0(t, Y)| \leq 2|X - Y|, \tag{4.3.8}$$

and such that

$$\{X \in \mathbb{R}^{2n}, \delta_0(t, X) = 0\} \subset \{X \in \mathbb{R}^{2n}, q(t, X) = 0\}, \tag{4.3.9}$$

$$\{X \in \mathbb{R}^{2n}, \pm q(t, X) > 0\} \subset \{X, \pm\delta_0(t, X) > 0\} \subset \{X, \pm q(t, X) \geq 0\}. \tag{4.3.10}$$

[3] If the distances of X to both $\mathbb{X}_\pm(t)$ are less than κ, we have $\sigma(t, X, \kappa) = |X - \mathbb{X}_-(t)| - |X - \mathbb{X}_+(t)|$.

4.3.3. Some classes of symbols. The following lemma is elementary and is a good introduction to the Calderón-Zygmund methods. This is lemma 2.1.5 in [L7]. The general definition of the classes $S(m,g)$ is given in the appendix 5.3.

Lemma 4.3.5. *Let f be a symbol in $S(h^{-m}, h\Gamma)$ where m is a positive real number. We define*

$$\lambda(X) = 1 + \max_{\substack{0\le j<2m\\ j\in\mathbb{N}}} \big(\|f^{(j)}(X)\|_{\Gamma}^{\frac{2}{2m-j}}\big). \tag{4.3.11}$$

Then $f \in S(\lambda^m, \lambda^{-1}\Gamma)$ and the mapping from $S(\Lambda^m, \Lambda^{-1}\Gamma)$ to $S(\lambda^m, \lambda^{-1}\Gamma)$ is continuous. Moreover, with $\gamma = \max_{\substack{0\le j<2m\\ j\in\mathbb{N}}} \gamma_j^{\frac{2}{2m-j}}$, where the γ_j are the semi-norms of f, we have for all $X \in \mathbb{R}^{2n}$,

$$1 \le \lambda(X) \le 1 + \gamma\Lambda. \tag{4.3.12}$$

The metric $\lambda^{-1}\Gamma$ is admissible (def.5.3.1), with structure constants depending only on γ. It will be called the m-proper metric of f. The function λ above is a weight for the metric $\lambda^{-1}\Gamma$ and will be called the m-proper weight of f.

The following two lemmas are more involved and appear as lemmas 2.1.6-7 in [L7].

Lemma 4.3.6. *Let $q(t,X)$ and $\delta_0(t,X)$ be as above. We define, with $\langle s\rangle = (1+s^2)^{1/2}$,*

$$\mu(t,X) = \langle\delta_0(t,X)\rangle^2 + |\Lambda^{1/2} q'_X(t,X)| + |\Lambda^{1/2} q''_{XX}(t,X)|^2. \tag{4.3.13}$$

The metric $\mu^{-1}(t,\cdot)\Gamma$ is slowly varying with structure constants depending only on a finite number of semi-norms of q in $S(\Lambda, \Lambda^{-1}\Gamma)$. Moreover, there exists $C>0$, depending only on a finite number of semi-norms of q, such that

$$\mu(t,X) \le C\Lambda, \quad \frac{\mu(t,X)}{\mu(t,Y)} \le C(1+|X-Y|^2), \tag{4.3.14}$$

and we have

$$\Lambda^{1/2} q(t,X) \in S(\mu(t,X)^{3/2}, \mu^{-1}(t,\cdot)\Gamma), \tag{4.3.15}$$

so that the semi-norms depend only the semi-norms of q in $S(\Lambda, \Lambda^{-1}\Gamma)$.

Lemma 4.3.7. *Let $q(t,X), \delta_0(t,X), \mu(t,X)$ be as above. We define,*

$$\nu(t,X) = \langle\delta_0(t,X)\rangle^2 + |\Lambda^{1/2} q'_X(t,X)\mu(t,X)^{-1/2}|^2. \tag{4.3.16}$$

The metric $\nu^{-1}(t,\cdot)\Gamma$ is slowly varying with structure constants depending only on a finite number of semi-norms of q in $S(\Lambda, \Lambda^{-1}\Gamma)$. There exists $C>0$, depending only on a finite number of semi-norms of q, such that

$$\nu(t,X) \le 2\mu(t,X) \le C\Lambda, \quad \frac{\nu(t,X)}{\nu(t,Y)} \le C(1+|X-Y|^2), \tag{4.3.17}$$

and we have

$$\Lambda^{1/2} q(t,X) \in S(\mu(t,X)^{1/2}\nu(t,X), \nu(t,\cdot)^{-1}\Gamma), \tag{4.3.18}$$

so that the semi-norms of this symbol depend only on the semi-norms of q in $S(\Lambda, \Lambda^{-1}\Gamma)$. Moreover the function $\mu(t,X)$ is a weight for the metric $\nu(t,\cdot)^{-1}\Gamma$.

We wish now to discuss the normal forms attached to the metric $\nu^{-1}(t,\cdot)\Gamma$ for the symbol $q(t,\cdot)$. In the sequel of this section, we consider that t is fixed.

Definition 4.3.8. Let $0 < r_1 \le 1/2$ be given. With ν defined in (4.3.16), we shall say that

(i) Y is a nonnegative (resp. nonpositive) point at level t if

$$\delta_0(t,Y) \ge r_1 \nu(t,Y)^{1/2}, \quad (\text{resp.} \quad \delta_0(t,Y) \le -r_1\nu(t,Y)^{1/2}).$$

(ii) Y is a gradient point at level t if

$$|\Lambda^{1/2} q'_Y(t,Y)\mu(t,Y)^{-1/2}|^2 \ge \nu(t,Y)/4 \quad \text{and} \quad \delta_0(t,Y)^2 < r_1^2 \nu(t,Y).$$

(iii) Y is a negligible point in the remaining cases

$$|\Lambda^{1/2} q'_Y(t,Y)\mu(t,Y)^{-1/2}|^2 < \nu(t,Y)/4 \quad \text{and} \quad \delta_0(t,Y)^2 < r_1^2 \nu(t,Y).$$

Note that this implies $\nu(t,Y) \le 1 + r_1^2\nu(t,Y) + \nu(t,Y)/4 \le 1 + \nu(t,Y)/2$ and thus $\nu(t,Y) \le 2$.

Note that if Y is a nonnegative point, from (4.3.8) we get, for $T \in \mathbb{R}^{2n}$, $|T| \le 1, 0 \le r \le r_1/4$

$$\delta_0\big(t, Y + r\nu^{1/2}(t,Y)T\big) \ge \delta_0(t,Y) - 2r\nu^{1/2}(t,Y) \ge \frac{r_1}{2}\nu^{1/2}(t,Y)$$

and from (4.3.10), this implies that $q(t,X) \ge 0$ on the ball $B(Y, r\nu^{1/2}(t,Y))$. Similarly if Y is a nonpositive point, $q(t,X) \le 0$ on the ball $B(Y, r\nu^{1/2}(t,Y))$. Moreover if Y is a gradient point, we have $|\delta_0(t,Y)| < r_1\nu(t,Y)^{1/2}$ so that, if $Y \in \mathbb{X}_+(t)$, we have $\min(|Y - \mathbb{X}_-(t)|, \Lambda^{1/2}) < r_1\nu(t,Y)^{1/2}$ and if r_1 is small enough, since $\nu \lesssim \Lambda$, we get that $|Y - \mathbb{X}_-(t)| < r_1\nu(t,Y)^{1/2}$ which implies that there exists $Z_1 \in \mathbb{X}_-(t)$ such that $|Y - Z_1| < r_1\nu(t,Y)^{1/2}$. On the segment $[Y, Z_1]$, the Lipschitz continuous function is such that $\delta_0(t,Y) > 0$ ($Y \in \mathbb{X}_+(t)$ cf. Lemma 4.3.3) and $\delta_0(t,Z_1) < 0$ ($Z_1 \in \mathbb{X}_-(t)$); as a result, there exists a point Z (on that segment) such that $\delta_0(t,Z) = 0$ and thus $q(t,Z) = 0$. Naturally the discussion for a gradient point Y in $\mathbb{X}_-(t)$, is analogous. If the gradient point Y belongs to $\mathbb{X}_0(t)$, we get right away $q(t,Y) = 0$, also from the lemma 4.3.3. The function

$$f(T) = \Lambda^{1/2} q\Big(t, Y + r_1\nu^{1/2}(t,Y)T\Big)\mu(t,Y)^{-1/2}\nu(t,Y)^{-1} \tag{4.3.19}$$

satisfies for r_1 small enough with respect to the semi-norms of q and c_0, C_0, C_1, C_2 fixed positive constants, $|T| \le 1$, from (4.3.18),

$$|f(T)| \le |S-T|C_0 r_1 \le C_1 r_1^2, \quad |f'(T)| \ge r_1 c_0, \quad |f''(T)| \le C_2 r_1^2.$$

The standard analysis (see the appendix A.7 in [L7]) of the Beals-Fefferman metric [BF] shows that, on $B(Y, r_1 \nu^{1/2}(t,Y))$

$$q(t,X) = \Lambda^{-1/2} \mu^{1/2}(t,Y) \nu^{1/2}(t,Y) e(t,X) \beta(t,X), \tag{4.3.20}$$

$$1 \le e \in S(1, \nu(t,Y)^{-1}\Gamma), \ \beta \in S(\nu(t,Y)^{1/2}, \nu(t,Y)^{-1}\Gamma), \tag{4.3.21}$$

$$\beta(t,X) = \nu(t,Y)^{1/2}(X_1 + \alpha(t,X')), \alpha \in S(\nu(t,Y)^{1/2}, \nu(t,Y)^{-1}\Gamma). \tag{4.3.22}$$

Lemma 4.3.9. *Let $q(t,X)$ be a smooth function satisfying* (4.3.1-2) *and let $t \in [-1,1]$ be given. The metric g_t on $\mathbb{R}^{2n}$ is defined as $\nu(t,X)^{-1}\Gamma$ where ν is defined in* (4.3.16)*. There exists $r_0 > 0$, depending only on a finite number of semi-norms of q in* (4.3.1) *such that, for any $r \in]0, r_0]$, there exists a sequence of points (X_k) in $\mathbb{R}^{2n}$, and sequences of functions $(\chi_k), (\psi_k)$ satisfying the properties in the lemma 5.3.3 such that there exists a partition of $\mathbb{N}$,*

$$\mathbb{N} = E_+ \cup E_- \cup E_0 \cup E_{00}$$

so that, according to the definition 4.3.8, $k \in E_+$ means that X_k is a nonnegative point, ($k \in E_-$:X_k nonpositive point; $k \in E_0$:X_k gradient point, $k \in E_{00}$:X_k negligible point).

This lemma is an immediate consequence of the definitions 4.3.8 and 5.3.1 and of lemma 4.3.7, asserting that the metric g_t is admissible.

4.3.4. Some lemmas on C^3 functions. We give in this section a key result on the second derivative f''_{XX} of a real-valued smooth function $f(t,X)$ such that $\tau + if(t,x,\xi)$ satisfies condition $(\overline{\psi})$. The following claim gives a good qualitative version of what is needed for our estimates. Although we shall not use that (very simple) result, proving the following claim may serve as a good warm-up exercise for the more difficult sequel.

Claim. Let f_1, f_2 be two real-valued twice differentiable functions defined on an open set Ω of $\mathbb{R}^N$ and such that $f_1^{-1}(\mathbb{R}_+^*) \subset f_2^{-1}(\mathbb{R}_+)$ (i.e. $f_1(x) > 0 \Longrightarrow f_2(x) \ge 0$). If for some $\omega \in \Omega$, the conditions $f_1(\omega) = f_2(\omega) = 0, \quad df_1(\omega) \ne 0, df_2(\omega) = 0$ are satisfied, we have $f_2''(\omega) \ge 0$ (as a quadratic form).

This claim has the following consequence: take three functions f_1, f_2, f_3, twice differentiable on Ω, such that, for $1 \le j \le k \le 3$, $f_j(x) > 0 \Longrightarrow f_k(x) \ge 0$. Assume that, at some point ω we have $f_1(\omega) = f_2(\omega) = f_3(\omega) = 0, \quad df_1(\omega) \ne 0, df_3(\omega) \ne 0, df_2(\omega) = 0$. Then one has $f_2''(\omega) = 0$: indeed, the previous claim gives $f_2''(\omega) \ge 0$ and it can be applied to the couple $(-f_3, -f_2)$ to get $-f_2''(\omega) \ge 0$.

Notation. The open Euclidean ball of $\mathbb{R}^N$ with center 0 and radius r will be denoted by B_r. For a k-multilinear symmetric form A on $\mathbb{R}^N$, we shall note $\|A\| = \max_{|T|=1} |AT^k|$ which is easily seen to be equivalent to the norm $\max_{|T_1|=\cdots=|T_k|=1} |A(T_1, \ldots, T_k)|$ since the symmetrized $T_1 \otimes \cdots \otimes T_k$ can be written a sum of k^{th} powers.

The next statement is a precise quantitative version of the previous claim and is lemma 2.2.2 in [L7].

Lemma 4.3.10. *Let $R_0 > 0$ and f_1, f_2 be real-valued functions defined in $\bar{B}_{R_0}$. We assume that f_1 is C^2, f_2 is C^3 and for $x \in \bar{B}_{R_0}$,*

$$f_1(x) > 0 \Longrightarrow f_2(x) \geq 0. \tag{4.3.23}$$

We define the non-negative numbers ρ_1, ρ_2, by

$$\rho_1 = \max\big(|f_1(0)|^{\frac{1}{2}}, |f_1'(0)|\big), \quad \rho_2 = \max\big(|f_2(0)|^{\frac{1}{3}}, |f_2'(0)|^{\frac{1}{2}}, |f_2''(0)|\big), \tag{4.3.24}$$

and we assume that, with a positive C_0,

$$0 < \rho_1, \quad \rho_2 \leq C_0 \rho_1 \leq R_0. \tag{4.3.25}$$

We define the non-negative numbers C_1, C_2, C_3, by

$$C_1 = 1 + C_0 \|f_1''\|_{L^\infty(\bar{B}_{R_0})}, \quad C_2 = 4 + \frac{1}{3}\|f_2'''\|_{L^\infty(\bar{B}_{R_0})}, \quad C_3 = C_2 + 4\pi C_1. \tag{4.3.26}$$

Assume that for some $\kappa_2 \in [0, 1]$, with $\kappa_2 C_1 \leq 1/4$,

$$\rho_1 = |f_1'(0)| > 0, \tag{4.3.27}$$

$$\max\big(|f_2(0)|^{1/3}, |f_2'(0)|^{1/2}\big) \leq \kappa_2 |f_2''(0)|, \tag{4.3.28}$$

$$B(0, \kappa_2^2 \rho_2) \cap \{x \in \bar{B}_{R_0}, f_1(x) \geq 0\} \neq \emptyset. \tag{4.3.29}$$

Then we have

$$|f_2''(0)_-| \leq C_3 \kappa_2 \rho_2, \tag{4.3.30}$$

where $f_2''(0)_-$ stands for the negative part of the quadratic form $f_2''(0)$. Note that, whenever (4.3.29) is violated, we get $B(0, \kappa_2^2\rho_2) \subset \{x \in \bar{B}_{R_0}, f_1(x) < 0\}$ (note that $\kappa_2^2 \rho_2 \leq \rho_2 \leq R_0$) and thus

$$\textit{distance}\big(0, \{x \in \bar{B}_{R_0}, f_1(x) \geq 0\}\big) \geq \kappa_2^2 \rho_2. \tag{4.3.31}$$

4.3.5. Inequalities for symbols. The next statement (theorem 2.3.1 in [L7]) is a (not-so-easy) consequence of the previous lemmas. A slightly weaker version of this theorem appeared for the first time in the Dencker's preprint [D2] and is certainly one of the main novelties brought forward by this author.

Theorem 4.3.11. *Let q be a symbol satisfying* (4.3.1-2) *and δ_0, μ, ν as defined above in* (4.3.7), (4.3.13) *and* (4.3.16). *For the real numbers t', t, t'', and $X \in \mathbb{R}^{2n}$, we define*

$$N(t', t'', X) = \frac{\langle \delta_0(t', X)\rangle}{\nu(t', X)^{1/2}} + \frac{\langle \delta_0(t'', X)\rangle}{\nu(t'', X)^{1/2}}, \tag{4.3.32}$$

$$R(t, X) = \Lambda^{-1/2}\mu(t, X)^{1/2}\nu(t, X)^{-1/2}\langle \delta_0(t, X)\rangle. \tag{4.3.33}$$

Then there exists a constant $C_0 \ge 1$, depending only on a finite number of semi-norms of q in (4.3.1), *such that, for $t' \le t \le t''$, we have*

$$C_0^{-1}R(t, X) \le N(t', t'', X) + \frac{\delta_0(t'', X) - \delta_0(t, X)}{\nu(t'', X)^{1/2}} + \frac{\delta_0(t, X) - \delta_0(t', X)}{\nu(t', X)^{1/2}}. \tag{4.3.34}$$

A differentiable function ψ of one variable is said to be quasi-convex on $\mathbb{R}$ if $\dot{\psi}(t)$ does not change sign from $+$ to $-$ for increasing t (see [H8]). In particular, a differentiable convex function is such that $\dot{\psi}(t)$ is increasing and is thus quasi-convex.

Definition 4.3.12. Let $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be an increasing function, $C_1 > 0$ and let $\rho_1 : \mathbb{R} \to \mathbb{R}_+$. We shall say that ρ_1 is quasi-convex with respect to (C_1, σ_1) if for $t_1, t_2, t_3 \in \mathbb{R}$,

$$t_1 \le t_2 \le t_3 \Longrightarrow \rho_1(t_2) \le C_1 \max\bigl(\rho_1(t_1), \rho_1(t_3)\bigr) + \sigma_1(t_3) - \sigma_1(t_1).$$

When σ_1 is a constant function and $C_1 = 1$, this is the definition of quasi-convexity. Let $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be an increasing function and let $\omega : \mathbb{R} \to \mathbb{R}_+$. We define

$$\rho_1(t) = \inf_{t' \le t \le t''}\Bigl(\omega(t') + \omega(t'') + \sigma_1(t'') - \sigma_1(t')\Bigr). \tag{4.3.35}$$

Then the function ρ_1 is quasi-convex with respect to $(2, \sigma_1)$.

The following lemma (lemma 2.4.3 in [L7]) is due to L.Hörmander [H11].

Lemma 4.3.13. *Let $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be an increasing function and let $\omega : \mathbb{R} \to \mathbb{R}_+$. Let $T > 0$ be given. We consider the function ρ_1 as given in definition 4.3.12 and we define*

$$\Theta_T(t) = \sup_{-T \le s \le t}\left\{\sigma_1(s) - \sigma_1(t) + \frac{1}{2T}\int_s^t \rho_1(r)dr - \rho_1(s)\right\}. \tag{4.3.36}$$

Then we have

$$2T\partial_t(\Theta_T + \sigma_1) \ge \rho_1, \quad \text{and for } |t| \le T, \quad |\Theta_T(t)| \le \rho_1(t). \tag{4.3.37}$$

Definition 4.3.14. For $T>0, X\in\mathbb{R}^{2n}, |t|\le T$, we define

$$\omega(t,X)=\frac{\langle\delta_0(t,X)\rangle}{\nu(t,X)^{1/2}},\ \sigma_1(t,X)=\delta_0(t,X),\ \eta(t,X)=\int_{-T}^{t}\delta_0(s,X)\Lambda^{-1/2}ds+2T,\quad (4.3.38)$$

where δ_0,ν are defined in (4.3.7), (4.3.16). For $T>0$, $(t,X)\in\mathbb{R}\times\mathbb{R}^{2n}$, we define $\Theta(t,X)$ by the formula (4.3.36),

$$\Theta(t,X)=\sup_{-T\le s\le t}\Big\{\sigma_1(s,X)-\sigma_1(t,X)+\frac{1}{2T}\int_s^t\rho_1(r,X)dr-\rho_1(s,X)\Big\},\qquad (4.3.39)$$

where ρ_1 is defined by (4.3.35). We define also

$$m(t,X)=\delta_0(t,X)+\Theta(t,X)+T^{-1}\delta_0(t,X)\eta(t,X).\qquad (4.3.40)$$

The next statement is theorem 2.4.5 in [L7]. The reader may be interested in checking that it is indeed the term η, defined above in (4.3.38), which allows us to cut the loss of derivatives from 2 to 3/2.

Theorem 4.3.15. *With the notations above for Θ,ρ_1,m, with R and C_0 defined in Theorem 4.3.11, we have for $T>0$, $|t|\le T$, $X\in\mathbb{R}^{2n},\Lambda\ge 1$,*

$$|\Theta(t,X)|\le\rho_1(t,X)\le 2\frac{\langle\delta_0(t,X)\rangle}{\nu(t,X)^{1/2}},\quad |\sigma_1(t,X)|=|\delta_0(t,X)|,\qquad (4.3.41)$$

$$C_0^{-1}R(t,X)\le\rho_1(t,X)\le 2T\frac{\partial}{\partial t}\Big(\Theta(t,X)+\sigma_1(t,X)\Big),\qquad (4.3.42)$$

$$0\le\eta(t,X)\le 4T,\quad \frac{d}{dt}\big(\delta_0\eta\big)\ge\delta_0^2\Lambda^{-1/2},\quad |\eta'_X(t,X)|\le 4T\Lambda^{-1/2},\qquad (4.3.43)$$

$$T\frac{d}{dt}m\ge\frac{1}{2}\rho_1+\delta_0^2\Lambda^{-1/2}\ge\frac{1}{2C_0}R+\delta_0^2\Lambda^{-1/2}\ge\frac{1}{2^{3/2}C_0}\langle\delta_0\rangle^2\Lambda^{-1/2}.\qquad (4.3.44)$$

4.3.6. Stationary estimates for the model cases.

Definition 4.3.16. Let $T>0$ be given. With m defined in (4.3.40), we define for $|t|\le T$,

$$M(t)=m(t,X)^{\mathrm{Wick}},\qquad (4.3.45)$$

where the Wick quantization is given in (3.2.1).

Let $T>0$ be given and $Q(t)=q(t)^w$ given by (4.3.1-2). We define $M(t)$ according to (4.3.45). We consider

$$\operatorname{Re}\big(Q(t)M(t)\big)=\frac{1}{2}Q(t)M(t)+\frac{1}{2}M(t)Q(t)=P(t).\qquad (4.3.46)$$

We have, omitting now the variable t fixed here,

$$P = \operatorname{Re}\Big[q^w\big(\delta_0(1+T^{-1}\eta)\big)^{\mathrm{Wick}} + q^w\Theta^{\mathrm{Wick}}\Big]. \tag{4.3.47}$$

Following the section 3.2 in [L7], we discuss now the various model cases that could occur for the symbol $q(t,X)$ when t is fixed.

The gradient points. Let us assume first that

$$q = \Lambda^{-1/2}\mu^{1/2}\nu^{1/2}\beta e_0$$

with $\beta \in S(\nu^{1/2},\nu^{-1}\Gamma), 1 \le e_0 \in S(1,\nu^{-1}\Gamma)$ and $\delta_0 = \beta$. Moreover, we assume

$$0 \le T^{-1}\eta \le 4, T^{-1}|\eta'| \le 4\Lambda^{-1/2}, \quad |\Theta| \le C\langle\delta_0\rangle\nu^{-1/2}.$$

Here Λ,μ,ν are assumed to be positive constants such that $\Lambda \ge \mu \ge \nu \ge 1$. After a rather simple but delicate discussion involving various properties of the Wick quantization, we get

$$\operatorname{Re}(QM) + S(\Lambda^{-1/2}\mu^{1/2}\nu^{-1/2},\Gamma)^w \ge 0. \tag{4.3.48}$$

The nonnegative points. Let us assume now that

$$q \ge 0, \quad q \in S(\Lambda^{-1/2}\mu^{1/2}\nu,\nu^{-1}\Gamma), \quad \gamma_0\nu^{1/2} \le \delta_0 \le \gamma_0^{-1}\nu^{1/2},$$

with a positive fixed constant γ_0. Moreover, we assume $0 \le T^{-1}\eta \le 4, T^{-1}|\eta'| \le 4\Lambda^{-1/2}$, $|\Theta(X)| \le C$, Θ real-valued. Here Λ,μ,ν are assumed to be positive constants such that $\Lambda \ge \mu \ge \nu \ge 1$. We start over our discussion from the identity (4.3.47):

$$P = \operatorname{Re}\Big[q^w\Big(\delta_0(1+T^{-1}\eta) + \Theta\Big)^{\mathrm{Wick}}\Big]. \tag{4.3.49}$$

Some arguments of symbolic calculus and the Fefferman-Phong inequality ([FP]) yield

$$\operatorname{Re}(QM) + S(\Lambda^{-1/2}\mu^{1/2},\Gamma)^w \ge 0. \tag{4.3.50}$$

The discussion is analogous for the nonpositive points and the negligible points.

Following the section 3.3 in [L7], we get the following result as a consequence of the previous discussion.

Lemma 4.3.17. *Let p be the Weyl symbol of P defined in* (4.3.46) *and* $\widetilde\Theta = \Theta * 2^n \exp -2\pi\Gamma$, *where Θ is defined in* (4.3.39) *(and satisfies* (4.3.41)*). Then we have*

$$p(t,X) \equiv p_0(t,X) = q(t,X)\Big(\delta_0(1+T^{-1}\eta) * 2^n \exp -2\pi\Gamma\Big) + q(t,X)\widetilde\Theta(t,X), \tag{4.3.51}$$

modulo $S(\Lambda^{-1/2}\mu^{1/2}\nu^{-1/2}\langle\delta_0\rangle,\Gamma)$.

Now, we shall use a partition of unity $1 = \sum_k \chi_k^2$ related to the metric $\nu(t,X)^{-1}\Gamma$ and a sequence (ψ_k) as in the lemma 5.3.3. We have, omitting the variable t, with p_0 defined in the previous lemma,

$$p_0(X) = \sum_k \chi_k(X)^2 q(X) \int \delta_0(Y)\big(1 + T^{-1}\eta(Y)\big)2^n \exp -2\pi\Gamma(X-Y)dY$$
$$+ \sum_k \chi_k(X)^2 q(X) \int \Theta(Y) 2^n \exp -2\pi\Gamma(X-Y)dY.$$

We obtain, assuming $\delta_0 = \delta_{0k}, \Theta = \Theta_k, q = q_k$ on U_k, that

$$p_0 = \sum_k \chi_k^2 q_k \big(\delta_{0k}(1+T^{-1}\eta) * 2^n \exp -2\pi\Gamma\big) + \sum_k \chi_k^2 q_k \big(\Theta_k * 2^n \exp -2\pi\Gamma\big)$$
$$+ S(\Lambda^{-1/2}\mu^{1/2}\nu^{-\infty}, \Gamma). \quad (4.3.52)$$

It is then rather straightforward to get the following lemma (cf. lemma 3.3.3 in [L7]).

Lemma 4.3.18. *With* $\widetilde{\Theta}_k = \Theta_k * 2^n \exp -2\pi\Gamma, \quad d_k = \delta_{0k}(1+T^{-1}\eta) * 2^n \exp -2\pi\Gamma$ *and* q_k, χ_k *defined above, we have*

$$\sum_k \chi_k \sharp q_k d_k \sharp \chi_k + \sum_k \chi_k \sharp q_k \widetilde{\Theta}_k \sharp \chi_k = p_0 + S(\Lambda^{-1/2}\mu^{1/2}\nu^{-1/2}\langle\delta_0\rangle, \Gamma). \quad (4.3.53)$$

From this, we can obtain the following result (cf. proposition 3.3.4 in [L7]).

Proposition 4.3.19. *Let* $T > 0$ *be given and* $Q(t) = q(t)^w$ *given by* (4.3.1-2). *We define* $M(t)$ *according to* (4.3.45). *Then, with a partition of unity* $1 = \sum_k \chi_k^2$ *related to the metric* $\nu(t,X)^{-1}\Gamma$ *we have*

$$\operatorname{Re}(Q(t)M(t)) = \sum_k \chi_k^w \operatorname{Re}\big(q_k^w d_k^w + q_k^w \widetilde{\Theta}_k^w\big)\chi_k^w + S(\Lambda^{-1/2}\mu^{1/2}\langle\delta_0\rangle\nu^{-1/2}, \Gamma)^w$$

$$\text{and} \qquad \operatorname{Re}(Q(t)M(t)) + S(\Lambda^{-1/2}\mu^{1/2}\langle\delta_0\rangle\nu^{-1/2}, \Gamma)^w \ge 0.$$

4.3.7. The multiplier method.

Theorem 4.3.20. *Let* $T > 0$ *be given and* $Q(t) = q(t)^w$ *given by* (4.3.1-2). *We define* $M(t)$ *according to* (4.3.45). *There exist* $T_0 > 0$ *and* $c_0 > 0$ *depending only on a finite number of* γ_k *in* (4.3.1) *such that, for* $0 < T \le T_0$, *with* $D(t,X) = \langle\delta_0(t,X)\rangle$, *(D is Lipschitz continuous with Lipschitz constant* 2, *as* δ_0 *in* (4.3.8) *and thus a* Γ*-weight),*

$$\frac{d}{dt}M(t) + 2\operatorname{Re}\big(Q(t)M(t)\big) \ge T^{-1}(D^2)^{\text{Wick}}\Lambda^{-1/2}c_0. \quad (4.3.54)$$

Moreover we have with m defined in (4.3.40)*,* $\widetilde{m}(t,\cdot) = m(t,\cdot) * 2^n \exp -2\pi\Gamma$,

$$M(t) = m(t,X)^{\text{Wick}} = \widetilde{m}(t,X)^w, \text{ with } \widetilde{m} \in S_1(D, D^{-2}\Gamma) + S(1,\Gamma), \tag{4.3.55}$$

where the set of symbols $S_1(D, D^{-2}\Gamma)$ is defined as symbols c satisfying the estimates of $S_1(D, D^{-2}\Gamma)$ for the function and the first derivatives and such that $c'' \in S(1,\Gamma)$. We have also

$$m(t,X) = a(t,X) + b(t,X), \ |a/D| + |a'_X| + |b| \text{ bounded}, \ \dot{m} \geq 0, \tag{4.3.56}$$
$$a = \delta_0(1 + T^{-1}\eta), \quad b = \widetilde{\Theta}.$$

This theorem is a direct consequence of the previous lemmas and propositions and is Theorem 3.4.1 in [L7]. We shall not give its complete proof here, but we wish to make a few points about the loss of derivatives in a semi-classical framework.

Remark 4.3.21. Let us check that this theorem gives an estimate with loss of $3/2$ derivatives for

$$L = D_t + iQ(t). \tag{4.3.57}$$

We compute for $u \in C^1_c(\mathbb{R}, L^2(\mathbb{R}^n))$, $\operatorname{supp} u \subset [-T_0, T_0]$, the quantity $\langle Lu, iMu\rangle$ and we use (4.3.54):

$$2\operatorname{Re}\langle Lu, iMu\rangle = \langle \dot{M}u, u\rangle + 2\operatorname{Re}\langle Qu, Mu\rangle \geq c_0 T^{-1}\Lambda^{-1/2}\langle\big(1+\delta_0^2(t,\cdot)\big)^{\text{Wick}}u, u\rangle.$$

We get, for all positive α,

$$c_0 T^{-1}\Lambda^{-1/2}\langle\big(1+\delta_0^2(t,\cdot)\big)^{\text{Wick}}u, u\rangle \leq \alpha^{-1}\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + \alpha\|Mu\|^2_{L^2(\mathbb{R}^{n+1})}$$

and from the lemma A.1.4 in [L7], with a positive fixed constant C_1, we obtain

$$c_0 T^{-1}\Lambda^{-1/2}\langle\big(1+\delta_0^2(t,\cdot)\big)^{\text{Wick}}u, u\rangle \leq \alpha^{-1}\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + \alpha C_1\langle\big(1+\delta_0^2(t,\cdot)\big)^{\text{Wick}}u, u\rangle.$$

Choosing now $\alpha = \frac{c_0}{2C_1 T\Lambda^{1/2}}$, we obtain

$$\frac{1}{2}c_0 T^{-1}\Lambda^{-1/2}\langle\big(1+\delta_0^2(t,\cdot)\big)^{\text{Wick}}u, u\rangle \leq \frac{2C_1 T}{c_0}\Lambda^{1/2}\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} \tag{4.3.58}$$

and thus with a fixed positive constant c_1, $\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} \geq c_1^2 T^{-2}\Lambda^{-1}\|u\|^2_{L^2(\mathbb{R}^{n+1})}$, yielding

$$\|Lu\|_{L^2(\mathbb{R}^{n+1})} \geq c_1 T^{-1}\Lambda^{-1/2}\|u\|_{L^2(\mathbb{R}^{n+1})}, \tag{4.3.59}$$

which is indeed an estimate with loss of $3/2$ derivatives with respect to the elliptic estimate $\|Lu\| \gtrsim \Lambda\|u\|$. We can notice also, that in the region where $\langle\delta_0\rangle \sim \Lambda^{1/2}$, the estimate (4.3.58) looses just one derivative and is an $L^2 - L^2$ estimate.

The following result follows from theorem 4.1.9 in [L7].

Theorem 4.3.22. *Let $f(t,x,\xi,h)$ be a smooth real-valued function defined on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times(0,1]$, satisfiying* (4.3.2) *and*

$$\sup_{\substack{t\in\mathbb{R}\\(x,\xi)\in\mathbb{R}^{2n}}} |(\partial_x^\alpha\partial_\xi^\beta f)(t,x,\xi,h)|h^{-|\beta|}=C_{\alpha\beta}<\infty. \tag{4.3.60}$$

Let $f_0(t,x,\xi,h)$ be a smooth complex-valued function defined on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times(0,1]$, such that $f_0(t,x,\xi,h)$ satisfies (4.3.60)*. Then there exists $T_0>0, c_0>0, h_0\in(0,1]$ depending on a finite number of seminorms of f, f_0, such that, for all $T\le T_0, h\in(0,h_0]$ and all $u\in C_c^\infty\big((-T,T);\mathcal{S}(\mathbb{R}^n)\big)$*

$$\|hD_tu+if(t,x,\xi,h)^wu+hf_0(t,x,\xi,h)^wu\|_{L^2(\mathbb{R}^{n+1})}\ge h^{3/2}c_0T^{-1}\left\|u\right\|_{L^2(\mathbb{R}^{n+1})}.$$

Remark 4.3.27. The previous theorem is indeed a version of (1.2.1) with $\mu=3/2$. However the deduction of a solvability result from this theorem is not completely obvious, because of the complications triggered by the loss of derivatives strictly larger than 1. The details are given in sections 4.2-3-4 of [L7].

5. Appendix

5.1. Homogeneous classes of pseudodifferential operators.

Definition 5.1.1. Let $n \geq 1$ be an integer and $m \in \mathbb{R}$. We shall say that a C^∞ function $a : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ is in S^m if for all multi-indices α, β,

$$\sup_{\mathbb{R}^n \times \mathbb{R}^n} |(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi)|(1+|\xi|)^{|\beta|-m} = \gamma_{\alpha\beta}(a) < \infty. \tag{5.1.1}$$

A function $p : \mathbb{R}^{2n} \longrightarrow \mathbb{C}$ will be said positively-homogeneous of degree k if,

$$\text{for } |\xi| \geq 1, \theta \geq 1, \quad p(x, \theta\xi) = \theta^k p(x,\xi). \tag{5.1.2}$$

Definition 5.1.2. Let $n \geq 1$ be an integer and $m \in \mathbb{R}$. We shall say that a function $a : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ is in S^m_{phg} if there exists a sequence $(p_j)_{j\in\mathbb{N}}$ of smooth positively-homogeneous functions of degree $m - j$ such that

$$a \sim \sum_{j \geq 0} p_j, \tag{5.1.3}$$
$$\text{i.e. for all } N \in \mathbb{N}, \quad a - \sum_{0 \leq j < N} p_j \in S^{m-N}.$$

Note that $S^m_{phg} \subset S^m$ and also that, given a family $(p_j)_{j\in\mathbb{N}}$ of functions of C^∞ functions such that, for each j, $p_j \in S^{m-j}$, there exists $a \in S^m_{phg}$ such that (5.1.3) is satisfied; the function p_0 above is called the principal symbol of the operator a^w.

5.2. Confinement-set and semi-classical wave-front set. The following definition is taken from the thesis of K.Pravda-Starov ([P2], p.134).

Notation. Let $h_0 \in (0, 1]$ be given. When a family of functions $\big(\chi(\cdot, \cdot, h)\big)_{0<h\leq h_0}$ is uniformly in $C_b^\infty(\mathbb{R}^{2n})$, i.e. for all $h \in (0, h_0]$, $(x, \xi) \mapsto \chi(x, \xi, h)$ is $C_b^\infty(\mathbb{R}^{2n})$ and for all α, β, we have $\sup_{\mathbb{R}^n\times\mathbb{R}^n\times(0,h_0]} |\partial_x^\alpha \partial_\xi^\beta \chi(x, \xi, h)| < \infty$, we shall still say that χ belongs to $C_b^\infty(\mathbb{R}^{2n})$.

Definition 5.2.1. Let F be a closed subset of $\mathbb{R}^{2n}$. Let $h_0 \in (0, 1]$ and $(u_h)_{0<h\leq h_0}$ be a family of functions in L^2. We shall say that F is a confinement-set for the family (u_h) if for all open neighborhoods V of F, for all $\chi \in C_b^\infty(\mathbb{R}^{2n})$ such that $\operatorname{supp}_{x,\xi} \chi \subset V^c$, we have

$$\|\chi(x, h\xi, h)^w u_h\|_{L^2(\mathbb{R}^n)} = O(h^\infty). \tag{5.2.1}$$

Remark 5.2.2. Note that if the family (u_h) satisfies the requirements in the definition 1.1.4, and if F is a confinement-set for the family (u_h) then

$$WF_{sc}(u_h) \subset F. \tag{5.2.2}$$

In fact if $(x_0,\xi_0)\notin F$, since F is closed, there exists a neighborhood V of F and $\chi_0 \in C_b^\infty(\mathbb{R}^{2n})$, such that $\chi_0(x_0,\xi_0)=1$ and $\operatorname{supp}\chi_0\subset V^c$ so that (5.2.1) is satisfied for χ_0 and thus (1.1.14). The main reason for introducing that notion is the following property. Take a family $(u_h)_{0<h\le h_0}$ of functions in $L^2(\mathbb{R}^n)$, $p,q\in C_b^\infty(\mathbb{R}^{2n})$. Assume now that $p=q$ on the support of a function $\psi\in C_b^\infty(\mathbb{R}^{2n})$, so that $\psi=1$ on a neighborhood of a confinement-set F of (u_h): then

$$\|p(x,h\xi)^w u_h\|_{L^2(\mathbb{R}^n)}=O(h^\nu)\Longleftrightarrow \|q(x,h\xi)^w u_h\|_{L^2(\mathbb{R}^n)}=O(h^\nu). \tag{5.2.3}$$

In fact, with $\psi\in C_b^\infty(\mathbb{R}^{2n})$, $\psi=1$ on V open $\supset F$, $\operatorname{supp}\psi\subset\{p=q\}$, $\operatorname{supp}(1-\psi)\subset V^c$, we have

$$p=\psi p+(1-\psi)p=\psi q+(1-\psi)p=q+(1-\psi)(p-q).$$

Since $\operatorname{supp}((1-\psi)(p-q))\subset V^c$, we get that $\|((1-\psi)(p-q))^{w_h}u_h\|_{L^2(\mathbb{R}^n)}=O(h^\infty)$ and (5.2.3).

5.3. Some standard facts about metrics on the phase space.

Definition 5.3.1. Let g be a metric on $\mathbb{R}^{2n}$, i.e. a mapping $X\mapsto g_X$ from $\mathbb{R}^{2n}$ to the cone of positive definite quadratic forms on $\mathbb{R}^{2n}$. Let M be a positive function defined on $\mathbb{R}^{2n}$.

(i) The metric g is said to be slowly varying whenever $\exists C>0,\exists r>0,\forall X,Y,T\in\mathbb{R}^{2n}$,

$$g_X(Y-X)\le r^2\Longrightarrow C^{-1}g_Y(T)\le g_X(T)\le Cg_Y(T).$$

(ii) The symplectic dual metric g^σ is defined as $g_X^\sigma(T)=\sup_{g_X(U)=1}[T,U]^2$, where $[,]$ is the symplectic form (1.1.5). The parameter of g is defined as

$$\lambda_g(X)=\inf_{T\ne0}\big(g_X^\sigma(T)/g_X(T)\big)^{1/2}$$

and we shall say that g satisfies the uncertainty principle if $\inf_X\lambda_g(X)\ge1$.

(iii) The metric g is said to be temperate when $\exists C>0,\exists N\ge0,\forall X,Y,T\in\mathbb{R}^{2n}$,

$$g_X^\sigma(T)\le Cg_Y^\sigma(T)\big(1+g_X^\sigma(X-Y)\big)^N.$$

When the three properties above are satisfied, we shall say that g is admissible. The constants appearing in (i) and (iii) will be called the structure constants of the metric g.

(iv) The function M is said to be g-slowly varying if $\exists C>0,\exists r>0,\forall X,Y\in\mathbb{R}^{2n}$,

$$g_X(Y-X)\le r^2\Longrightarrow C^{-1}\le\frac{M(X)}{M(Y)}\le C.$$

(v) The function M is said to be g-temperate if $\exists C>0,\exists N\ge0,\ \forall X,Y\in\mathbb{R}^{2n}$,

$$\frac{M(X)}{M(Y)}\le C\big(1+g_X^\sigma(X-Y)\big)^N.$$

When M satisfies (iv) and (v), we shall say that M is a g-weight.

Definition 5.3.2. Let g be a metric on $\mathbb{R}^{2n}$ and M be a positive function defined on $\mathbb{R}^{2n}$. The set $S(M,g)$ is defined as the set of functions $a \in C^\infty(\mathbb{R}^{2n})$ such that, for all $l \in \mathbb{N}$, $\sup_X \|a^{(l)}(X)\|_{g_X} M(X)^{-1} < \infty$, where $a^{(l)}$ is the l-th derivative. It means that $\forall l \in \mathbb{N}, \exists C_l, \forall X \in \mathbb{R}^{2n}, \forall T_1, \dots, T_l \in \mathbb{R}^{2n}$,

$$|a^{(l)}(X)(T_1, \dots, T_l)| \le C_l M(X) \prod_{1 \le j \le l} g_X(T_j)^{1/2}.$$

We discuss now some basic facts about partitions of unity. We refer the reader to the chapter 18 in [H7] for the basic properties of admissible metrics as well as for the following lemma.

Lemma 5.3.3. *Let g be an admissible metric on $\mathbb{R}^{2n}$. There exists a sequence $(X_k)_{k\in\mathbb{N}}$ of points in the phase space $\mathbb{R}^{2n}$ and positive numbers r_0, N_0, such that the following properties are satisfied. We define U_k, U_k^*, U_k^{**} as the $g_k = g_{X_k}$ balls with center X_k and radius $r_0, 2r_0, 4r_0$. There exist two families of non-negative smooth functions on $\mathbb{R}^{2n}$, $(\chi_k)_{k\in\mathcal{N}}$, $(\psi_k)_{k\in\mathbb{N}}$ such that*

$$\sum_k \chi_k(X) = 1, \ \textit{supp}\chi_k \subset U_k, \ \psi_k \equiv 1 \quad \textit{on } U_k^*, \ \textit{supp}\psi_k \subset U_k^{**}.$$

Moreover, $\chi_k, \psi_k \in S(1, g_k)$ with semi-norms bounded independently of k. The overlap of the balls U_k^{**} is bounded, i.e.

$$\bigcap_{k\in\mathcal{N}} U_k^{**} \neq \emptyset \implies \#\mathcal{N} \le N_0.$$

Also we have $g_X \sim g_k$ all over U_k^{**} (i.e. the ratios $g_X(T)/g_k(T)$ are bounded above and below by a fixed constant, provided that $X \in U_k^{**}$).

The next lemma in proved in [BC](see also lemma 6.3 in [L5]).

Lemma 5.3.4. *Let g be an admissible metric on $\mathbb{R}^{2n}$ and $\sum_k \chi_k(x,\xi) = 1$ be a partition of unity related to g as in the previous lemma. There exists a positive constant C such that for all $u \in L^2(\mathbb{R}^n)$*

$$C^{-1}\|u\|^2_{L^2(\mathbb{R}^n)} \le \sum_k \|\chi_k^w u\|^2_{L^2(\mathbb{R}^n)} \le C\|u\|^2_{L^2(\mathbb{R}^n)},$$

where a^w stands for the Weyl quantization of the symbol a.

The following lemma is proved in [BL].

Lemma 5.3.5. *Let g be an admissible metric on $\mathbb{R}^{2n}$, m be a weight for g, U_k and g_k as in lemma 5.3.3. Let (a_k) be a sequence of bounded symbols in $S\big(m(X_k), g_k\big)$ such that, for all non-negative integers l, N*

$$\sup_{k\in\mathbb{N}, T\in\mathbb{R}^{2n}} |m(X_k)^{-1} a_k^{(l)}(X) T^l \big(1 + g_k^\sigma(X - U_k)\big)^N g_k(T)^{-l/2}| < +\infty.$$

Then the symbol $a = \sum_k a_k$ makes sense and belongs to $S(m, g)$. The important point here is that no support condition is required for the a_k, but instead some decay estimates with respect to g^σ. The sequence (a_k) will be called a confined-sequence in $S(m, g)$.

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