Metrics on the Phase Space and Non-Selfadjoint Pseudodifferential Operators

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Preface

This is a three-chapter-book on the topic of pseudodifferential operators, with special emphasis on non-selfadjoint operators, a priori estimates and localization in the phase space.

The first chapter, *Basic Notions of Phase Space Analysis*, is introductory and gives a presentation of very classical classes of pseudodifferential operators, along with some basic properties. As an illustration of the power of these methods, we give a proof of propagation of singularities for real-principal type operators (using a priori estimates, and not Fourier integral operators), and we introduce the reader to local solvability problems. That chapter will be hopefully useful for a reader, say at the graduate level in analysis, eager to learn some basics on pseudodifferential operators.

The second chapter, *Metrics on the Phase Space* begins with a review of symplectic algebra, Wigner functions, quantization formulas, metaplectic group and is intended to set the basic study of the phase space. We move forward to the more general setting of metrics on the phase space, following essentially the basic assumptions of L. Hörmander (Chapter 18 in the book [73]) on this topic. We use the notion of confinement, introduced by J.-M. Bony and the author and we follow the initial part of the paper [20] on these topics. We expose as well some elements of the so-called Wick calculus. We present some key examples related to the Calderón-Zygmund decompositions such that the Fefferman-Phong inequality and we prove that the analytic functional calculus works for admissible metrics. We give a description of the construction of Sobolev spaces attached to a pseudo-differential calculus, following the paper by J.-M. Bony and J.-Y. Chemin [19]; this construction of Sobolev spaces has been discussed in the aforementioned paper and also in several articles of R. Beals such as [6] (see also the paper [7] for a key lemma of characterization of pseudodifferential operators).

The third and last chapter, *Estimates for Non-Selfadjoint Operators*, is devoted to the more difficult and less classical topic of non-selfadjoint pseudo-differential operators. We discuss the details of the various types of estimates that can be proved or disproved, depending on the geometry of the symbols. We start with a rather elementary section containing examples and various classical models such as the Hans Lewy example. Next, we move forward with a quite easy discussion on the analysis of the first Poisson bracket of the imaginary and real part. The following sections are more involved; in particular we start a discussion on the geometry of condition (Ψ), with some known facts on flow-invariant sets, but we expose also the contribution of N. Dencker in the understanding of that geometric condition, with various inequalities satisfied by symbols. The next two sections are concerned respectively with the proof of the necessity of condition (Ψ) for local solvability and also with subelliptic estimates: on these two topics, we refer essentially to the existing literature, but we mention the results to hopefully provide the reader with some continuous overview of the subject. Then we enter into the discussion of estimates with loss of one derivative; we start with a detailed proof of the Beals-Fefferman result on local solvability with loss of one derivative under condition (P). Although this proof is classical, it seems useful to review its main arguments based on Calderón-Zygmund decompositions to understand how this type of cutting and stopping procedure works in a rather simple setting (at any rate simpler than in the section devoted to condition (Ψ)). We show, following the author's counterexample, that an estimate with loss of one derivative is not a consequence of condition (Ψ). Finally, we give a proof of an estimate with loss of $3/2$ derivatives under condition (Ψ), following the articles of N. Dencker [35] and the author's [98]. We end that chapter with a short historical account of solvability questions and also with a list of open questions.

There is also a lengthy appendix to this book. Some topics of this appendix are simply very classical material whose re-exposition might benefit to the reader by providing an immediate access to a reference for some calculations or formulas: it is the case of the first two sections of that appendix *Some elements of Fourier analysis, Some remarks of algebra* and also of the fourth one *On the symplectic and metaplectic groups*. Other parts of the appendix are devoted to technical questions, which would have impeded the reader in his progression: this is the case in particular of the very last section *More on symbolic calculus*.

It is our hope that the first two parts of the book are accessible to graduate students with a decent background in Analysis. The third chapter is directed more to researchers but should also be accessible to the readers able to get some good familiarity with the first two chapters, in which the main tools for the proofs of Chapter 3 are provided.

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