

INTRODUCTION TO THE WEYL-HÖRMANDER CALCULUS OF PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. In this series of lectures, we introduce the basic elements for the understanding of the Weyl-Hörmander calculus of pseudodifferential operators. We begin with introducing a few elements of symplectic algebra and the basic facts about the Hermann Weyl quantization of Hamiltonians. Next we discuss the notion of metrics on the phase space, along with several examples and give a version of the Heisenberg uncertainty principle, linked to these metrics. We introduce the notion of confinement, and give a detailed proof of the biconfinement estimates, a basic tool established in a joint paper with J.-M.Bony. We give the main results of the symbolic calculus and a version of Cotlar's lemma, which allows us to prove a L^2 -boundedness result.

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CHAPTER I

THE STRUCTURE OF THE PHASE SPACE

1.1. Basic facts

Symplectic algebra. Let us consider a finite dimensional real vector space E (the configuration¹ space $\mathbb{R}_x^n$) and its dual space E^* (the momentum space $\mathbb{R}_\xi^n$). The phase space is defined as $\Phi = E \oplus E^*$; its running point will be denoted in general by a capital letter ($X = (x, \xi), Y = (y, \eta)$). The symplectic form on Φ is given by

$$(1.1.1) \quad [(x, \xi), (y, \eta)] = \langle \xi, y \rangle_{E^*, E} - \langle \eta, x \rangle_{E^*, E}$$

where $\langle \cdot, \cdot \rangle_{E^*, E}$ stands for the bracket of duality. The symplectic group $\text{Sp}(n)$ is the subgroup of the linear group of Φ preserving (1.1.1). With

$$\sigma = \begin{pmatrix} 0 & \text{Id}(E^*) \\ -\text{Id}(E) & 0 \end{pmatrix},$$

we have for $X, Y \in \Phi$, $[X, Y] = \langle \sigma X, Y \rangle_{\Phi^*, \Phi}$, so that the equation of the symplectic group is

$$A^* \sigma A = \sigma.$$

One can describe a set of generators for the symplectic group $\text{Sp}(n)$, identifying Φ with $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$: the mappings

- (i) $(x, \xi) \mapsto (Tx, {}^t T^{-1} \xi)$, where T is an automorphism of E ,
- (ii) $(x_k, \xi_k) \mapsto (\xi_k, -x_k)$, and the other coordinates fixed,
- (iii) $(x, \xi) \mapsto (x, \xi + Sx)$, where S is symmetric from E to E^* .

Fourier transform. For a function u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions, we shall use the following definition for the Fourier transform,

$$(1.1.2) \quad \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi \langle x, \xi \rangle} u(x) dx.$$

¹We shall identify the configuration space E with $\mathbb{R}^n$, but the reader may keep in mind that all the theory can be developed without this identification.

The Fourier transform is an isomorphism of the space $\mathcal{S}(\mathbb{R}^n)$ and the Fourier inversion formula gives

$$(1.1.3) \quad u(x) = \int_{\mathbb{R}^n} e^{2i\pi\langle x, \xi \rangle} \hat{u}(\xi) d\xi.$$

One of the advantages of this normalization, due to Laurent Schwartz, is that the function $u_A(x) = \exp -\pi\langle Ax, x \rangle$, where A is a positive definite matrix, has the Fourier transform

$$(1.1.4) \quad \widehat{u_A}(\xi) = (\det A)^{-1/2} \exp -\pi\langle A^{-1}\xi, \xi \rangle.$$

Also, the same formula works, mutatis mutandis, for the Fourier transform of the temperate distribution $v_B(x) = \exp i\pi\langle Bx, x \rangle$, where B is a real symmetric nondegenerate matrix,

$$\widehat{v_B}(\xi) = |\det B|^{-1/2} e^{i\frac{\pi}{4} \text{sign} B} \exp -i\pi\langle B^{-1}\xi, \xi \rangle,$$

where the signature of B , $\text{sign} B$ is the number of positive eigenvalues minus the number of negative eigenvalues. The formula (1.1.4) can also be extended by analytic continuation to symmetric and complex-valued matrices A whose real part is positive definite. The Fourier transform is also a unitary isomorphism of $L^2(\mathbb{R}^n)$ and can be extended to the temperate distributions $\mathcal{S}'(\mathbb{R}^n)$ with the duality formula

$$\langle \hat{T}, \varphi \rangle_{\mathcal{S}', \mathcal{S}} = \langle T, \hat{\varphi} \rangle_{\mathcal{S}', \mathcal{S}}.$$

The Fourier inversion formula

$$\check{\hat{T}} = T$$

holds for $T \in \mathcal{S}'(\mathbb{R}^n)$: here the operator $\check{}$ stands for $\check{u}(x) = u(-x)$ which can be extended by duality to $\mathcal{S}'(\mathbb{R}^n)$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, denoting $|\alpha| = \sum_1^n \alpha_j$, we define

$$D_x^\alpha = (2i\pi)^{-|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$$

so that we have, with $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$, for $u \in \mathcal{S}'(\mathbb{R}^n)$

$$(1.1.5) \quad \widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi).$$

Wigner function. For $u, v \in \mathcal{S}(\mathbb{R}^n)$, we define the Wigner function $\mathcal{H}(u, v)$ on the phase space by the formula

$$(1.1.6) \quad \mathcal{H}(u, v)(x, \xi) = \int u\left(x + \frac{y}{2}\right) \bar{v}\left(x - \frac{y}{2}\right) e^{-2i\pi y \cdot \xi} dy.$$

The mapping $(u, v) \mapsto \mathcal{H}(u, v)$ is sesquilinear continuous from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$. Since $\mathcal{H}(u, v)$ is the partial Fourier transform of the function $(x, y) \mapsto u(x+y/2)\bar{v}(x-y/2)$, the Wigner function also satisfies,

$$(1.1.7) \quad \begin{aligned} \|\mathcal{H}(u, v)\|_{L^2(\mathbb{R}^{2n})} &= \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}, \\ \mathcal{H}(u, v)(x, \xi) &= 2^n \langle \sigma_{x, \xi} u, v \rangle_{L^2(\mathbb{R}^n)}, \end{aligned}$$

$$\text{with } (\sigma_{x, \xi} u)(y) = u(2x - y) \exp -4i\pi(x - y) \cdot \xi$$

and the so-called “phase symmetries” σ_X are unitary and selfadjoint on $L^2(\mathbb{R}^n)$.

1.2. Quantization formulas

Let a be temperate distribution on $\mathbb{R}^{2n}$. The Weyl quantization rule associates to this function the operator a^w defined formally on functions $u \in \mathcal{S}(\mathbb{R}^n)$ by

$$(1.2.1) \quad (a^w u)(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

The previous formula does not obviously make sense, so we use the Wigner function (1.1.6) to provide a weak definition,

$$(a^w u, v)_{\mathcal{S}^*(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)} = \langle a, \mathcal{H}(u, v) \rangle_{\mathcal{S}'(\mathbb{R}^{2n}), \mathcal{S}(\mathbb{R}^{2n})},$$

where the $\mathcal{S}^*(\mathbb{R}^n)$ is the antidual of $\mathcal{S}(\mathbb{R}^n)$. For instance we have

$$(x \cdot \xi)^w = (x \cdot D_x + D_x \cdot x)/2,$$

with $D_x = \frac{1}{2i\pi} \frac{\partial}{\partial x}$ whereas the *classical* quantization rule would map the Hamiltonian $x \cdot \xi$ to the operator $x \cdot D_x$. A nice feature of the Weyl quantization rule, introduced in 1928 by Hermann Weyl in [Wy], is the fact that real Hamiltonians get quantized by (formally) self-adjoint operators. Let us recall that the classical quantization of the Hamiltonian $a(x, \xi)$ is given by the operator $\text{Op}(a)$ acting on functions $u(x)$ by

$$(1.2.2) \quad (\text{Op}(a)u)(x) = \int e^{2i\pi x \cdot \xi} a(x, \xi) \widehat{u}(\xi) d\xi.$$

In fact, introducing the following one-parameter group $J^t = \exp 2i\pi t D_x \cdot D_\xi$, given by the integral formula

$$(J^t a)(x, \xi) = |t|^{-n} \iint e^{-2i\pi t^{-1} y \cdot \eta} a(x + y, \xi + \eta) dy d\eta,$$

we see that

$$(\text{Op}((J^t a))u)(x) = \iint e^{2i\pi(x-y)\cdot\xi} a((1-t)x + ty, \xi) u(y) dy d\xi.$$

In particular one gets $a^w = \text{Op}(J^{1/2}a)$. Moreover since $(\text{Op}(a))^* = \text{Op}(J\bar{a})$ we obtain

$$(a^w)^* = \text{Op}(J(\overline{J^{1/2}a})) = \text{Op}(J^{1/2}\bar{a}) = (\bar{a})^w,$$

yielding formal self-adjointness for real a . Formula (1.2.1) can be written as

$$(1.2.3) \quad (a^w u, v) = \iint a(x, \xi) \mathcal{H}(u, v)(x, \xi) dx d\xi,$$

where the Wigner function $\mathcal{H}$ is defined by (1.1.6). We have also ([Un], [Wy]),

$$(1.2.4) \quad a^w = \int_{\mathbb{R}^{2n}} a(X) 2^n \sigma_X dX = \int_{\mathbb{R}^{2n}} \widehat{a}(\Xi) \exp(2i\pi \Xi \cdot M) d\Xi,$$

where $\Xi \cdot M = \hat{x} \cdot x + \hat{\xi} \cdot D_x$ (here $\Xi = (\hat{x}, \hat{\xi})$). These formulas give in particular

$$(1.2.5) \quad \|a^w\|_{\mathcal{L}(L^2)} \leq \min(2^n \|a\|_{L^1(\mathbb{R}^{2n})}, \|\widehat{a}\|_{L^1(\mathbb{R}^{2n})}),$$

where $\mathcal{L}(L^2)$ stands for the space of bounded linear maps from $L^2(\mathbb{R}^n)$ into itself.

We then describe the metaplectic group, introduced by André Weil [Wi]. The metaplectic group $Mp(n)$ is the subgroup of the group of unitary transformations of $L^2(\mathbb{R}^n)$ generated by

- (j) $(M_T u)(x) = |\det T|^{-1/2} u(T^{-1}x)$, where T is an automorphism of E ,
- (jj) Partial Fourier transformation, with respect to x_k for $k = 1, \dots, n$,
- (jjj) Multiplication by $\exp(i\pi \langle Sx, x \rangle)$, where S is symmetric from E to E^* .

There exists a two-fold covering (the π_1 of both $Mp(n)$ and $Sp(n)$ is $\mathbb{Z}$)

$$\pi : Mp(n) \rightarrow Sp(n)$$

such that, if $\chi = \pi(M)$ and u, v are in $L^2(\mathbb{R}^n)$, $\mathcal{H}(u, v)$ is their Wigner function,

$$\mathcal{H}(Mu, Mv) = \mathcal{H}(u, v) \circ \chi^{-1}.$$

This is Segal formula [Se] which could be rephrased as follows. Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\chi \in Sp(n)$. There exists M in the fiber of χ such that

$$(1.2.6) \quad (a \circ \chi)^w = M^* a^w M.$$

In particular, the images by π of the transformations (j), (jj), (jjj) are respectively (i), (ii), (iii) in section 1.1. Moreover, if χ is the phase translation, $\chi(x, \xi) = (x + x_0, \xi + \xi_0)$, (1.2.6) is fulfilled with $M = \tau_{x_0, \xi_0}$, the phase translation given by

$$(\tau_{x_0, \xi_0} u)(y) = u(y - x_0) e^{2i\pi \langle y - \frac{x_0}{2}, \xi_0 \rangle}.$$

If χ is the symmetry with respect to (x_0, ξ_0) , M in (1.2.6) is, up to a unit factor, the phase symmetry σ_{x_0, ξ_0} defined above. This yields the following composition formula $a^w b^w = (a \sharp b)^w$ with

$$(1.2.7) \quad (a \sharp b)(X) = 2^{2n} \iint e^{-4i\pi[X-Y, X-Z]} a(Y) b(Z) dY dZ,$$

with an integral on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. We can compare this with the classical composition formula,

$$\text{Op}(a)\text{Op}(b) = \text{Op}(a \circ b)$$

(cf.(1.2.2)) with

$$(1.2.8) \quad (a \circ b)(x, \xi) = \iint e^{-2i\pi y \cdot \eta} a(x, \xi + \eta) b(y + x, \xi) dy d\eta,$$

with an integral on $\mathbb{R}^n \times \mathbb{R}^n$.

REMARK. The symplectic invariance of the Weyl quantization given by formula (1.2.6) is its most important property. The fact that the Weyl formula quantizes real Hamiltonians into formally selfadjoint operators is shared with other quantizations such as the Feynman quantization, given by

$$\begin{aligned} a^F &= \frac{1}{2}(\text{Op}(a) + \text{Op}(J\bar{a})), \quad \text{that is} \\ (a^F u)(x) &= \iint e^{2i\pi(x-y)\xi} \frac{1}{2}(a(x, \xi) + \overline{a(y, \xi)}) u(y) dy d\xi, \quad \text{so that} \\ (a^F)^* &= \frac{1}{2}(\text{Op}(J\bar{a}) + \text{Op}(J\overline{J\bar{a}})) = \frac{1}{2}(\text{Op}(J\bar{a}) + \text{Op}(a)) = a^F. \end{aligned}$$

It is convenient to give an asymptotic version of these compositions formulae, e.g. in the semi-classical² case. Let m be a real number. A smooth function $a(x, \xi, \lambda)$ defined on $\mathbb{R}_x^n \times \mathbb{R}_\xi^n \times [1, +\infty)$ is in the symbol class S_{scl}^m if

$$\sup_{(x, \xi) \in \mathbb{R}^{2n}, \lambda \geq 1} |D_x^\alpha D_\xi^\beta a(x, \xi, \lambda)| \lambda^{-m+|\beta|} < \infty.$$

Then one has for $a \in S_{\text{scl}}^{m_1}$ and $b \in S_{\text{scl}}^{m_2}$, the expansion

$$(1.2.9) \quad (a \sharp b)(x, \xi) = \sum_{0 \leq k < N} 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha! \beta!} D_\xi^\alpha \partial_x^\beta a D_\xi^\beta \partial_x^\alpha b + r_N(a, b),$$

with $r_N(a, b) \in S_{\text{scl}}^{m_1+m_2-N}$. The beginning of this expansion is thus $ab + \frac{1}{2i}\{a, b\}$, where

$$(1.2.10) \quad \{a, b\} = \sum_{1 \leq j \leq n} \partial_{\xi_j} a \partial_{x_j} b - \partial_{x_j} a \partial_{\xi_j} b$$

is the Poisson bracket and $i = 2\pi i$. The sums inside (1.2.9) with k even are symmetric in a, b and skew-symmetric for k odd. This can be compared to the classical expansion formula

$$(a \circ b)(x, \xi) = \sum_{|\alpha| < N} \frac{1}{\alpha!} D_\xi^\alpha a \partial_x^\alpha b + t_N(a, b),$$

²We have developed a taste for using a *large* parameter λ instead of a small Planck constant h . writing $\lambda = 1/h$ will give back the more familiar picture.

with $t_N(a, b) \in S_{\text{scl}}^{m_1+m_2-N}$. The proof of these expansion formulas will be given later on in a general context. We can nevertheless note the following formulas, for a, b as above,

$$(1.2.11) \quad a \sharp b = ab + \frac{1}{2\ell} \{a, b\} + S_{\text{scl}}^{m_1+m_2-2},$$

$$(1.2.12) \quad a \sharp b + b \sharp a = 2ab + S_{\text{scl}}^{m_1+m_2-2},$$

$$(1.2.13) \quad a \sharp b - b \sharp a = \frac{1}{\ell} \{a, b\} + S_{\text{scl}}^{m_1+m_2-3}.$$

It will turn out that these formulas are true for very general class of symbols. They can also be compared to the less symmetric ordinary quantization with

$$\begin{aligned} a \circ b &= ab + \frac{1}{\ell} \partial_\xi a \cdot \partial_x b + S_{\text{scl}}^{m_1+m_2-2}, \\ a \circ b + b \circ a &= 2ab + \frac{1}{\ell} (\partial_\xi a \cdot \partial_x b + \partial_\xi b \cdot \partial_x a) + S_{\text{scl}}^{m_1+m_2-2}, \\ a \circ b - b \circ a &= \frac{1}{\ell} \{a, b\} + S_{\text{scl}}^{m_1+m_2-2}. \end{aligned}$$

The formula (1.2.10) can be precised by the computation of the term where $k = 2$ in (1.2.8). We get

$$a \sharp b = ab + \frac{1}{2\ell} \{a, b\} + \frac{1}{(2\ell)^2} \frac{1}{2!} \left(\partial_\xi^2 a \partial_x^2 b - 2\partial_\xi \partial_x a \partial_\xi \partial_x b + \partial_x^2 a \partial_\xi^2 b \right) + S_{\text{scl}}^{m_1+m_2-3},$$

where

$$\begin{aligned} \frac{1}{2!} \partial_\xi^2 a \partial_x^2 b &= \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b = \frac{1}{2} \sum_{1 \leq j \leq n} \frac{\partial^2 a}{\partial \xi_j^2} \frac{\partial^2 b}{\partial x_j^2} + 2 \sum_{1 \leq j < k \leq n} \frac{\partial^2 a}{\partial \xi_j \partial \xi_k} \frac{\partial^2 b}{\partial x_j \partial x_k}, \\ \partial_\xi \partial_x a \partial_\xi \partial_x b &= \sum_{|\alpha|=1=|\beta|} \partial_\xi^\alpha \partial_x^\beta a \partial_\xi^\beta \partial_x^\alpha b = \sum_{1 \leq j, k \leq n} \frac{\partial^2 a}{\partial \xi_j \partial x_k} \frac{\partial^2 b}{\partial x_j \partial \xi_k}, \\ \frac{1}{2!} \partial_x^2 a \partial_\xi^2 b &= \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_x^\alpha a \partial_\xi^\alpha b = \frac{1}{2} \sum_{1 \leq j \leq n} \frac{\partial^2 a}{\partial x_j^2} \frac{\partial^2 b}{\partial \xi_j^2} + 2 \sum_{1 \leq j < k \leq n} \frac{\partial^2 a}{\partial x_j \partial x_k} \frac{\partial^2 b}{\partial \xi_j \partial \xi_k}. \end{aligned}$$

Also, for a, b in $\mathcal{S}(\mathbb{R}^{2n})$, the formula (1.2.7) can be written as

$$(1.2.14) \quad (a \sharp b)(X) = \left(\exp\left(\frac{\ell}{2} [D_X, D_Y]\right) (a(X) \otimes b(Y)) \right) \Big|_{X=Y}.$$

As a matter of fact, the expansion (1.2.9) is the value on the diagonal $x = y, \xi = \eta$ of

$$\begin{aligned} ab + \frac{\ell}{2} \frac{1}{\ell^2} (\partial_\xi \partial_y - \partial_\eta \partial_x) a(x, \xi) b(y, \eta) + \frac{1}{2!} \frac{\ell^2}{2^2} \frac{1}{\ell^4} (\partial_\xi \partial_y - \partial_\eta \partial_x)^2 a(x, \xi) b(y, \eta) + \dots \\ = ab + \frac{1}{2\ell} \{a, b\} + \frac{1}{(2\ell)^2} \frac{1}{2!} \left(\partial_\xi^2 a \partial_x^2 b - 2\partial_\xi \partial_x a \partial_\xi \partial_x b + \partial_x^2 a \partial_\xi^2 b \right) + \dots \end{aligned}$$

Note that

$$\partial_\xi^2 a \partial_x^2 b = \left(\sum_j \partial_{\xi_j} \partial_{y_j} \right)^2 (a \otimes b) = \sum_{j,k} \partial_{\xi_j \xi_k}^2 a \partial_{x_j x_k}^2 b$$

which corresponds to the formulas given above.

CHAPTER II

METRICS ON THE PHASE SPACE

2.1. Introduction

The standard class. Let us first review a few classical examples of pseudodifferential operators. The most classical case is to consider smooth functions $a(x, \xi)$ defined on $\mathbb{R}^{2n}$ such that, for all multi-indices α, β , there exists $C_{\alpha\beta}$ such that for all $(x, \xi) \in \mathbb{R}^{2n}$,

$$(2.1.1) \quad |(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|)^{m-|\beta|},$$

where m is a given real number. This class of functions is usually called $S_{1,0}^m$. In particular, it contains the smooth functions homogeneous of degree m for $|\xi| \geq 1$, i.e. functions such that, for all ξ with $|\xi| \geq 1$ and $\lambda \geq 1$,

$$a(x, \lambda\xi) = \lambda^m a(x, \xi).$$

That class is sufficient to construct parametrices of differential operators with smooth coefficients and to prove that, if P is a differential operator with smooth coefficients in an open set Ω of $\mathbb{R}^n$

$$(2.1.2) \quad WFPu \subset WF(u) \subset WF(Pu) \cup \text{char } P$$

where

$$\text{char } P = \{(x, \xi) \in \dot{T}^*(\Omega), p_m(x, \xi) = 0\},$$

with p_m the principal symbol of the operator P and $\dot{T}^*(\Omega)$ the cotangent bundle minus the zero section. The second inclusion is most interesting, since Pu is known, and one obtains an information on the location of the singularities of u . Pseudodifferential operators are useful to prove the second inclusion in (2.1.2), since taking $(x_0, \xi_0) \notin WF(Pu)$ and such that $p_m(x_0, \xi_0) \neq 0$ will be enough to get that $(x_0, \xi_0) \notin WF(u)$. In fact, there is a conic neighborhood of (x_0, ξ_0) on which p_m does not vanish and thus the function

$$a(x, \xi) = \frac{\chi(x, \xi)}{p_m(x, \xi)}$$

belongs to the class $S_{1,0}^{-m}$, if χ is an appropriate cut-off function. One can use the operator $\text{Op}(a)$ as a first step to invert the operator P so that

$$AP\psi u = u + Ru$$

where R is an operator with symbol $r \in S_{1,0}^0$ such that $r \in S_{1,0}^{-1}$ in a conic neighborhood of (x_0, ξ_0) . Then it is not difficult to iterate that procedure and to obtain (2.1.2). It means that this class of pseudodifferential operators is useful to invert elliptic or microelliptic operators. In fact the parametrices of these elliptic operators belong to that class, which remains one of the most useful tool in classical analysis.

Parametrices of hypoelliptic operators with constant coefficients. An operator with smooth coefficients is said to be microhypoelliptic if

$$WF(u) = WF(Pu)$$

i.e. according to (2.1.2), if $WF(u) \subset WF(Pu)$. The operator P is said to be hypoelliptic if $\text{singsupp } u = \text{singsupp } Pu$, that is $\text{singsupp } u \subset \text{singsupp } Pu$. Clearly, since the projection of the wave-front-set is the singular support, microhypoellipticity implies hypoellipticity (the converse is not true). When P is elliptic, the inclusions (2.1.2) prove that it is hypoelliptic. However some non-elliptic operators are indeed hypoelliptic,³ such as the heat operator,

$$\partial_t - \Delta_x, \quad t \in \mathbb{R}, x \in \mathbb{R}^d.$$

In the late fifties, Hörmander gave a characterization of hypoellipticity of constant coefficients differential operators: $P(D)$ is hypoelliptic if and only if

$$\text{Im } \zeta \rightarrow \infty \quad \text{if } \zeta \rightarrow \infty \text{ on } \{\zeta \in \mathbb{C}^n, P(\zeta) = 0\}.$$

The properties of symbols of hypoelliptic operators led to the introduction of the following class of pseudodifferential operators. Let $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1, \delta < 1$. A function $a \in C^\infty(\mathbb{R}^{2n})$ belongs to $S_{\rho,\delta}^m$ means that

$$(2.1.3) \quad |(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|)^{m - \rho|\beta| + \delta|\alpha|}.$$

The parametrices of hypoelliptic operators with constant coefficients have a symbol in such a class.

REMARK. The study of hypoelliptic operators with variable coefficients is of a considerable difficulty, and no simple geometric characterization is in sight. In fact, the

³We speak here about the C^∞ hypoellipticity. Other notions of hypoellipticity can be considered such as the analytic hypoellipticity. For operators with constant coefficients, analytic hypoellipticity is equivalent to ellipticity.

parametrix of these operators are not in general pseudodifferential operators, even for quite simple examples such as

$$\partial_{x_1}^2 + x_1^4 \partial_{x_2}^2$$

which is hypoelliptic in $\mathbb{R}^2$. In fact, even the simpler hypoelliptic operator $\partial_{x_1}^2 + x_1^2 \partial_{x_2}^2$ does not have a parametrix in a simple class of pseudodifferential operators. The most widely known hypoelliptic operators with variable coefficients are the Hörmander sum of squares [Hö1]

$$\sum_{1 \leq j \leq m} X_j^2$$

with X_j real vector fields such that the Lie algebra generated by the X_j spans the whole tangent space at each point. L.Boutet de Monvel in [Bo] introduced a specific class of pseudodifferential operators for studying hypoelliptic operators with double characteristics and the paper [BGH] introduces a class of hypoelliptic operators with a pseudodifferential parametrix. Also pseudodifferential operators are useful for proving the key estimates needed to get hypoellipticity, but not always for the direct construction of a parametrix.

The class $S_{\rho, \delta}^m$ defined in (2.1.3) is already quite difficult to handle, and the L^2 -boundedness of operators with symbols in the class $S_{\delta, \delta}^0$ ($\delta < 1$) is due to A.Calderón and R.Vaillancourt [CV].

Global classes of pseudodifferential operators. For applications in spectral theory, were introduced the following classes of symbols, denoted by Σ_ρ^m , $m \in \mathbb{R}, \rho \in (0, 1]$: a smooth function on $\mathbb{R}^{2n}$ belongs to Σ_ρ^m means that

$$(2.1.4) \quad |(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi)| \leq C_{\alpha\beta} (1 + |x| + |\xi|)^{m - \rho|\alpha| - \rho|\beta|}.$$

This class is useful for studying the spectral properties of globally defined operators and can be generalized in different directions, with applications in global analysis.

The Beals-Fefferman revolution. In 1973, R.Beals and C.Fefferman proved the sufficiency of condition (P) for the local solvability of differential equation. The main tool in their proof was a new microlocalization scheme, going much beyond what has been done before. Their classes of symbols were defined with properties like

$$(2.1.5) \quad |(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi)| \leq C_{\alpha\beta} \varphi(x, \xi)^{m - |\alpha|} \Phi(x, \xi)^{M - |\beta|}$$

where φ, Φ are positive functions satisfying various requirements. The most original idea in their work was that, starting with a classical homogeneous symbol, a Calderón-Zygmund procedure would lead to a class with optimal properties for the given symbol. For instance, if $a \in S_{1,0}^1$, they define a weight depending on a by

$$(2.1.6) \quad \lambda(x, \xi) = |a(x, \xi)| + \langle \xi \rangle |\partial_\xi a(x, \xi)|^2 + \langle \xi \rangle^{-1} |\partial_x a(x, \xi)|^2 + 1, \quad \langle \xi \rangle = 1 + |\xi|.$$

Then one has obviously the following estimates

$$|a| \leq \lambda, \quad |\partial_x a| \leq \lambda^{1/2} \langle \xi \rangle^{1/2}, \quad |\partial_\xi a| \leq \lambda^{1/2} \langle \xi \rangle^{-1/2},$$

and more generally

$$(2.1.7) \quad |(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi)| \leq C_{\alpha\beta} \lambda^{1 - \frac{|\alpha|}{2} - \frac{|\beta|}{2}} \langle \xi \rangle^{\frac{|\alpha|}{2} - \frac{|\beta|}{2}}.$$

Note that (2.1.7) for $|\alpha| + |\beta| \geq 2$ is a direct consequence of $a \in S_{1,0}^1$: we know that $|(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{1-|\beta|}$ and thus it suffices to prove that

$$\langle \xi \rangle^{1-|\beta|} \leq C \lambda^{1 - \frac{|\alpha|}{2} - \frac{|\beta|}{2}} \langle \xi \rangle^{\frac{|\alpha|}{2} - \frac{|\beta|}{2}}, \quad \text{for } |\alpha| + |\beta| \geq 2,$$

that is

$$\lambda^{\frac{|\alpha|+|\beta|-2}{2}} \leq C \langle \xi \rangle^{\frac{|\alpha|+|\beta|-2}{2}}$$

which is true since, from (2.1.6) and $a \in S_{1,0}^1$, we have $\lambda \leq C \langle \xi \rangle$. Now due to the properties of λ , one is essentially reduced to study some normal forms, depending on which term is the largest in the rhs of (2.1.6). We shall go back to this later on, but the examples above show that, at the end of seventies, there was a need for some synthesis on the notion of pseudodifferential operator which is such a versatile tool. It was certainly necessary to think about general conditions that make a pseudodifferential calculus possible. We shall describe below the approach provided by L.Hörmander in 1979 in [Hö2], with some generalizations of the paper [BL]. The works of R.Beals and C.Fefferman [BF] and by A.Unterberger[Un] were also important for the understanding of the conditions to impose to get a useful pseudodifferential calculus.

2.2. Symbol classes defined by metrics

Preliminaries. We have already seen several examples of classes of symbols, and we can try to get some unified understanding of these examples. The class $S_{\rho,\delta}^m$ defined in (2.1.3) is related to a metric on the phase space

$$(2.2.1) \quad g_{x,\xi} = \frac{|dx|^2}{\langle \xi \rangle^{-2\delta}} + \frac{|d\xi|^2}{\langle \xi \rangle^{2\rho}}, \quad \text{i.e. for } (t, \tau) \in \mathbb{R}^n \times \mathbb{R}^n, \quad g_{x,\xi}(t, \tau) = \frac{|t|^2}{\langle \xi \rangle^{-2\delta}} + \frac{|\tau|^2}{\langle \xi \rangle^{2\rho}},$$

since, with $k = |\alpha| + |\beta|$, $e_j = ((\delta_{j,k})_{1 \leq k \leq n}, 0_{\mathbb{R}^n})$, $\epsilon_j = (0_{\mathbb{R}^n}, (\delta_{j,k})_{1 \leq k \leq n})$, we have

$$(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi) = \underbrace{a^{(k)}(x, \xi)}_{k\text{-multilinear form}} e_1^{\alpha_1} \dots e_n^{\alpha_n} \epsilon_1^{\beta_1} \dots \epsilon_n^{\beta_n}$$

so that the conditions (2.1.3) can be written as

$$\begin{aligned} |a^{(k)}(x, \xi) e_1^{\alpha_1} \dots e_n^{\alpha_n} \epsilon_1^{\beta_1} \dots \epsilon_n^{\beta_n}| &\leq C_{\alpha\beta} \langle \xi \rangle^m \prod_{1 \leq j \leq n} g_{x,\xi}(e_j)^{\alpha_j/2} \prod_{1 \leq j \leq n} g_{x,\xi}(\epsilon_j)^{\beta_j/2} \\ &= C_{\alpha\beta} \langle \xi \rangle^m \langle \xi \rangle^{\delta|\alpha|} \langle \xi \rangle^{-\rho|\beta|} \end{aligned}$$

and using the notation $X = (x, \xi)$ for the running point of the phase space, we can reformulate (2.1.3) as

$$(2.2.2) \quad |a^{(k)}(X)(T_1, \dots, T_k)| \leq C_k M(X) \prod_{1 \leq j \leq k} g_X(T_j)^{1/2},$$

with $M(x, \xi) = \langle \xi \rangle^m$. The metric linked to the class (2.1.4) is

$$g_{x, \xi} = \frac{|dx|^2 + |d\xi|^2}{1 + |x|^2 + |\xi|^2}$$

and the metric linked to (2.1.7) is

$$(2.2.3) \quad G_{x, \xi} = \frac{|dx|^2}{\lambda \langle \xi \rangle^{-1}} + \frac{|d\xi|^2}{\lambda \langle \xi \rangle} = \left(\frac{|dx|^2}{1} + \frac{|d\xi|^2}{\langle \xi \rangle^2} \right) \frac{\langle \xi \rangle}{\lambda} \gtrsim \frac{|dx|^2}{1} + \frac{|d\xi|^2}{\langle \xi \rangle^2}.$$

The initial strategy to define a general class of pseudodifferential operator is thus to use some Riemannian structure on the phase space and define the symbols by conditions (2.2.2). The metric g will have to satisfy some requirements as well as the “weight” M in (2.2.2). The next section describes these conditions.

Slowly varying metrics. Let us consider a metric on $\mathbb{R}^{2n}$, i.e. a measurable mapping from $\mathbb{R}^{2n}$ to $\mathcal{C}$, the cone of positive definite quadratic forms on $\mathbb{R}^{2n}$. For $X \in \mathbb{R}^{2n}$, we shall note that quadratic form g_X : for $T \in \mathbb{R}^{2n}$, $g_X(T)$ is a positive definite quadratic form in the variable T , with coefficients depending on X . The first important property that we shall require is the *slow variation* of the metric g : $\exists C > 0, \exists r > 0, \forall X, Y, T \in \mathbb{R}^{2n}$,

$$(2.2.4) \quad g_X(Y - X) \leq r^2 \implies C^{-1} g_Y(T) \leq g_X(T) \leq C g_Y(T).$$

REMARK 2.2.1. Note that for checking (2.2.4), it is enough to verify that

$$(2.2.5) \quad \exists C > 0, \forall X, Y, T \in \mathbb{R}^{2n}, g_X(Y - X) \leq C^{-1} \implies g_Y(T) \leq C g_X(T).$$

In fact, once (2.2.5) is proven, we see that assuming $g_X(Y - X) \leq C^{-2}$ implies $g_Y(Y - X) \leq C^{-1}$ and from (2.2.5), we get the missing $g_X(T) \leq C g_Y(T)$.

A SIMPLE EXAMPLE. If we consider for a positive Lipschitz function ρ defined on $\mathbb{R}^{2n}$, and an Euclidean norm $\|\cdot\|$ on $\mathbb{R}^{2n}$, the metric defined by

$$g_X(T) = \frac{\|T\|^2}{\rho(X)^2},$$

one can check easily that g is slowly varying: assuming $g_X(Y - X) \leq r^2$ means that $\|Y - X\| \leq r \rho(X)$ so that, if L is the Lipschitz constant of ρ , we have

$$\rho(X) \leq \rho(Y) + L \|X - Y\| \leq \rho(Y) + L r \rho(X).$$

Choosing $r = 1/2L$ gives the desired property.

Using remark 2.2.1, it is very easy to verify that the metric of $S_{1,0}^m$ is slowly varying. Assuming $|\xi - \eta| \leq \langle \xi \rangle / 2$, we get

$$\langle \xi \rangle = 1 + |\xi| \leq 1 + |\xi - \eta| + |\eta| \leq \langle \eta \rangle + \langle \xi \rangle / 2 \implies \langle \xi \rangle \leq 2\langle \eta \rangle,$$

To prove that the metric (2.2.1) of $S_{\rho,\delta}^m$ is slowly varying if $\rho \leq 1$, we assume $|\xi - \eta| \leq \langle \xi \rangle^\rho / 2$, and we get, using $\rho \leq 1$,

$$\langle \xi \rangle = 1 + |\xi| \leq 1 + |\xi - \eta| + |\eta| \leq \langle \eta \rangle + \langle \xi \rangle^\rho / 2 \leq \langle \eta \rangle + \langle \xi \rangle / 2 \implies \langle \xi \rangle \leq 2\langle \eta \rangle.$$

$$\langle \eta \rangle = 1 + |\eta| \leq 1 + |\xi - \eta| + |\xi| \leq \langle \xi \rangle + \langle \xi \rangle^\rho / 2 \leq 3\langle \xi \rangle / 2.$$

Note that the metric (2.2.1) with $\rho > 1$ is not slowly varying since

$$\frac{\langle \eta \rangle^\rho}{\langle \xi \rangle^\rho} \leq \sup_{(t,\tau)} \frac{g_{x,\xi}(t,\tau)^{1/2}}{g_{y,\eta}(t,\tau)^{1/2}}$$

and (assuming $n = 1$), for $k > 1$ with $\xi = k, \eta = k^\rho / \ln k$, we have $|\xi - \eta| \sim k^\rho / \ln k \ll k^\rho \sim \langle \xi \rangle^\rho$, so that

$$\frac{|\xi - \eta|}{\langle \xi \rangle^\rho} = g_{x,\xi}(x - x, \xi - \eta)^{1/2} \xrightarrow[k \rightarrow +\infty]{} 0 \quad \text{but} \quad \frac{\langle \eta \rangle}{\langle \xi \rangle} \sim k^{\rho-1} / \ln k \xrightarrow[k \rightarrow +\infty]{} +\infty.$$

Let us prove that (2.2.3) is slowly varying. We want to find $r, C > 0$ such that

$$G_{x,\xi}(x - y, \xi - \eta) = \frac{|x - y|^2}{\lambda(x, \xi) \langle \xi \rangle^{-1}} + \frac{|\xi - \eta|^2}{\lambda(x, \xi) \langle \xi \rangle} \leq r^2 \implies \frac{\lambda(x, \xi) \langle \xi \rangle^{-1}}{\lambda(y, \eta) \langle \eta \rangle^{-1}} + \frac{\lambda(x, \xi) \langle \xi \rangle}{\lambda(y, \eta) \langle \eta \rangle} \leq C.$$

Since $G_{x,\xi} \gtrsim \frac{|dx|^2}{1} + \frac{|d\xi|^2}{\langle \xi \rangle^2}$, the fact that the smaller metric is slowly varying implies that $r > 0$ can be chosen so that $\langle \xi \rangle \approx \langle \eta \rangle$, that is the ratio $\langle \xi \rangle / \langle \eta \rangle$ is bounded above and below by fixed constants. We write now the Taylor expansion

$$\begin{aligned} a(x, \xi) &= a(y, \eta) + \partial_x a(y, \eta)(x - y) + \partial_\xi a(y, \eta)(\xi - \eta) \\ &\quad + O(\langle \xi \rangle |x - y|^2 + |x - y| |\xi - \eta| + \langle \xi \rangle^{-1} |\xi - \eta|^2) \end{aligned}$$

which implies

$$\begin{aligned} |a(x, \xi)| &\lesssim |a(y, \eta)| + \langle \eta \rangle^{-1/2} |\partial_x a(y, \eta)| |x - y| \langle \xi \rangle^{1/2} + \langle \eta \rangle^{1/2} |\partial_\xi a(y, \eta)| |\xi - \eta| \langle \xi \rangle^{-1/2} \\ &\quad + \langle \xi \rangle r^2 \lambda(x, \xi) \langle \xi \rangle^{-1} + r \lambda(x, \xi)^{1/2} \langle \xi \rangle^{-1/2} r \lambda(x, \xi)^{1/2} \langle \xi \rangle^{1/2} \\ &\quad + \langle \xi \rangle^{-1} r^2 \lambda(x, \xi) \langle \xi \rangle \end{aligned}$$

so that

$$|a(x, \xi)| \lesssim |a(y, \eta)| + \langle \eta \rangle^{-1} |\partial_x a(y, \eta)|^2 + r^2 \lambda(x, \xi) + \langle \eta \rangle |\partial_\xi a(y, \eta)|^2 \leq \lambda(y, \eta) + r^2 \lambda(x, \xi).$$

Similarly, using the Taylor expansions

$$\begin{aligned} \partial_x a(x, \xi) &= \partial_x a(y, \eta) + O(\langle \xi \rangle |x - y| + |\xi - \eta|), \\ \partial_\xi a(x, \xi) &= \partial_\xi a(y, \eta) + O(|x - y| + \langle \xi \rangle^{-1} |\xi - \eta|), \end{aligned}$$

we get

$$\begin{aligned} \langle \xi \rangle^{-1} |\partial_x a(x, \xi)|^2 &\lesssim \langle \eta \rangle^{-1} |\partial_x a(y, \eta)|^2 + \langle \xi \rangle^{-1} \langle \xi \rangle^2 r^2 \lambda(x, \xi) \langle \xi \rangle^{-1} + \langle \xi \rangle^{-1} r^2 \lambda(x, \xi) \langle \xi \rangle \\ &\lesssim \lambda(y, \eta) + r^2 \lambda(x, \xi) \end{aligned}$$

and

$$\begin{aligned} \langle \xi \rangle |\partial_\xi a(x, \xi)|^2 &\lesssim \langle \eta \rangle |\partial_\xi a(y, \eta)|^2 + \langle \xi \rangle r^2 \lambda(x, \xi) \langle \xi \rangle^{-1} + \langle \xi \rangle \langle \xi \rangle^{-2} r^2 \lambda(x, \xi) \langle \xi \rangle \\ &\lesssim \lambda(y, \eta) + r^2 \lambda(x, \xi). \end{aligned}$$

Eventually, we obtain, with a constant C_0 depending only on the dimension,

$$\lambda(x, \xi) \leq C_0 \lambda(y, \eta) + C_0 r^2 \lambda(x, \xi).$$

We can then choose r such that $C_0 r^2 \leq 1/2$ and obtain the desired result. We invite the reader to check carefully in the reasoning above that the constant C_0 depend only on the dimension: in particular, this is the case in the constants occurring above with the $O()$ and our signs $a \lesssim b$ meaning that $a \leq C_1 b$ where C_1 is a constant depending only on the dimension.

We define also the notion of *slowly varying weight*: a measurable function $M : \mathbb{R}^{2n} \rightarrow]0, +\infty[$ is a slowly varying weight for the metric g if

$$(2.2.6) \quad \exists C > 0, \forall X, Y \in \mathbb{R}^{2n}, g_X(Y - X) \leq C^{-1} \implies C^{-1} \leq \frac{M(X)}{M(Y)} \leq C.$$

It is easy to verify that $\langle \xi \rangle$ is a slowly varying weight for the metric (2.2.1) when $\rho \leq 1$ and also to check that $\lambda(x, \xi)$ is a slowly varying weight for the metric (2.2.3). Note also that products of slowly varying weights for g are still slowly varying weights for g , and more generally that M^θ is a slowly varying weight if M is a slowly varying weight and $\theta \in \mathbb{R}$.

Symbol classes defined by a metric on the phase space.

DEFINITION 2.2.2. Let g be a metric defined on the phase space $\mathbb{R}^{2n}$ and let M be a slowly varying weight for g . The space of symbols $S(M, g)$ is defined as the set of C^∞ functions on $\mathbb{R}^{2n}$ such that $\forall k \in \mathbb{N}, \exists C_k, \forall X, T_1, \dots, T_k \in \mathbb{R}^{2n}$,

$$(2.2.7) \quad |a^{(k)}(X)(T_1, \dots, T_k)| \leq C_k M(X) \prod_{1 \leq j \leq k} g_X(T_j)^{1/2}.$$

For $a \in S(M, g)$, we shall denote

$$(2.2.7)' \quad \|a^{(k)}\|_{g, M} = \sup_{\substack{X, T \\ g_X(T_j)=1}} |a^{(k)}(X)(T_1, \dots, T_k)| M(X)^{-1}.$$

We can check readily that

$$\begin{aligned} S_{\rho, \delta}^m &= S(\langle \xi \rangle^m, \frac{|dx|^2}{\langle \xi \rangle^{-2\delta}} + \frac{|d\xi|^2}{\langle \xi \rangle^{2\rho}}) \quad (\text{see (2.2.1)}), \\ \Sigma_\rho^m &= S((1 + |x| + |\xi|)^m, \frac{|dx|^2 + |d\xi|^2}{(1 + |x|^2 + |\xi|^2)^\rho}) \quad (\text{see (2.1.4)}), \\ (2.1.5) \text{ defines } S(\Phi^M \varphi^m, \frac{|dx|^2}{\varphi^2} + \frac{|d\xi|^2}{\Phi^2}), \\ (2.1.7) \text{ defines } S(\lambda, G), \quad \text{where } G \text{ is defined by (2.2.3)}. \end{aligned}$$

SOME COMMENTS. It is not difficult to illustrate the meaning of the localization via a slowly varying metric. Around each point X , one may draw the ball

$$B_g(X, r) = \{Y, g_X(Y - X) \leq r^2\}$$

and if r is chosen small enough, *uniformly with respect to X* the shape and size of $B_g(Y, r)$ will be equivalent to the shape and size of $B_g(X, r)$ for $Y \in B_g(X, r)$. R.Beals and C.Fefferman in [BF] preferred to speak about “boxes”, since their metrics were defined by (2.1.5) and they would consider

$$\beta(x, \xi; r) = \left\{ (y, \eta), \max\left(\frac{|y - x|}{\varphi(x, \xi)}, \frac{|\eta - \xi|}{\Phi(x, \xi)}\right) \leq r \right\}.$$

The picture on next page shows the metric related to the simple $S_{1,0}$ class.

The uncertainty principle. Let us start with some more symplectic algebra. Let Γ be a positive definite quadratic form on the symplectic space $\mathbb{R}^{2n}$. We define the dual quadratic Γ^σ with respect to the symplectic form σ form by

$$(2.2.8) \quad \Gamma^\sigma(T) = \sup_{U \neq 0} \frac{[T, U]^2}{\Gamma(U)} = \sup_{U \neq 0} \frac{\langle \sigma T, U \rangle^2}{\langle \Gamma U, U \rangle},$$

PICTURE 1. THE METRIC OF THE CLASS $S_{1,0}$. THE TYPICAL BOX IS $\{(x, \xi), |x - x_0| \lesssim 1, |\xi - \xi_0| \lesssim |\xi_0|\}$.

where the brackets are brackets of duality between the phase space and its dual (which is canonically isomorphic). We can consider the phase space Φ as an Euclidean vector space with the dot product $X \cdot Y = \langle \Gamma X, Y \rangle_{\Phi^*, \Phi}$, and the symplectic form σ can be identified to an antisymmetric nondegenerate matrix Ω : defining Ω with the identity

$$\langle \sigma X, Y \rangle_{\Phi^*, \Phi} = \Omega X \cdot Y = \langle \Gamma \Omega X, Y \rangle_{\Phi^*, \Phi}, \quad \text{that is } \sigma = \Gamma \Omega,$$

we have

$$\Omega X \cdot Y = \langle \sigma X, Y \rangle_{\Phi^*, \Phi} = -\langle \sigma Y, X \rangle_{\Phi^*, \Phi} = -\Omega Y \cdot X == -{}^t\Omega X \cdot Y \implies {}^t\Omega = -\Omega.$$

Then, we have

$$\Gamma^\sigma(T) = \sup_{U \neq 0} \frac{\langle \sigma T, U \rangle^2}{\langle \Gamma U, U \rangle} = \sup_{U \neq 0} \frac{\langle \Gamma \Omega T, U \rangle^2}{\langle \Gamma U, U \rangle} = \sup_{U \neq 0} \frac{(\Omega T \cdot U)^2}{U \cdot U} = \Omega T \cdot \Omega T$$

and with $\sigma T = \Gamma S$, $\langle \sigma^* \Gamma^{-1} \sigma T, T \rangle = \langle S, \Gamma S \rangle = S \cdot S$. But we know that $\sigma T = \Gamma \Omega T$, so that $S = \Omega T$. Thus we obtain

$$\Gamma^\sigma = \sigma^* \Gamma^{-1} \sigma.$$

We have seen that, considering the phase space Φ as an Euclidean vector space with the dot product $X \cdot Y = \langle \Gamma X, Y \rangle_{\Phi^*, \Phi}$, the symplectic form σ can be identified to an antisymmetric nondegenerate matrix. Then there exists an orthonormal basis (for the dot product above) $(e_j, \epsilon_j)_{1 \leq j \leq n}$ such that $\sigma(\epsilon_j, e_k) = \lambda_j \delta_{j,k}$ with $\lambda_j > 0$, $\sigma(\epsilon_j, \epsilon_k) = 0$, $\sigma(e_j, e_k) = 0$. Eventually, we have, for $X = \sum_{1 \leq j \leq n} x_j \underbrace{\lambda_j^{-1/2} e_j}_{\tilde{e}_j} + \xi_j \underbrace{\lambda_j^{-1/2} \epsilon_j}_{\tilde{\epsilon}_j}$,

$$\langle \Gamma X, X \rangle = \sum_{1 \leq j \leq n} \lambda_j^{-1} (x_j^2 + \xi_j^2).$$

We note that the Γ -orthogonal basis $(\tilde{e}_j, \tilde{\epsilon}_j)_{1 \leq j \leq n}$ is also symplectic, i.e. such that

$$\sigma(\tilde{\epsilon}_j, \tilde{e}_k) = \delta_{j,k}, \quad \sigma(\tilde{\epsilon}_j, \tilde{\epsilon}_k) = 0, \quad \sigma(\tilde{e}_j, \tilde{e}_k) = 0.$$

In that set of coordinates, we have, using (2.2.8),

$$\langle \Gamma^\sigma X, X \rangle = \sum_{1 \leq j \leq n} \lambda_j (x_j^2 + \xi_j^2).$$

Note also that, for Γ given with symplectic coordinates by $\sum_{1 \leq j \leq n} h_j x_j^2 + k_j \xi_j^2$, then Γ^σ is $\sum_{1 \leq j \leq n} k_j^{-1} x_j^2 + h_j^{-1} \xi_j^2$. We can remark also that, for Γ_1, Γ_2 positive definite quadratic forms on $\mathbb{R}^{2n}$, we have

$$\Gamma_1 \leq \Gamma_2 \implies \Gamma_2^\sigma \leq \Gamma_1^\sigma.$$

The uncertainty principle is asserting some limitations on the localization in a symplectic plane. Our formulation, will thus be that, for all j , $\lambda_j \geq 1$, namely the positive definite quadratic form Γ will be said to satisfy the uncertainty principle whenever

$$(2.2.9) \quad \Gamma \leq \Gamma^\sigma.$$

When dealing with a metric g_X on the phase space, we shall always require that

$$(2.2.10) \quad \forall X \in \mathbb{R}^{2n}, \quad g_X \leq g_X^\sigma,$$

and define

$$(2.2.11) \quad \lambda_g(X) = \inf_{T \neq 0} \left(\frac{g_X^\sigma(T)}{g_X(T)} \right)^{1/2} (\geq 1 \text{ from (2.2.10)}).$$

For instance for the (constant with respect to X) metric Γ above, we have $\lambda_\Gamma = \min_j \lambda_j$.

Going back to our examples of metrics above, we check easily that for

$$\begin{aligned} g &= \frac{|dx|^2}{\langle \xi \rangle^{-2\delta}} + \frac{|d\xi|^2}{\langle \xi \rangle^{2\rho}}, \quad g^\sigma = \frac{|dx|^2}{\langle \xi \rangle^{-2\rho}} + \frac{|d\xi|^2}{\langle \xi \rangle^{2\delta}}, \quad \lambda = \langle \xi \rangle^{\rho-\delta} \quad (\text{see (2.2.1)}), \\ g &= \frac{|dx|^2 + |d\xi|^2}{1 + |x|^2 + |\xi|^2}, \quad g^\sigma = (1 + |x|^2 + |\xi|^2)(|dx|^2 + |d\xi|^2), \quad \lambda = 1 + |x|^2 + |\xi|^2, \\ g &= \frac{|dx|^2}{\varphi^2} + \frac{|d\xi|^2}{\Phi^2}, \quad g^\sigma = \frac{|dx|^2}{\Phi^{-2}} + \frac{|d\xi|^2}{\phi^{-2}}, \quad \lambda = \Phi\varphi \quad (\text{or } \min_j \Phi_j\varphi_j), \\ G^\sigma &= \frac{|dx|^2}{\lambda^{-1}\langle \xi \rangle^{-1}} + \frac{|d\xi|^2}{\lambda^{-1}\langle \xi \rangle}, \quad \text{where } G \text{ is defined by (2.2.3),} \quad \lambda_G = \lambda. \end{aligned}$$

THEOREM 2.2.3. *Let g be a metric on the phase space such that the uncertainty principle (2.2.10) is satisfied. Let m_1, m_2 be two slowly varying weights for g (cf. (2.2.6)) and $a_j \in S(m_j, g)$. Then, with λ defined in (2.2.11), we have*

$$a_1 a_2 \in S(m_1 m_2, g), \quad \{a_1, a_2\} \in S(m_1 m_2 \lambda^{-1}, g)$$

and more generally

$$(2.2.12) \quad \omega_N(a_1, a_2) = \left(\frac{\iota}{2} [D_{X_1}, D_{X_2}] \right)^N (a_1(X_1) \otimes a_2(X_2))|_{X_1=X_2} \in S(m_1 m_2 \lambda^{-N}, g).$$

PROOF. Using the fact that there exists a symplectic basis (possibly depending on X) such that

$$g_X = \sum_{1 \leq j \leq n} \lambda_j^{-1} (|dy_j|^2 + |d\eta_j|^2), \quad \min_j \lambda_j = \lambda(X),$$

it is enough to check $\partial_{y_j} a_1 \partial_{\eta_j} a_2, \partial_{\eta_j} a_1 \partial_{y_j} a_2 \in S(m_1 m_2 \lambda^{-1}, g)$, which is obvious since we have

$$|\partial_{y_j} a_1 \partial_{\eta_j} a_2| \lesssim m_1 \lambda_j^{-1/2} m_2 \lambda_j^{-1/2} = m_1 m_2 \lambda_j^{-1} \leq m_1 m_2 \lambda^{-1},$$

and similar estimates hold for the derivatives. $\square$

NOTA BENE. This theorem also gives an asymptotic composition formula

$$\begin{aligned} S(m_1, g)/S(m_1 \lambda^{-\infty}, g) \times S(m_2, g)/S(m_2 \lambda^{-\infty}, g) &\longrightarrow S(m_1 m_2, g)/S(m_1 m_2 \lambda^{-\infty}, g) \\ (a_1, a_2) &\longmapsto \sum_{k \geq 0} \omega_k(a_1, a_2) \end{aligned}$$

where $S(m \lambda^{-\infty}, g) = \cap_{N \geq 0} S(m \lambda^{-N}, g)$ and the infinite sum on the right should be understood as a symbol a such that

$$\forall N \geq 0, \quad a - \sum_{0 \leq k < N} \omega_k(a_1, a_2) \in S(m_1 m_2 \lambda^{-N}, g).$$

The existence of such a symbol is obtained with a Borel-type argument. However, this result is uninteresting if λ is bounded above, and does not replace the study of the actual composition formula given by the integral (1.2.7).

Temperance. Let us start again with some linear algebra. We consider Γ_1, Γ_2 two positive definite quadratic forms on a finite dimensional real vector space. We define, the “infimum” of Γ_1, Γ_2 by the formula

$$(2.2.13) \quad (\Gamma_1 \wedge \Gamma_2)(T) = 2 \inf_{T_1+T_2=T} (\Gamma_1(T_1) + \Gamma_2(T_2)) \leq 2\Gamma_j(T), \quad j = 1, 2.$$

Note that

$$\Gamma_1 \wedge \Gamma_2 = 2(\Gamma_1^{-1} + \Gamma_2^{-1})^{-1}, \quad (\text{so that } \Gamma \wedge \Gamma = \Gamma),$$

that is the harmonic mean of Γ_1, Γ_2 . In fact, for $T = (\Gamma_1^{-1} + \Gamma_2^{-1})\Xi$, we have

$$\begin{aligned} 2(\Gamma_1^{-1} + \Gamma_2^{-1})^{-1}(T) &= 2\langle (\Gamma_1^{-1} + \Gamma_2^{-1})^{-1}T, T \rangle \\ &= 2\langle \Xi, (\Gamma_1^{-1} + \Gamma_2^{-1})\Xi \rangle = 2\langle \Xi, \Gamma_1^{-1}\Xi \rangle + 2\langle \Xi, \Gamma_2^{-1}\Xi \rangle \\ &= 2\langle \Gamma_1 T_1, T_1 \rangle + 2\langle \Gamma_2 T_2, T_2 \rangle, \end{aligned}$$

and $T = (\Gamma_1^{-1} + \Gamma_2^{-1})\Xi$, $\Xi = \Gamma_1 T_1 = \Gamma_2 T_2$, and thus $T = T_1 + T_2$. Moreover, we have

$$\begin{aligned} &2\langle \Gamma_1(T_1 + S), (T_1 + S) \rangle + 2\langle \Gamma_2(T_2 - S), (T_2 - S) \rangle \\ &= \underbrace{2\langle \Gamma_1 T_1, T_1 \rangle + 2\langle \Gamma_2 T_2, T_2 \rangle}_{=2(\Gamma_1^{-1} + \Gamma_2^{-1})^{-1}(T)} + \underbrace{4\langle \Gamma_1 T_1, S \rangle - 4\langle \Gamma_2 T_2, S \rangle}_{=0} + \underbrace{2\langle \Gamma_1 S, S \rangle + 2\langle \Gamma_2 S, S \rangle}_{\geq 0}, \end{aligned}$$

so that $(\Gamma_1 \wedge \Gamma_2)(T) = 2(\Gamma_1^{-1} + \Gamma_2^{-1})^{-1}(T)$, qed. Defining

$$(2.2.14) \quad \Gamma_1 \vee \Gamma_2 = \frac{\Gamma_1 + \Gamma_2}{2},$$

we see that $(\Gamma_1 \vee \Gamma_2)^{-1} = 2(\Gamma_1 + \Gamma_2)^{-1} = \Gamma_1^{-1} \wedge \Gamma_2^{-1}$ and, going back to (2.2.8), we obtain, on the symplectic $\mathbb{R}^{2n}$,

$$(2.2.15) \quad \Gamma_1^\sigma \wedge \Gamma_2^\sigma = \sigma^* \Gamma_1^{-1} \sigma \wedge \sigma^* \Gamma_2^{-1} \sigma = \sigma^* (\Gamma_1^{-1} \wedge \Gamma_2^{-1}) \sigma = (\Gamma_1 \vee \Gamma_2)^\sigma.$$

Apart from (2.2.4) and (2.2.11), we shall need a global condition on the metrics g .

DEFINITION 2.2.4. TEMPERATE METRIC. Let g be a metric on $\mathbb{R}^{2n}$ (see (2.2.4)). We shall say that g is temperate if

$$(2.2.16) \quad \exists C > 0, \exists N \geq 0, \forall X, Y, T \in \mathbb{R}^{2n}, \quad \frac{g_X(T)}{g_Y(T)} \leq C(1 + (g_X^\sigma \wedge g_Y^\sigma)(X - Y))^N.$$

REMARK 2.2.5. The definition in [Hö2] looked less symmetrical and weaker, since the rhs of the inequality in (2.2.16) was $(1 + g_X^\sigma(X - Y))^N$. The definition (2.2.16) was

introduced in [BL] to handle pseudodifferential calculus on an open set of $\mathbb{R}^{2n}$. Let us show now that these conditions are equivalent when considering metrics defined on the whole $\mathbb{R}^{2n}$. For $X, Y \in \mathbb{R}^{2n}$, there exists $Z \in \mathbb{R}^{2n}$ such that

$$(g_X^\sigma \wedge g_Y^\sigma)(X - Y) = 2g_X^\sigma(X - Z) + 2g_Y^\sigma(Z - Y).$$

Now from the weak version of (2.2.16), we know that

$$g_X \leq C_1 g_Z (1 + g_X^\sigma(X - Z))^{N_1} \quad \text{and also} \quad g_Z^\sigma \leq C_1 g_Y^\sigma (1 + g_Y^\sigma(Y - Z))^{N_1}.$$

Then we get

$$g_Z^\sigma(Z - Y) \leq C_1 (1 + g_Y^\sigma(Y - Z))^{N_1+1} \quad \text{and thus} \quad g_Z \leq C_2 g_Y (1 + g_Y^\sigma(Z - Y))^{N_1(N_1+1)}.$$

We obtain

$$\begin{aligned} \frac{g_X(T)}{g_Y(T)} &\leq C_1 \frac{g_Z(T)}{g_Y(T)} (1 + g_X^\sigma(X - Z))^{N_1} \\ &\leq C_1 C_2 (1 + g_Y^\sigma(Z - Y))^{N_1(N_1+1)} (1 + g_X^\sigma(X - Z))^{N_1} \\ &\leq C_3 (1 + 2g_X^\sigma(X - Z) + 2g_Y^\sigma(Z - Y))^{N_1(N_1+1)} \\ &= C_3 (1 + (g_X^\sigma \wedge g_Y^\sigma)(X - Y))^{N_1(N_1+1)}, \text{ qed.} \end{aligned}$$

We can go back to our examples and check that, for the metric (2.2.1) the temperance holds whenever $\delta \leq \rho \leq 1, \delta < 1$. In particular, the metric of the class $S_{1,1}$ is not temperate since that would imply

$$\langle \xi \rangle \leq C \langle \eta \rangle (1 + |\xi - \eta| / \langle \xi \rangle)^N,$$

which cannot hold for $\eta = 0$ and unbounded ξ .

Admissible metric.

DEFINITION 2.2.6. ADMISSIBLE METRIC AND WEIGHTS. Let g be a metric on $\mathbb{R}^{2n}$, which is slowly varying (see (2.2.4)), satisfies the uncertainty principle (see 2.2.10) and is temperate (see (2.2.16)). We shall say then that g is an admissible metric. Let m be a slowly varying weight for g (see (2.2.6)) such that

$$(2.2.17) \quad \exists C, N, \forall X, Y \in \mathbb{R}^{2n}, \quad \frac{m(X)}{m(Y)} \leq C (1 + (g_X^\sigma \wedge g_Y^\sigma)(X - Y))^N.$$

then we shall say that m is an admissible weight for g .

REMARK 2.2.7. Note that, for an admissible metric g , the function λ_g defined in (2.2.11) is an admissible weight: first λ_g is a slowly varying weight for g , since for $r > 0$ small enough, $g_X(Y - X) \leq r^2$ implies $g_X \sim g_Y$ and thus $g_X^\sigma \sim g_Y^\sigma$, so that

$$\frac{g_X^\sigma(T)}{g_X(T)} \sim \frac{g_Y^\sigma(T)}{g_Y(T)} \implies \lambda_g(X) \sim \lambda_g(Y).$$

Moreover we have

$$\begin{aligned} \frac{g_X^\sigma(T)}{g_X(T)} &\leq C_1 (1 + (g_X^\sigma \wedge g_Y^\sigma)(X - Y))^{N_1} \frac{g_Y^\sigma(T)}{g_Y(T)} \\ &\implies \lambda_g(X)^2 \leq \lambda_g(Y)^2 C_1 (1 + (g_X^\sigma \wedge g_Y^\sigma)(X - Y))^{N_1}. \end{aligned}$$

Naturally, the powers of λ_g are also admissible weights.

NOTA BENE. The class $S_{\rho,\delta}^m$ is $S(\langle \xi \rangle^m, \frac{|dx|^2}{\langle \xi \rangle^{-2\delta}} + \frac{|d\xi|^2}{\langle \xi \rangle^{2\rho}})$ and, for this metric $\lambda = \langle \xi \rangle^{\rho-\delta}$. The weights are multiplicative whereas the “orders” of standard classes are additive ; although we shall not use this terminology, it would be natural to say that the symbols of order μ related to the metric g are those in $S(\lambda^\mu, g)$.

CHAPTER III

THE KEY ESTIMATES

3.1. More results on admissible metrics

Partitions of unity. Let g be a metric on $\mathbb{R}^{2n}$. We note

$$(3.1.1) \quad U_{X,r} = \{Y \in \mathbb{R}^{2n}, g_X(Y - X) \leq r^2\},$$

the closed g_X -ball of center X and radius r .

THEOREM 3.1.1. *Let g be a slowly varying metric on $\mathbb{R}^{2n}$ (see (2.2.4)). Then there exists $r_0 > 0$ such that, for all $r \in (0, r_0]$, there exists a family $(\varphi_Y)_{Y \in \mathbb{R}^{2n}}$ of functions uniformly in $S(1, g)$ (i.e. according to (2.2.7)', $\forall k, \sup_Y \|\varphi_Y^{(k)}\|_{g,M} < \infty$) supported in $U_{Y,r}$ such that, for all $X \in \mathbb{R}^{2n}$,*

$$\int_{\mathbb{R}^{2n}} \varphi_Y(X) |g_Y|^{1/2} dY = 1,$$

where $|g_Y|$ is the determinant of g_Y (with respect to the standard Euclidean norm).

PROOF. Let $\chi_0 \in C^\infty(\mathbb{R}; [0, 1])$ nonincreasing such that $\chi_0(t) = 1$ on $t \leq 1/2$, $\chi_0(t) = 0$ on $t \geq 1$. Let us assume that, with $r_0 > 0$, for all X, Y, T , we have

$$g_Y(Y - X) \leq r_0^2 \implies C_0^{-1} \leq \frac{g_X(T)}{g_Y(T)} \leq C_0.$$

We define, for $0 < r \leq r_0$, $\omega(X, r) = \int_{\mathbb{R}^{2n}} \chi_0(r^{-2} g_Y(X - Y)) |g_Y|^{1/2} dY$. We have

$$\omega(X, r) \geq \int_{\mathbb{R}^{2n}} \chi_0(r^{-2} C_0 g_X(X - Y)) |g_X|^{1/2} C_0^{-n} dY = \underbrace{\int_{\mathbb{R}^{2n}} \chi_0(C_0 |Z|^2) C_0^{-n} r^{2n} dZ}_{\text{a positive constant}},$$

and an estimate from above of the same type. Also we see that, using the notation $\|T\|_Y = g_Y(T)^{1/2}$ and $\langle, \rangle_Y$ for the associated dot product, we have

$$\omega'(X, r)T = \int_{\mathbb{R}^{2n}} \chi_0'(r^{-2} \|X - Y\|_Y^2) r^{-2} 2 \langle (X - Y), T \rangle_Y |g_Y|^{1/2} dY,$$

which entails immediately for all k , $|\omega^{(k)}(X, r)T^k| \leq C_k g_X(T)^{k/2}$, so that $\omega(\cdot, r)^{-1}$ and $\omega(\cdot, r)$ belong to $S(1, g)$. As a consequence, we get that

$$1 = \int_{\mathbb{R}^{2n}} \underbrace{\chi_0(r^{-2}g_Y(X - Y))\omega(X, r)^{-1}}_{=\varphi_Y(X)} |g_Y|^{1/2} dY,$$

where the family $(\varphi_Y)_{Y \in \mathbb{R}^{2n}}$ satisfies the requirements of the theorem. $\square$

An immediate consequence of Theorem 3.1.1 is the following

THEOREM 3.1.2. *Let g be an admissible metric on $\mathbb{R}^{2n}$ and M be an admissible weight for g . Let a be a symbol in $S(m, g)$. Then there exists $r_0 > 0$ such that, for all $0 < r \leq r_0$,*

$$a(X) = \int_{\mathbb{R}^{2n}} a_Y(X) |g_Y|^{1/2} dY$$

where $a_Y \in S(m(Y), g_Y)$ uniformly and $\text{supp } a_Y \subset U_{Y,r}$.

Main distance function. Let g be an admissible metric on $\mathbb{R}^{2n}$. We define, for $X, Y \in \mathbb{R}^{2n}$ and $r > 0$,

$$(3.1.2) \quad \delta_r(X, Y) = 1 + (g_X^\sigma \wedge g_Y^\sigma)(U_{X,r} - U_{Y,r}).$$

We note that

$$\delta_r(X, Y) \leq 1 + 2 \min(g_X^\sigma(U_{X,r} - U_{Y,r}), g_Y^\sigma(U_{X,r} - U_{Y,r})),$$

and since g is temperate, for $X' \in U_{X,r}, Y' \in U_{Y,r}$, we have, for r small enough,

$$g_X^\sigma \wedge g_Y^\sigma \sim g_{X'}^\sigma \wedge g_{Y'}^\sigma \gtrsim g_{X'}^\sigma (1 + (g_{X'}^\sigma \wedge g_{Y'}^\sigma)(X' - Y'))^{-N} \sim g_X^\sigma (1 + (g_X^\sigma \wedge g_Y^\sigma)(X' - Y'))^{-N}$$

which implies $g_X^\sigma \wedge g_Y^\sigma \gtrsim g_X^\sigma \delta_r(X, Y)^{-N}$ and thus $g_X^\sigma(U_{X,r} - U_{Y,r}) \lesssim \delta_r(X, Y)^{N+1}$.

LEMMA 3.1.3. *Let g and δ_r be as above, and (2.2.16) satisfied with $N = N_0$. Then, there exist C_1, C_2 and $r_0 > 0$ such that, for $0 < r \leq r_0$,*

$$(3.1.3) \quad \begin{aligned} \delta_r(X, Y) &\leq C_1 \left(1 + \min(g_X^\sigma(U_{X,r} - U_{Y,r}), g_Y^\sigma(U_{X,r} - U_{Y,r})) \right) \\ &\leq C_1 \left(1 + \max(g_X^\sigma(U_{X,r} - U_{Y,r}), g_Y^\sigma(U_{X,r} - U_{Y,r})) \right) \leq C_2 \delta_r(X, Y)^{N_0+1}. \end{aligned}$$

There exists C such that for $0 < r \leq r_0$ and for all $X, Y, T \in \mathbb{R}^{2n}$,

$$(3.1.4) \quad \frac{g_X(T)}{g_Y(T)} \leq C \delta_r(X, Y)^{N_0}.$$

There exists N such that

$$(3.1.5) \quad \sup_{X \in \mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \delta_r(X, Y)^{-N} |g_Y|^{1/2} dY < \infty.$$

For $r \leq r_0$ and $g_X(Y - X) \geq Cr^2$,

$$(3.1.6) \quad \lambda(X) + \lambda(Y) \leq C\delta_r(X, Y)^{N_1}.$$

PROOF. The estimates (3.1.3) are proven above. We note that, with the same notations as before the lemma,

$$\frac{g_X}{g_Y} \sim \frac{g_{X'}}{g_{Y'}} \lesssim (1 + (g_{X'}^\sigma \wedge g_{Y'}^\sigma)(X' - Y'))^{N_0} \sim (1 + (g_X^\sigma \wedge g_Y^\sigma)(X' - Y'))^{N_0}$$

which gives (3.1.4). Moreover, using the same notations, we have

$$\begin{aligned} 1 + g_X(X - Y) &\lesssim 1 + g_X(X - X') + g_X(X' - Y') + g_X(Y' - Y) \\ &\lesssim \delta_r(X, Y)^{N_0} (1 + g_X(X - X') + g_X(X' - Y') + g_Y(Y' - Y)) \\ &\lesssim \delta_r(X, Y)^{N_0} (1 + g_X^\sigma(X' - Y')) \lesssim \delta_r(X, Y)^{2N_0} (1 + g_X^\sigma \wedge g_Y^\sigma(X' - Y')) \\ &\lesssim \delta_r(X, Y)^{2N_0+1}, \end{aligned}$$

and since $\frac{|g_Y|^{1/2}}{|g_X|^{1/2}} \lesssim \delta_r(X, Y)^{nN_0}$ we obtain (3.1.5). We have

$$\lambda(X)^2 g_X(X' - Y') \leq g_X^\sigma(X' - Y') \lesssim \delta_r(X, Y)^{N_0} g_X^\sigma \wedge g_Y^\sigma(X' - Y') \leq \delta_r(X, Y)^{N_0+1}.$$

As a consequence, for C large enough, if $g_X(X - Y) \geq Cr^2$ we get $g_X(X' - Y') \geq r^2$ (otherwise $g_X(X' - Y') < r^2$ and $g_X \sim g_Y$ with $g_X(X - Y) \leq C_0 r^2$). The previous inequality gives $\lambda(X) \lesssim \delta_r(X, Y)^{N_1}$ and the other follows from (3.1.4). $\square$

3.2. Confinement estimates

Preliminaries. We need now to attack the hardest part of our job, namely, we should provide an argument to prove that the composition formula (1.2.7) holds for $a \in S(m_1, g), b \in S(m_2, g)$ and that $a \sharp b \in S(m_1 m_2, g)$. Our idea is to use Theorem 3.1.2 above so that the problem is reduced to handle

$$\iint_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} (a_Y \sharp b_Z) |g_Y|^{1/2} |g_Z|^{1/2} dY dZ.$$

In fact, to show a convergence result for that integral, it will be enough to study carefully the compositions $a_Y \sharp b_Z$. The first thing that we realize is that $a_Y \sharp b_Z$ is not compactly supported, since it is an entire function on $\mathbb{C}^{2n}$, as it is apparent on (1.2.7). Nevertheless, we should expect some localization properties of some sort for that symbol. In particular, if $Y = Z$, that symbol should be “small” far away from Y . In the next section, we give a precise quantitative version of these assertions.

DEFINITION 3.2.1. **CONFINED SYMBOLS.** Let g be a positive definite quadratic form on $\mathbb{R}^{2n}$ such that $g \leq g^\sigma$. Let a be a smooth function in $\mathbb{R}^{2n}$ and $U \subset \mathbb{R}^{2n}$. We shall say that a is g -confined in U if $\forall k \in \mathbb{N}, \forall N \geq 0, \exists C_{k,N}$ such that, for all $X, T \in \mathbb{R}^{2n}$

$$(3.2.1) \quad |a^{(k)}(X)T^k| \leq C_{k,N} g(T)^{k/2} (1 + g^\sigma(X - U))^{-N/2}.$$

We shall set

$$(3.2.2) \quad \|a\|_{\text{Conf}(g,U)}^{k,N} = \sup_{\substack{X,T \in \mathbb{R}^{2n} \\ g(T)=1}} |a^{(k)}(X)T^k| (1 + g^\sigma(X - U))^{N/2}$$

and

$$(3.2.3) \quad \|a\|_{g,U}^{(l)} = \max_{k \leq l, N \leq l} \|a\|_{\text{Conf}(g,U)}^{k,N}.$$

When U is compact, any smooth function supported in U is confined in U .

REMARKS. Let us note first that it is a trivial point of commutative algebra to check that $T_1 \otimes \cdots \otimes T_k$ is a linear combination of powers $S \otimes \cdots \otimes S$ with k factors; this means in particular that the estimates above are equivalent to estimates controlling $a^{(k)}(X)T_1 \otimes \cdots \otimes T_k$. We see also that, as a set, $\text{Conf}(g,U)$ coincides with the Schwartz class. Naturally, the set of semi-norms (3.2.2) will be most important for our purpose as well as uniform estimates with respect to g .

Biconfinement estimates. Our most important technical result is the following.

THEOREM 3.2.2. *Let g_1, g_2 be two positive definite quadratic forms on $\mathbb{R}^{2n}$ such that $g_j \leq g_j^\sigma$ and let $a_j, j = 1, 2$ be confined in U_j , a g_j -ball of radius ≤ 1 . Then, for all k, l , for all $X, T \in \mathbb{R}^{2n}$,*

$$(3.2.4) \quad |(a_1 \sharp a_2)^{(l)} T^l| \leq A_{k,l} (g_1 + g_2)(T)^{l/2} \left(1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1) + (g_1^\sigma \wedge g_2^\sigma)(X - U_2) \right)^{-k/2},$$

with

$$A_{k,l} = \gamma(n, k, l) \|a_1\|_{g_1, U_1}^{(m)} \|a_2\|_{g_2, U_2}^{(m)}, \quad m = 2n + 1 + k + l.$$

COROLLARY 3.2.3. *Under the assumptions of theorem 3.2.2, for all (p, N) , there exists $q = q(p, N)$ and $C = C(p, N)$, such that*

$$(3.2.5) \quad \|a_1 \sharp a_2\|_{g_1 + g_2, U_1}^{(p)} + \|a_1 \sharp a_2\|_{g_1 + g_2, U_2}^{(p)} \leq C \|a_1\|_{g_1, U_1}^{(q)} \|a_2\|_{g_2, U_2}^{(q)} \left(1 + (g_1^\sigma \wedge g_2^\sigma)(U_1 - U_2) \right)^{-N}.$$

Corollary 3.2.3 follows from Theorem 3.2.2 by applying the triangle inequality to the quadratic form $g_1^\sigma \wedge g_2^\sigma$. $\square$

PROOF OF THEOREM 3.2.2. We start from the formula (1.2.7)

$$(a_1 \sharp a_2)(X) = \iint a_1(Y_1) a_2(Y_2) e^{-2\iota[Y_1 - X, Y_2 - X]} dY_1 dY_2 2^{2n}$$

and for Y_1, X given in $\mathbb{R}^{2n}$ we define $\theta = \theta_{Y_1, X} \in \mathbb{R}^{2n}$ such that

$$g_2(\theta) = 1, \quad [\theta, Y_1 - X] = g_2^\sigma(Y_1 - X)^{1/2} \quad (\text{see (2.2.8)}).$$

Then for $k \in \mathbb{N}$,

$$\left(\frac{1}{2}\langle\theta, D_{Y_2}\rangle + 1\right)^k (\exp -2\iota[Y_1 - X, Y_2 - X]) = (1 + g_2^\sigma(Y_1 - X)^{1/2})^k \exp -2\iota[Y_1 - X, Y_2 - X].$$

Consequently, we have

$$\begin{aligned} (a_1 \sharp a_2)(X) &= \iint a_1(Y_1) \sum_{0 \leq l \leq k} a_2^{[l]}(Y_2) e^{-2\iota[Y_1 - X, Y_2 - X]} (1 + g_2^\sigma(Y_1 - X)^{1/2})^{-k} dY_1 dY_2 2^{2n} \end{aligned}$$

with $a_2^{[l]} = \binom{k}{l} a_2^{(l)} \theta^l$. As a consequence, for all k, k_1, N , we get

$$\begin{aligned} |(a_1 \sharp a_2)(X)| &\leq \iint \|a_1\|_{g_1, U_1}^{(0, k_1)} (1 + g_1^\sigma(Y_1 - U_1))^{-k_1/2} (1 + g_2^\sigma(Y_1 - X))^{-k/2} \\ &\quad \|a_2\|_{g_2, U_2}^{(k, N)} (1 + g_2^\sigma(Y_2 - U_2))^{-N/2} dY_1 dY_2 2^{2n+k}. \end{aligned}$$

We notice then that

$$\begin{aligned} (1 + g_1^\sigma(Y_1 - U_1))^{k_1/2} (1 + g_2^\sigma(Y_1 - X))^{k_1/2} &\geq (1 + g_1^\sigma(Y_1 - U_1) + g_2^\sigma(Y_1 - X))^{k_1/2} \\ &\geq 2^{-k_1/2} (1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1))^{k_1/2}, \end{aligned}$$

so that for $k = k_1 + k_2$,

$$\begin{aligned} |(a_1 \sharp a_2)(X)| &\leq \|a_1\|_{g_1, U_1}^{(0, k_1)} \|a_2\|_{g_2, U_2}^{(k, N)} (1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1))^{-k_1/2} \\ &\quad \iint (1 + g_2^\sigma(Y_1 - X))^{-k_2/2} (1 + g_2^\sigma(Y_2 - U_2))^{-N/2} dY_1 dY_2 2^{2n+k+\frac{k_1}{2}}. \end{aligned}$$

Since U_2 is a g_2 -ball of radius ≤ 1 with center X_2 , we have for $Z \in U_2$,

$$\begin{aligned} 1 + g_2(Y_2 - X_2) &\leq 1 + 2g_2(Y_2 - Z) + 2g_2(Z - X_2) \leq 3(1 + g_2(Y_2 - Z)) \\ &\leq 3(1 + g_2^\sigma(Y_2 - Z)) \end{aligned}$$

which implies $1 + g_2(Y_2 - X_2) \leq 3(1 + g_2^\sigma(Y_2 - U_2))$. As a consequence, we get

$$|(a_1 \sharp a_2)(X)| \leq \|a_1\|_{g_1, U_1}^{(0, k_1)} \|a_2\|_{g_2, U_2}^{(k, N)} (1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1))^{-k_1/2} \\ \iint (1 + g_2^\sigma(Y_1 - X))^{-k_2/2} (1 + g_2(Y_2 - X_2))^{-N/2} dY_1 dY_2 2^{2n+k+\frac{k_1}{2}} 3^{N/2}.$$

Since the product of the determinants of g_2 and g_2^σ is 1, choosing $k_2 = N = 2n + 1$, we obtain

$$|(a_1 \sharp a_2)(X)| \leq \|a_1\|_{g_1, U_1}^{(0, k_1)} \|a_2\|_{g_2, U_2}^{(2n+1+k_1, 2n+1)} (1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1))^{-k_1/2} C(k_1, n).$$

Analogously, we get

$$|(a_1 \sharp a_2)(X)| \leq \|a_2\|_{g_1, U_1}^{(0, k_1)} \|a_1\|_{g_2, U_2}^{(2n+1+k_1, 2n+1)} (1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_2))^{-k_1/2} C(k_1, n).$$

Taking the minimum of the rhs in the two previous inequalities, we obtain with $k_1 = k$,

$$(3.2.6) \quad |(a_1 \sharp a_2)(X)| \leq C(k, n) 2^{k/2} \|a_1\|_{g_1, U_1}^{(2n+1+k)} \|a_2\|_{g_2, U_2}^{(2n+1+k)} \\ \times (1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1) + (g_1^\sigma \wedge g_2^\sigma)(X - U_2))^{-k/2},$$

which is indeed (3.2.4) for $l = 0$. Writing (1.2.7) as

$$(3.2.7) \quad (a_1 \sharp a_2)(X) = 2^{2n} \iint e^{-2\iota[Y_1, Y_2]} a_1(Y_1 + X) a_2(Y_2 + X) dY_1 dY_2,$$

we realize that if D a derivation, we have $D(a_1 \sharp a_2) = Da_1 \sharp a_2 + a_1 \sharp Da_2$, so that applying (3.2.6) to the previous formula gives (3.2.4) for $l \geq 1$. The proof of Theorem 3.2.2 is complete. $\square$

We need now to get the analogous result for the expansion of $a_1 \sharp a_2$.

THEOREM 3.2.4. *Let a_1, a_2 as in Theorem 3.2.2. For $\nu \in \mathbb{N}$, we define*

$$R_\nu(a_1, a_2)(X) = a_1 \sharp a_2 - \sum_{0 \leq j < \nu} \frac{1}{j!} \left(\frac{\iota}{2} [D_{X_1}, D_{X_2}] \right)^j (a_1 \otimes a_2)|_{\text{diagonal}}.$$

Then, for all k, l, ν , for all $X, T \in \mathbb{R}^{2n}$,

$$(3.2.8) \quad |(R_\nu(a_1, a_2))^{(l)} T^l| \leq A_{k, l, \nu} (g_1 + g_2)(T)^{l/2} \Lambda_{1,2}^{-\nu} \\ \times \left(1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1) + (g_1^\sigma \wedge g_2^\sigma)(X - U_2) \right)^{-k/2},$$

with $A_{k, l, \nu} = C(k, n, l, \nu) \|a_1\|_{g_1, U_1}^{(2n+1+k+l+\nu)} \|a_2\|_{g_2, U_2}^{(2n+1+k+l+\nu)}$ and

$$(3.2.9) \quad \Lambda_{1,2} = \inf_T \left(\frac{g_1^\sigma(T)}{g_2(T)} \right)^{1/2} = \inf_T \left(\frac{g_2^\sigma(T)}{g_1(T)} \right)^{1/2}.$$

REMARKS. First of all, note that we can get a result similar to the corollary 3.2.3: for all (p, N, ν) , there exists $q = q(p, N, \nu)$ and $C = C(p, N, \nu)$, such that

$$(3.2.10) \quad \|R_\nu(a_1, a_2)\|_{g_1+g_2, U_1}^{(p)} + \|R_\nu(a_1, a_2)\|_{g_1+g_2, U_2}^{(p)} \\ \leq C \|a_1\|_{g_1, U_1}^{(q)} \|a_2\|_{g_2, U_2}^{(q)} \Lambda_{1,2}^{-\nu} \left(1 + (g_1^\sigma \wedge g_2^\sigma)(U_1 - U_2)\right)^{-N}.$$

Next, let us give a proof of the identity (3.2.9). We have, using (2.2.8),

$$\frac{g_1^\sigma(T)}{g_2(T)} = \frac{[T, U_T]^2}{g_1(U_T)g_2(T)} \leq \frac{g_2(T)g_2^\sigma(U_T)}{g_1(U_T)g_2(T)} = \frac{g_2^\sigma(U_T)}{g_1(U_T)}$$

so that $\inf_T \frac{g_1^\sigma(T)}{g_2(T)} \leq \inf_T \frac{g_2^\sigma(U_T)}{g_1(U_T)}$ and since the mapping $T \mapsto U_T$ is an isomorphism,⁴ we get (3.2.9).

PROOF OF THEOREM 2.3.4. We have from (1.2.7),

$$(a_1 \sharp a_2)(X) = \iint a_1(X + Y_1) a_2(Y_2) e^{-\iota \langle 2\sigma Y_1, Y_2 - X \rangle} dY_1 dY_2 2^{2n} \\ = \iint a_1(X + \frac{1}{2}\sigma^{-1}\Xi) a_2(Y_2) e^{-\iota \langle \Xi, Y_2 - X \rangle} d\Xi dY_2 \\ = \int a_1(X + \frac{1}{2}\sigma^{-1}\Xi) \widehat{a}_2(\Xi) e^{\iota \langle \Xi, X \rangle} d\Xi,$$

and consequently,

$$(a_1 \sharp a_2)(X) = \sum_{0 \leq j < \nu} a_1^{(j)}(X) \frac{1}{j!2^j} \int (\sigma^{-1}\Xi)^j \widehat{a}_2(\Xi) e^{\iota \langle \Xi, X \rangle} d\Xi \\ + \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!2^\nu} \int (\sigma^{-1}\Xi)^\nu a_1^{(\nu)}(X + \frac{\theta}{2}\sigma^{-1}\Xi) \widehat{a}_2(\Xi) e^{\iota \langle \Xi, X \rangle} d\Xi d\theta \\ = \sum_{0 \leq j < \nu} a_1^{(j)}(X) \frac{1}{j!2^j} (\sigma^{-1}D_X)^j a_2(X) + R_\nu(a_1, a_2)(X),$$

with

$$R_\nu(a_1, a_2)(X) = \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!2^\nu} \int (\sigma^{-1}\Xi)^\nu a_1^{(\nu)}(X + \frac{\theta}{2}\sigma^{-1}\Xi) \widehat{a}_2(\Xi) e^{\iota \langle \Xi, X \rangle} d\Xi d\theta,$$

and thus

$$R_\nu(a_1, a_2)(X) = \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} \int Z^\nu a_1^{(\nu)}(X + \theta Z) \widehat{a}_2(2\sigma Z) e^{\iota \langle 2\sigma Z, X \rangle} dZ d\theta 2^{2n} \\ = \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} \int a_1^{(\nu)}(X + \theta Z) Z^\nu a_2(Y_2) e^{\iota \langle 2\sigma Z, X \rangle - \iota \langle Y_2, 2\sigma Z \rangle} dZ d\theta 2^{2n} \\ = \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} \iint a_1^{(\nu)}(X + \theta(Y_1 - X))(Y_1 - X)^\nu a_2(Y_2) \\ e^{-2\iota[Y_1 - X, Y_2 - X]} dY_1 dY_2 2^{2n} d\theta.$$

⁴For a positive definite quadratic form Γ , using the notations of the discussion following (2.2.8), we have $\Gamma^\sigma(T) = \sup_U \frac{\langle \sigma T, U \rangle^2}{\Gamma(U)} = \sup_U \frac{(\Omega T \cdot U)^2}{U \cdot U} = \Omega T \cdot \Omega T$ so that $U_T = \Omega T$ and since $\sigma = \Gamma\Omega$, Ω is an isomorphism.

We consider now $T = T_{Y_1, X}$ such that $g_2(T) = 1$ and $[T, Y_1 - X] = g_2^\sigma(Y_1 - X)^{1/2}$. Then for $k \in \mathbb{N}$,

$$\left(\frac{1}{2}\langle T, D_{Y_2} \rangle + 1\right)^k e^{-2\iota[Y_1 - X, Y_2 - X]} = e^{-2\iota[Y_1 - X, Y_2 - X]} (1 + g_2^\sigma(Y_1 - X)^{1/2})^k.$$

As a consequence we get

$$\begin{aligned} & |R_\nu(a_1, a_2)(X)| \\ & \leq \iiint_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} 2^{2n+k} \|a_1^{(\nu)}\|_{g_1, U_1}^{(0, k_1)} (1 + g_1^\sigma(X + \theta(Y_1 - X) - U_1))^{-k_1/2} \\ & \quad g_1(Y_1 - X)^{\nu/2} (1 + g_2^\sigma(Y_1 - X))^{-k/2} \\ & \quad \|a_2\|_{g_2, U_2}^{(k, N)} (1 + g_2^\sigma(Y_2 - U_2))^{-N/2} dY_1 dY_2 d\theta. \end{aligned}$$

Since $\theta \in [0, 1]$, we have $g_2^\sigma(Y_1 - X) \geq g_2^\sigma(\theta(Y_1 - X))$ and for $k = k_1 + k_2 + \nu$ and $N = 2n + 1 = k_2$,

$$\begin{aligned} & |R_\nu(a_1, a_2)(X)| \\ & \leq \iiint_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} 2^{2n+k} \|a_1\|_{g_1, U_1}^{(\nu, k_1)} (1 + g_1^\sigma(X + \theta(Y_1 - X) - U_1))^{-k_1/2} \\ & \quad \Lambda_{12}^{-\nu} g_2^\sigma(Y_1 - X)^{\nu/2} (1 + g_2^\sigma(\theta(Y_1 - X)))^{-\frac{k_1}{2}} (1 + g_2^\sigma(Y_1 - X))^{-\frac{k_2 + \nu}{2}} \\ & \quad \|a_2\|_{g_2, U_2}^{(k, N)} (1 + g_2^\sigma(Y_2 - U_2))^{-(2n+1)/2} dY_1 dY_2 d\theta. \end{aligned}$$

We use now that

$$2g_1^\sigma(X + \theta(Y_1 - X) - U_1) + 2g_2^\sigma(\theta(Y_1 - X)) \geq (g_1^\sigma \wedge g_2^\sigma)(X - U_1)$$

and we obtain

$$\begin{aligned} |R_\nu(a_1, a_2)(X)| & \leq C(k, n, \nu) \|a_1\|_{g_1, U_1}^{(\nu, k_1)} \|a_2\|_{g_2, U_2}^{(k, 2n+1)} \Lambda_{12}^{-\nu} (1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1))^{-k_1/2} \\ & \quad \iint (1 + g_2^\sigma(Y_1 - X))^{-\frac{k_2}{2}} (1 + g_2^\sigma(Y_2 - U_2))^{-(2n+1)/2} dY_1 dY_2. \end{aligned}$$

Reasoning as in the proof of Theorem 3.2.2, we get (3.2.8) for $l = 0$, which is easy to generalize to $l \geq 1$. The proof of Theorem 3.2.4 is complete. $\square$

CHAPTER IV
THE CALCULUS

4.1. Symbolic calculus

THEOREM 4.1.1. *Let g be an admissible metric on $\mathbb{R}^{2n}$, m_1, m_2 two admissible weights for g , $a_j \in S(m_j, g)$, $j = 1, 2$. Then, for all $\nu \in \mathbb{N}$,*

$$a_1 \sharp a_2 - \sum_{0 \leq j < \nu} \frac{1}{j!} \left(\frac{\iota}{2} [D_{X_1}, D_{X_2}] \right)^j (a_1 \otimes a_2)|_{\text{diagonal}} \in S(m_1 m_2 \lambda_g^{-\nu}, g)$$

PROOF. We shall first consider

$$\iint (a_{1,Y_1} \sharp a_{2,Y_2})(X) |g_{Y_1}|^{1/2} |g_{Y_2}|^{1/2} dY_1 dY_2$$

and notice that, from Theorem 3.2.2,

$$\begin{aligned} |(a_{1,Y_1} \sharp a_{2,Y_2})(X)| &\leq C_{N,M} \delta_r(Y_1, Y_2)^{-N} (1 + (g_{Y_1}^\sigma \wedge g_{Y_2}^\sigma)(X - U_{Y_1,r}))^{-M} m_1(Y_1) m_2(Y_2) \\ \frac{m_2(Y_2)}{m_2(Y_1)} &\leq C_0 \delta_r(Y_1, Y_2)^{N_0}, \quad \frac{g_{Y_1}^\sigma}{g_{Y_2}^\sigma} \leq C_0 \delta_r(Y_1, Y_2)^{N_0} \end{aligned}$$

so that

$$\begin{aligned} |(a_{1,Y_1} \sharp a_{2,Y_2})(X)| &\lesssim \delta_r(Y_1, Y_2)^{-N+N_0} (1 + (g_{Y_1}^\sigma \wedge g_{Y_2}^\sigma)(X - U_{Y_1,r}))^{-M} (m_1 m_2)(Y_1) \\ &\lesssim (m_1 m_2)(Y_1) (1 + g_{Y_1}^\sigma(X - U_{Y_1,r}))^{-M} \delta_r(Y_1, Y_2)^{-N+N_0+N_0 M} \\ &\lesssim (m_1 m_2)(Y_1) \delta_r(Y_1, X)^{-M} \delta_r(Y_1, Y_2)^{-N+N_0+N_0 M} \\ &\lesssim (m_1 m_2)(X) \delta_r(Y_1, X)^{-M+N_0} \delta_r(Y_1, Y_2)^{-N+N_0+N_0 M}. \end{aligned}$$

Choosing now $M - N_0$ large enough, then $N - N_0 - N_0 M$ large enough, we get $|a_1 \sharp a_2| \leq m_1 m_2$. To obtain the estimate for the derivatives is analogous as well as the estimate for the expansion, using Theorem 3.2.4. $\square$

4.2. L^2 boundedness

THEOREM 4.2.1. *Let g be an admissible metric on $\mathbb{R}^{2n}$, $a \in S(1, g)$. Then a^w is bounded on $L^2(\mathbb{R}^n)$ and $\|a^w\|_{\mathcal{L}(L^2)}$ can be estimated by a seminorm of the symbol a .*

We start with a precised version of Cotlar's lemma

LEMMA 4.2.2. *Let $\mathbb{H}$ be a Hilbert space and $(\Omega, \mathcal{A}, \nu)$ a measured space such that ν is a σ -finite positive measure. Let $(A_y)_{y \in \Omega}$ be a measurable family of bounded operators on $\mathbb{H}$ such that*

$$\sup_{y \in \Omega} \int_{\Omega} \|A_y^* A_z\|^{1/2} d\nu(z) \leq M, \quad \sup_{y \in \Omega} \int_{\Omega} \|A_y A_z^*\|^{1/2} d\nu(z) \leq M.$$

Then for all $u \in \mathbb{H}$

$$(4.2.1) \quad \iint_{\Omega \times \Omega} |\langle A_y u, A_z u \rangle_{\mathbb{H}}| d\nu(y) d\nu(z) \leq M^2 \|u\|_{\mathbb{H}}^2$$

which implies the strong convergence of

$$A = \int_{\Omega} A_y d\nu(y) \quad \text{and} \quad \|A\| \leq M.$$

REMARKS. We shall use this result here with $\Omega = \mathbb{R}^{2n}$ and $d\nu(Z) = |g_Z|^{1/2} dZ$. The novelty of this lemma does not rely on the generalization to a measured space, but on the estimate (4.2.1), which is optimal. In the more familiar discrete framework, the lemma implies that for a sequence (A_j) of bounded operators on $\mathbb{H}$, the conditions

$$\sup_j \|A_j^* A_k\|^{1/2} \leq M, \quad \sup_j \|A_j A_k^*\|^{1/2} \leq M$$

imply $\sum_{j,k} |\langle A_j u, A_k u \rangle| \leq M^2 \|u\|^2$ and the strong convergence of $A = \sum_j A_j$ with $\|A\| \leq M$.

PROOF. From the σ -finiteness of ν , we know that Ω is a countable union of sets ω with $\nu(\omega) < \infty$. Moreover since $y \mapsto \|A_y\|$ is everywhere finite, we have $\cup_{k \in \mathbb{N}} \{y \in \omega, \|A_y\| \leq k\} = \omega$. Then Ω is a countable union of sets K such that $\nu(K) < \infty$ and $\sup_{y \in K} \|A_y\| < \infty$. We consider then such a K and for a measurable function $\theta(y, z)$ with modulus smaller than 1, we define

$$T_{K, \theta} = \int_{K \times K} A_y^* A_z \theta(y, z) d\nu(y) d\nu(z).$$

We have $\|T\|^2 = \|T^*T\|$, $\|T\|^4 = \|T^*TT^*T\|$, $\dots$, $\|T\|^{2m} = \|(T^*T)^m\|$ for m (e.g. a power of 2), so that

$$\|T\|^{2m} = \left\| \int_{K^{4m}} A_{z_1}^* A_{y_1} A_{y_1'}^* A_{z_1'} \dots A_{z_m}^* A_{y_m} A_{y_m'}^* A_{z_m'} \prod_{1 \leq j \leq m} \bar{\theta}(y_j, z_j) \theta(y_j', z_j') d\nu(y_j) d\nu(z_j) d\nu(y_j') d\nu(z_j') \right\|.$$

Using then the standard Cotlar's trick, we take the above operators two by two, beginning with the first or the second, we get inside the integral the estimates from above

$$\prod_{1 \leq j \leq m} \|A_{z_j}^* A_{y_j}\| \|A_{y_j'}^* A_{z_j'}\|, \quad \|A_{z_1}^*\| \prod_{1 \leq j < m} \|A_{y_j} A_{y_j'}^* A_{z_j'} A_{z_{j+1}}^*\| \|A_{y_m} A_{y_m'}^* A_{z_m'}\|$$

so that, taking the geometric mean, we have

$$\begin{aligned} \|T_{K,\theta}\|^{2m} &\leq \sup_K \|A_y\| \int_{K^{2m}} \prod_{1 \leq j \leq m} \|A_{z_j}^* A_{y_j}\|^{1/2} \|A_{y_j'}^* A_{z_j'}\|^{1/2} \\ &\quad \prod_{1 \leq j < m} \|A_{y_j} A_{y_j'}^*\|^{1/2} \|A_{z_j'} A_{z_{j+1}}^*\|^{1/2} \|A_{y_m} A_{y_m'}^*\|^{1/2} \\ &\quad d\nu(y_j) d\nu(z_j) d\nu(y_j') d\nu(z_j'). \end{aligned}$$

Integrating with respect to z_m' , will give a quantity depending on y_m' but $\leq M$,

Integrating with respect to y_m' will give a quantity depending on y_m but $\leq M$,

Integrating with respect to y_m will give a quantity depending on z_m but $\leq M$,

Integrating with respect to z_m will give a quantity depending on z_{m-1}' but $\leq M$,

and iterating this process up to $j = 2$, we get

$$\begin{aligned} \|T_{K,\theta}\|^{2m} &\leq \sup_K \|A_y\| M^{4m-4} \\ &\quad \times \int_{K^4} \|A_{z_1}^* A_{y_1}\|^{1/2} \|A_{y_1'}^* A_{z_1'}\|^{1/2} \|A_{y_1} A_{y_1'}^*\|^{1/2} d\nu(y_1) d\nu(z_1) d\nu(y_1') d\nu(z_1'). \\ &\leq \sup_K \|A_y\| M^{4m-1} \nu(K), \end{aligned}$$

and taking the $2m$ -th root, letting m go to infinity, we obtain $\|T_{K,\theta}\| \leq M^2$. We consider now $u_0 \in \mathbb{H}$ and $\theta(y, z)$ measurable with modulus one such that

$$\langle A_y u_0, A_z u_0 \rangle \theta(y, z) = |\langle A_y u_0, A_z u_0 \rangle|.$$

We have thus

$$\begin{aligned} M^2 \|u_0\|^2 &\geq \langle T u_0, u_0 \rangle = \int_{K^2} \langle A_y u_0, A_z u_0 \rangle \theta(y, z) d\nu(y) d\nu(z) \\ &= \int_{K^2} |\langle A_y u_0, A_z u_0 \rangle| d\nu(y) d\nu(z) \end{aligned}$$

and then (4.2.1), completing the proof of the lemma. $\square$

LEMMA 4.2.3. *Let g be a positive definite quadratic form on $\mathbb{R}^{2n}$ such that $g \leq g^\sigma$. Let a be a smooth function in $\mathbb{R}^{2n}$ and U be a g -ball with radius ≤ 1 such that a is g -confined in U (see definition 3.2.1). Then*

$$(4.2.2) \quad \|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C(n) \|a\|_{g,U}^{(2n+1)}.$$

PROOF. For k integer, we have

$$\hat{a}(-\Xi) = \int a(X) (1 + \langle g^{-1} D_X, \Xi \rangle)^k e^{\iota X \cdot \Xi} dX (1 + \langle g^{-1} \Xi, \Xi \rangle)^{-k}.$$

With X_0 standing for the center of U , we have, with $X' \in U$,

$$1 + g(X - X_0) \leq 1 + 2g(X - X') + 2g(X' - X_0) \leq 3 + 2g^\sigma(X - X')$$

and consequently $1 + g(X - X_0) \leq 3(1 + g^\sigma(X - U))$, so that

$$\begin{aligned} |\hat{a}(-\Xi)| &\leq (1 + \langle g^{-1} \Xi, \Xi \rangle)^{-2n-2} \|a\|_{g,U}^{(2n+2, 2n+2)} \\ &\quad \times \int (1 + g^\sigma(X - U))^{-n-1} dX \langle g g^{-1} \Xi, g^{-1} \Xi \rangle^{n+1} \\ &\leq (1 + \langle g^{-1} \Xi, \Xi \rangle)^{-n-1} \|a\|_{g,U}^{(2n+2, 2n+2)} 3^{n+1} |g|^{-1/2} c(n). \end{aligned}$$

As a consequence, the L^1 -norm of $\hat{a}$ is bounded above by the rhs of (4.2.2), and using the estimate (1.2.5) we get (4.2.2). $\square$

PROOF OF THEOREM 4.2.1. It is now a simple matter to obtain the L^2 -boundedness. We write

$$a^w = \int a_Y^w |g_Y|^{1/2} dY$$

and we note that from Lemma 4.2.3, $\sup_Y \|a_Y^w\|_{\mathcal{L}(L^2)} < \infty$. Moreover, the estimate (3.2.4) implies that, for all $l, k = k_1 + k_2$,

$$\begin{aligned} |(\bar{a}_Y \sharp a_Z)^{(l)} T^l| &\leq A_{k,l} (g_Y + g_Z) (T)^{l/2} \left(1 + (g_Y^\sigma \wedge g_Z^\sigma)(X - U_Y) + (g_Y^\sigma \wedge g_Z^\sigma)(X - U_Z) \right)^{-k/2} \\ &\leq A_{k,l} g_Y (T)^{l/2} \delta_r(Y, Z)^{N_0 l} \delta_r(Y, Z)^{-k_1} (1 + g_Y^\sigma(X - U_Y))^{-k_2/2} \delta_r(Y, Z)^{N_0 k_2} \\ &\leq A_{k,l} g_Y (T)^{l/2} (1 + g_Y^\sigma(X - U_Y))^{-k_2/2} \delta_r(Y, Z)^{N_0 k_2 + N_0 l - k_1} \end{aligned}$$

so that, for l, N, M given one can choose

$$k_2 = N, \quad k_1 - N_0 N - N_0 l \geq M$$

and get that $\bar{a}_Y \sharp a_Z$ is g_Y -confined in U_Y and with lemma 4.2.3,

$$\max(\|\bar{a}_Y^w a_Z^w\|_{\mathcal{L}(L^2)}^{1/2}, \|a_Y^w \bar{a}_Z^w\|_{\mathcal{L}(L^2)}^{1/2}) \leq C_M \delta(Y, Z)^{-M}$$

for all M . To conclude, we note that (3.1.5) implies that the assumptions of Lemma 4.2.2 are fulfilled. The proof of Theorem 4.2.1 is complete.

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