Chapter 2

Metrics on the Phase Space

In this chapter, we describe a general version of the pseudodifferential calculus, due to L. Hörmander ([69], chapter 18 in [73]). That version followed some earlier generalizations due to R. Beals and C. Fefferman ([8]) and to R. Beals ([6]). It was followed by some other generalizations due to A. Unterberger [145] and to a joint work of J.-M. Bony and N. Lerner ([20]), whose we follow the presentation. We also give a precised version of the Fefferman-Phong inequality, following [100] where we provided an upper bound for the number of derivatives sufficient to obtain that inequality. Finally, we study the Sobolev spaces naturally attached to an admissible metric on the phase space, essentially following the paper by J.-M. Bony and J.-Y. Chemin [19].

2.1 The structure of the phase space

2.1.1 Symplectic algebra

Let us consider a finite dimensional real vector space E (the configuration[1] space $\mathbb{R}^n_x$) and its dual space E^* (the momentum space $\mathbb{R}^n_\xi$). The phase space is defined as $\mathcal{P} = E \oplus E^*$; its running point will be denoted in general by a upper case letter ($X = (x, \xi), Y = (y, \eta)$). The symplectic form on $\mathcal{P}$ is given by

$$[(x, \xi), (y, \eta)] = \langle \xi, y \rangle_{E^*,E} - \langle \eta, x \rangle_{E^*,E} \tag{2.1.1}$$

where $\langle \cdot, \cdot \rangle_{E^*,E}$ stands for the bracket of duality. The symplectic group $Sp(2n)$ is the subgroup of the linear group of $\mathcal{P}$ preserving (2.1.1). With

$$\sigma = \begin{pmatrix} 0 & \mathrm{Id}(E^*) \\ -\mathrm{Id}(E) & 0 \end{pmatrix}, \tag{2.1.2}$$

[1] We shall identify the configuration space E with $\mathbb{R}^n$, but the reader may keep in mind that all the theory can be developed without this identification.

we have for $X, Y \in \mathcal{P}$, $[X,Y] = \langle \sigma X, Y\rangle_{\mathcal{P}^*,\mathcal{P}}$, so that the equation of the symplectic group is

$$A^* \sigma A = \sigma.$$

One can describe (see the section 4.4.1 in the appendix) a set of generators for the symplectic group $Sp(2n)$, identifying $\mathcal{P}$ with $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$: the following mappings are generating the symplectic group,

$$\begin{aligned}
&(x,\xi) \mapsto (Tx, {}^t T^{-1}\xi), \text{ where } T \text{ is an automorphism of } E, && (i)\\
&(x_k,\xi_k) \mapsto (\xi_k\ , -x_k), \text{ and the other coordinates fixed,} && (ii)\\
&(x,\xi) \mapsto (x, \xi + Sx), \text{ where } S \text{ is symmetric from } E \text{ to } E^*. && (iii)
\end{aligned}$$

2.1.2 Wigner function

For $u, v \in \mathscr{S}(\mathbb{R}^n)$, we define the Wigner function $\mathcal{H}(u,v)$ on the phase space by the formula

$$\mathcal{H}(u,v)(x,\xi) = \int u(x + \frac{y}{2})\bar{v}(x - \frac{y}{2}) e^{-2i\pi y\cdot\xi} dy. \tag{2.1.3}$$

The mapping $(u,v) \mapsto \mathcal{H}(u,v)$ is sesquilinear continuous from $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^{2n})$. Since $\mathcal{H}(u,v)$ is the partial Fourier transform of the function $(x,y) \mapsto u(x+y/2)\bar{v}(x-y/2)$, the Wigner function also satisfies,

$$\|\mathcal{H}(u,v)\|_{L^2(\mathbb{R}^{2n})} = \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}, \tag{2.1.4}$$

$$\mathcal{H}(u,v)(x,\xi) = 2^n \langle \sigma_{x,\xi} u, v\rangle_{L^2(\mathbb{R}^n)}, \tag{2.1.5}$$

$$\text{with} \quad (\sigma_{x,\xi} u)(y) = u(2x-y) \exp -4i\pi(x-y)\cdot\xi \tag{2.1.6}$$

and the so-called "phase symmetries" σ_X are unitary and selfadjoint on $L^2(\mathbb{R}^n)$.

2.1.3 Quantization formulas

Let a be temperate distribution on $\mathbb{R}^{2n}$. We have seen in the section 1.1.2 and in the definition 1.1.9 that the Weyl quantization rule associates to this distribution the operator a^w defined formally on functions $u \in \mathscr{S}(\mathbb{R}^n)$ by

$$(a^w u)(x) = \iint e^{2i\pi(x-y)\cdot\xi}\ a(\frac{x+y}{2}, \xi)\ u(y) dy d\xi. \tag{2.1.7}$$

Since the previous formula does not obviously make sense, we use the Wigner function (2.1.3) to provide a weak definition,

$$\langle a^w u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \prec a, \mathcal{H}(u,v) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}, \tag{2.1.8}$$

where $\mathscr{S}^*(\mathbb{R}^n)$ is the antidual of $\mathscr{S}(\mathbb{R}^n)$ and the second bracket is for real duality. For instance we have

$$(x\cdot\xi)^w = (x\cdot D_x + D_x \cdot x)/2,$$

with $D_x = \frac{1}{2i\pi}\frac{\partial}{\partial x}$ whereas the *classical* quantization rule would map the Hamiltonian $x \cdot \xi$ to the operator $x \cdot D_x$. The formula (2.1.8) can be written as

$$(a^w u, v) = \iint a(x,\xi)\mathcal{H}(u,v)(x,\xi)dxd\xi,$$

where the Wigner function $\mathcal{H}$ is defined by (2.1.3). We have indeed

$$\begin{aligned}\langle a^w u, v\rangle &= \iiint a(\frac{x+y}{2},\xi)e^{2i\pi(x-y)\xi}u(y)\bar{v}(x)dydxd\xi \\ &= \iiint a(x',\xi)e^{2i\pi x''\xi}u(x'-\frac{x''}{2})\bar{v}(x'+\frac{x''}{2})dx'dx''d\xi \\ &= \iint a(x,\xi)\mathcal{H}(u,v)(x,\xi)dxd\xi,\end{aligned}$$

as claimed in (2.1.8). Now, we have also from that formula

$$\mathcal{H}(u,v)(x,\xi) = 2^n\int u(2x-z)\bar{v}(z)e^{-4i\pi(x-z)\xi}dz = 2^n\langle \sigma_{x,\xi}u, v\rangle_{L^2(\mathbb{R}^n)}$$

which can be written ([145], [150]), as

$$a^w = \int_{\mathbb{R}^{2n}} a(X)2^n\sigma_X dX = \int_{\mathbb{R}^{2n}} \widehat{a}(\Xi)\exp(2i\pi\Xi\cdot M)d\Xi, \tag{2.1.9}$$

where $\Xi \cdot M = \hat{x}\cdot x + \hat{\xi}\cdot D_x$ (here $\Xi = (\hat{x},\hat{\xi})$). The last formula can be proven by applying Plancherel's formula to (2.1.8): for $u, v \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\prec a, \mathcal{H}(u,v)\succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} = \prec \widehat{a}, \check{\widehat{\mathcal{H}(u,v)}}\succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$$

and $\check{\widehat{\mathcal{H}(u,v)}}(\hat{x},\hat{\xi}) = \iint e^{2i\pi(x\cdot\hat{x}+\xi\cdot\hat{\xi})}\overbrace{\mathcal{H}(u,v)(x,\xi)}^{\in\mathscr{S}(\mathbb{R}^{2n})}dxd\xi$ is also, according to (2.1.8), equal to $\langle b^w u, v\rangle$ with $b(x,\xi) = e^{2i\pi(x\cdot\hat{x}+\xi\cdot\hat{\xi})}$, that is the exponential of a purely imaginary linear form.

Lemma 2.1.1. *If L is a linear form on the phase space $\mathbb{R}^{2n}$, $a \in \mathscr{S}'(\mathbb{R}^{2n})$, we have*

$$L^w L^w = (L^2)^w, \quad \forall k\in\mathbb{N},\ (L^w)^k = (L^k)^w \tag{2.1.10}$$

$$L^w a^w = \Big(La + \frac{1}{4i\pi}\{L,a\}\Big)^w, \quad a^w L^w = \Big(La - \frac{1}{4i\pi}\{L,a\}\Big)^w \tag{2.1.11}$$

where the Poisson bracket $\{L,a\} = H_L(a)$ is given by (1.1.23). *If L is real-valued, L^w is an (unbounded) selfadjoint operator on $L^2(\mathbb{R}^n)$ with domain $\{u \in L^2(\mathbb{R}^n), L^w u \in L^2(\mathbb{R}^n)\}$ and we have $\exp(iL^w) = (e^{iL})^w$.*

Proof. Using (2.1.7) for the linear forms ξ_j and x_j, one gets at once (2.1.11) by linearity and this implies readily (2.1.10). When the linear form L is real-valued, the function e^{iL} belongs to $\mathscr{S}'(\mathbb{R}^{2n})$ and for $t \in \mathbb{R}$ (using (2.1.11) for the second equation),

$$\frac{d}{dt}(e^{itL})^w = (iLe^{itL})^w = iL^w(e^{itL})^w = (e^{itL})^w iL^w$$

which gives $(e^{itL})^w = \exp itL^w$ and the last assertion of the lemma. □

Note also that for a linear form L given by $L(X) = [X, T]$, (2.1.1) and (2.1.11) give

$$[L^w, a^w] = \frac{1}{2i\pi}(a' \cdot T)^w. \tag{2.1.12}$$

The operator with Weyl symbol e^{iL} is an isomorphism of $\mathscr{S}(\mathbb{R}^n)$ whose inverse has Weyl symbol e^{-iL} since

$$\frac{d}{dt}\Big((e^{itL})^w(e^{-itL})^w\Big) = \Big((e^{itL})^w iL^w(e^{-itL})^w - (e^{itL})^w iL^w(e^{-itL})^w\Big) = 0$$

so that $(e^{itL})^w(e^{-itL})^w = \mathrm{Id}$. Moreover $(e^{iL})^w$ can be extended to a unitary operator of $L^2(\mathbb{R}^n)$ since, according to the remark 1.1.11, for $u, v \in \mathcal{S}(\mathbb{R}^n)$

$$\langle (e^{iL})^w u, (e^{iL})^w v\rangle_{L^2(\mathbb{R}^n)} = \langle (e^{-iL})^w(e^{iL})^w u, v\rangle_{L^2(\mathbb{R}^n)} = \langle u, v\rangle_{L^2(\mathbb{R}^n)},$$

and the rôle of $L, -L$ can be exchanged so that $(e^{-iL})^w(e^{iL})^w = (e^{iL})^w(e^{-iL})^w = \mathrm{Id}_{L^2(\mathbb{R}^n)}$. On the other hand, one may also consider the unbounded operator L^w on $L^2(\mathbb{R}^n)$ with domain $D = \{u \in L^2(\mathbb{R}^n), L^w u \in L^2(\mathbb{R}^n)\}$ and show directly that, for L real-valued linear form, L^w is a selfadjoint operator. (cf. e.g. the section 4.7 in the appendix). The formulas (2.1.9) give in particular

$$\|a^w\|_{\mathcal{L}(L^2)} \leq \min(2^n \|a\|_{L^1(\mathbb{R}^{2n})}, \|\widehat{a}\|_{L^1(\mathbb{R}^{2n})}), \tag{2.1.13}$$

where $\mathcal{L}(L^2)$ stands for the space of bounded linear maps from $L^2(\mathbb{R}^n)$ into itself.

2.1.4 The metaplectic group

The metaplectic group $Mp(n)$ is the subgroup of the group of unitary transformations of $L^2(\mathbb{R}^n)$ generated by the multiplication by complex numbers with modulus 1 and

$(M_T u)(x) = |\det T|^{-1/2} u(T^{-1}x)$, where T is an automorphism of E, *(j)*

Partial Fourier transformation, with respect to x_k for $k = 1, \ldots, n$, *(jj)*

Multiplication by $\exp(i\pi\langle Sx, x\rangle)$, where S is symmetric from E to E^*. *(jjj)*

A detailed study of the symplectic and metaplectic groups is given in the appendix 4.4.1. In particular, one can define a surjective homomorphism of groups

$$\Pi : Mp(n) \to Sp(2n),$$

with kernel $\mathbb{S}^1\,\mathrm{Id}_{L^2(\mathbb{R}^n)}$ and such that, if $\chi = \Pi(M)$ and u, v are in $L^2(\mathbb{R}^n)$, $\mathcal{H}(u,v)$ is their Wigner function,

$$\mathcal{H}(Mu, Mv) = \mathcal{H}(u,v) \circ \chi^{-1}.$$

This is Segal formula [128] which could be rephrased as follows.

Theorem 2.1.2. *Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ and $\chi \in Sp(2n)$. There exists M in the fiber of χ (M is a unitary transformation which is also an automorphism of $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$) such that*

$$(a \circ \chi)^w = M^* a^w M. \tag{2.1.14}$$

In particular, the images by Π of the transformations (j), (jj), (jjj) are respectively (i), (ii), (iii) in page 58 of section 2.1. Moreover, if χ is the phase translation, $\chi(x,\xi) = (x+x_0, \xi+\xi_0)$, (2.1.14) is fulfilled with $M = \tau_{x_0,\xi_0}$, the phase translation given by

$$(\tau_{x_0,\xi_0} u)(y) = u(y - x_0)\; e^{2i\pi\langle y - \frac{x_0}{2}, \xi_0\rangle}. \tag{2.1.15}$$

If $\chi(x,\xi) = (2x - x_0, 2\xi - \xi_0)$ is the symmetry with respect to (x_0,ξ_0), M in (2.1.14) is, up to a unit factor, the phase symmetry σ_{x_0,ξ_0} defined in (2.1.6).

Proof. The proof is a consequence of the remark 4.4.19 of the appendix. □

The following lemma is describing the natural link between the phase translations and the phase symmetries.

Lemma 2.1.3. *The phase symmetry σ_X given in (2.1.6) is unitary and selfadjoint on $L^2(\mathbb{R}^n)$, the phase translation τ_X given in (2.1.15) is unitary and $\tau_X^* = \tau_{-X}$. Moreover, we have for $Y, Z \in \mathbb{R}^{2n}$,*

$$\sigma_Y \sigma_Z = \tau_{2Y-2Z} e^{4i\pi[Z,Y]} = \int_{\mathbb{R}^{2n}} e^{-4i\pi[X-Y, X-Z]} 2^n \sigma_X dX, \tag{2.1.16}$$

$$\tau_{2Z}\sigma_Y = \sigma_{Y+Z} e^{4i\pi[Z,Y]},\;\; \sigma_Y \tau_{2Z} = \sigma_{Y-Z} e^{4i\pi[Z,Y]},\;\; \tau_Y \tau_Z = \tau_{Y+Z} e^{i\pi[Y,Z]}. \tag{2.1.17}$$

Proof. The first statements are already proven. Let us note that

$$T \mapsto 2Z - T \mapsto 2Y - (2Z - T)$$

is the composition of the symmetry w.r.t. Z followed by the symmetry w.r.t. Y, which gives the translation by $2Y - 2Z$, so that the first formula in (2.1.16) is natural; as well as (2.1.17), it can be checked directly by an elementary computation left to the reader. Let us verify the second formula in (2.1.16). For $u \in \mathscr{S}(\mathbb{R}^n)$, we have, with $Y = (y,\eta), Z = (z,\zeta)$, both in $\mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned}(\sigma_{y,\eta}\sigma_{z,\zeta} u)(x) &= e^{4i\pi(x-y)\cdot\eta}(\sigma_{z,\zeta} u)(2y - x) \\ &= e^{4i\pi(x-y)\cdot\eta} e^{4i\pi(2y-x-z)\cdot\zeta} u(2z - 2y + x).\end{aligned}$$

Since we have

$$[X-Y,X-Z]=[X,Y-Z]+[Y,Z],$$

the following integral can be considered as a Fourier transform of the tempered distribution (depending on the parameter x) $X'\mapsto(\sigma_{X'}u)(x)$. We calculate

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} e^{-4i\pi[X'-Y,X'-Z]}2^n(\sigma_{X'}u)(x)dX'\\
&=\iint e^{-4i\pi(\xi'-\eta)\cdot(x'-z)+4i\pi(x'-y)\cdot(\xi'-\zeta)}2^n u(2x'-x)e^{4i\pi(x-x')\cdot\xi'}dx'd\xi'\\
&=\iint e^{-4i\pi(\xi'-\eta)\cdot(\frac{x}{2}+\frac{x''}{2}-z)+4i\pi(\frac{x}{2}+\frac{x''}{2}-y)\cdot(\xi'-\zeta)}u(x'')e^{4i\pi(\frac{x}{2}-\frac{x''}{2})\cdot\xi'}dx''d\xi'\\
&=\iint e^{4i\pi\xi'\cdot(z-y+\frac{x-x''}{2})}e^{4i\pi\eta\cdot(\frac{x}{2}+\frac{x''}{2}-z)+4i\pi(\frac{x}{2}+\frac{x''}{2}-y)\cdot(-\zeta)}u(x'')dx''d\xi'\\
&=\int\delta(2z-2y+x-x'')e^{4i\pi\eta\cdot(\frac{x}{2}+\frac{x''}{2}-z)+4i\pi(\frac{x}{2}+\frac{x''}{2}-y)\cdot(-\zeta)}u(x'')dx''\\
&=u(2z-2y+x)e^{4i\pi\eta\cdot(x-y)+4i\pi(x+z-2y)\cdot(-\zeta)}.
\end{aligned}$$

□

2.1.5 Composition formula

According to the theorem 1.1.2 and to the lemma 4.1.2, the operator a^w is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself whenever $a\in C_b^\infty(\mathbb{R}^{2n})$ and thus in particular when $a\in\mathscr{S}(\mathbb{R}^{2n})$. Using for $a,b\in\mathscr{S}(\mathbb{R}^{2n})$ the first formula in (2.1.9), we obtain

$$a^wb^w=\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}}a(Y)b(Z)2^{2n}\sigma_Y\sigma_Z dYdZ.$$

Using the last formula of (2.1.16) we get $a^wb^w=(a\sharp b)^w$ with

$$(a\sharp b)(X)=2^{2n}\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}}e^{-4i\pi[X-Y,X-Z]}a(Y)b(Z)dYdZ. \tag{2.1.18}$$

We can compare this with the classical composition formula, as studied in the theorem 1.1.5, namely $\mathrm{op}(a)\mathrm{op}(b)=\mathrm{op}(a\diamond b)$ with

$$(a\diamond b)(x,\xi)=\iint_{\mathbb{R}^n\times\mathbb{R}^n}e^{-2i\pi y\cdot\eta}a(x,\xi+\eta)b(y+x,\xi)dyd\eta.$$

Another method to perform that calculation would be to use the kernels of the operators a^w,b^w, in the spirit of the proof of the theorem 1.1.5. For future reference, we note that the distribution kernel k_a of the operator $a(x,D)$ (for $a\in\mathscr{S}'(\mathbb{R}^{2n})$) as defined by (1.1.1) is

$$k_a(x,y)=\int e^{2i\pi(x-y)\cdot\xi}a(x,\xi)d\xi=\widehat{a}^2(x,y-x) \tag{2.1.19}$$

so that $\widehat{a}^2(x,y) = k_a(x, y+x)$ and in the distribution sense

$$a(x,\xi) = \int e^{2i\pi y\cdot\xi} k_a(x, y+x)dy. \tag{2.1.20}$$

The distribution kernel κ_a of the operator a^w (for $a \in \mathscr{S}'(\mathbb{R}^{2n})$) as defined by (2.1.8) is (in the distribution sense)

$$\kappa_a(x,y) = \int e^{2i\pi(x-y)\cdot\xi} a(\frac{x+y}{2},\xi)d\xi \tag{2.1.21}$$

so that $\kappa_a(x-\frac{t}{2}, x+\frac{t}{2}) = \int e^{-2i\pi t\cdot\xi} a(x,\xi)d\xi = \widehat{a}^2(x,t)$ and thus

$$a(x,\xi) = \int \kappa_a(x-\frac{t}{2}, x+\frac{t}{2}) e^{2i\pi t\xi} dt. \tag{2.1.22}$$

The formulas (2.1.19), (2.1.20), (2.1.21) and (2.1.22) are proved above for a general distribution symbol $a \in \mathscr{S}'(\mathbb{R}^{2n})$ whereas the composition formula (2.1.18) (as well as (1.1.4)) require some more hypotheses to make sense (even as "oscillatory" integrals as in the section 7.8 of [71] or in a weak distribution sense), which is quite natural since we need to compose the operators a^w, b^w; at this point, we have proven the composition formula (2.1.18) for a, b both in $\mathscr{S}(\mathbb{R}^{2n})$, a case in which a^w and b^w send (continuously) $\mathscr{S}(\mathbb{R}^n)$ into itself; note also that the theorem 1.1.20 has already provided a composition formula along with an asymptotic expansion in the classical quantization framework for the class $S^m_{1,0}$ of symbols studied in the section 1.1.3. One of the goals of this chapter is to show that it is possible to construct various algebras of pseudodifferential operators while considerably relaxing the assumptions on the symbol classes.

Remark 2.1.4. The symplectic invariance of the Weyl quantization given by the formula (2.1.14) is its most important property. The fact that the Weyl formula quantizes real Hamiltonians into formally selfadjoint operators is shared with other quantizations such as the Feynman quantization, given by $a^F = \frac{1}{2}\big(\mathrm{op}(a)+\mathrm{op}(Ja)\big)$ (see the remark 1.1.11). The reader can check the section 4.4.3 of our appendix to see that the Feynman quantization does not satisfy the symplectic invariance property.

For $a_1, a_2 \in \mathscr{S}(\mathbb{R}^{2n})$, the formula (2.1.18) can be written as well as (we use the symplectic form σ given in (2.1.2))

$$(a_1 \sharp a_2)(X) = \iint e^{-2i\pi\langle\sigma Y_1, Y_2\rangle} a_1(2^{-1/2}Y_1 + X) a_2(2^{-1/2}Y_2 + X) dY_1 dY_2$$

so that using (see proposition 4.1.1) that the Fourier transform of

$$\mathbb{R}^{2n}\times\mathbb{R}^{2n} \ni (X_1, X_2) \mapsto \exp -2i\pi\langle\sigma X_1, X_2\rangle = \exp -i\pi \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

is $\exp i\pi \begin{pmatrix}\Xi_1 & \Xi_2\end{pmatrix} \begin{pmatrix} 0 & \sigma^{-1} \\ -\sigma^{-1} & 0 \end{pmatrix} \begin{pmatrix}\Xi_1 \\ \Xi_2\end{pmatrix} = \exp 2i\pi\langle \Xi_1, \sigma^{-1}\Xi_2\rangle$: in fact the $4n \times 4n$ matrix

$$\begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -I_E \\ 0 & 0 & I_{E^*} & 0 \\ 0 & I_E & 0 & 0 \\ -I_{E^*} & 0 & 0 & 0 \end{pmatrix}$$

is symmetric with determinant 1 and signature 0 and its inverse is

$$\begin{pmatrix} 0 & 0 & 0 & -I_{E^*} \\ 0 & 0 & I_E & 0 \\ 0 & I_{E^*} & 0 & 0 \\ -I_E & 0 & 0 & 0 \end{pmatrix}.$$

We obtain, after applying Plancherel's formula and performed a simple affine change of variables,

$$(a_1 \sharp a_2)(X) = \iint \exp i\pi\langle \Xi_1, \sigma^{-1}\Xi_2\rangle \\ \left(\iint \exp 2i\pi\big((X - Y_1)\cdot \Xi_1 + (X - Y_2)\cdot \Xi_2\big) a_1(Y_1)a_2(Y_2) dY_1 dY_2 \right) d\Xi_1 d\Xi_2$$

which is

$$(a_1 \sharp a_2)(X) = \exp(i\pi[D_{X_1}, D_{X_2}])\big(a_1(X_1)a_2(X_2)\big)_{|X_1 = X_2 = X}, \tag{2.1.23}$$

where the symplectic form on the dual of the phase space is naturally given by $[\Xi_1, \Xi_2] = \langle -\sigma^{-1}\Xi_1, \Xi_2\rangle$ so that

$$[D_{X_1}, D_{X_2}] = [(D_{x_1}, D_{\xi_1}), (D_{x_2}, D_{\xi_2})] = D_{\xi_1}\cdot D_{x_2} - D_{x_1}\cdot D_{\xi_2}. \tag{2.1.24}$$

Remark 2.1.5. For future reference, it will be useful to notice that

$$\big(i\pi[D_{X_1}, D_{X_2}]\big)^j (a_1 \otimes a_2)_{|X_1 = X_2 = X} = 2^{-j} a_1^{(j)}(X)((\sigma^{-1}D_X)^j a_2)(X).$$

In fact, the lhs is equal to

$$\begin{aligned} &\iint \big(i\pi\langle -\sigma^{-1}\Xi_1, \Xi_2\rangle\big)^j \widehat{a_1}(\Xi_1)\widehat{a_2}(\Xi_2) e^{2i\pi(X\cdot(\Xi_1+\Xi_2))} d\Xi_1 d\Xi_2 \\ &\quad = \iint \big(i\pi\langle \Xi_1, \sigma^{-1}\Xi_2\rangle\big)^j \widehat{a_1}(\Xi_1)\widehat{a_2}(\Xi_2) e^{2i\pi(X\cdot(\Xi_1+\Xi_2))} d\Xi_1 d\Xi_2 \\ &\quad = \iint \big(i\pi \mathtt{Fourier}\big(\langle D, \sigma^{-1}\Xi_2\rangle\big)^j a_1\big)(\Xi_1)\widehat{a_2}(\Xi_2) e^{2i\pi(X\cdot(\Xi_1+\Xi_2))} d\Xi_1 d\Xi_2 \\ &\quad = \int \big(\frac{1}{2}\big(\langle \partial, \sigma^{-1}\Xi_2\rangle\big)^j a_1\big)(X)\widehat{a_2}(\Xi_2) e^{2i\pi X\cdot\Xi_2} d\Xi_2 \\ &\quad = 2^{-j}\int a_1^{(j)}(X)(\sigma^{-1}\Xi)^j \widehat{a_2}(\Xi) e^{2i\pi X\cdot\Xi} d\Xi \quad \text{which is the rhs.} \end{aligned}$$

Remark 2.1.6. We can observe that the expression

$$a_1 \boxtimes a_2 = \exp(i\pi[D_{X_1}, D_{X_2}])\big(a_1 \otimes a_2\big) = \exp \frac{1}{4i\pi}[\partial_{X_1}, \partial_{X_2}]\big(a_1 \otimes a_2\big) \quad (2.1.25)$$

makes sense for all tempered distributions $a_j \in \mathscr{S}'(\mathbb{R}^{2n})$ and provides a distribution in $\mathscr{S}'(\mathbb{R}^{4n})$ although the formula (2.1.23) requires a restriction to the diagonal of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ of the distribution $a_1 \boxtimes a_2$, restriction which is naturally subjected to some conditions on the symbols a_1, a_2.

It is convenient to give right now an "asymptotic" version of the composition formulas (2.1.18) and (1.1.4), as it is already done with (1.1.18). We use

$$\begin{aligned}
\frac{1}{k!}\left(\frac{1}{4i\pi}[\partial_X, \partial_Y]\right)^k &= \frac{1}{k!}\Big(\frac{1}{4i\pi}\sum_{1\le j\le n}(\partial_{\xi_j}\partial_{y_j} - \partial_{x_j}\partial_{\eta_j})\Big)^k \\
&= \frac{1}{2^k k!}\Big(\sum_{1\le j\le n}(D_{\xi_j}\partial_{y_j} - \partial_{x_j}D_{\eta_j})\Big)^k \\
&= \frac{1}{2^k}\sum_{\gamma_1+\cdots+\gamma_n=k}\prod_{1\le j\le n}\frac{1}{\gamma_j!}(D_{\xi_j}\partial_{y_j} - \partial_{x_j}D_{\eta_j})^{\gamma_j} \\
&= \frac{1}{2^k}\sum_{\substack{\gamma_1+\cdots+\gamma_n=k \\ \gamma_j=\alpha_j+\beta_j}}\prod_{1\le j\le n}\frac{1}{\alpha_j!\beta_j!}(D_{\xi_j}\partial_{y_j})^{\alpha_j}(-1)^{|\beta_j|}(\partial_{x_j}D_{\eta_j})^{\beta_j} \\
&= \frac{1}{2^k}\sum_{|\alpha|+|\beta|=k}\frac{1}{\alpha!\beta!}D_\xi^\alpha\partial_y^\alpha(-1)^{|\beta|}\partial_x^\beta D_\eta^\beta,
\end{aligned}$$

and we get from (2.1.25)

$$(a\sharp b)(x,\xi) = \sum_{0\le k<\nu}2^{-k}\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{|\beta|}}{\alpha!\beta!}D_\xi^\alpha\partial_x^\beta a\ D_\xi^\beta\partial_x^\alpha b + r_\nu(a,b), \quad (2.1.26)$$

$$\text{with} \quad r_\nu(a,b)(X) = R_\nu\big(a(X)\otimes b(Y)\big)_{|X=Y}, \quad (2.1.27)$$

$$R_\nu = \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!}\exp\frac{\theta}{4i\pi}[\partial_X,\partial_Y]d\theta\Big(\frac{1}{4i\pi}[\partial_X,\partial_Y]\Big)^\nu. \quad (2.1.28)$$

Defining

$$\omega_k(a,b) = 2^{-k}\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{|\beta|}}{\alpha!\beta!}D_\xi^\alpha\partial_x^\beta a\ D_\xi^\beta\partial_x^\alpha b, \quad (2.1.29)$$

we get that

$$\omega_0 = ab, \quad \omega_1 = \frac{1}{4i\pi}\{a,b\},$$

so that the beginning of this expansion is thus

$$ab + \frac{1}{4i\pi}\{a,b\},$$

where the Poisson bracket $\{a,b\}$ is given by (1.1.23). The $\omega_k(a,b)$ with k even are symmetric in a,b and skew-symmetric for k odd: this is obvious from the above expression coming from $[\partial_X,\partial_Y]^k$. Also, when a,b are real-valued the $\omega_k(a,b)$ with k even are real and purely imaginary for k odd. This can be compared to the classical expansion formula (1.1.18), which is less symmetrical. We can prove as well, using the Taylor expansion of the exponential function, another expression of R_ν in (2.1.33),

$$(a_1\sharp a_2)(X)=\sum_{0\leq j<\nu}\overbrace{\frac{(i\pi)^j}{j!}[D_{X_1},D_{X_2}]^j(a_1\otimes a_2)_{|X_1=X_2=X}}^{\omega_j(a_1,a_2)}$$
$$+\int_0^1\frac{(1-\theta)^{\nu-1}}{(\nu-1)!}2^{2n}\theta^{-2n}\iint e^{-\frac{4i\pi}{\theta}[X-Y_1,X-Y_2]}(i\pi)^\nu[D_{Y_1},D_{Y_2}]^\nu(a_1\otimes a_2)(Y_1,Y_2)$$
$$dY_1dY_2d\theta. \quad (2.1.30)$$

Remark 2.1.7. The expansion formula above are proven true at this point only for $a,b\in\mathscr{S}(\mathbb{R}^{2n})$, but we shall see later on that for rather general symbols, these formulas can actually be used as true asymptotic expansions. To build-up this from the previous sections, we give the result for the class of symbols (1.1.13) introduced in the first section (a proof for very general classes of symbols will be given later on, so we omit the proof). For $a_j\in S^{m_j}_{1,0}, j=1,2$ we have

$$a_1\sharp a_2=a_1a_2+\frac{1}{4i\pi}\{a_1,a_2\}\mod S^{m_1+m_2-2}_{1,0}, \quad (2.1.31)$$
$$a_1\sharp a_2+a_2\sharp a_1=2a_1a_2\mod S^{m_1+m_2-2}_{1,0}, \quad (2.1.32)$$
$$a_1\sharp a_2-a_2\sharp a_1=\frac{1}{2i\pi}\{a_1,a_2\}\mod S^{m_1+m_2-3}_{1,0}. \quad (2.1.33)$$

Let us give a more explicit expression for the third term ω_2 in the expansion (2.1.26). We have

$$(4i\pi)^2\omega_2(a,b)=\sum_{|\alpha|=2}\frac{1}{\alpha!}\left(\partial_\xi^\alpha a\partial_x^\alpha b+\partial_x^\alpha a\partial_\xi^\alpha b\right)-\sum_{|\alpha|=|\beta|=1}\partial_\xi^\alpha\partial_x^\beta a\;\partial_x^\alpha\partial_\xi^\beta b.$$

Remark 2.1.8. We have implicitely considered that our symbols were scalar-valued, but it is of course interesting, e.g. for the study of systems of PDE, to consider the case where a is a tempered distribution on $\mathbb{R}^{2n}$ valued in some Banach algebra of operators. For instance a may be a smooth function, bounded as well as all its derivatives, from $\mathbb{R}^{2n}$ into $\mathscr{B}(\mathbb{H})$ where $\mathbb{H}$ is a Hilbert space and $\mathscr{B}(\mathbb{H})$ stands for the bounded operators on $\mathbb{H}$. Even the finite-dimensional case is of interest, but one has to pay a careful attention to the composition formula, because $\mathscr{B}(\mathbb{H})$ is not commutative. We see that part of the discussion above remains valid but that the product $a(Y)b(Z)$ in (2.1.18) cannot be replaced by $b(Z)a(Y)$. In particular, the

formula (2.1.31) remains true in the non-commutative case, provided the Poisson bracket is defined as in (1.1.23), respecting the order of the products; however, the formulas following (2.1.31) must be modified in the non-commutative case: in particular we have, say for finite dimensional square systems a_1, a_2 with entries in $S^{m_j}_{1,0}$,

$$a_1 \sharp a_2 - a_2 \sharp a_1 = a_1 a_2 - a_2 a_1 + \frac{1}{4i\pi}\{a_1, a_2\} - \frac{1}{4i\pi}\{a_2, a_1\} \mod S^{m_1+m_2-2}_{1,0}$$

so that the symbol of the commutator $[a_1^w, a_2^w]$ is, modulo $S^{m_1+m_2-2}_{1,0}$,

$$\underbrace{\sqsubset a_1(x,\xi), a_2(x,\xi) \sqsupset}_{\substack{\text{commutator of the matrices} \\ a_1(x,\xi), a_2(x,\xi)}} + \frac{1}{4i\pi} \sum_{1\le j\le n} \frac{\partial a_1}{\partial \xi_j}\frac{\partial a_2}{\partial x_j} + \frac{\partial a_2}{\partial x_j}\frac{\partial a_1}{\partial \xi_j} - \frac{\partial a_1}{\partial x_j}\frac{\partial a_2}{\partial \xi_j} - \frac{\partial a_2}{\partial \xi_j}\frac{\partial a_1}{\partial x_j}.$$

2.2 Admissible metrics

2.2.1 A short review of examples of pseudodifferential calculi

We have already seen some examples of symbol classes, such as
· $S^m_{\rho,\delta}$: C^∞ functions a defined on $\mathbb{R}^{2n}$ satisfying (1.1.32). The conditions

$$\delta < 1, 0 \le \delta \le \rho \le 1$$

are required. The classes $S^m_{1,0}$ provide a good framework for the analysis of PDE in an open set and in particular are useful to construct parametrices of (micro)elliptic operators and to prove some results of propagation of singularities for real principal type operators (see e.g. our section 1.2.3). The classes $S^m_{\rho,\delta}$ were introduced in [65] as the natural space containing the parametrices of hypoelliptic operators with constant coefficients.
· S^m_{scl}: C^∞ functions a defined on $\mathbb{R}^{2n} \times (0,1]$ satisfying (1.1.33). These classes constitute the natural framework of semi-classical analysis, where the operators depend on a small parameter h, likened to the Planck constant.
· The classes Σ^m: C^∞ functions a defined on $\mathbb{R}^{2n}$ satisfying (1.1.48). These classes are extensively used in the book [129] to study the spectral properties of globally defined pseudodifferential operators on $\mathbb{R}^n$.

Further steps in the generalization, essentially motivated by the proof of solvability of principal-type differential operators satisfying Nirenberg-Treves' condition (P) in [8], were taken by R. Beals and C. Fefferman in their 1973 paper [9], in which the authors introduced symbols a which are C^∞ functions on $\mathbb{R}^{2n}$ such that

$$|(\partial^\alpha_\xi \partial^\beta_x a)(x,\xi)| \le C_{\alpha\beta} \Phi(x,\xi)^{M-|\alpha|} \varphi(x,\xi)^{m-|\beta|} \tag{2.2.1}$$

where the positive functions Φ, φ should satisfy some conditions. The article [69] by L. Hörmander in 1979 (see also the chapter 18 in [73]), introduced the more

flexible notion of *metric* on the phase space, a notion also used in the paper [145] of A. Unterberger and in a more general context by J.-M. Bony and N. Lerner in [20]. It is easy to see that the conditions (2.2.1) provide a more general framework than the three examples at the beginning of this section and one can consider the following Riemannian structure on the phase space $\mathbb{R}^{2n}$:

$$g_{x,\xi} = \frac{|dx|^2}{\varphi(x,\xi)^2} + \frac{|d\xi|^2}{\Phi(x,\xi)^2} : \text{ for } (t,\tau) \in \mathbb{R}^n \times \mathbb{R}^n, \quad g_{x,\xi}(t,\tau) = \frac{|t|^2}{\varphi(x,\xi)^2} + \frac{|\tau|^2}{\Phi(x,\xi)^2}.$$

The estimates (2.2.1) for $|\alpha| + |\beta| = 1$ (when $M = m = 0$), can be written as well as $|(\nabla a)(X) \cdot T| \leq C g_X(T)^{1/2}$ ($X = (x,\xi), T = (t,\tau)$) and for higher order derivatives as

$$|(\nabla^k a)(X)(T_1, \ldots, T_k)| \leq C g_X(T_1)^{1/2} \ldots g_X(T_k)^{1/2}.$$

As a particular case, the class $S^m_{\rho,\delta}$ defined above is related to the following metric on the phase space,

$$g_{x,\xi} = \frac{|dx|^2}{\langle\xi\rangle^{-2\delta}} + \frac{|d\xi|^2}{\langle\xi\rangle^{2\rho}} : \quad \text{for } (t,\tau) \in \mathbb{R}^n \times \mathbb{R}^n,\, g_{x,\xi}(t,\tau) = \frac{|t|^2}{\langle\xi\rangle^{-2\delta}} + \frac{|\tau|^2}{\langle\xi\rangle^{2\rho}}, \tag{2.2.2}$$

with $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. These examples will provide a guideline for the definitions in the sequel. A pseudodifferential calculus will be essentially the datum of a metric g that should satisfy some local and some global conditions. The symplectic structure and the uncertainty principle will also play a natural rôle in the constraints imposed to the metric g.

2.2.2 Slowly varying metrics on $\mathbb{R}^{2n}$

Definition 2.2.1. Let g be a (measurable) mapping from $\mathbb{R}^{2n}$ into $\mathcal{C}$, the cone of positive definite quadratic forms on $\mathbb{R}^{2n}$: for each $X \in \mathbb{R}^{2n}$, g_X is a positive-definite quadratic form on $\mathbb{R}^{2n}$. We shall say that g is a slowly varying metric on $\mathbb{R}^{2n}$ if

$$\exists C > 0, \exists r > 0, \forall X, Y, T \in \mathbb{R}^{2n},$$

$$g_X(Y - X) \leq r^2 \Longrightarrow C^{-1} g_Y(T) \leq g_X(T) \leq C g_Y(T). \tag{2.2.3}$$

Remark 2.2.2. The previous property will be satisfied if we ask only

$$\exists C > 0, \exists r > 0, \forall X, Y, T \in \mathbb{R}^{2n}, \quad g_X(Y-X) \leq r^2 \Longrightarrow g_Y(T) \leq C g_X(T). \tag{2.2.4}$$

In fact, assuming (2.2.4) gives that whenever $g_X(Y - X) \leq C^{-1} r^2$(which is $\leq r^2$ since $C \geq 1$ from (2.2.4) with $X = Y$) this implies $g_Y(X - Y) \leq r^2$, and thus $g_X(T) \leq C g_Y(T)$, so that (2.2.3) is satisfied with C_0, r_0 such that $r_0 = C^{-1/2} r, C_0 = C$.

Examples 2.2.3. If we consider for a positive Lipschitz continuous function ρ defined on $\mathbb{R}^{2n}$, and an Euclidean norm $\|\|$ on $\mathbb{R}^{2n}$, the metric defined by

$$g_X(T) = \frac{\|T\|^2}{\rho(X)^2},$$

one can check easily that g is slowly varying: assuming $g_X(Y-X) \leq r^2$ means that $\|Y-X\| \leq r\rho(X)$ so that, if L is the Lipschitz constant of ρ, we have

$$\rho(X) \leq \rho(Y) + L\|X-Y\| \leq \rho(Y) + Lr\rho(X),$$

so that choosing $0 < r < \frac{1}{L}$ gives the desired property (2.2.4) with $C = (1-Lr)^{-2}$. Let us verify that the metric (2.2.2) is slowly varying when $\rho \leq 1$. We assume that $g_{x,\xi}\big((y,\eta)-(x,\xi)\big) \leq r^2$ so that $|\eta-\xi|^2 \leq r^2(1+|\xi|^2)^\rho$ and we obtain

$$|\xi|^2 \leq 2r^2(1+|\xi|^2) + 2|\eta|^2$$

and thus $(1+|\xi|^2)(1-2r^2) \leq 2(1+|\eta|^2)$ which proves (2.2.4) if $r < 2^{-1/2}$. Note that the metric (2.2.2) with $\rho > 1$ is not slowly varying since

$$\frac{\langle\eta\rangle^\rho}{\langle\xi\rangle^\rho} \leq \sup_{(t,\tau)} \frac{g_{x,\xi}(t,\tau)^{1/2}}{g_{y,\eta}(t,\tau)^{1/2}}$$

and for $k > 1$ with $|\xi| = k, |\eta| = k^\rho/\ln k$, we have for $k \to +\infty$, $|\xi-\eta| \sim k^\rho/\ln k \ll k^\rho \sim \langle\xi\rangle^\rho$, so that

$$|\xi-\eta|\langle\xi\rangle^{-\rho} = g_{x,\xi}(x-x,\xi-\eta)^{1/2} \to 0, \text{ but } \langle\eta\rangle\langle\xi\rangle^{-1} \sim k^{\rho-1}/\ln k \to +\infty.$$

Definition 2.2.4. Let g be a slowly varying metric on $\mathbb{R}^{2n}$. A function $M : \mathbb{R}^{2n} \to]0,+\infty)$ is a g-slowly varying weight if

$$\exists C > 0, \exists r > 0, \forall X, Y \in \mathbb{R}^{2n}, g_X(Y-X) \leq r^2 \Longrightarrow C^{-1} \leq \frac{M(X)}{M(Y)} \leq C. \quad (2.2.5)$$

Remark 2.2.5. It is easy to verify that $\langle\xi\rangle$ is a slowly varying weight for the metrics (2.2.2) when $\rho \leq 1$: we have

$$\frac{\langle\xi\rangle^2}{\langle\eta\rangle^2} \leq \frac{1+2|\xi-\eta|^2}{1+|\eta|^2} + 2 \leq 3 + 2\frac{|\xi-\eta|^2}{\langle\xi\rangle^{2\rho}}\frac{\langle\xi\rangle^{2\rho}}{\langle\eta\rangle^2} \leq 3 + 2g_{x,\xi}\big((y,\eta)-(x,\xi)\big)\frac{\langle\xi\rangle^2}{\langle\eta\rangle^2}$$

which gives the answer provided $g_X(Y-X) < 1/2$. Note also that products of slowly varying weights for g are still slowly varying weights for g, and more generally that M^θ is a slowly varying weight if M is a slowly varying weight and $\theta \in \mathbb{R}$.

Definition 2.2.6. Let g be a slowly varying metric[2] on $\mathbb{R}^{2n}$ and let M be a g-slowly varying weight. The space of symbols $S(M,g)$ is defined as the set of C^∞ functions on $\mathbb{R}^{2n}$ such that $\forall k \in \mathbb{N}, \exists C_k, \forall X, T_1, \dots T_k \in \mathbb{R}^{2n}$,

$$|a^{(k)}(X)(T_1,\dots,T_k)| \leq C_k M(X) \prod_{1\leq j\leq k} g_X(T_j)^{1/2}. \tag{2.2.6}$$

For $a \in S(M,g), l \in \mathbb{N}$, we shall note

$$\|a\|^{(l)}_{S(M,g)} = \max_{0\leq k\leq l} \sup_{\substack{X,T_j \\ g_X(T_j)=1}} |a^{(k)}(X)(T_1,\dots,T_k)|M(X)^{-1}. \tag{2.2.7}$$

The space $S(M,g)$, with the countable family of semi-norms $(\|\cdot\|^{(l)}_{S(M,g)})_{l\in\mathbb{N}}$ is a Fréchet space.

Looking back at the couple of examples above, we can check readily that

$$\begin{aligned}
S^m_{\rho,\delta} &= S\big(\langle\xi\rangle^m, \frac{|dx|^2}{\langle\xi\rangle^{-2\delta}} + \frac{|d\xi|^2}{\langle\xi\rangle^{2\rho}}\big) \quad \text{(see (1.1.32))},\\
S^m_{scl} &= S\big(h^{-m}, |dx|^2 + h^2|d\xi|^2\big) \quad \text{(see (1.1.33))},\\
\Sigma^m &= S\big((1+|x|^2+|\xi|^2)^m, \frac{|dx|^2+|d\xi|^2}{1+|x|^2+|\xi|^2}\big) \quad \text{(see (1.1.48))},
\end{aligned}$$

and (2.2.1) defines $S(\Phi^M\varphi^m, \frac{|dx|^2}{\varphi^2} + \frac{|d\xi|^2}{\Phi^2})$. It is not difficult to illustrate the meaning of the localization via a slowly varying metric: around each point X, one may draw the ball

$$U_{X,r} = \{Y, g_X(Y-X) \leq r^2\} \tag{2.2.8}$$

and if r is chosen small enough, *uniformly with respect to* X, the shape and size of $U_{Y,r}$ will be equivalent to the shape and size of $U_{X,r}$ for $Y \in U_{X,r}$. R. Beals and C. Fefferman in [9] preferred to speak about "boxes", since their metrics were defined by (2.2.1) and they would consider instead of the Euclidean ball U the following product of (Euclidean) balls

$$B_{x,\xi,r} = \Big\{(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n, \max\Big(\frac{|y-x|}{\varphi(x,\xi)}, \frac{|\eta-\xi|}{\Phi(x,\xi)}\Big) \leq r\Big\}.$$

The main feature of a slowly varying metric is that it is possible to introduce some partitions of unity related to the metric in a way made precise in the following theorem. The reader will find in the appendix 4.3.4 a construction of a partition of unity for a metric defined only on some open subset of $\mathbb{R}^N$.

[2]The reader may notice that these slow variations properties are not needed for setting the definition which makes sense for a metric g and a positive function M.

Theorem 2.2.7. *Let g be a slowly varying metric on $\mathbb{R}^{2n}$. Then there exists $r_0 > 0$ such that, for all $r \in (0, r_0]$, there exists a family $(\varphi_Y)_{Y\in\mathbb{R}^{2n}}$ of functions uniformly in $S(1,g)$ (i.e. $\forall k, \sup_Y \|\varphi_Y\|^{(k)}_{S(1,g)} < \infty$) supported in $U_{Y,r}$ such that, for all $X \in \mathbb{R}^{2n}$,*

$$\int_{\mathbb{R}^{2n}} \varphi_Y(X)|g_Y|^{1/2}dY = 1,$$

where $|g_Y|$ is the determinant of g_Y (with respect to the standard Euclidean norm).

Proof. Let $\chi_0 \in C^\infty(\mathbb{R};[0,1])$ non-increasing such that $\chi_0(t) = 1$ on $t \le 1/2$, $\chi_0(t) = 0$ on $t \ge 1$. Let us assume that the property (2.2.3) is satisfied with some positive constants $C_0, r_0 > 0$. We define, for $r \in (0, r_0]$, the function

$$\omega(X,r) = \int_{\mathbb{R}^{2n}} \chi_0\big(r^{-2}g_Y(X-Y)\big)|g_Y|^{1/2}dY.$$

Since χ_0 is non-increasing and $X \mapsto \chi_0\big(r^{-2}g_Y(X-Y)\big)$ is supported in $U_{Y,r}$, we have

$$\omega(X,r) \ge \int_{\mathbb{R}^{2n}} \chi_0\big(r^{-2}C_0 g_X(X-Y)\big)|g_X|^{1/2}C_0^{-n}dY = \underbrace{\int_{\mathbb{R}^{2n}} \chi_0(|Z|^2)dZ C_0^{-2n}}_{\text{a positive constant}} r^{2n}, \tag{2.2.9}$$

and an estimate from above of the same type. Also we see that, using the notation $\|T\|_Y = g_Y(T)^{1/2}$ and $\langle , \rangle_Y$ for the associated dot-product, we have

$$\omega'(X,r)T = \int_{\mathbb{R}^{2n}} \chi_0'\big(r^{-2}\|X-Y\|_Y^2\big)r^{-2}2\langle (X-Y), T\rangle_Y |g_Y|^{1/2}dY,$$

and by induction, for $k \in \mathbb{N}^*, T \in \mathbb{R}^{2n}$, $\omega^{(k)}(X,r)T^k$ is a finite sum of terms of type

$$c_{p,k}\int_{\mathbb{R}^{2n}} \chi_0^{(p)}\big(r^{-2}\|X-Y\|_Y^2\big)r^{-2p}\langle X-Y, T\rangle_Y^{2p-k} g_Y(T)^{k-p}|g_Y|^{1/2}dY, \tag{2.2.10}$$

where $c_{p,k}$ is a constant depending only on p, k and $p \in [k/2, k] \cap \mathbb{N}$. The support of $\chi_0^{(p)}$, the condition $0 < r \le r_0$, show as above that (2.2.10) is bounded above by

$$c(p,k,r,C_0)\|\chi_0^{(p)}\|_{L^\infty}\int_{|Z|\le 1}|Z|^{2p-k}dZ g_X(T)^{k/2},$$

so that for all k, $|\omega^{(k)}(X,r)T^k| \le C_k g_X(T)^{k/2}$. According to the section 4.2.2, this implies that the multilinear forms $\omega^{(k)}(X,r)$ have a bounded norm with respect to g_X. Thus we obtain that $\omega(\cdot,r)$ belongs to $S(1,g)$ and the estimate from below (2.2.9) shows that $1/\omega(\cdot,r)$ belongs as well to $S(1,g)$. As a consequence, we get that

$$1 = \int_{\mathbb{R}^{2n}} \underbrace{\chi_0\big(r^{-2}g_Y(X-Y)\big)\omega(X,r)^{-1}}_{=\varphi_Y(X)}|g_Y|^{1/2}dY,$$

where the family $(\varphi_Y)_{Y\in\mathbb{R}^{2n}}$ satisfies the requirements of the theorem. □

N.B. A discrete version of the theorem 2.2.7 is given in the appendix with the theorem 4.3.7.

Remark 2.2.8. The slow variation property does not require any local regularity, not even continuity[3]. Once a slowly varying metric g is given, the partition of unity allows us to find a smooth metric $\widetilde{g}$ somewhat equivalent to g: we define

$$\widetilde{g}_X(T) = \int_{\mathbb{R}^{2n}} \varphi_Y(X) g_Y(T) |g_Y|^{1/2} dY$$

which is a positive-definite quadratic form such that for $T \neq 0$,

$$\frac{\widetilde{g}_X(T)}{g_X(T)} = \int_{\mathbb{R}^{2n}} \varphi_Y(X) \frac{g_Y(T)}{g_X(T)} |g_Y|^{1/2} dY \in [C^{-1}, C]$$

since for all $X \in \operatorname{supp}\varphi_Y$, $\frac{g_Y(T)}{g_X(T)} \in [C^{-1}, C]$. The same can be done with a g-slowly varying weight M (definition 2.2.4): we define

$$\widetilde{M}(X) = \int_{\mathbb{R}^{2n}} \varphi_Y(X) M(Y) |g_Y|^{1/2} dY$$

and we see that the ratio $\widetilde{M}(X)/M(X)$ belongs to $[C^{-1}, C]$ where C is the fixed constant given in (2.2.5). Moreover, using the notations of the proof of the theorem 2.2.7 and $\chi_1 = 1$ on the support of χ_0, χ_1 supported in $(-\infty, 2]$, we get that $\widetilde{M}$ is smooth with $|\widetilde{M}^{(k)}(X)T^k| \leq M(X) g_X(T)^{k/2} C_k \int_{\mathbb{R}^{2n}} \chi_1(r^{-2} g_Y(X-Y)) |g_Y|^{1/2} dY$, so that

$$\widetilde{M} \in S(M, g), \quad \forall X \in \mathbb{R}^{2n}, \quad \frac{\widetilde{M}(X)}{M(X)} \in [C^{-1}, C]. \tag{2.2.11}$$

An immediate consequence of the theorem 2.2.7 is the following

Proposition 2.2.9. *Let g be a slowly varying metric on $\mathbb{R}^{2n}$, M be a g-slowly varying weight. and a be a symbol in $S(M, g)$. Then there exists $r_0 > 0$ such that, for all $0 < r \leq r_0$,*

$$a(X) = \int_{\mathbb{R}^{2n}} a_Y(X) |g_Y|^{1/2} dY$$

where $a_Y \in S(M(Y), g_Y)$ uniformly and $\operatorname{supp} a_Y \subset U_{Y,r}$.

2.2.3 The uncertainty principle for metrics

The slow variation properties exposed in the previous section do not use the symplectic structure of the phase space $\mathbb{R}^{2n}$. It is quite natural to expect to have some contraints linking the metric g with the symplectic form (2.1.1). Let us first

[3]Note that the integral version of the partition requires some measurability assumption.

consider a slightly more abstract framework with the configuration space E given as an n-dimensional real vector space and the phase space $\mathcal{P} = E \oplus E^*$ equipped with the symplectic form σ given by (4.4.1). We refer the reader to the appendix 4.4.1 for some basic facts on symplectic algebra.

Let us consider a metric on $\mathcal{P}$, i.e. a mapping $\mathcal{P} \ni X \mapsto g_X \in \mathcal{C}$, where $\mathcal{C}$ is the cone of positive-definite quadratic forms on $\mathcal{P}$. We shall always assume that

$$\forall X \in \mathcal{P}, \quad g_X \leq g_X^\sigma, \tag{2.2.12}$$

where g_X^σ is given by (4.4.18). Moreover we define, for $X \in \mathcal{P}$,

$$\lambda_g(X) = \inf_{T \neq 0} \left(\frac{g_X^\sigma(T)}{g_X(T)} \right)^{1/2} \tag{2.2.13}$$

and we note from the previous hypotheses that

$$g_X \leq \lambda_g(X)^{-2} g_X^\sigma, \quad \lambda_g(X) \geq 1. \tag{2.2.14}$$

Examples 2.2.10. Let us review the couple of examples above; for the example (2.2.2), that we write

$$g_{x,\xi} = \langle \xi \rangle^{2\delta} |dx|^2 + \langle \xi \rangle^{-2\rho} |d\xi|^2,$$

the lemma 4.4.25 shows that $g_{x,\xi}^\sigma = \langle \xi \rangle^{2\rho} |dx|^2 + \langle \xi \rangle^{-2\delta} |d\xi|^2$, $\lambda_g(x,\xi) = \langle \xi \rangle^{\rho-\delta}$, so that the condition (2.2.12) is equivalent to $\rho \geq \delta$. The semi-classical metric satisfies

$$g_{x,\xi} = |dx|^2 + h^2 |d\xi|^2, \quad g_{x,\xi}^\sigma = h^{-2}|dx|^2 + |d\xi|^2, \quad \lambda = h^{-1},$$

and the condition $h \in (0,1]$ appears to be the expression of (2.2.12). For the global metric defined for $X \in \mathbb{R}^{2n}$ by

$$g_X = \langle X \rangle^{-2} g_0, \quad g_0 = g_0^\sigma, \quad g_X^\sigma = \langle X \rangle^2 g_0, \quad \lambda_g(X) = \langle X \rangle^2.$$

For the metric related to the estimates (2.2.1) we have

$$g_{x,\xi} = \varphi(x,\xi)^{-2}|dx|^2 + \Phi(x,\xi)^{-2}|d\xi|^2, \; g_{x,\xi}^\sigma = \Phi(x,\xi)^2|dx|^2 + \varphi(x,\xi)^2|d\xi|^2, \; \lambda = \Phi\varphi.$$

Looking now at the lemma 4.4.25, we see that for each $(x,\xi) \in \mathcal{P}$, there exists some symplectic coordinates (y,η) such that

$$g_{x,\xi}(y,\eta) = \sum_{1 \leq j \leq n} \lambda_j^{-1}(y_j^2 + \eta_j^2), \quad g_{x,\xi}^\sigma(y,\eta) = \sum_{1 \leq j \leq n} \lambda_j (y_j^2 + \eta_j^2), \tag{2.2.15}$$

with $\lambda(x,\xi) = \min_{1 \leq j \leq n} \lambda_j$, so that (2.2.12) can be expressed as

$$\min_{1 \leq j \leq n} \lambda_j \geq 1.$$

Comment. Checking the ellipsoid $\{(y,\eta), g_{x,\xi}(y,\eta) \leq 1\}$, we see that the conjugate axes have size $\lambda_j^{1/2}$, and we can write symbolically that $\Delta y_j \Delta \eta_j \sim \lambda_j$. Now the Heisenberg uncertainty principle is requiring a lowerbound on these products, so that it is natural to require as well a lower bound for the λ_j, that is to set a rule which prevents the localization to be too sharp on conjugate axes. Although the analogy with this physical principle is only heuristical at this point, the next lemma is providing a mathematical reason to require (2.2.12).

Lemma 2.2.11. *Let g be a metric on $\mathbb{R}^{2n}$, m_1, m_2 be two positive functions on $\mathbb{R}^{2n}$ and $a_j \in S(m_j, g)$, $j = 1, 2$ (cf. definition 2.2.6). We have*

$$\omega_k(a_1, a_2) \in S(m_1 m_2 \lambda_g^{-k}, g)$$

where $\omega_k(a_1, a_2)$ is defined in (2.1.29).

Proof. Using the lemma 4.4.25, we may assume that for a given $X \in \mathbb{R}^{2n}$, the metric g_X has the form (2.2.15) and since the linear change of variables used to achieve that form is symplectic, it leaves unchanged the brackets $[\partial_{X_1}, \partial_{X_2}]$ involved in the formula defining ω_k. We check for $X, T \in \mathbb{R}^{2n}$,

$$\begin{aligned}
&\frac{1}{l!}\omega_k(a_1,a_2)^{(l)}(X)T^l \\
&\quad = \frac{1}{k!}\Big(\frac{1}{4i\pi}\Big)^k \sum_{l_1+l_2=l} \frac{1}{l_1!l_2!}[\partial_{X_1},\partial_{X_2}]^k \big(a_1^{(l_1)}(X_1)T^{l_1} \otimes a_2^{(l_2)}(X_2)T^{l_2}\big)_{X_1=X_2=X},
\end{aligned}$$

and we see that we have a linear combination of terms

$$a_1^{(l_1+k)}(X)T^{l_1}S_1' \dots S_k' a_2^{(l_2+k)}(X)T^{l_2}S_1'' \dots S_k''.$$

with $\Gamma(S_{j'}') = \Gamma(S_{j''}'') = 1, \Gamma = \sum_{1\leq j\leq n} |dy_j|^2 + |d\eta_j|^2$. As a result, we get

$$|\omega_k(a_1,a_2)^{(l)}(X)T^l| \leq C g_X(T)^{l/2} \left(\sup_S \frac{g_X(S)}{\Gamma(S)}\right)^k$$

and since

$$\sup_S \frac{g_X(S)}{\Gamma(S)} = \max(\lambda_j^{-1}) = (\min \lambda_j)^{-1} = \lambda_g(X)^{-1},$$

this gives the result. □

Comment. Although we have not proven yet a composition formula, it is not difficult to guess that we expect that (2.1.26) provides some asymptotic expansion and for instance that the weight of the product $a_1 a_2$ (say $m_1 m_2$) should be larger than the weight of the Poisson bracket $\{a_1, a_2\}$ which is $m_1 m_2 \lambda_g^{-1}$, which requires that $\lambda_g \geq 1$. It will be important for our calculi to deal with large regions of the phase space where $\lambda_g = 1$: at these places no asymptotic expansion is valid (e.g. for the example (2.2.2) when $\rho = \delta$), since all terms in (2.1.26) are potentially of the same size but we should certainly avoid to have regions where λ_g is not bounded from below since the divergence of (2.1.26) will get out of control.

2.2.4 Temperate metrics

We begin with a remark, stressing the non-locality of the composition formulas of pseudodifferential operators.

Remark 2.2.12. Looking at the composition formula (2.1.18), we see that, whenever a, b are compactly supported (and say in $L^1(\mathbb{R}^{2n})$), $a\sharp b$ is an entire function; if in addition, the supports of a and b are disjoint with a, b smooth, all the terms $\omega_k(a,b)$ (given in (2.1.29)) are identically 0. Nevertheless, taking $a \in C_c^\infty(\mathbb{R}^{2n})$(not identically 0), and b defined by

$$b(X) = \overline{\hat{a}(-2\sigma X)}\chi(X),$$

where χ is a non-negative C_c^∞ function with $\operatorname{supp}\chi \cap \operatorname{supp} a = \emptyset, \operatorname{supp}\chi \neq \emptyset$ (so that $a, b \in C_c^\infty(\mathbb{R}^{2n})$ with disjoint supports), we get from (2.1.18)

$$(a\sharp b)(0) = 2^{2n}\int \hat{a}(-2\sigma Z)b(Z)dZ = 2^{2n}\int |\hat{a}(-2\sigma Z)|^2\chi(Z)dZ > 0$$

since the function $\hat{a}(-2\sigma\cdot)$ is analytic and cannot vanish on the non-empty open set $\operatorname{int}(\operatorname{supp}\chi)$. As a consequence, the composition formula (2.1.18) is not local, and the values of $a\sharp b$ at a given point may depend on the values of a, b far away from that point. This indicates that, for a given metric g defined on $\mathbb{R}^{2n}$, some assumption should be made to relate g_X to g_Y for any $X, Y \in \mathbb{R}^{2n}$.

Definition 2.2.13. Let g be a metric on $\mathbb{R}^{2n}$. We shall say that g is temperate if

$$\exists C > 0, \exists N \geq 0, \forall X, Y, T \in \mathbb{R}^{2n}, \quad \frac{g_X(T)}{g_Y(T)} \leq C\big(1+(g_X^\sigma \wedge g_Y^\sigma)(X-Y)\big)^N, \tag{2.2.16}$$

where g^σ is given by (4.4.18) and $\wedge$ in (4.4.23).

The definition in [73] looked less symmetrical and weaker, since the rhs of the inequality in (2.2.16) was replaced by $(1+g_X^\sigma(X-Y))^N$. The definition 2.2.13 was introduced in [20] to handle pseudodifferential calculus on an open set of $\mathbb{R}^{2n}$. Let us show now that these conditions are equivalent when considering metrics defined on the whole[4] $\mathbb{R}^{2n}$.

Lemma 2.2.14. *Let g be a metric on $\mathbb{R}^{2n}$ such that*

$$\exists C > 0, \exists N \geq 0, \forall X, Y, T \in \mathbb{R}^{2n}, \quad \frac{g_X(T)}{g_Y(T)} \leq C\big(1 + g_X^\sigma(X-Y)\big)^N. \tag{2.2.17}$$

Then g is temperate, i.e. satisfies (2.2.16).

[4]This equivalence does not hold in general when the metric is defined only on a proper open subset of $\mathbb{R}^{2n}$ and that was the reason for introducing the formulation above.

Proof. For $X, Y \in \mathbb{R}^{2n}$, according to (4.4.27) there exists $Z \in \mathbb{R}^{2n}$ such that

$$(g_X^\sigma \wedge g_Y^\sigma)(X-Y) = 2g_X^\sigma(X-Z) + 2g_Y^\sigma(Z-Y).$$

Now from (2.2.17), we know that

$$g_X \leq C_1 g_Z\big(1+g_X^\sigma(X-Z)\big)^{N_1}, \quad g_Z^\sigma \leq C_1 g_Y^\sigma\big(1+g_Y^\sigma(Y-Z)\big)^{N_1}.$$

Then we get $g_Z^\sigma(Z-Y) \leq C_1\big(1+g_Y^\sigma(Y-Z)\big)^{N_1+1}$ and thus

$$g_Z \leq C_2 g_Y\big(1+g_Y^\sigma(Z-Y)\big)^{N_1(N_1+1)}.$$

We obtain

$$\begin{aligned}\frac{g_X(T)}{g_Y(T)} &\leq C_1\frac{g_Z(T)}{g_Y(T)}\big(1+g_X^\sigma(X-Z)\big)^{N_1}\\ &\leq C_1C_2\big(1+g_Y^\sigma(Z-Y)\big)^{N_1(N_1+1)}\big(1+g_X^\sigma(X-Z)\big)^{N_1}\\ &\leq C_3\big(1+2g_X^\sigma(X-Z)+2g_Y^\sigma(Z-Y)\big)^{N_1(N_1+1)}\\ &= C_3\big(1+(g_X^\sigma\wedge g_Y^\sigma)(X-Y)\big)^{N_1(N_1+1)}, \text{qed.}\end{aligned}$$

□

2.2.5 Admissible metric and weights

Definition 2.2.15. Let g be a metric on $\mathbb{R}^{2n}$, which is slowly varying (see (2.2.3)), satisfies the uncertainty principle (see (2.2.12)) and is temperate (see (2.2.16)). We shall say then that g is an admissible metric. Let M be a slowly varying weight for g (see (2.2.5)) such that

$$\exists C, N, \forall X, Y \in \mathbb{R}^{2n}, \quad \frac{M(X)}{M(Y)} \leq C\big(1+(g_X^\sigma \wedge g_Y^\sigma)(X-Y)\big)^N. \tag{2.2.18}$$

then we shall say that M is an admissible weight for g.

Remark 2.2.16. The previous property is equivalent to the weaker

$$\exists C, N, \forall X, Y \in \mathbb{R}^{2n}, \quad \frac{M(X)}{M(Y)} \leq C\big(1+g_X^\sigma(X-Y)\big)^N. \tag{2.2.19}$$

Let X, Y be given in $\mathbb{R}^{2n}$ and Z such that $(g_X^\sigma \wedge g_Y^\sigma)(X-Y) = 2g_X^\sigma(X-Z) + 2g_Y^\sigma(Z-Y)$. We have

$$\frac{M(X)}{M(Y)} = \frac{M(X)}{M(Z)}\frac{M(Z)}{M(Y)} \leq C^2\big(1+g_X^\sigma(X-Z)\big)^N\big(1+g_Z^\sigma(Y-Z)\big)^N,$$

and since g satisfies (2.2.16), we can estimate from above the $\big(1+g_Z^\sigma(Y-Z)\big)^N$ by $\big(1+g_Y^\sigma(Y-Z)\big)^{N_1}$ and this gives the property of the definition 2.2.15.

Remark 2.2.17. Note that, for an admissible metric g, the function λ_g defined in (2.2.13) is an admissible weight. First λ_g is a slowly varying weight for g, since for $r > 0$ small enough, $g_X(Y - X) \leq r^2$ implies[5] $g_X \sim g_Y$ and thus $g_X^\sigma \sim g_Y^\sigma$, so that

$$\frac{g_X^\sigma(T)}{g_X(T)} \sim \frac{g_Y^\sigma(T)}{g_Y(T)} \Longrightarrow \lambda_g(X) \sim \lambda_g(Y).$$

Moreover we have

$$\frac{g_X^\sigma(T)}{g_X(T)} \leq C_1\big(1 + (g_X^\sigma \wedge g_Y^\sigma)(X - Y)\big)^{N_1} \frac{g_Y^\sigma(T)}{g_Y(T)}$$
$$\Longrightarrow \lambda_g(X)^2 \leq \lambda_g(Y)^2 C_1\big(1 + (g_X^\sigma \wedge g_Y^\sigma)(X - Y)\big)^{N_1}.$$

Naturally, the powers of λ_g are also admissible weights. The weights are multiplicative whereas the "orders" of standard classes are additive ; although we shall not use this terminology, it would be natural to say that the symbols of order μ related to the metric g are those in $S(\lambda_g^\mu, g)$.

Lemma 2.2.18. *The metric* (2.2.2) *is admissible when* $0 \leq \delta \leq \rho \leq 1, \delta < 1$. *The metric* (2.2.2) *is not admissible when* $\rho = \delta = 1$. *The metrics* $|dx|^2 + h^2|d\xi|^2$, *(see* (1.1.33)*,* $h \in (0, 1]$*),* $\frac{|dx|^2+|d\xi|^2}{1+|x|^2+|\xi|^2}$ *(see* (1.1.48)*) are admissible.*

Proof. We have already seen with the examples 2.2.3 that the slow variation of (2.2.2) is equivalent to $\rho \leq 1$. Also the study of the examples 2.2.10 shows that the uncertainty principle for these metrics is equivalent to $\delta \leq \rho$. The metric of the class $S_{1,1}$ is not temperate since that would imply

$$\exists C,\ \exists N,\ \forall \xi, \forall \eta, \quad \langle \xi \rangle \leq C \langle \eta \rangle \left(1 + |\xi - \eta|/\langle \xi \rangle\right)^N,$$

which cannot hold for $\eta = 0$ and unbounded ξ. Let us show that the metric of the class $S_{\rho,\delta}$ is temperate whenever $\delta < 1, \delta \leq \rho \leq 1$: we check

$$\frac{g_{x,\xi}(t,\tau)}{g_{y,\eta}(t,\tau)} = \frac{\langle \xi \rangle^{2\delta}|t|^2 + \langle \xi \rangle^{-2\rho}|\tau|^2}{\langle \eta \rangle^{2\delta}|t|^2 + \langle \eta \rangle^{-2\rho}|\tau|^2} \leq \max\big(\langle \xi \rangle^{2\delta}\langle \eta \rangle^{-2\delta}, \langle \xi \rangle^{-2\rho}\langle \eta \rangle^{2\rho}\big)$$

and we also have $g_{x,\xi}^\sigma(x - y, \xi - \eta) = \langle \xi \rangle^{2\rho}|x - y|^2 + \langle \xi \rangle^{-2\delta}|\xi - \eta|^2 \geq \langle \xi \rangle^{-2\delta}|\xi - \eta|^2$. According to the lemma 2.2.14, it is enough to check

$$\frac{1 + |\xi|}{1 + |\eta|} + \frac{1 + |\eta|}{1 + |\xi|} \leq 2C\big(1 + (1 + |\xi|)^{-\delta}|\xi - \eta|\big)^{\frac{1}{1-\delta}},$$

which is a consequence of the symmetrical

$$\frac{1 + |\eta|}{1 + |\xi|} \leq C\big(1 + (1 + |\xi| + |\eta|)^{-\delta}|\xi - \eta|\big)^{\frac{1}{1-\delta}}, \tag{2.2.20}$$

[5]Here and thereafter, for a, b positive quadratic forms, we shall use the abuse of language $a \lesssim b$ (resp. $a \gtrsim b$, $a \sim b$) to mean that the ratio $a(T)/b(T)$ is bounded above (resp. below, resp. above and below) by "fixed" positive constants which depend only on the "structure" constants occurring in (2.2.3), (2.2.16), (2.2.5).

which we prove now. The estimate (2.2.20) is obvious for $|\eta| \leq 2|\xi|$ and for $|\eta| \leq 1$. If $|\eta| \geq 2|\xi|, |\eta| \geq 1$, the rhs of (2.2.20) is bounded below by $C'|\eta|$, implying (2.2.20). The metric $|dx|^2 + h^2|d\xi|^2$ is constant and the condition (2.2.12) means $h \leq 1$. The admissibility of the last metric of the lemma is an easy exercise left to the reader. □

Definition 2.2.19. Let g be a metric on $\mathbb{R}^{2n}$. The metric g is said to be *symplectic* whenever for all $X \in \mathbb{R}^{2n}$, $g_X = g^\sigma_X$, where g^σ_X is defined in (4.4.18). We define the *symplectic intermediate metric* $g^\natural$ by

$$g^\natural_X = \sqrt{g_X \cdot g^\sigma_X}. \tag{2.2.21}$$

The metric $g^\natural$ is symplectic, according to (4.4.28) and, if (2.2.12) holds, we have $g_X \leq g^\natural_X = (g^\natural_X)^\sigma \leq g^\sigma_X$.

Assuming (2.2.12) for g we get from the lemma 4.4.25 that

$$g_X \leq \lambda_g(X)^{-1} g^\natural_X \leq g^\natural_X = (g^\natural_X)^\sigma \leq \lambda_g(X) g^\natural_X \leq g^\sigma_X. \tag{2.2.22}$$

The following very useful observation is due to J. Toft [137].

Proposition 2.2.20. *Let g be an admissible metric on $\mathbb{R}^{2n}$. Then the metric $g^\natural$ is admissible and there exists C, N such that, for all $X, Y \in \mathbb{R}^{2n}$,*

$$g_X \leq C g_Y \big(1 + (g^\natural_X \wedge g^\natural_Y)(Y - X)\big)^N. \tag{2.2.23}$$

Moreover if M is an admissible weight for g, it is also an admissible weight for $g^\natural$.

Proof. The properties (2.2.3) and (2.2.17) are satisfied for g: the constants occurring there will be denoted by r_0, C_0, N_0. Let X be given in $\mathbb{R}^{2n}$. According to the lemma 4.4.25, there exists a basis $(e_1, \ldots, e_n, \epsilon_1, \ldots, \epsilon_n)$ of $\mathbb{R}^{2n}$ such that

the matrix of the symplectic form σ is $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$,

the matrix of g_X is $\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$ with $h = \mathrm{diag}(h_1, \ldots, h_n)$, $(h_j \in (0, 1])$,

so that the matrix of g^σ_X is $\begin{pmatrix} h^{-1} & 0 \\ 0 & h^{-1} \end{pmatrix}$ and the matrix of $g^\natural_X$ is I_{2n}.

For $T \in \mathbb{R}^{2n}$, we shall write $T = \sum_{1 \leq j \leq n} (t_j e_j + \tau_j \epsilon_j)$, $T_j = t_j$ for $1 \leq j \leq n$ and $T_j = \tau_{j-n}$ for $n+1 \leq j \leq 2n$. For $Y \in \mathbb{R}^{2n}$, $r > 0$ we define

$$J = \{j \in \{1, \ldots, 2n\}, h_j |X_j - Y_j|^2 \leq r^2\}, \quad Z_j = \begin{cases} Y_j, & \text{if } j \in J, \\ X_j, & \text{if } j \notin J. \end{cases}$$

We have $g_X(X - Z) = \sum_{j \in J} h_j |X_j - Y_j|^2 \leq 2nr^2$ and if $2nr^2 \leq r_0^2$, we obtain that the ratios $g_X(S)/g_Z(S)$ (and thus from the remark 4.4.24, $g^\sigma_X(S)/g^\sigma_Z(S)$) belong

to $[C_0^{-1}, C_0]$. Using the temperance of g, we get

$$\begin{aligned} g_X \le C_0 g_Z &\le C_0^2 g_Y \big(1 + g_Z^\sigma(Y-Z)\big)^{N_0} \\ &\le C_0^2 g_Y \big(1 + C_0 g_X^\sigma(Y-Z)\big)^{N_0} = C_0^2 g_Y \big(1 + C_0 \sum_{j\notin J} h_j^{-1} |Y_j - X_j|^2\big)^{N_0} \\ &\le C_0^2 g_Y \big(1 + C_0 \sum_{j\notin J} r^{-2} |Y_j - X_j|^4\big)^{N_0} \\ &\le C_0^2 g_Y \big(1 + C_0 r^{-2} g_X^\natural (Y-X)^2\big)^{N_0}, \end{aligned}$$

that is

$$g_X \le C_1 g_Y \big(1 + g_X^\natural (Y-X)\big)^{2N_0}. \tag{2.2.24}$$

We can repeat the previous argument, replacing g by g^σ (and also replacing g by a weight M, admissible for g):

$$\begin{aligned} g_X^\sigma \le C_0 g_Z^\sigma &\le C_0^2 g_Y^\sigma \big(1 + g_Z^\sigma(Y-Z)\big)^{N_0} \\ &\le C_0^2 g_Y^\sigma \big(1 + C_0 g_X^\sigma(Y-Z)\big)^{N_0} = C_0^2 g_Y^\sigma \big(1 + C_0 \sum_{j\notin J} h_j^{-1} |Y_j - X_j|^2\big)^{N_0} \\ &\le C_0^2 g_Y^\sigma \big(1 + C_0 \sum_{j\notin J} r^{-2} |Y_j - X_j|^4\big)^{N_0} \\ &\le C_0^2 g_Y^\sigma \big(1 + C_0 r^{-2} g_X^\natural (Y-X)^2\big)^{N_0}, \end{aligned}$$

which implies

$$g_Y \le C_1 g_X \big(1 + g_X^\natural (Y-X)\big)^{2N_0}. \tag{2.2.25}$$

Let X, Y be given in $\mathbb{R}^{2n}$. There exists $T \in \mathbb{R}^{2n}$ such that

$$(g_X^\natural \wedge g_Y^\natural)(X-Y) = 2g_X^\natural (X-T) + 2g_Y^\natural (T-Y)$$

and we have thus from (2.2.24), (2.2.25),

$$g_X \le C_1 g_T \big(1 + g_X^\natural (X-T)\big)^{2N_0}, \quad g_T \le C_1 g_Y \big(1 + g_Y^\natural (T-Y)\big)^{2N_0}.$$

providing $g_X \le C_2 g_Y \big(1 + g_X^\natural (X-T) + g_Y^\natural (T-Y)\big)^{4N_0}$, which is (2.2.23). On the other hand, the metric $g^\natural$ is slowly varying since it is larger than the slowly varying g and $g^\natural = \sqrt{g \cdot g^\sigma}$; the latter formula and the now proven (2.2.23) give the temperance of $g^\natural$ which also satisfies trivially (2.2.12). The proof of the proposition is complete. □

The following lemma will be useful in the sequel.

Lemma 2.2.21. *Let g be an admissible metric on $\mathbb{R}^{2n}$ and let G be a slowly varying metric on $\mathbb{R}^{2n}$. We assume that $G = \mu g$ is conformal to g, with $\mu \le \lambda_g$. Then the metric G is admissible.*

Proof. We have $G^\sigma = \mu^{-1} g^\sigma \geq \mu^{-1}\lambda_g^2 g \geq \mu g = G$, so that G satisfies (2.2.12) and we get also $G^\natural = g^\natural$. Since we have assumed that G is slowly varying, we may assume to prove (2.2.17), that for X, Y given in $\mathbb{R}^{2n}$, $G_Y(Y-X) \geq r_0^2$ with some positive r_0. Using now (2.2.22) for G (for which (2.2.12) holds), we get, since $G \leq \lambda_G^{-1} G^\natural$,

$$g_Y^\natural(Y-X) = G_Y^\natural(Y-X) \geq \lambda_G(Y) r_0^2 = r_0^2 \lambda_g(Y)\mu(Y)^{-1}.$$

Then for $T \in \mathbb{R}^{2n}$, using the temperance of g (which implies (2.2.23)), we get

$$\frac{G_X(T)}{G_Y(T)} = \frac{\mu(X)}{\mu(Y)}\frac{g_X(T)}{g_Y(T)} \leq \frac{\lambda_g(X)}{r_0^2\lambda_g(Y)} g_Y^\natural(Y-X) C\big(1 + (g_X^\natural \wedge g_Y^\natural)(Y-X)\big)^N,$$

and the rhs of this inequality is bounded above by a power of $1 + g_X^\natural(X-Y)$, yielding (2.2.17) and the lemma. □

Lemma 2.2.22. *Let g be an admissible metric on $\mathbb{R}^{2n}$ and m be an admissible weight for g. Let $f \in C^\infty(\mathbb{R}_+^*; \mathbb{R}_+^*)$ such that*

$$\forall r \in \mathbb{N}, \quad \sup_{t>0} |f^{(r)}(t)| f(t)^{-1} t^r < \infty. \tag{2.2.26}$$

Then there exists $C > 0$ and $\tilde{m} \in S(m, g)$ such that $\forall X$, $C^{-1} m(X) \leq \tilde{m}(X) \leq C m(X)$, $f(\tilde{m})$ is an admissible weight for g and $f(\tilde{m}) \in S(f(\tilde{m}), g)$.

N.B. The assumption on f is satisfied in particular for the function $t \mapsto t^\theta$ where $\theta \in \mathbb{R}$.

Proof. Remark 2.2.8 gives the existence of $\tilde{m}$ and $C > 0$ satisfying $\tilde{m} \in S(m, g)$ and (2.2.11), which gives also that $\tilde{m}$ is admissible for g. For $T \in \mathbb{R}^{2n}$, we check with $M = f(\tilde{m})$, using the Faà di Bruno formula (see e.g. section 4.3.1),

$$M^{(k)}(X) T^k = k! \sum_{\substack{k_1+\cdots+k_r=k \\ 1\leq r\leq k,\ k_j \geq 1}} \frac{1}{r!} f^{(r)}(\tilde{m}) \frac{1}{k_1! \ldots k_r!} \tilde{m}^{(k_1)} T^{k_1} \ldots m^{(k_r)} T^{k_r},$$

which entails, thanks to (2.2.26),

$$|M^{(k)}(X) T^k| \leq C \sum_{1\leq r\leq k} f(\tilde{m}) \tilde{m}(X)^{-r} m(X)^r g_X(T)^{k/2} \leq k C g_X(T)^{k/2} M(X),$$

which is the sought result. □

2.2.6 The main distance function

Definition 2.2.23. Let g be an admissible metric on $\mathbb{R}^{2n}$ (see the definition 2.2.15). We define, for $X, Y \in \mathbb{R}^{2n}$ and $r > 0$, using the notations (4.4.18), (4.4.23), (2.2.8),

$$\delta_r(X, Y) = 1 + (g_X^\sigma \wedge g_Y^\sigma)(U_{X,r} - U_{Y,r}). \tag{2.2.27}$$

Lemma 2.2.24. *Let g and δ_r be as above, with* (2.2.16) *satisfied with N_0. Then, there exist some positive constants $(C_j)_{1\le j\le 5}, (N_j)_{1\le j\le 2}$ and $r_0>0$ such that, for $0<r\le r_0$, $\forall X,Y,T\in\mathbb{R}^{2n}$,*

$$\begin{aligned}\delta_r(X,Y)&\le C_1\Big(1+\min\big(g_X^\sigma(U_{X,r}-U_{Y,r}),g_Y^\sigma(U_{X,r}-U_{Y,r})\big)\Big)\\&\le C_1\Big(1+\max\big(g_X^\sigma(U_{X,r}-U_{Y,r}),g_Y^\sigma(U_{X,r}-U_{Y,r})\big)\Big)\\&\le C_2\delta_r(X,Y)^{N_0+1},\end{aligned}\tag{2.2.28}$$

$$\frac{g_X(T)}{g_Y(T)}\le C_3\delta_r(X,Y)^{N_0},\quad \sup_{X\in\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\delta_r(X,Y)^{-N_1}|g_Y|^{1/2}dY<\infty,\tag{2.2.29}$$

$$\text{and for } g_X(Y-X)\ge C_4r^2,\quad \lambda_g(X)+\lambda_g(Y)\le C_5\delta_r(X,Y)^{N_2}.\tag{2.2.30}$$

Proof. We note that from the second inequality in (4.4.26),

$$\delta_r(X,Y)\le 1+2\min\big(g_X^\sigma(U_{X,r}-U_{Y,r}),g_Y^\sigma(U_{X,r}-U_{Y,r})\big),$$

and since g is temperate, for $X'\in U_{X,r}, Y'\in U_{Y,r}$, we have, for r small enough,

$$\begin{aligned}g_X^\sigma\wedge g_Y^\sigma\sim g_{X'}^\sigma\wedge g_{Y'}^\sigma\gtrsim g_{X'}^\sigma\big(1+(g_{X'}^\sigma\wedge g_{Y'}^\sigma)(X'-Y')\big)^{-N_0}\\\sim g_X^\sigma\big(1+(g_X^\sigma\wedge g_Y^\sigma)(X'-Y')\big)^{-N_0}\end{aligned}$$

which implies $g_X^\sigma\wedge g_Y^\sigma\gtrsim g_X^\sigma\delta_r(X,Y)^{-N_0}$ and thus $g_X^\sigma(U_{X,r}-U_{Y,r})\lesssim\delta_r(X,Y)^{N_0+1}$, yielding (2.2.28); moreover, with the same notations, we have

$$\frac{g_X}{g_Y}\sim\frac{g_{X'}}{g_{Y'}}\lesssim\big(1+(g_{X'}^\sigma\wedge g_{Y'}^\sigma)(X'-Y')\big)^{N_0}\sim\big(1+(g_X^\sigma\wedge g_Y^\sigma)(X'-Y')\big)^{N_0},$$

which gives the first inequality in (2.2.29). Still with the same notations, we have

$$\begin{aligned}1+g_X(X-Y)&\le 1+3g_X(X-X')+3g_X(X'-Y')+3g_X(Y'-Y)\\&\lesssim\delta_r(X,Y)^{N_0}\big(1+g_X(X-X')+g_X(X'-Y')+g_Y(Y'-Y)\big)\\&\lesssim\delta_r(X,Y)^{N_0}\big(1+g_X^\sigma(X'-Y')\big)\\&\lesssim\delta_r(X,Y)^{2N_0}\big(1+(g_X^\sigma\wedge g_Y^\sigma)(X'-Y')\big)\\&\lesssim\delta_r(X,Y)^{2N_0+1},\end{aligned}$$

and since $\frac{|g_Y|^{1/2}}{|g_X|^{1/2}}\lesssim\delta_r(X,Y)^{nN_0}$ we obtain the integrability property in (2.2.29). We have also from (2.2.14)

$$\begin{aligned}\lambda_g(X)^2g_X(X'-Y')\le g_X^\sigma(X'-Y')\lesssim\delta_r(X,Y)^{N_0}(g_X^\sigma\wedge g_Y^\sigma)(X'-Y')\\\le\delta_r(X,Y)^{N_0+1}.\end{aligned}$$

As a consequence, for C large enough, if $g_X(X-Y)\ge Cr^2$ we get $g_X(X'-Y')\ge r^2$ (otherwise $g_X(X'-Y')<r^2$ and $g_X\sim g_Y$ with $g_X(X-Y)\le C_0r^2$). The previous inequality gives $\lambda_g(X)\lesssim\delta_r(X,Y)^{N_1}$ and (2.2.30). □

Lemma 2.2.25. *Let g be an admissible metric on $\mathbb{R}^{2n}$ and M be an admissible weight for g (cf. definition 2.2.15). There exist $r_0 > 0, C > 0, N_0 \geq 0$, such that for $0 < r \leq r_0$, $X, Y \in \mathbb{R}^{2n}$,*

$$\frac{M(X)}{M(Y)} \leq C\delta_r(X,Y)^{N_0}. \tag{2.2.31}$$

Proof. Since M is admissible for g, it satisfies (2.2.18) and using that M is slowly varying for g and g is slowly varying, for r small enough, we have for $X' \in U_{X,r}, Y' \in U_{Y,r}$,

$$\frac{M(X)}{M(Y)} \sim \frac{M(X')}{M(Y')} \leq C\big(1 + (g^\sigma_{X'} \wedge g^\sigma_{Y'})(X' - Y')\big)^N \sim \delta_r(X,Y)^N,$$

which completes the proof. □

The remaining part of that section is a small variation on the definition 2.2.23, and the lemmas 2.2.24, 2.2.25, based upon the property given by the proposition 2.2.20: that result allows us to replace $\delta_r(X,Y)$ by a smaller quantity $\Delta_r(X,Y)$ defined below while keeping the same properties (2.2.29).

Definition 2.2.26. Let g be an admissible metric on $\mathbb{R}^{2n}$ (see the definition 2.2.15). We define, for $X, Y \in \mathbb{R}^{2n}$ and $r > 0$, using the notations (2.2.21), (4.4.18), (4.4.23), (2.2.8),

$$\Delta_r(X,Y) = 1 + (g^\natural_X \wedge g^\natural_Y)(U_{X,r} - U_{Y,r}). \tag{2.2.32}$$

Lemma 2.2.27. *Let g and Δ_r be as above, with (2.2.23) satisfied with N_0. Then, there exist some positive constants $(C_j)_{1\leq j\leq 5}, (N_j)_{1\leq j\leq 2}$ and $r_0 > 0$ such that, for $0 < r \leq r_0$, $\forall X, Y, T \in \mathbb{R}^{2n}$,*

$$\begin{aligned}\Delta_r(X,Y) &\leq C_1\Big(1 + \min\big(g^\natural_X(U_{X,r} - U_{Y,r}), g^\natural_Y(U_{X,r} - U_{Y,r})\big)\Big)\\ &\leq C_1\Big(1 + \max\big(g^\natural_X(U_{X,r} - U_{Y,r}), g^\natural_Y(U_{X,r} - U_{Y,r})\big)\Big)\\ &\leq C_2\Delta_r(X,Y)^{N_0+1},\end{aligned} \tag{2.2.33}$$

$$\frac{g_X(T)}{g_Y(T)} \leq C_3\Delta_r(X,Y)^{N_0}, \quad \sup_{X\in\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \Delta_r(X,Y)^{-N_1}|g_Y|^{1/2}dY < \infty, \tag{2.2.34}$$

$$\text{and for } g_X(Y - X) \geq C_4 r^2, \quad \lambda_g(X) + \lambda_g(Y) \leq C_5\Delta_r(X,Y)^{N_2}. \tag{2.2.35}$$

Proof. We note that from the second inequality in (4.4.26),

$$\Delta_r(X,Y) \leq 1 + 2\min\big(g^\natural_X(U_{X,r} - U_{Y,r}), g^\natural_Y(U_{X,r} - U_{Y,r})\big),$$

and since g is slowly varying and satisfies (2.2.23), for $X' \in U_{X,r}, Y' \in U_{Y,r}$, we have, for r small enough,

$$g_X^\natural \wedge g_Y^\natural \sim g_{X'}^\natural \wedge g_{Y'}^\natural \gtrsim g_{X'}^\natural \big(1 + (g_{X'}^\natural \wedge g_{Y'}^\natural)(X' - Y')\big)^{-N_0} \\ \sim g_X^\natural \big(1 + (g_X^\natural \wedge g_Y^\natural)(X' - Y')\big)^{-N_0}$$

which implies $g_X^\natural \wedge g_Y^\natural \gtrsim g_X^\natural \Delta_r(X,Y)^{-N_0}$ and thus

$$g_X^\natural (U_{X,r} - U_{Y,r}) \lesssim \Delta_r(X,Y)^{N_0+1},$$

yielding (2.2.33); moreover, with the same notations, we have

$$\frac{g_X}{g_Y} \sim \frac{g_{X'}}{g_{Y'}} \lesssim \big(1 + (g_{X'}^\natural \wedge g_{Y'}^\natural)(X' - Y')\big)^{N_0} \sim \big(1 + (g_X^\natural \wedge g_Y^\natural)(X' - Y')\big)^{N_0},$$

which gives the first inequality in (2.2.34). Still with the same notations, we have

$$\begin{aligned} 1 + g_X(X - Y) &\le 1 + 3g_X(X - X') + 3g_X(X' - Y') + 3g_X(Y' - Y) \\ &\lesssim \Delta_r(X,Y)^{N_0}\big(1 + g_X(X - X') + g_X(X' - Y') + g_Y(Y' - Y)\big) \\ &\lesssim \Delta_r(X,Y)^{N_0}\big(1 + g_X^\natural(X' - Y')\big) \\ &\lesssim \Delta_r(X,Y)^{2N_0}\big(1 + (g_X^\natural \wedge g_Y^\natural)(X' - Y')\big) \\ &\lesssim \Delta_r(X,Y)^{2N_0+1}, \end{aligned}$$

and since $\frac{|g_Y|^{1/2}}{|g_X|^{1/2}} \lesssim \Delta_r(X,Y)^{nN_0}$ we obtain the integrability property in (2.2.34). We have also from (2.2.22)

$$\lambda_g(X) g_X(X' - Y') \le g_X^\natural(X' - Y') \lesssim \Delta_r(X,Y)^{N_0}(g_X^\natural \wedge g_Y^\natural)(X' - Y') \\ \le \Delta_r(X,Y)^{N_0+1}.$$

As a consequence, for C large enough, if $g_X(X-Y) \ge Cr^2$ we get $g_X(X'-Y') \ge r^2$ (otherwise $g_X(X'-Y') < r^2$ and $g_X \sim g_Y$ with $g_X(X-Y) \le C_0 r^2$). The previous inequality gives $\lambda_g(X) \lesssim \Delta_r(X,Y)^{N_1}$ and (2.2.35). □

Lemma 2.2.28. *Let g be an admissible metric on $\mathbb{R}^{2n}$ and M be an admissible weight for g (cf. definition 2.2.15). There exist $r_0 > 0, C > 0, N_0 \ge 0$, such that for $0 < r \le r_0$, $X, Y \in \mathbb{R}^{2n}$,*

$$\frac{M(X)}{M(Y)} \le C\Delta_r(X,Y)^{N_0}. \tag{2.2.36}$$

Proof. Since M is admissible for g, it is an admissible weight for $g^\natural$, thanks to the proposition 2.2.20, and using that M is slowly varying for g and g is slowly varying, for r small enough, we have for $X' \in U_{X,r}, Y' \in U_{Y,r}$,

$$\frac{M(X)}{M(Y)} \sim \frac{M(X')}{M(Y')} \le C\big(1 + (g_{X'}^\natural \wedge g_{Y'}^\natural)(X' - Y')\big)^N \sim \Delta_r(X,Y)^N,$$

which completes the proof. □

2.3 General principles of pseudodifferential calculus

We need now to attack the hardest part of our task, namely, we should provide an argument to prove that the composition formula (2.1.23) holds for $a \in S(m_1, g), b \in S(m_2, g)$ and that $a\sharp b \in S(m_1 m_2, g)$. Our idea is to use the partition of unity of the proposition 2.2.9 above so that the problem is reduced to handle

$$\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} (a_Y \sharp b_Z)|g_Y|^{1/2}|g_Z|^{1/2} dY dZ.$$

In fact, to show a convergence result for that integral, it will be enough to study carefully the compositions $a_Y\sharp b_Z$. The first thing that we realize is that $a_Y\sharp b_Z$ is not compactly supported, since it is an entire function on $\mathbb{C}^{2n}$ (see the remark 2.2.12). Nevertheless, we should expect some localization properties of some sort for that symbol. In particular, if $Y = Z$, that symbol should be "small" far away from Y. In the next section, we give a precise quantitative version of these assertions.

2.3.1 Confinement estimates

Definition 2.3.1 (Confined symbols)**.** Let g be a positive definite quadratic form on $\mathbb{R}^{2n}$ such that $g \leq g^\sigma$. Let a be a smooth function in $\mathbb{R}^{2n}$ and $U \subset \mathbb{R}^{2n}$. We shall say that a is g-confined in U if $\forall k \in \mathbb{N}, \forall N \geq 0, \exists C_{k,N}$ such that, for all $X, T \in \mathbb{R}^{2n}$

$$|a^{(k)}(X)T^k| \leq C_{k,N} g(T)^{k/2}\big(1 + g^\sigma(X-U)\big)^{-N/2}. \tag{2.3.1}$$

We shall set

$$\|a\|^{(k,N)}_{g,U} = \sup_{\substack{X,T\in\mathbb{R}^{2n}\\ g(T)=1}} |a^{(k)}(X)T^k|\big(1 + g^\sigma(X-U)\big)^{N/2} \tag{2.3.2}$$

$$\text{and} \qquad \|\!|a|\!\|^{(l)}_{g,U} = \max_{k\leq l} \|a\|^{(k,l)}_{g,U}. \tag{2.3.3}$$

When U is compact, any smooth function supported in U is confined in U. The set of the g-confined symbols in U will be denoted by $\mathrm{Conf}(g, U)$.

N.B. Let us note that it is an easy point of commutative algebra to check that $T_1 \otimes \cdots \otimes T_k$ is a linear combination of powers $S \otimes \cdots \otimes S$ with k factors (see 4.2.4 in the appendix); this means in particular that the estimates above are equivalent to estimates controlling $a^{(k)}(X)T_1 \otimes \cdots \otimes T_k$. We see also that, as a set, $\mathrm{Conf}(g, U)$ coincides with the Schwartz class on $\mathbb{R}^{2n}$. Naturally, the set of semi-norms (2.3.3) will be most important for our purpose as well as uniform estimates with respect to g.

2.3.2 Biconfinement estimates

Our most important technical result is the following.

Theorem 2.3.2. *Let g_1, g_2 be two positive definite quadratic forms on $\mathbb{R}^{2n}$ such that $g_j \leq g_j^\sigma$ and let $a_j, j = 1, 2$ be g_j-confined in U_j, a g_j-ball of radius ≤ 1. Then, for all k, l, for all $X, T \in \mathbb{R}^{2n}$,*

$$|(a_1 \sharp a_2)^{(l)} T^l| \leq A_{k,l}(g_1+g_2)(T)^{l/2}\Big(1+(g_1^\sigma \wedge g_2^\sigma)(X-U_1)+(g_1^\sigma \wedge g_2^\sigma)(X-U_2)\Big)^{-k/2}, \tag{2.3.4}$$

with $A_{k,l} = \gamma(n,k,l)\|a_1\|^{(m)}_{g_1,U_1}\|a_2\|^{(m)}_{g_2,U_2}$, $\quad m = 2n+1+k+l$.

Corollary 2.3.3. *Under the assumptions of the previous theorem, for all nonnegative integers (p, N), there exists $q = q(p, N)$ and $C = C(p, N)$, such that*

$$\|a_1 \sharp a_2\|^{(p)}_{g_1+g_2,U_1} + \|a_1 \sharp a_2\|^{(p)}_{g_1+g_2,U_2} \leq C\|a_1\|^{(q)}_{g_1,U_1}\|a_2\|^{(q)}_{g_2,U_2}\Big(1+(g_1^\sigma \wedge g_2^\sigma)(U_1-U_2)\Big)^{-N}, \tag{2.3.5}$$

$$\|a_1 \sharp a_2\|^{(p)}_{g_1,U_1} + \|a_1 \sharp a_2\|^{(p)}_{g_2,U_2} \leq C\|a_1\|^{(q)}_{g_1,U_1}\|a_2\|^{(q)}_{g_2,U_2} \times \big(1+\sup_{g_1(T)=1} g_2(T) + \sup_{g_2(T)=1} g_1(T)\big)^p\Big(1+(g_1^\sigma \wedge g_2^\sigma)(U_1-U_2)\Big)^{-N}. \tag{2.3.6}$$

Proof that the theorem implies the corollary. The first inequality follows from the previous theorem by applying the triangle inequality to the quadratic form $g_1^\sigma \wedge g_2^\sigma$. We note that (2.3.4) implies

$$|(a_1 \sharp a_2)^{(l)} T^l| \leq g_1(T)^{l/2} A_{k,l}\big(1+\sup_{g_1(T)=1} g_2(T)\big)^{l/2} \times \Big(1+(g_1^\sigma \wedge g_2^\sigma)(X-U_1)+(g_1^\sigma \wedge g_2^\sigma)(X-U_2)\Big)^{-k/2},$$

and thus, using the inequality (4.4.37), we get

$$|(a_1 \sharp a_2)^{(l)} T^l| \leq g_1(T)^{l/2} A_{k_1+k_2,l}\big(1+\sup_{g_1(T)=1} g_2(T)\big)^{l/2} \times\Big(1+(g_1^\sigma \wedge g_2^\sigma)(U_1-U_2)\Big)^{-k_1/2}\big(1+g_1^\sigma(X-U_1)\big)^{-k_2/2}2^{k_2/2}\big(1+\sup_{g_1^\sigma(T)=1} g_2^\sigma(T)\big)^{k_2/2},$$

and[6]

$$|(a_1 \sharp a_2)^{(l)} T^l| g_1(T)^{-l/2}\big(1+g_1^\sigma(X-U_1)\big)^{k_2/2} \leq A_{k_1+k_2,l}2^{k_2/2} \times \big(1+\sup_{g_1(T)=1} g_2(T)\big)^{l/2}\Big(1+(g_1^\sigma \wedge g_2^\sigma)(U_1-U_2)\Big)^{-k_1/2}\big(1+\sup_{g_2(T)=1} g_1(T)\big)^{k_2/2},$$

[6] We use here the implication (with $\alpha_{12} > 0$) $g_1 \leq \alpha_{12} g_2 \Longrightarrow g_2^\sigma \leq \alpha_{12} g_1^\sigma$, which follows from the remark 4.4.24.

yielding

$$\|a_1\sharp a_2\|^{(p)}_{g_1,U_1} \le \max_{l\le p} A_{k_1+p,l}2^{p/2} \times \Big(1+(g_1^\sigma\wedge g_2^\sigma)(U_1-U_2)\Big)^{-k_1/2}\big(1+\sup_{g_1(T)=1} g_2(T)+\sup_{g_2(T)=1} g_1(T)\big)^p$$

which gives the second inequality of the corollary, completing its proof. □

Comment. If for instance each a_j is supported in the ball U_j, the composition $a_1\sharp a_2$ is "confined" in both balls U_1, U_2 in the precise quantitative sense given by the decay estimates (2.3.4). In particular, when $a_1 = a_2 = a$ is supported in the g-ball U, the composition $a\sharp a$ is no longer supported in U but is "confined" in U. Obviously the class of functions supported in U is not stable by composition (the operation $\sharp$), but the class of confined functions (in fact the Schwartz class) is indeed stable by composition.

Proof of the theorem 2.3.2. We start from the formula (2.1.18)

$$(a_1\sharp a_2)(X) = \iint a_1(Y_1)a_2(Y_2)e^{-4i\pi[Y_1-X,Y_2-X]}dY_1dY_2 2^{2n}$$

and for Y_1, X given in $\mathbb{R}^{2n}$ we define $\theta = \theta_{Y_1,X} \in \mathbb{R}^{2n}$ such that

$$g_2(\theta) = 1, \quad [\theta, Y_1 - X] = g_2^\sigma(Y_1-X)^{1/2} \qquad \text{(see (4.4.21))}.$$

Then for $k \in \mathbb{N}$, since θ does not depend on Y_2,

$$\Big(\frac{1}{2}\langle\theta, D_{Y_2}\rangle + 1\Big)^k(\exp -4i\pi[Y_1-X, Y_2-X]) = \big(1+g_2^\sigma(Y_1-X)^{1/2}\big)^k \exp -4i\pi[Y_1-X,Y_2-X].$$

Consequently, we have

$$(a_1\sharp a_2)(X) = \iint a_1(Y_1)\sum_{0\le l\le k} a_2^{[l]}(Y_2)e^{-4i\pi[Y_1-X,Y_2-X]}\big(1+g_2^\sigma(Y_1-X)^{1/2}\big)^{-k}dY_1dY_2 2^{2n}$$

with $a_2^{[l]} = (-4i\pi)^{-l}\binom{k}{l}a_2^{(l)}\theta^l$. As a consequence, for all k, k_1, N, we get

$$|(a_1\sharp a_2)(X)| \le \iint \|a_1\|^{(0,k_1)}_{g_1,U_1}\big(1+g_1^\sigma(Y_1-U_1)\big)^{-k_1/2}\big(1+g_2^\sigma(Y_1-X)\big)^{-k/2} \|a_2\|^{(k,N)}_{g_2,U_2}\big(1+g_2^\sigma(Y_2-U_2)\big)^{-N/2}dY_1dY_2 2^{2n+k}. \quad (2.3.7)$$

We notice then that

$$\big(1+g_1^\sigma(Y_1-U_1)\big)^{k_1/2}\big(1+g_2^\sigma(Y_1-X)\big)^{k_1/2} \geq \big(1+g_1^\sigma(Y_1-U_1)+g_2^\sigma(Y_1-X)\big)^{k_1/2} \geq 2^{-k_1/2}\big(1+(g_1^\sigma\wedge g_2^\sigma)(X-U_1)\big)^{k_1/2},$$

so that for $k=k_1+k_2$, the estimate (2.3.7) gives

$$|(a_1\sharp a_2)(X)| \leq \|a_1\|_{g_1,U_1}^{(0,k_1)}\|a_2\|_{g_2,U_2}^{(k,N)}\big(1+(g_1^\sigma\wedge g_2^\sigma)(X-U_1)\big)^{-k_1/2} \iint \big(1+g_2^\sigma(Y_1-X)\big)^{-k_2/2}\big(1+g_2^\sigma(Y_2-U_2)\big)^{-N/2}dY_1dY_2 2^{2n+k+\frac{k_1}{2}}.$$

Since U_2 is a g_2-ball of radius ≤ 1 with center X_2, we have for $Z\in U_2$,

$$1+g_2(Y_2-X_2) \leq 1+2g_2(Y_2-Z)+2g_2(Z-X_2) \leq 3\big(1+g_2(Y_2-Z)\big) \leq 3\big(1+g_2^\sigma(Y_2-Z)\big)$$

which implies $1+g_2(Y_2-X_2)\leq 3\big(1+g_2^\sigma(Y_2-U_2)\big)$. As a consequence, we get

$$|(a_1\sharp a_2)(X)| \leq \|a_1\|_{g_1,U_1}^{(0,k_1)}\|a_2\|_{g_2,U_2}^{(k,N)}\big(1+(g_1^\sigma\wedge g_2^\sigma)(X-U_1)\big)^{-k_1/2} \iint \big(1+g_2^\sigma(Y_1-X)\big)^{-k_2/2}\big(1+g_2(Y_2-X_2)\big)^{-N/2}dY_1dY_2 2^{2n+k+\frac{k_1}{2}}3^{N/2}.$$

Since the product of the determinants of g_2 and g_2^σ is 1, choosing $k_2=N=2n+1$, we obtain

$$|(a_1\sharp a_2)(X)| \leq \|a_1\|_{g_1,U_1}^{(0,k_1)}\|a_2\|_{g_2,U_2}^{(2n+1+k_1,2n+1)}\big(1+(g_1^\sigma\wedge g_2^\sigma)(X-U_1)\big)^{-k_1/2}C(k_1,n).$$

Analogously, we get

$$|(a_1\sharp a_2)(X)| \leq \|a_2\|_{g_1,U_1}^{(0,k_1)}\|a_1\|_{g_2,U_2}^{(2n+1+k_1,2n+1)}\big(1+(g_1^\sigma\wedge g_2^\sigma)(X-U_2)\big)^{-k_1/2}C(k_1,n).$$

Taking the minimum of the rhs in the two previous inequalities, we obtain with $k_1=k$,

$$|(a_1\sharp a_2)(X)| \leq C(k,n)2^{k/2}\|\!|a_1|\!\|_{g_1,U_1}^{(2n+1+k)}\|\!|a_2|\!\|_{g_2,U_2}^{(2n+1+k)} \times\big(1+(g_1^\sigma\wedge g_2^\sigma)(X-U_1)+(g_1^\sigma\wedge g_2^\sigma)(X-U_2)\big)^{-k/2}, \quad (2.3.8)$$

which is indeed (2.3.4) for $l=0$. Writing (2.1.18) as

$$(a_1\sharp a_2)(X)=2^{2n}\iint e^{-4i\pi[Y_1,Y_2]}a_1(Y_1+X)a_2(Y_2+X)dY_1dY_2,$$

we realize that if $D=\partial_{X_j}$, we have

$$D(a_1\sharp a_2)=Da_1\sharp a_2+a_1\sharp Da_2, \quad (2.3.9)$$

so that applying (2.3.8) to the previous formula gives (2.3.4) for $l\geq 1$. The proof of the theorem 2.3.2 is complete. □

We need now to get the analogous result for the expansion of $a_1 \sharp a_2$.

Theorem 2.3.4. *Let g_1, g_2, a_1, a_2 as in Theorem 2.3.2. For $\nu \in \mathbb{N}$, we recall from the definitions* (2.1.26), (2.1.29),

$$r_\nu(a_1,a_2)(X) = a_1 \sharp a_2 - \sum_{0\le j<\nu} \frac{1}{j!}\big(i\pi[D_{X_1}, D_{X_2}]\big)^j (a_1 \otimes a_2)_{|_{diagonal}}.$$

Then, for all k, l, ν, for all $X, T \in \mathbb{R}^{2n}$, we have

$$\begin{aligned} |\big(r_\nu(a_1,a_2)\big)^{(l)} T^l| &\le A_{k,l,\nu}(g_1+g_2)(T)^{l/2}\Lambda_{1,2}^{-\nu} \\ &\times \Big(1 + (g_1^\sigma \wedge g_2^\sigma)(X-U_1) + (g_1^\sigma \wedge g_2^\sigma)(X-U_2)\Big)^{-k/2}, \quad (2.3.10) \end{aligned}$$

with $A_{k,l,\nu} = C(k,n,l,\nu)\|a_1\|_{g_1,U_1}^{(2n+1+k+l+\nu)} \|a_2\|_{g_2,U_2}^{(2n+1+k+l+\nu)}$ and

$$\Lambda_{1,2} = \inf_T \left(\frac{g_1^\sigma(T)}{g_2(T)}\right)^{1/2} = \inf_T \left(\frac{g_2^\sigma(T)}{g_1(T)}\right)^{1/2}. \quad (2.3.11)$$

Remark 2.3.5. First of all, note that we can get a result similar to the corollary 2.3.3: for all (p, N, ν), there exists $q = q(p,N,\nu)$ and $C = C(p,N,\nu)$, such that

$$\begin{aligned} &\|R_\nu(a_1,a_2)\|_{g_1+g_2,U_1}^{(p)} + \|R_\nu(a_1,a_2)\|_{g_1+g_2,U_2}^{(p)} \\ &\qquad \le C\|a_1\|_{g_1,U_1}^{(q)} \|a_2\|_{g_2,U_2}^{(q)} \Lambda_{1,2}^{-\nu}\Big(1 + (g_1^\sigma \wedge g_2^\sigma)(U_1-U_2)\Big)^{-N}. \quad (2.3.12) \end{aligned}$$

Next, let us give a proof of the identity (2.3.11). We have, using (4.4.22),

$$\frac{g_1^\sigma(T)}{g_2(T)} = \frac{[T,U_T]^2}{g_1(U_T)g_2(T)} \le \frac{g_2(T)g_2^\sigma(U_T)}{g_1(U_T)g_2(T)} = \frac{g_2^\sigma(U_T)}{g_1(U_T)}, \quad \text{with } U_T = g_1^{-1}\sigma T,$$

so that $\inf_T \frac{g_1^\sigma(T)}{g_2(T)} \le \inf_T \frac{g_2^\sigma(U_T)}{g_1(U_T)}$ and since the mapping $T \mapsto U_T$ is an isomorphism, we get (2.3.11).

Proof of the theorem. We have from (2.1.18),

$$\begin{aligned} (a_1\sharp a_2)(X) &= \iint a_1(X+Y_1)a_2(Y_2)e^{-4i\pi\langle \sigma Y_1, Y_2 - X\rangle} dY_1 dY_2 2^{2n} \\ &= \iint a_1(X+\frac{1}{2}\sigma^{-1}\Xi)a_2(Y_2)e^{-2i\pi\langle \Xi, Y_2 - X\rangle} d\Xi dY_2 \\ &= \int a_1(X+\frac{1}{2}\sigma^{-1}\Xi)\widehat{a_2}(\Xi)e^{2i\pi\langle \Xi, X\rangle} d\Xi, \quad (2.3.13) \end{aligned}$$

and[7] consequently,

$$
\begin{aligned}
(a_1\sharp a_2)(X) &= \sum_{0\le j<\nu} a_1^{(j)}(X)\frac{1}{j!2^j}\int(\sigma^{-1}\Xi)^j\widehat{a_2}(\Xi)e^{2i\pi\langle\Xi,X\rangle}d\Xi \\
&\quad+\int_0^1\frac{(1-\theta)^{\nu-1}}{(\nu-1)!2^\nu}\int a_1^{(\nu)}(X+\frac{\theta}{2}\sigma^{-1}\Xi)(\sigma^{-1}\Xi)^\nu\widehat{a_2}(\Xi)e^{2i\pi\langle\Xi,X\rangle}d\Xi d\theta \\
&= \sum_{0\le j<\nu} a_1^{(j)}(X)\frac{1}{j!2^j}(\sigma^{-1}D_X)^j a_2(X)+r_\nu(a_1,a_2)(X),
\end{aligned}
$$

which corresponds to the expansion of the theorem according to the remark 2.1.5 with

$$
r_\nu(a_1,a_2)(X)=\int_0^1\frac{(1-\theta)^{\nu-1}}{(\nu-1)!2^\nu}\int a_1^{(\nu)}(X+\frac{\theta}{2}\sigma^{-1}\Xi)(\sigma^{-1}\Xi)^\nu\widehat{a_2}(\Xi)e^{2i\pi\langle\Xi,X\rangle}d\Xi d\theta,
$$

so that

$$
\begin{aligned}
&r_\nu(a_1,a_2)(X)\\
&=\int_0^1\frac{(1-\theta)^{\nu-1}}{(\nu-1)!}\int a_1^{(\nu)}(X+\theta Z)Z^\nu\widehat{a_2}(2\sigma Z)e^{2i\pi\langle 2\sigma Z,X\rangle}dZd\theta 2^{2n}\\
&=\int_0^1\frac{(1-\theta)^{\nu-1}}{(\nu-1)!}\iint a_1^{(\nu)}(X+\theta Z)Z^\nu a_2(Y_2)e^{2i\pi\langle 2\sigma Z,X\rangle-2i\pi\langle Y_2,2\sigma Z\rangle}dZdY_2d\theta 2^{2n}\\
&=\int_0^1\frac{(1-\theta)^{\nu-1}}{(\nu-1)!}\iint a_1^{(\nu)}(X+\theta(Y_1-X))(Y_1-X)^\nu a_2(Y_2)\\
&\qquad\qquad e^{-4i\pi[Y_1-X,Y_2-X]}dY_1dY_2 2^{2n}d\theta.
\end{aligned}
$$

We consider now (using 4.4.22) $T=T_{Y_1,X}$ such that $g_2(T)=1$ and $[T,Y_1-X]=g_2^\sigma(Y_1-X)^{1/2}$. Then for $k\in\mathbb{N}$,

$$
\Big(\frac{1}{2}\langle T,D_{Y_2}\rangle+1\Big)^k e^{-4i\pi[Y_1-X,Y_2-X]}=e^{-4i\pi[Y_1-X,Y_2-X]}(1+g_2^\sigma(Y_1-X)^{1/2})^k.
$$

As a consequence we get

$$
\begin{aligned}
&|r_\nu(a_1,a_2)(X)|\\
&\le\iiint_0^1\frac{(1-\theta)^{\nu-1}}{(\nu-1)!}2^{2n+k}\|a_1^{(\nu)}\|_{g_1,U_1}^{(0,k_1)}\Big(1+g_1^\sigma\big(X+\theta(Y_1-X)-U_1\big)\Big)^{-k_1/2}\\
&\qquad g_1(Y_1-X)^{\nu/2}(1+g_2^\sigma(Y_1-X))^{-k/2}\\
&\qquad\qquad\|a_2\|_{g_2,U_2}^{(k,N)}\big(1+g_2^\sigma(Y_2-U_2\big)^{-N/2}dY_1dY_2d\theta.
\end{aligned}
$$

[7]The expression (2.3.13) is interesting of its own and can be written as

$$
(a_1\sharp a_2)(X)=(a_1(X+\frac{1}{2}\sigma^{-1}D_X)a_2)(X). \tag{2.3.14}
$$

Since $\theta \in [0,1]$, we have $g_2^\sigma(Y_1 - X) \geq g_2^\sigma(\theta(Y_1 - X))$ and for $k = k_1 + k_2 + \nu$ and $N = 2n + 1 = k_2$,

$$
\begin{aligned}
&|r_\nu(a_1, a_2)(X)| \\
&\quad \leq \iiint_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} 2^{2n+k} \, \|a_1\|_{g_1,U_1}^{(\nu,k_1)} \left(1 + g_1^\sigma\big(X + \theta(Y_1 - X) - U_1\big)\right)^{-k_1/2} \\
&\qquad \Lambda_{12}^{-\nu} g_2^\sigma(Y_1 - X)^{\nu/2} \big(1 + g_2^\sigma\big(\theta(Y_1 - X)\big)\big)^{-\frac{k_1}{2}} (1 + g_2^\sigma(Y_1 - X))^{-\frac{k_2+\nu}{2}} \\
&\qquad\qquad \|a_2\|_{g_2,U_2}^{(k,N)} \left(1 + g_2^\sigma(Y_2 - U_2)\right)^{-(2n+1)/2} dY_1 dY_2 d\theta.
\end{aligned}
$$

We use now that

$$
2g_1^\sigma(X + \theta(Y_1 - X) - U_1) + 2g_2^\sigma(\theta(X - Y_1)) \geq (g_1^\sigma \wedge g_2^\sigma)(X - U_1)
$$

and we obtain

$$
\begin{aligned}
&|r_\nu(a_1, a_2)(X)| \\
&\quad \leq C(k, n, \nu) \, \|a_1\|_{g_1,U_1}^{(\nu,k_1)} \, \|a_2\|_{g_2,U_2}^{(k,2n+1)} \, \Lambda_{12}^{-\nu} \big(1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1)\big)^{-k_1/2} \\
&\qquad \iint (1 + g_2^\sigma(Y_1 - X))^{-\frac{k_2}{2}} \big(1 + g_2^\sigma(Y_2 - U_2\big)^{-(2n+1)/2} dY_1 dY_2. \quad (2.3.15)
\end{aligned}
$$

Reasoning as in the proof of the theorem 2.3.2, we get (2.3.10) for $l = 0$, which is easy to generalize to $l \geq 1$. The proof of the theorem 2.3.4 is complete. □

The next lemma is an important remark which will be useful in the sequel.

Lemma 2.3.6. *Let g be a positive-definite quadratic form on $\mathbb{R}^{2n}$ such that $g \leq g^\sigma$, let $c \in S(1, g)$ and $\psi \in \mathrm{Conf}(g, U_r)$ where U_r is a g-ball with radius $r \leq 1$. Then $\psi\sharp c$ belongs to $\mathrm{Conf}(g, U_r)$, and the mapping $\mathrm{Conf}(g, U_r) \times S(1, g) \ni (\psi, c) \mapsto \psi\sharp c \in \mathrm{Conf}(g, U_r)$ is continuous.*

Proof. Using a partition of unity $1 = \int \phi_T |g|^{1/2} dT$ for the constant metric g, with $\phi_T \in S(1, g)$ supported in a g-ball with center T and radius r, we get, assuming that U_r has center 0, $\psi\sharp c = \int_{\mathbb{R}^{2n}} \psi\sharp(c\phi_T)|g|^{1/2} dT$, and the biconfinement estimates of the theorem 2.3.2 give

$$
|(\psi\sharp c)^{(l)}(X) S^l| \leq C_{k,l} g(S)^{l/2} \int_{\mathbb{R}^{2n}} \big(1 + g^\sigma(X - U_r) + g^\sigma(X - U_r - T)\big)^{-k/2} |g|^{1/2} dT,
$$

with $C_{k,l}$ seminorms of ψ, c, the inequalities

$$
\begin{aligned}
&\sup_{g(S)\leq 1} |(\psi\sharp c)^{(l)}(X) S^l| \big(1 + g^\sigma(X - U_r)\big)^{N/2} \\
&\qquad\qquad \leq C_{k+N,l} \int_{\mathbb{R}^{2n}} \big(1 + g^\sigma(X - U_r - T)\big)^{-k/2} |g|^{1/2} dT.
\end{aligned}
$$

For $g(T') \leq r^2 \leq 1$, we have

$$g(X-T) \leq 2g(X-T-T') + 2g(T') \leq 2g^\sigma(X-T-T') + 2r^2$$

and thus $1 + g(X-T) \leq 3 + 2g^\sigma(X-T-U_r)$ so that for $k = 2n+1$,

$$\sup_{g(S)\leq 1} |(\psi\sharp c)^{(l)}(X)S^l| \big(1+g^\sigma(X-U_r)\big)^{N/2}$$
$$\leq C_{2n+1+N,l} 3^{n+1/2} \int_{\mathbb{R}^{2n}} \big(1+g(X-T)\big)^{-n-\frac{1}{2}} |g|^{1/2} dT = C_{2n+1+N,l} 3^{n+1/2} \alpha(n),$$

completing the proof of the lemma. □

2.3.3 Symbolic calculus

The previous section dealt with the study of the composition formula (2.1.18) for two symbols in the Schwartz class attached to two constant metrics. That will be essentially enough to tackle the more interesting case where each symbol belongs to a symbol class attached to a general admissible metric.

Theorem 2.3.7. *Let g be an admissible metric on $\mathbb{R}^{2n}$ (see the definition 2.2.15), m_1, m_2 two admissible weights for g, $a_j \in S(m_j, g), j = 1, 2$. Then, for all $\nu \in \mathbb{N}$,*

$$a_1\sharp a_2 - \sum_{0\leq j<\nu} \frac{1}{j!} \big(i\pi[D_{X_1}, D_{X_2}]\big)^j (a_1 \otimes a_2)_{|\text{diagonal}} \in S(m_1 m_2 \lambda_g^{-\nu}, g). \quad (2.3.16)$$

In particular, we have $a_1\sharp a_2 = a_1 a_2 + \frac{1}{4i\pi}\{a_1, a_2\} + r_2, \quad r_2 \in S(m_1 m_2 \lambda_g^{-2}, g)$. The mappings $S(m_1, g) \times S(m_2, g) \ni (a_1, a_2) \mapsto a_1\sharp a_2 \in S(m_1 m_2, g)$ as well as (2.3.16) *are continuous for the Fréchet space structure given by the seminorms* (2.2.7).

Proof. We shall first use the partition of unity of the proposition 2.2.9 and consider

$$\iint (a_{1,Y_1}\sharp a_{2,Y_2})(X) |g_{Y_1}|^{1/2} |g_{Y_2}|^{1/2} dY_1 dY_2$$

and notice that, from Theorem 2.3.2, we get

$$|(a_{1,Y_1}\sharp a_{2,Y_2})(X)|$$
$$\leq C_{N,M} \delta_r(Y_1, Y_2)^{-N} \big(1 + (g^\sigma_{Y_1} \wedge g^\sigma_{Y_2})(X - U_{Y_1,r})\big)^{-M} m_1(Y_1) m_2(Y_2).$$

On the other hand, the lemmas 2.2.25–2.2.24 give

$$\frac{m_2(Y_2)}{m_2(Y_1)} \leq C_0 \delta_r(Y_1, Y_2)^{N_0}, \quad \frac{g^\sigma_{Y_1}}{g^\sigma_{Y_2}} \leq C_0 \delta_r(Y_1, Y_2)^{N_0},$$

so that

$$
\begin{aligned}
&|(a_{1,Y_1}\sharp a_{2,Y_2})(X)| \\
&\quad\lesssim \delta_r(Y_1,Y_2)^{-N+N_0}\big(1+(g^\sigma_{Y_1}\wedge g^\sigma_{Y_2})(X-U_{Y_1,r})\big)^{-M}(m_1m_2)(Y_1) \\
&\quad\lesssim (m_1m_2)(Y_1)\big(1+g^\sigma_{Y_1}(X-U_{Y_1,r})\big)^{-M}\delta_r(Y_1,Y_2)^{-N+N_0+N_0M} \\
&\quad\lesssim (m_1m_2)(Y_1)\delta_r(Y_1,X)^{-M}\delta_r(Y_1,Y_2)^{-N+N_0+N_0M} \\
&\quad\lesssim (m_1m_2)(X)\delta_r(Y_1,X)^{-M+N_0}\delta_r(Y_1,Y_2)^{-N+N_0+N_0M}.
\end{aligned}
$$

Choosing now M so that $M-N_0$ is large enough, then choosing N so that $N-N_0-N_0M$ is large enough, we can use the integrability condition in (2.2.29) to secure $|a_1\sharp a_2|\le m_1m_2$. To obtain the estimate for the derivatives is analogous (using (2.3.9)) as well as the estimate for the expansion, using Theorem 2.3.4. □

In fact the previous theorem can be extended to the following statement, quite useful in the third chapter. The reader may wait to hit that chapter to pay more attention to these results, which look more technical than the previous one.

Theorem 2.3.8. *Let g_1,g_2 be two admissible metrics on $\mathbb{R}^{2n}$, m_j an admissible weight for g_j, $a_j\in S(m_j,g_j), j=1,2$. According to the identity* (2.3.11)*, we define*

$$
\Lambda_{12}(X)=\inf_T\left(\frac{g^\sigma_{1,X}(T)}{g_{2,X}(T)}\right)^{1/2}=\inf_T\left(\frac{g^\sigma_{2,X}(T)}{g_{1,X}(T)}\right)^{1/2},
$$

and we assume that $\inf_{X\in\mathbb{R}^{2n}}\Lambda_{1,2}(X)\ge 1$. *We also assume that there exist $C>0, N\ge 0$ such that for all $X,Y,T\in\mathbb{R}^{2n}$,*

$$
\frac{g_{1,X}(T)+g_{2,X}(T)}{g_{1,Y}(T)+g_{2,Y}(T)}\le C\big(1+(g^\sigma_{1,X}\wedge g^\sigma_{2,X})(X-Y)\big)^N, \tag{2.3.17}
$$

$$
\frac{m_j(X)}{m_j(Y)}\le C\big(1+(g^\sigma_{1,X}\wedge g^\sigma_{2,X})(X-Y)\big)^N,\quad j=1,2. \tag{2.3.18}
$$

Then the metric $g_1\vee g_2=\frac12(g_1+g_2)$ is admissible, the product m_1m_2 is an admissible weight for $g_1\vee g_2$ and, for all $\nu\in\mathbb{N}$,

$$
a_1\sharp a_2-\sum_{0\le j<\nu}\frac{1}{j!}\big(i\pi[D_{X_1},D_{X_2}]\big)^j(a_1\otimes a_2)_{|_{\text{diagonal}}}\in S(m_1m_2\Lambda_{12}^{-\nu},g_1\vee g_2).
$$

Moreover, the term $\omega_j=\frac{1}{j!}\big(i\pi[D_{X_1},D_{X_2}]\big)^j(a_1\otimes a_2)_{|_{\text{diagonal}}}\in S(m_1m_2\Lambda_{12}^{-j},g_1\vee g_2)$.

Proof. The metric $g_1\vee g_2$ is obviously slowly varying since the condition

$$
(g_{1,X}\vee g_{2,X})(Y-X)\le r^2
$$

implies that for both $j=1,2$ we have $g_{j,X}(Y-X)\le 2r^2$, so that the slow variation of the g_j implies $g_{j,Y}\le Cg_{j,X}$ and $g_{1,Y}\vee g_{2,Y}\le C(g_{1,X}\vee g_{2,X})$ which gives the

slow variation of $g_1 \vee g_2$, according to the remark 2.2.2. We have also with $\lambda_j = \lambda_{g_j}$, the inequalities

$$g_j^\sigma \geq \lambda_j^2 g_j, \quad g_1^\sigma \geq \Lambda_{12}^2 g_2, \quad g_2^\sigma \geq \Lambda_{12}^2 g_1,$$

so that

$$g_1^\sigma \wedge g_2^\sigma \geq \frac{1}{2}\min(\Lambda_{12}^2, \lambda_1^2) g_1 + \frac{1}{2}\min(\Lambda_{12}^2, \lambda_2^2) g_2 \geq \min(\Lambda_{12}^2, \lambda_1^2, \lambda_2^2)(g_1 \vee g_2),$$

and the metric $g_1 \vee g_2$ satisfies the uncertainty principle (2.2.12) since $(g_1 \vee g_2)^\sigma = g_1^\sigma \wedge g_2^\sigma$ and $\min(\Lambda_{12}^2, \lambda_1^2, \lambda_2^2) \geq 1$. The temperance of $g_1 \vee g_2$ follows from the lemma 2.2.14 since our hypothesis (2.3.17) is precisely (2.2.17) for $g_1 \vee g_2$, which is now proven admissible. The weight m_j is slowly varying for g_j and a fortiori for $g_1 \vee g_2 \geq g_j/2$; it is also admissible for $g_1 \vee g_2$ since our hypothesis (2.3.18) is (2.2.19). Since $a_j \in S(m_j, g_j) \subset S(m_j, g_1 \vee g_2)$, $g_1 \vee g_2$ is admissible, m_j is an admissible weight for $g_1 \vee g_2$, we can apply the theorem 2.3.7 to obtain $a_1 \sharp a_2 \in S(m_1 m_2, g_1 \vee g_2)$. To obtain the last statements, we have to consider the integral formula issued of the proposition 2.2.9

$$(a_1 \sharp a_2)(X) = \iint (a_{1,Y_1} \sharp a_{2,Y_2})(X) |g_{Y_1}|^{1/2} |g_{Y_2}|^{1/2} dY_1 dY_2$$

and to apply the theorem 2.3.4 to $a_{1,Y_1} \sharp a_{2,Y_2}$ with the same proof as Theorem 2.3.7. The very last assertion of the theorem follows from the previous one since we have

$$\omega_\nu + \overbrace{a_1 \sharp a_2 - \sum_{0 \leq j \leq \nu} \omega_\nu}^{S(m_1 m_2 \Lambda_{12}^{-\nu-1}, g_1 \vee g_2)} = a_1 \sharp a_2 - \sum_{0 \leq j < \nu} \omega_j \in S(m_1 m_2 \Lambda_{12}^{-\nu}, g_1 \vee g_2),$$

which completes the proof. □

Remark 2.3.9. One important point of this theorem is that the quantity Λ_{12} can be much larger than $\min(\lambda_1, \lambda_2)$, as it is the case when $g_1 = g_1^\sigma$ so that $\lambda_1 = 1$ and $g_2 = \lambda_2^{-1} g_1$. In this situation, $\min(\lambda_1, \lambda_2) = 1$ whereas $\Lambda_{12} = \lambda_2^{1/2}$. The next corollary says a little bit more.

Corollary 2.3.10. *Let g_1, g_2 be two admissible metrics on $\mathbb{R}^{2n}$ such that $g_1^\natural = g_2^\natural$ (note that this is the case when g_2 and g_1 are conformal, cf. the remark 4.4.30). Let m_j be an admissible weight for g_j, $a_j \in S(m_j, g_j), j = 1, 2$. With the notation of the previous theorem, we have*

$$\Lambda_{12} \geq \sqrt{\lambda_1 \lambda_2}, \quad \textit{(with equality when } g_2 \textit{ and } g_1 \textit{ are conformal).}$$

The metric $g_1 \vee g_2 = \frac{1}{2}(g_1 + g_2)$ is admissible, the product $m_1 m_2$ is an admissible weight for $g_1 \vee g_2$ and, for all $\nu \in \mathbb{N}$, the last statement of the previous theorem holds.

Proof. We have, according to the lemma 4.4.25,

$$\lambda_j(X)g_{j,X} \leq g_{j,X}^\natural = \Gamma_X = \Gamma_X^\sigma \leq \lambda_j(X)^{-1} g_{j,X}^\sigma$$

so[8] that $\Lambda_{12} \geq \sqrt{\lambda_1\lambda_2}$. Using now the proposition 2.2.20, we check

$$\frac{g_{1,X}(T)+g_{2,X}(T)}{g_{1,Y}(T)+g_{2,Y}(T)} \leq C\big(1+(\Gamma_X \wedge \Gamma_Y)(X-Y)\big)^N$$

and we have a similar estimate for the ratios $m_j(X)/m_j(Y)$. All the assumptions of the theorem 2.3.8 are satisfied and we obtain the corollary. □

2.3.4 Additional remarks

In that section, we prove some technical results on confined symbols which shall be useful in the study of Sobolev spaces.

Lemma 2.3.11. *Let g be a positive-definite quadratic form on $\mathbb{R}^{2n}$ such that $g \leq g^\sigma$, U be a closed g-ball of radius r, and χ be a linear symplectic mapping of $\mathbb{R}^{2n}$. Then if $a \in$ Conf(g,U) (cf. definition 2.3.1), the function $a \circ \chi$ belongs to Conf$(\tilde{g},\tilde{U})$ with*

$$\tilde{g} = \chi^* g\chi, \quad \tilde{U} = \chi^{-1}U, \quad \|a\|_{g,U}^{(k,N)} = \|a\circ\chi\|_{\tilde{g},\tilde{U}}^{(k,N)}.$$

Note that $\tilde{g}$ is the positive-definite quadratic form given by the pull-back by χ of g and that $\tilde{U}$ is the $\tilde{g}$-ball of radius r.

Proof. We define $b(Y) = a(\chi Y)$ and we get

$$|b^{(k)}(Y)S^k| = |a^{(k)}(\chi Y)(\chi S)^k| \leq \|a\|_{g,U}^{k,N} g(\chi S)^{k/2}\Big(1+g^\sigma\big(\chi(Y-\tilde{U})\big)\Big)^{-N/2},$$

so that using (4.4.18) and $\chi^*\sigma\chi = \sigma$ (cf. (4.4.2)), we infer

$$(\tilde{g})^\sigma = \sigma^*(\tilde{g})^{-1}\sigma = \sigma^*\chi^{-1}g^{-1}(\chi^*)^{-1}\sigma = \chi^*\sigma^* g^{-1}\sigma\chi = \chi^* g^\sigma \chi,$$

so that $g^\sigma\big(\chi(Y-\tilde{U})\big) = (\tilde{g})^\sigma(Y-\tilde{U})$, implying that $b \in \mathrm{Conf}(\tilde{g},\tilde{U})$ and the inequality $\|a\circ\chi\|_{\tilde{g},\tilde{U}}^{k,N} \leq \|a\|_{g,U}^{k,N}$; the reverse inequality follows from the same proof with a replaced by b and χ by χ^{-1}. □

Lemma 2.3.12. *Let g be a positive-definite quadratic form on $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$ such that $g \leq g^\sigma$ and $g(x,\xi) = g(x,-\xi)$. Let U_r be a g-ball with radius $r \leq 1$. Then for all $t \in \mathbb{R}$, the operator $J^t = \exp 2i\pi t D_x \cdot D_\xi$ (see the lemma 4.1.2) is continuous from Conf(g,U_r) into itself and more precisely, for all $l,\nu \in \mathbb{N}$, there exists $l'(l,\nu) \in \mathbb{N}, C(l,\nu) > 0, \mu(l,\nu) \geq 0$, such that for all $t \in \mathbb{R}$,*

$$\|\!| J^t a - \sum_{0\leq j<\nu} (2i\pi t D_x \cdot D_\xi)^j a |\!\|_{g,U_r}^{(l)} \leq C(l,\nu)\|\!| a |\!\|_{g,U_r}^{(l')} \lambda^{-\nu}(1+|t|)^{\mu(l,\nu)}. \quad (2.3.19)$$

[8] When the g_j are conformal, $g_2 = \mu g_1$, we have $\Lambda_{12}^2 = \mu^{-1}\lambda_1^2 = \mu\lambda_2^2 \Longrightarrow \Lambda_{12} = \sqrt{\lambda_1\lambda_2}$.

Proof. Since $g(x,\xi) = g(x,-\xi)$, we have $g(x,\xi) = \gamma_1(x) + \gamma_2(\xi)$ where the γ_j are positive-definite quadratic forms on $\mathbb{R}^n$, identified with $n \times n$ symmetric matrices with positive eigenvalues. Then there exists $V \in O(n)$, i.e. such that $V^*V = \mathrm{Id}$, with

$$g(Vy, V^{*-1}\eta) = g(Vy, V\eta) = \langle \gamma_1 Vy, Vy\rangle + \langle \gamma_2 V\eta, V\eta\rangle = \sum_{1\le j\le n} \mu_j y_j^2 + \langle \gamma_2 V\eta, V\eta\rangle.$$

With $M = \mathrm{diag}(\mu_j^{-1/2})$, we get $g(VMy, VM^{-1}\eta) = \|y\|^2 + \langle \gamma\eta, \eta\rangle$ where γ is a positive symmetric matrix and $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^n$. Let $U \in O(n)$ such that $U^*\gamma U = \mathrm{diag}(\nu_j)$: we obtain

$$g(VMUy, VM^{-1}U\eta) = \|Uy\|^2 + \sum_{1\le j\le n} \nu_j\eta_j^2 = \sum_{1\le j\le n} (y_j^2 + \nu_j\eta_j^2).$$

We note also that $\big((VMU)^*\big)^{-1} = \big(U^*M\ V^*\big)^{-1} = VM^{-1}U$ so that the mapping $(y,\eta) \mapsto (VMUy, VM^{-1}U\eta)$ is symplectic (cf. Theorem 4.4.3). Furthermore, we can use the symplectic mapping $(y_j,\eta_j) \mapsto (\nu_j^{1/4}y_j, \nu_j^{-1/4}\eta_j)$ to reach the identity $\sum_{1\le j\le n} h_j(y_j^2+\eta_j^2) = g(B^{-1}y, B^*\eta)$ with some $B \in Gl(n,\mathbb{R})$ (and positive h_j); the condition $g \le g^\sigma$ is expressed by $\max_{1\le j\le n} h_j \le 1$. Moreover that type of symplectic mapping is leaving invariant[9] the quadratic form $x\cdot\xi$, so that, with $b(x,\xi) = a(B^{-1}x, B^*\xi)$, we have

$$(J^t b)(x,\xi) = (J^t a)(B^{-1}x, B^*\xi).$$

Using the lemma 2.3.11, we may indeed assume that

$$g = \sum_{1\le j\le n} h_j(|dx_j|^2 + |d\xi_j|^2), \quad \lambda^{-1} = \max_{1\le j\le n} h_j \le 1.$$

It is also enough to check the result only for $J_1^t = \exp itD_{x_1}D_{\xi_1}$ since $h_1 \le \lambda^{-1}$ and J^t is the commutative product of $\exp itD_{x_j}D_{\xi_j}$. We have then with $s \in \mathbb{R}$, $h_1 = \lambda_1^{-1}$, using the formula of Lemma 4.1.2,

$$(J_1^{s^2}a)(x,\xi) = \int_{\mathbb{R}^2} e^{-2i\pi y_1\eta_1} a(sy_1 + x_1, x', s\eta_1 + \xi_1, \xi')dy_1 d\eta_1.$$

We use the identity $(i + \lambda_1^{1/2}y_1)^{-1}(i - \lambda_1^{1/2}D_{\eta_1})e^{-2i\pi y_1\eta_1} = e^{-2i\pi y_1\eta_1}$, which gives for $k \in \mathbb{N}$,

$$(J_1^{s^2}a)(x,\xi) =$$
$$\int_{\mathbb{R}^2} e^{-2i\pi y_1\eta_1}(i+\lambda_1^{1/2}y_1)^{-k}\big((i + s\lambda_1^{1/2}D_{\xi_1})^k a\big)(sy_1 + x_1, x', s\eta_1 + \xi_1, \xi')dy_1 d\eta_1.$$

[9] This is not the case of all symplectic mappings: the mappings *(iii), (jjj)* in the proposition 4.4.4 do *not* leave invariant $x\cdot\xi$.

Similarly, the identity $(i+\lambda_1^{1/2}\eta_1)^{-1}(i-\lambda_1^{1/2}D_{y_1})e^{-2i\pi y_1\eta_1}=e^{-2i\pi y_1\eta_1}$, gives for $l\in\mathbb{N}$,

$$(J_1^{s^2}a)(x,\xi)=\int_{\mathbb{R}^2}e^{-2i\pi y_1\eta_1}(i+\lambda_1^{1/2}\eta_1)^{-l}$$
$$\times(i+\lambda_1^{1/2}D_{y_1})^l\Big[(i+\lambda_1^{1/2}y_1)^{-k}\big((i+s\lambda_1^{1/2}D_{\xi_1})^k a\big)(sy_1+x_1,x',s\eta_1+\xi_1,\xi')\Big]dy_1d\eta_1.$$

As a consequence, $(J_1^{s^2}a)(x,\xi)$ is a finite sum[10] of terms of type

$$\alpha_{l',l''}(x,\xi)=\int_{\mathbb{R}^2}e^{-2i\pi y_1\eta_1}(i+\lambda_1^{1/2}\eta_1)^{-l}(i+\lambda_1^{1/2}y_1)^{-k-l''}\lambda_1^{l''}$$
$$\times\big((i+s\lambda_1^{1/2}D_{x_1})^{l'}(i+s\lambda_1^{1/2}D_{\xi_1})^k a\big)(sy_1+x_1,x',s\eta_1+\xi_1,\xi')dy_1d\eta_1.$$

with $l',l''\in\mathbb{N}$ with $l'+l''\le l$. This implies that, assuming that 0 is the center of the ball U_r, for $q\in\mathbb{N}$,

$$|(J_1^{s^2}a)(x,\xi)|\le C_{k,l}\lambda_1^{N(k,l)}\|a\|_{g,U_r}^{(k+l,q)}\langle s\rangle^{k+l}\times$$
$$\int_{\mathbb{R}^2}\Big[1+\inf_{g(z,\zeta)\le r^2}\big(\lambda_1^{1/2}|sy_1+x_1-z_1|+\lambda_1^{1/2}|s\eta_1+\xi_1-\zeta_1|$$
$$+\sum_{2\le j\le n}\lambda_j^{1/2}(|x_j-z_j|+|\xi_j-\zeta_j|)\big)\Big]^{-q}$$
$$\times(1+\lambda_1^{1/2}|\eta_1|)^{-l}(1+\lambda_1^{1/2}|y_1|)^{-k}dy_1d\eta_1.$$

As a result, for $k,l\ge\max(2,q)$, if $|s|\le 1$, we get with $X=(x,\xi)$,

$$|(J_1^{s^2}a)(x,\xi)|\le C'_{k,l}\lambda_1^{N(k,l)}\|a\|_{g,U_r}^{(k+l,q)}\big(1+g^\sigma(X-U_r)\big)^{-q/2}. \tag{2.3.20}$$

If $|s|>1$ (and $k,l\ge\max(2,q)$), we have

$$(1+\lambda_1^{1/2}|\eta_1|)^{-l}(1+\lambda_1^{1/2}|y_1|)^{-k}$$
$$\le(1+\lambda_1^{1/2}|\eta_1|)^{-2}(1+\lambda_1^{1/2}|y_1|)^{-2}(1+\lambda_1^{1/2}|\eta_1|)^{-l+2}(1+\lambda_1^{1/2}|y_1|)^{-k+2}$$
$$\le(1+\lambda_1^{1/2}|\eta_1|)^{-2}(1+\lambda_1^{1/2}|y_1|)^{-2}\big(1+\lambda_1^{1/2}|\eta_1|+\lambda_1^{1/2}|y_1|\big)^{-\min(k-2,l-2)}$$
$$\le(1+\lambda_1^{1/2}|\eta_1|)^{-2}(1+\lambda_1^{1/2}|y_1|)^{-2}\big(1+|s|\lambda_1^{1/2}|\eta_1|+|s|\lambda_1^{1/2}|y_1|\big)^{-\min(k-2,l-2)}$$
$$\times|s|^{\min(k-2,l-2)},$$

entailing for $k,l\ge\max(2,q)+2$, $|s|>1$,

$$|(J_1^{s^2}a)(x,\xi)|\le C''_{k,l}\lambda_1^{N(k,l)}\|a\|_{g,U_r}^{(k+l,q)}\big(1+g^\sigma(X-U_r)\big)^{-q/2}\langle s\rangle^{2k+2l}. \tag{2.3.21}$$

[10]The number of terms depends only on l and the coefficient of each term is bounded above by a constant depending only on k,l.

Moreover we have

$$J_1^t a = \sum_{0\le j<\nu} (2i\pi t D_{x_1}\cdot D_{\xi_1})^j a + \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} J_1^{\theta t} d\theta (2i\pi t D_{x_1}\cdot D_{\xi_1})^\nu a, \quad (2.3.22)$$

and applying (2.3.20) or (2.3.21), we get for $k = l = \max(2,q)+2$,

$$|(J_1^t a)(X)| \le \big(1+g^\sigma(X-U_r)\big)^{-q/2} \times \Big(\sum_{0\le j<\nu} \|a\|_{g,U_r}^{(2j,q)} (2\pi|t|)^j + \frac{(2\pi|t|)^\nu}{\nu!} \|(\partial_{x_1}\cdot\partial_{\xi_1})^\nu a\|_{g,U_r}^{(k+l,q)} \langle t\rangle^{k+l} C'''_{k,l} \lambda_1^{N(k,l)} \Big).$$

Since $\|(\partial_{x_1}\cdot\partial_{\xi_1})^\nu a\|_{g,U_r}^{(p,q)} \le \lambda_1^{-\nu}\|a\|_{g,U_r}^{(2\nu+p,q)}$, for a given q, with $k=l=\max(2,q)+2$,we may choose $\nu \ge N(k,l)$ and obtain

$$|(J_1^t a)(X)| \le \big(1+g^\sigma(X-U_r)\big)^{-q/2} C_q \langle t\rangle^{\mu_q} \|a\|_{g,U_r}^{(m_q,q)}. \quad (2.3.23)$$

Since J_1^t commutes with the derivations, we get from (2.3.23), for $k\in\mathbb{N}, T\in\mathbb{R}^{2n}$,

$$|(J_1^t a)^{(k)}(X)T^k| = |(J_1^t a^{(k)}T^k)(X)| \le \big(1+g^\sigma(X-U_r)\big)^{-q/2} C_q\langle t\rangle^{\mu_q}\|a\|_{g,U_r}^{(m_q+k,q)} g(T)^{k/2}, \quad (2.3.24)$$

and using $J^t = J_1^t \dots J_n^t$ we get (2.3.19) for $\nu = 0$. To obtain the result for $\nu > 0$, we use the formula (2.3.22) for J^t instead of J_1^t and the estimate

$$\|(\partial_x\cdot\partial_\xi)^\nu a\|_{g,U_r}^{(p,q)} \le \lambda^{-\nu} C_\nu \|a\|_{g,U_r}^{(2\nu+p,q)}.$$

The proof of the lemma is complete. □

Lemma 2.3.13. *Let g be a positive quadratic form on $\mathbb{R}^{2n}$ such that $g \le g^\sigma$, U_r be a closed g-ball with radius $r \le 1$ and $a \in Conf(g,U_r)$. Then for all $R > r$, there exist $b,c \in Conf(g,U_R)$ such that $a = b\sharp c$, where U_R is the g-ball with radius R and same center as U_r. The semi-norms of b,c depend only on R,r and the semi-norms of a.*

Proof. Using Lemma 4.4.25 and Lemma 2.3.11, we see that it is enough to prove the lemma assuming

$$g = \sum_{1\le j\le n} h_j(|dx_j|^2 + |d\xi_j|^2), \quad \max_{1\le j\le n} h_j \le 1,$$

where $\{(x_j,\xi_j)\}$ is a set of linear symplectic coordinates. From Lemma 2.3.12, we see that is is enough to prove the result for the ordinary quantization $\mathrm{op}_0(a) = a(x,D)$ as given by Proposition 1.1.10. Also we may assume that the center of U_r is 0. We can now apply Lemma 4.3.20 to get that

$$a(x,\xi) = a_1(x)a_2(x,\xi), \quad \text{with } a_2 \in \mathrm{Conf}(g,U_r)$$

and with $\gamma = \sum_{1\leq j\leq n} h_j|dx_j|^2$, $\mathbf{b}_R$ the γ-ball with radius R and center 0

$$\forall k, N \in \mathbb{N}, \quad \sup_{t,\gamma(t)\leq 1} |a_1^{(k)}(x)t^k|\big(1+\gamma^{-1}(x-\mathbf{b}_R)\big)^N < +\infty. \tag{2.3.25}$$

Defining $a_3 = J\overline{a_2}$, we get that $a_3 \in \mathrm{Conf}(g, U_r)$ and applying again Lemma 4.3.20, we obtain $a_3(x,\xi) = a_4(x,\xi)a_5(\xi)$ with a_5 satisfying (2.3.25). As a result we have

$$a(x,D) = a_1(x)a_2(x,D), \quad a_2(x,D)^* = a_3(x,D) = a_4(x,D)a_5(D),$$
$$\text{and thus } a(x,D) = a_1(x)\overline{a_5}(D)(J\overline{a_4})(x,D).$$

The symbol $J\overline{a_4}$ belongs to $\mathrm{Conf}(g, U_r)$ whereas the classical symbol of the operator $a_1(x)\overline{a_5}(D)$ is $a_1(x)\overline{a_5}(\xi)$, which is g-confined in the "box" $\mathbf{b}_R \times \mathbf{b}_R$, which gives only the g-confinement in $U_{R\sqrt{2}}$. However, we can replace a by $J^{-1/2}a$ and get

$$a^w = (J^{-1/2}(b_1(x)b_2(\xi)))^w c^w, \quad b_1, b_2 \text{ satisfying (2.3.25)}, c \in \mathrm{Conf}(g, U_r).$$

We may consider the symplectic rotation χ_θ, for a real parameter θ, $(x_j, \xi_j) \mapsto (x_j\cos\theta - \xi_j\sin\theta, x_j\sin\theta + \xi_j\cos\theta)$, which leaves invariant g, and we may prove a factorization

$$(c\circ\chi_\theta)^w = b_3(x)b_4(D)d^w, \quad b_3, b_4 \text{ satisfying (2.3.25)}, d \in \mathrm{Conf}(g, U_r),$$

so that $a^w = \big(J^{-1/2}(b_1\otimes b_2)\big)^w (J^{-1/2}(b_3\otimes b_4))^w d^w$. Now the symbol

$$J^{-1/2}(b_1\otimes b_2))\sharp J^{-1/2}(b_3\otimes b_4)$$

is g-confined in $\mathbf{b}_R\times\mathbf{b}_R \cap \chi_\theta(\mathbf{b}_R\times\mathbf{b}_R)$. Factorizing again d^w, and varying θ, we can get the result of the lemma with R replaced by any $R' > R$, which gives the same result. □

Definition 2.3.14 (Uniformly confined family of symbols)**.** Let g be an admissible metric on $\mathbb{R}^{2n}$ (see the definition 2.2.15), r_0 be the positive constant given by the theorem 2.2.7, and let $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$ be a family of functions on $\mathbb{R}^{2n}$. We shall say that this family is a *uniformly confined family of symbols* if there exists $r \in]0, r_0[$ such that $\forall k, N \in \mathbb{N}$,

$$\sup_{Y\in\mathbb{R}^{2n}, T\neq 0} |\psi_Y^{(k)}(X)T^k| g_Y(T)^{-k/2}\big(1+g_Y^\sigma(X-U_{Y,r})\big)^{N/2} < +\infty, \tag{2.3.26}$$

where $U_{Y,r}$ is the closed g_Y-ball with radius r. In other words, each ψ_Y is g_Y-confined in $U_{Y,r}$ (according to the definition 2.3.1) and

$$\forall l \in \mathbb{N}, \quad \omega_l = \sup_{Y\in\mathbb{R}^{2n}} \|\psi_Y\|^{(l)}_{g_Y, U_{Y,r}} < +\infty. \tag{2.3.27}$$

The constants $(\omega_l)_{l\in\mathbb{N}}$ are the seminorms of the family $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$.

The next theorem is an immediate consequence of Lemma 2.3.13 and of Definition 2.3.14.

Theorem 2.3.15. *Let g be an admissible metric on $\mathbb{R}^{2n}$ (see Definition 2.2.15), and let $(\phi_Y)_{Y\in\mathbb{R}^{2n}}$ be a uniformly confined family of symbols. Then there exists $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$, $(\theta_Y)_{Y\in\mathbb{R}^{2n}}$ two uniformly confined family of symbols such that*

$$\phi_Y = \psi_Y \sharp \theta_Y.$$

The seminorms of the families $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$, $(\theta_Y)_{Y\in\mathbb{R}^{2n}}$ are bounded above by functions of the seminorms of $(\phi_Y)_{Y\in\mathbb{R}^{2n}}$.

Proposition 2.3.16. *Let g be an admissible metric on $\mathbb{R}^{2n}$, m be a g-admissible weight, and $(\phi_Y)_{Y\in\mathbb{R}^{2n}}$ be a uniformly confined family of symbols. Then the function a defined by*

$$a(X) = \int_{\mathbb{R}^{2n}} m(Y)\phi_Y(X)|g_Y|^{1/2}dY$$

belongs to $S(m,g)$; if $b \in S(m,g)$, the family $(m(Y)^{-1}\phi_Y \sharp b)_{Y\in\mathbb{R}^{2n}}$ is a uniformly confined family of symbols. Moreover, using the notation (2.1.29), defining

$$\rho_Y = m(Y)^{-1}\lambda_g(Y)^{\nu}\big(\phi_Y \sharp b - \sum_{0\le j<\nu} \omega_j(\phi_Y, b)\big),$$

the family $(\rho_Y)_{Y\in\mathbb{R}^{2n}}$ is a uniformly confined family of symbols.

Remark 2.3.17. For b as above, the family $\big(m(Y)^{-1}\lambda_g(Y)^j\omega_j(\varphi_Y, b)\big)_{Y\in\mathbb{R}^{2n}}$ is trivially a uniformly confined family of symbols.

Proof. We have for all k, N,

$$\begin{aligned}
&|a^{(k)}(X)T^k| = \\
&\left|\int_{\mathbb{R}^{2n}} m(Y)\phi_Y^{(k)}(X)T^k\big(1+g_Y^{\sigma}(X-U_{Y,r})\big)^{N/2}\big(1+g_Y^{\sigma}(X-U_{Y,r})\big)^{-N/2}|g_Y|^{1/2}dY\right| \\
&\le g_X(T)^{k/2}m(X)\int_{\mathbb{R}^{2n}}\frac{m(Y)}{m(X)}\|\phi_Y\|^{(\max(k,N))}_{g_Y,U_{Y,r}}\frac{g_Y(T)^{k/2}}{g_X(T)^{k/2}}\delta_r(X,Y)^{-N/2}|g_Y|^{1/2}dY,
\end{aligned}$$

so that using (2.2.29) and (2.2.31), choosing N large enough, we get $a \in S(m,g)$. On the other hand, if $b \in S(m,g)$, the family $(m(Y)^{-1}\phi_Y \sharp b)_{Y\in\mathbb{R}^{2n}}$ is a uniformly confined family of symbols since

$$m(Y)^{-1}(\phi_Y \sharp b)^{(k)}(X)S^k = \int m(Y)^{-1}\big(\phi_Y \sharp (b\varphi_T)\big)^{(k)}(X)S^k|g_T|^{1/2}dT$$

which is bounded above by (see Theorem 2.3.2, Lemmas 2.2.24, 2.2.25)

$$(1+g_Y^\sigma(X-U_{Y,r}))^{-N}g_Y(S)^{k/2}\int m(Y)^{-1}m(T)\frac{(g_Y+g_T)(S)^{k/2}}{g_Y(S)^{k/2}}\delta_r(Y,T)^{NN_0}$$
$$\times C_M\big(1+(g_Y^\sigma\wedge g_T^\sigma)(X-U_{Y,r})+(g_Y^\sigma\wedge g_T^\sigma)(X-U_{T,r})\big)^{-M+N}|g_T|^{1/2}dT$$
$$\leq C'\int\delta_r(Y,T)^{-M/2}|g_T|^{1/2}dT<\infty$$

for M large enough (N,k are given, N_0 is a constant given by (2.2.29), M can be chosen as large as we wish). To get the very last statement, we repeat the previous proof, using Theorem 2.3.4 instead of Theorem 2.3.2. □

2.3.5 Changing the quantization

Theorem 2.3.18. *Let g be an admissible metric on $\mathbb{R}^{2n}$ (see the definition 2.2.15) such that*

$$\forall(x,\xi)\in\mathbb{R}^{2n},\forall(y,\eta)\in\mathbb{R}^{2n},\quad g_{x,\xi}(y,\eta)=g_{x,\xi}(y,-\eta). \tag{2.3.28}$$

Let m be an admissible weight for g and $a\in S(m,g)$. Then for all $t\in\mathbb{R}$, the symbol $J^ta=\exp 2i\pi D_x\cdot D_\xi a$ belongs to $S(m,g)$ and the mapping $a\mapsto J^ta$ is an isomorphism of the Fréchet space $S(m,g)$, with polynomial bounds in the real variable t. Moreover, with the weight λ_g defined in (2.2.13)*, we have*

$$J^ta-\sum_{0\leq j<\nu}\frac{1}{j!}(2i\pi tD_x\cdot D_\xi)^ja\in S(m\lambda_g^{-\nu},g). \tag{2.3.29}$$

Proof. Using a partition of unity, as given by Proposition 2.2.9, Lemma 2.3.12 gives the result. □

Theorem 2.3.19. *Let g be an admissible metric on $\mathbb{R}^{2n}$ (see the definition 2.2.15) such that* (2.3.28) *is satisfied, let m_1,m_2 be two admissible weights for g, $a_j\in S(m_j,g),j=1,2$. With $a(x,D)=(J^{-1/2}a)^w$ (see also* (1.1.1)*), we have*

$$a_1(x,D)a_2(x,D)=(a_1\diamond a_2)(x,D),\quad a_1\diamond a_2\in S(m_1m_2,g), \tag{2.3.30}$$
$$\forall\nu\in\mathbb{N},\quad a_1\diamond a_2=\sum_{|\alpha|<\nu}\frac{1}{\alpha!}D_\xi^\alpha a_1\partial_x^\alpha a_2+r_\nu(a_1,a_2), \tag{2.3.31}$$

with $r_\nu(a_1,a_2)\in S(m_1m_2\lambda_g^{-\nu},g)$. In particular, we have

$$a_1\diamond a_2=a_1a_2+\frac{1}{2i\pi}\partial_\xi a_1\cdot\partial_x a_2+r_2,\quad r_2\in S(m_1m_2\lambda_g^{-2},g).$$

Proof. We have $a_1\diamond a_2=(J^{-1/2}a_1)\sharp(J^{-1/2}a_2)$ so that the result follows from the theorem 2.3.7 and the theorem 2.3.18. □

2.4 The Wick calculus of pseudodifferential operators

2.4.1 Wick quantization

We recall here some facts on the so-called Wick quantization, as used in [93], [94], [95].

Definition 2.4.1. Let $Y = (y, \eta)$ be a point in $\mathbb{R}^n \times \mathbb{R}^n$. The operator Σ_Y is defined as $\left[2^n e^{-2\pi|\cdot - Y|^2}\right]^w$. Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick quantization of a is defined as

$$a^{\mathrm{Wick}} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY. \tag{2.4.1}$$

Remark 2.4.2. The operator Σ_Y is a rank-one orthogonal projection: we have

$$\Sigma_Y u = (Wu)(Y)\tau_Y\varphi_0 \quad \text{with } (Wu)(Y) = \langle u, \tau_Y\varphi_0\rangle_{L^2(\mathbb{R}^n)}, \tag{2.4.2}$$

$$\text{where } \varphi_0(x) = 2^{n/4}e^{-\pi|x|^2} \text{ and } (\tau_{y,\eta}\varphi_0)(x) = \varphi_0(x-y)e^{2i\pi\langle x-\frac{y}{2},\eta\rangle}. \tag{2.4.3}$$

In fact we get from the definition of Σ_Y that, for $u \in \mathscr{S}(\mathbb{R}^n)$

$$\begin{aligned}
(\Sigma_{y,\eta}u)(x) &= \iint u(z)e^{2i\pi(x-z)\cdot\xi}2^n e^{-2\pi|\frac{x+z}{2}-y|^2}e^{-2\pi|\xi-\eta|^2}dzd\xi \\
&= \int u(z)e^{2i\pi(x-z)\cdot\eta}2^{n/2}e^{-2\pi|\frac{x+z}{2}-y|^2}e^{-\frac{\pi}{2}|x-z|^2}dz \\
&= \int u(z)e^{-2i\pi(z-\frac{y}{2})\cdot\eta}2^{n/4}e^{-\pi|z-y|^2}dz\ 2^{n/4}e^{-\pi|x-y|^2}e^{2i\pi(x-\frac{y}{2})\cdot\eta} \\
&= \langle u, \tau_{y,\eta}\varphi_0\rangle\tau_{y,\eta}\varphi_0.
\end{aligned}$$

Proposition 2.4.3.
(1) *Let a be in $L^\infty(\mathbb{R}^{2n})$. Then $a^{Wick} = W^*a^\mu W$ and $1^{Wick} = Id_{L^2(\mathbb{R}^n)}$ where W is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given above, and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_{\mathcal{H}} = WW^*$ is the orthogonal projection on a closed proper subspace $\mathcal{H}$ of $L^2(\mathbb{R}^{2n})$ and has the kernel*

$$\Pi(X, Y) = e^{-\frac{\pi}{2}|X-Y|^2}e^{-i\pi[X,Y]}, \tag{2.4.4}$$

where $[,]$ is the symplectic form (2.1.1). *Moreover, we have*

$$\|a^{\mathrm{Wick}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|a\|_{L^\infty(\mathbb{R}^{2n})}, \tag{2.4.5}$$

$$a(X) \geq 0 \text{ for all } X \text{ implies } a^{Wick} \geq 0. \tag{2.4.6}$$

(2) *Let m be a real number, and $p \in S(\Lambda^m, \Lambda^{-1}\Gamma)$, where Γ is the Euclidean norm on $\mathbb{R}^{2n}$. Then $p^{Wick} = p^w + r(p)^w$, with $r(p) \in S(\Lambda^{m-1}, \Lambda^{-1}\Gamma)$ so that the mapping $p \mapsto r(p)$ is continuous. More precisely, one has*

$$r(p)(X) = \int_0^1\int_{\mathbb{R}^{2n}} (1-\theta)p''(X+\theta Y)Y^2 e^{-2\pi\Gamma(Y)}2^n dY d\theta.$$

Note that $r(p) = 0$ *if* p *is affine and* $r(p) = \frac{1}{8\pi}\operatorname{trace} p''$ *if* p *is a polynomial with degree* ≤ 2.
(3) *For* $a \in L^\infty(\mathbb{R}^{2n})$, *the Weyl symbol of* a^{Wick} *is*

$$a * 2^n \exp -2\pi\Gamma, \text{ which belongs to } S(1,\Gamma) \text{ with } k^{th}\text{-seminorm } c(k)\|a\|_{L^\infty}. \quad (2.4.7)$$

(4) *Let* $\mathbb{R} \ni t \mapsto a(t,X) \in \mathbb{R}$ *such that, for* $t \leq s$, $a(t,X) \leq a(s,X)$. *Then, for* $u \in C^1_c\big(\mathbb{R}_t, L^2(\mathbb{R}^n)\big)$, *assuming* $a(t,\cdot) \in L^\infty(\mathbb{R}^{2n})$,

$$\int_{\mathbb{R}} \mathrm{Re}\langle D_t u(t), ia(t)^{Wick} u(t)\rangle_{L^2(\mathbb{R}^n)} dt \geq 0. \quad (2.4.8)$$

(5) *With the operator* Σ_Y *given in Definition 2.4.1, we have the estimate*

$$\|\Sigma_Y \Sigma_Z\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq 2^n e^{-\frac{\pi}{2}\Gamma(Y-Z)}. \quad (2.4.9)$$

(6) *More precisely, the Weyl symbol of* $\Sigma_Y\Sigma_Z$ *is, as a function of the variable* $X \in \mathbb{R}^{2n}$, *setting* $\Gamma(T) = |T|^2$

$$e^{-\frac{\pi}{2}|Y-Z|^2} e^{-2i\pi[X-Y,X-Z]} 2^n e^{-2\pi|X-\frac{Y+Z}{2}|^2}. \quad (2.4.10)$$

Comment. Part of this proposition is well summarized by the following diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}^{2n}) & \xrightarrow[\text{(multiplication by } a)]{a} & L^2(\mathbb{R}^{2n}) \\ {\scriptstyle W}\uparrow & & \downarrow{\scriptstyle W^*} \\ L^2(\mathbb{R}^n) & \xrightarrow[a^{\mathrm{Wick}}]{} & L^2(\mathbb{R}^n) \end{array}$$

Proof. For $u, v \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\langle a^{\mathrm{Wick}} u, v\rangle = \int_{\mathbb{R}^{2n}} a(Y)\langle \Sigma_Y u, v\rangle_{L^2(\mathbb{R}^n)} dY = \int_{\mathbb{R}^{2n}} a(Y)(Wu)(Y)\overline{(Wv)(Y)} dY,$$

which gives

$$a^{\mathrm{Wick}} = W^* a^\mu W. \quad (2.4.11)$$

Also we have from (2.4.1) that $1^{\mathrm{Wick}} = \mathrm{Id}$, since

$$1^{\mathrm{Wick}} = \int_{\mathbb{R}^{2n}} \Sigma_Y dY \quad \text{has Weyl symbol} \int_{\mathbb{R}^{2n}} 2^n e^{-2\pi|X-Y|^2} dY = 1.$$

This implies that

$$W^*W = \mathrm{Id},$$

i.e. W is isometric from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{2n})$. The operator WW^* is bounded selfadjoint and is a projection since $WW^*WW^* = WW^*$. Defining $\mathcal{H}$ as $\operatorname{ran} W$,

we get that WW^* is the orthogonal projection onto $\mathcal{H}$, since the range of WW^* is included in the range of W, and for $\Phi \in \mathcal{H}$, we have

$$\Phi = Wu = WW^*Wu \in \mathrm{ran}(WW^*).$$

Moreover $\mathrm{ran}\, W$ is closed since W is isometric, that latter property implying also, using (2.4.11), the property (2.4.5), whereas (2.4.6) follows from (2.4.1) and $\Sigma_Y \geq 0$ as an orthogonal projection. The kernel of the operator WW^* is, from (2.4.2), (2.4.3), with $X = (x,\xi), Y = (y,\eta)$,

$$\begin{aligned}
\Pi(X,Y) &= \langle \tau_Y\varphi_0, \tau_X\varphi_0\rangle_{L^2(\mathbb{R}^n)} \\
&= 2^{n/2}\int_{\mathbb{R}^n} e^{-\pi|t-x|^2}e^{-\pi|t-y|^2}e^{2i\pi(t-\frac{y}{2})\cdot\eta}e^{-2i\pi(t-\frac{x}{2})\cdot\xi}dt \\
&= e^{-\frac{\pi}{2}|x-y|^2}2^{n/2}\int_{\mathbb{R}^n} e^{-\frac{\pi}{2}|2t-x-y|^2}e^{2i\pi t\cdot(\eta-\xi)}dt e^{i\pi(x\cdot\xi-y\cdot\eta)} \\
&= e^{-\frac{\pi}{2}|x-y|^2}2^{n/2}\int_{\mathbb{R}^n} e^{-2\pi|t|^2}e^{2i\pi(t+\frac{x+y}{2})\cdot(\eta-\xi)}dt e^{i\pi(x\cdot\xi-y\cdot\eta)} \\
&= e^{-\frac{\pi}{2}|x-y|^2}e^{-\frac{\pi}{2}|\xi-\eta|^2}e^{i\pi(x+y)\cdot(\eta-\xi)}e^{i\pi(x\cdot\xi-y\cdot\eta)} \\
&= e^{-\frac{\pi}{2}|x-y|^2}e^{-\frac{\pi}{2}|\xi-\eta|^2}e^{i\pi(x\eta-y\xi)} = e^{-\frac{\pi}{2}|X-Y|^2}e^{-i\pi[X,Y]},
\end{aligned}$$

which is (2.4.4). Postponing the proof of $\mathcal{H} \neq L^2(\mathbb{R}^{2n})$ until after the proof of (2), we have proven (1). To obtain (2), we note that (2.4.1) gives directly that

$$a^{\mathrm{Wick}} = (a * 2^n \exp -2\pi\Gamma)^w$$

and the second order Taylor expansion gives (2) while (3) is obvious from the convolution formula. Note also that $u \in \mathscr{S}(\mathbb{R}^n)$ implies $Wu \in \mathscr{S}(\mathbb{R}^{2n})$ since $e^{-i\pi y\cdot\eta}(Wu)(y,\eta)$ is the partial Fourier transform with respect to x of $\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto u(x)2^{n/4}e^{-\pi|x-y|^2}$: this gives also another proof of W isometric since

$$\iint |u(x)|^2 2^{n/2}e^{-2\pi|x-y|^2}dxdy = \|u\|^2_{L^2(\mathbb{R}^n)}.$$

We calculate now, for $u \in \mathscr{S}(\mathbb{R}^n)$ with L^2 norm 1, using the already proven (2) on the Wick quantization of linear forms,

$$\begin{aligned}
2\,\mathrm{Re}\langle \pi_{\mathcal{H}}\xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} &= 2\,\mathrm{Re}\langle W^*\xi_1 Wu, iW^*x_1 Wu\rangle_{L^2(\mathbb{R}^n)} \\
&= 2\,\mathrm{Re}\langle \xi_1^{\mathrm{Wick}}u, ix_1^{\mathrm{Wick}}u\rangle_{L^2(\mathbb{R}^n)} = 2\,\mathrm{Re}\langle D_1u, ix_1u\rangle_{L^2(\mathbb{R}^n)} = 1/2\pi.
\end{aligned}$$

If $\mathcal{H}$ were the whole $L^2(\mathbb{R}^{2n})$, the projection $\pi_{\mathcal{H}}$ would be the identity and we would have

$$0 = 2\,\mathrm{Re}\langle \xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 2\,\mathrm{Re}\langle \pi_{\mathcal{H}}\xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 1/2\pi.$$

Let us prove (4). We have from the Lebesgue dominated convergence theorem,

$$\begin{aligned}
\alpha &= \int_{\mathbb{R}} \mathrm{Re}\langle D_t u(t), ia(t)^{\mathrm{Wick}} u(t)\rangle_{L^2(\mathbb{R}^n)} dt \\
&= -\lim_{h\to 0_+} \int_{\mathbb{R}} \frac{1}{2\pi h} \mathrm{Re}\langle u(t+h)-u(t), a(t)^{\mathrm{Wick}} u(t+h)\rangle_{L^2(\mathbb{R}^n)} dt \\
&= \lim_{h\to 0_+} \frac{1}{2\pi h}\Big(-\int_{\mathbb{R}} \mathrm{Re}\langle u(t), a(t-h)^{\mathrm{Wick}} u(t)\rangle_{L^2(\mathbb{R}^n)} dt \\
&\qquad\qquad + \int_{\mathbb{R}} \mathrm{Re}\langle u(t), a(t)^{\mathrm{Wick}} u(t+h)\rangle_{L^2(\mathbb{R}^n)} dt\Big) \\
&= \lim_{h\to 0_+} \Big\{ \underbrace{\frac{1}{2\pi h}\int_{\mathbb{R}} \mathrm{Re}\langle \big(a(t)-a(t-h)\big)^{\mathrm{Wick}} u(t), u(t)\rangle_{L^2(\mathbb{R}^n)} dt}_{=\beta(h)} \\
&\qquad\qquad + \underbrace{\int_{\mathbb{R}} \mathrm{Re}\langle \frac{-1}{2\pi h i}\big(u(t+h)-u(t)\big), ia(t)^{\mathrm{Wick}} u(t)\rangle_{L^2(\mathbb{R}^n)} dt}_{\text{with limit } -\alpha}\Big\}.
\end{aligned}$$

The previous calculation shows that $\beta(h)$ has a limit when $h \to 0_+$ and $2\alpha = \lim_{h\to 0_+} \beta(h)$. Since the function $a(t)-a(t-h)$ is non-negative, the already proven (2.4.6) implies that the operator $(a(t)-a(t-h))^{\mathrm{Wick}}$ is also non-negative, implying $\beta(h) \geq 0$ which gives $\alpha \geq 0$, i.e. (2.4.8)[11]. Since for the Weyl quantization, one has from (2.1.13), $\|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq 2^n \|a\|_{L^1(\mathbb{R}^{2n})}$, we get the result (2.4.9) from (2.4.10). Let us finally prove the latter formula. From the composition formula (2.1.18), we obtain that the Weyl symbol ω of $\Sigma_Y \Sigma_Z$ is

$$\begin{aligned}
\omega(X) &= 2^{2n} \iint e^{-4i\pi[X-X_1, X-X_2]} 2^{2n} e^{-2\pi|X_1-Y|^2} e^{-2\pi|X_2-Z|^2} dX_1 dX_2 \\
&= 2^{4n} \iint e^{-4i\pi[X-Y, X-X_2]} e^{-2i\pi\langle X_1, 2\sigma(X-X_2)\rangle} e^{-2\pi|X_1|^2} e^{-2\pi|X_2-Z|^2} dX_1 dX_2 \\
&= 2^{3n} \int e^{-4i\pi[X-Y, X-X_2]} e^{-2\pi|X-X_2|^2} e^{-2\pi|X_2-Z|^2} dX_2 \\
&= 2^{3n} e^{-\pi|X-Z|^2} \int e^{-4i\pi[X-Y, X-X_2]} e^{-\pi|X+Z-2X_2|^2} dX_2 \\
&= 2^{3n} e^{-\pi|X-Z|^2} e^{-2i\pi[X-Y, X-Z]} \int e^{-4i\pi[X-Y, -X_2]} e^{-4\pi|X_2|^2} dX_2 \\
&= 2^n e^{-\pi|X-Z|^2} e^{-2i\pi[X-Y, X-Z]} e^{-\pi|X-Y|^2} \\
&= 2^n e^{-2i\pi[X-Y, X-Z]} e^{-2\pi|X-\frac{Y+Z}{2}|^2} e^{-\frac{\pi}{2}|Y-Z|^2}.
\end{aligned}$$

□

[11] Note that (2.4.8) is simply a way of writing that $\frac{d}{dt}\big(a(t)^{\mathrm{Wick}}\big) \geq 0$, which is a consequence of (2.4.6) and of the non-decreasing assumption made on $t \mapsto a(t, X)$.

2.4.2 Fock-Bargmann spaces

There are also several links with the so-called Fock-Bargmann spaces (the space $\mathcal{H}$ above), that we can summarize with the following definitions and properties.

Proposition 2.4.4. *With $\mathcal{H}$ defined in Proposition 2.4.3 we have*

$$\mathcal{H} = \{\Phi \in L^2(\mathbb{R}^{2n}_{y,\eta}),\quad \Phi = f(z)\exp -\frac{\pi}{2}|z|^2,\ z = \eta + iy\ ,\ f\ \textit{entire}\},\qquad (2.4.12)$$

i.e. $\mathcal{H} = \mathrm{ran} W = L^2(\mathbb{R}^{2n}) \cap \ker(\bar\partial + \frac{\pi}{2}z)$.

Proof. For $v \in L^2(\mathbb{R}^n)$, we have, with the notation $z^2 = \sum_{1\le j\le n} z_j^2$ for $z \in \mathbb{C}^n$,

$$\begin{aligned}(Wv)(y,\eta) &= \int_{\mathbb{R}^n} v(x)2^{n/4}e^{-\pi(x-y)^2}e^{-2i\pi(x-\frac{y}{2})\eta}dx\\ &= \int_{\mathbb{R}^n} v(x)2^{n/4}e^{-\pi(x-y+i\eta)^2}dx e^{-\frac{\pi}{2}(y^2+\eta^2)}e^{-\frac{\pi}{2}(\eta+iy)^2}\end{aligned}\qquad (2.4.13)$$

and we see that $Wv \in L^2(\mathbb{R}^{2n}) \cap \ker(\bar\partial + \frac{\pi}{2}z)$. Conversely, if $\Phi \in L^2(\mathbb{R}^{2n}) \cap \ker(\bar\partial + \frac{\pi}{2}z)$, we have $\Phi(x,\xi) = e^{-\frac{\pi}{2}(x^2+\xi^2)}f(\xi + ix)$ with $\Phi \in L^2(\mathbb{R}^{2n})$ and f entire. This gives

$$\begin{aligned}(WW^*\Phi)(x,\xi) &= \iint e^{-\frac{\pi}{2}\left((\xi-\eta)^2+(x-y)^2+2i\xi y-2i\eta x\right)}\Phi(y,\eta)dyd\eta\\ &= e^{-\frac{\pi}{2}(\xi^2+x^2)}\iint e^{-\frac{\pi}{2}(\eta^2-2\xi\eta+y^2-2xy+2i\xi y-2i\eta x)}\Phi(y,\eta)dyd\eta\\ &= e^{-\frac{\pi}{2}(\xi^2+x^2)}\iint e^{-\frac{\pi}{2}\left(\eta^2+y^2+2iy(\xi+ix)-2\eta(\xi+ix)\right)}\Phi(y,\eta)dyd\eta\\ &= e^{-\frac{\pi}{2}(\xi^2+x^2)}\iint e^{-\pi(y^2+\eta^2)}e^{\pi(\eta-iy)(\xi+ix)}f(\eta+iy)dyd\eta\\ &= e^{-\frac{\pi}{2}|z|^2}\iint e^{-\pi|\zeta|^2}e^{\pi\bar\zeta z}f(\zeta)dyd\eta\quad(\zeta=\eta+iy,\ z=\xi+ix)\\ &= e^{-\frac{\pi}{2}|z|^2}\iint f(\zeta)\prod_{1\le j\le n}\frac{1}{\pi(z_j-\zeta_j)}\frac{\partial}{\partial\bar\zeta_j}\Big(e^{-\pi|\zeta|^2}e^{\pi\bar\zeta z}\Big)dyd\eta\\ &= e^{-\frac{\pi}{2}|z|^2}\langle f(\zeta)\prod_{1\le j\le n}\frac{\partial}{\partial\bar\zeta_j}\Big(\frac{1}{\pi(\zeta_j-z_j)}\Big), e^{-\pi|\zeta|^2}e^{\pi\bar\zeta z}\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}\\ &= e^{-\frac{\pi}{2}|z|^2}f(z),\end{aligned}$$

since f is entire. This implies $WW^*\Phi = \Phi$ and $\Phi \in \mathrm{ran}\, W$, completing the proof of the proposition. □

Proposition 2.4.5. *Defining*

$$\mathscr{H} = \ker(\bar\partial + \frac{\pi}{2}z) \cap \mathscr{S}'(\mathbb{R}^{2n}),\qquad (2.4.14)$$

the operator W given by (2.4.2) *can be extended as a continuous mapping from $\mathscr{S}'(\mathbb{R}^n)$ onto $\mathscr{H}$ (the $L^2(\mathbb{R}^n)$ dot-product is replaced by a bracket of (anti)duality). The operator $\widetilde{\Pi}$ with kernel Π given by* (2.4.4) *defines a continuous mapping from $\mathscr{S}(\mathbb{R}^{2n})$ into itself and can be extended as a continuous mapping from $\mathscr{S}'(\mathbb{R}^{2n})$ onto $\mathscr{H}$. It verifies*

$$\widetilde{\Pi}^2 = \widetilde{\Pi}, \quad \widetilde{\Pi}_{|\mathscr{H}} = \mathrm{Id}_{\mathscr{H}}\,. \tag{2.4.15}$$

Proof. As above we use that $e^{-i\pi y\eta}(Wv)(y,\eta)$ is the partial Fourier transform w.r.t. x of the tempered distribution on $\mathbb{R}^{2n}_{x,y}$

$$v(x)2^{n/4}e^{-\pi(x-y)^2}.$$

Since $e^{\pm i\pi y\eta}$ are in the space $\mathscr{O}_M(\mathbb{R}^{2n})$ of multipliers of $\mathscr{S}(\mathbb{R}^{2n})$, that transformation is continuous and injective from $\mathscr{S}'(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^{2n})$. Replacing in (2.4.13) the integrals by brackets of duality, we see that $W(\mathscr{S}'(\mathbb{R}^n)) \subset \mathscr{H}$. Conversely, if $\Phi \in \mathscr{H}$, the same calculations as above give (2.4.15) and (2.4.14). □

2.4.3 On the composition formula for the Wick quantization

In this section, we prove some formulas of composition for operators with very irregular Wick symbols.

Lemma 2.4.6. *For $p, q \in L^\infty(\mathbb{R}^{2n})$ real-valued with $p'' \in L^\infty(\mathbb{R}^{2n})$, we have*

$$\mathrm{Re}\big(p^{Wick}q^{Wick}\big) = \Big(pq - \frac{1}{4\pi}\nabla p\cdot\nabla q\Big)^{Wick} + R, \tag{2.4.16}$$

$$\|R\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C(n)\|p''\|_{L^\infty}\|q\|_{L^\infty}. \tag{2.4.17}$$

The product $\nabla p\cdot\nabla q$ above makes sense (see our appendix section 4.3.2) as a tempered distribution since ∇p is a Lipschitz continuous function and ∇q is the derivative of an L^∞ function: in fact, we shall use as a definition (see section 4.3.2) $\nabla p\cdot\nabla q = \nabla\cdot(\underbrace{q}_{L^\infty}\underbrace{\nabla p}_{Lip.}) - \underbrace{q}_{L^\infty}\underbrace{\Delta p}_{L^\infty}$.

Proof. Using the definition (2.4.1) , we see that

$$\begin{aligned} p^{\mathrm{Wick}}q^{\mathrm{Wick}} &= \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} p(Y)q(Z)\Sigma_Y\Sigma_Z dYdZ \\ &= \iint \Big(p(Z) + p'(Z)(Y-Z) + p_2(Z,Y)(Y-Z)^2\Big)q(Z)\Sigma_Y\Sigma_Z dYdZ \\ &= \int (pq)(Z)\Sigma_Z dZ + \iint p'(Z)(Y-Z)\Sigma_Y dY q(Z)\Sigma_Z dZ + R_0, \end{aligned}$$

with $R_0 = \iint_0^1 (1-\theta)p''(Z+\theta(Y-Z))(Y-Z)^2 q(Z)\Sigma_Y\Sigma_Z dYdZd\theta$.

Remark 2.4.7. *Let ω be a measurable function defined on $\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ such that*

$$|\omega(Y,Z)|\leq\gamma_0\big(1+|Y-Z|\big)^{N_0}.$$

Then the operator $\iint\omega(Y,Z)\Sigma_Y\Sigma_Z dYdZ$ is bounded on $L^2(\mathbb{R}^n)$ with $\mathcal{L}(L^2(\mathbb{R}^n))$ norm bounded above by a constant depending on γ_0, N_0. This is an immediate consequence of Cotlar's lemma 4.7.1 and of the estimate (2.4.9).

Using that remark, we obtain that

$$\|R_0\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\leq C_1(n)\|p''\|_{L^\infty(\mathbb{R}^{2n})}\|q\|_{L^\infty(\mathbb{R}^{2n})}. \tag{2.4.18}$$

We check now $\int(Y-Z)\Sigma_Y dY$ whose Weyl symbol is, as a function of X,

$$\int(Y-Z)2^ne^{-2\pi|X-Y|^2}dY=\int(X-Z)2^ne^{-2\pi|X-Y|^2}dY=X-Z.$$

So with $L_Z(X)=X-Z$, we have $\int(Y-Z)\Sigma_Y dY\Sigma_Z=(X-Z)^w\Sigma_Z=L_Z^w\Sigma_Z$ and thus

$$\begin{aligned}\operatorname{Re}\int(Y-Z)\Sigma_Y dY\Sigma_Z=\operatorname{Re}(L_Z^w\Sigma_Z)=&\big((X-Z)2^ne^{-2\pi|X-Z|^2}\big)^w\\ &=\frac{1}{4\pi}\partial_Z(2^ne^{-2\pi|X-Z|^2})^w,\end{aligned} \tag{2.4.19}$$

so that

$$\operatorname{Re}\int(Y-Z)\Sigma_Y dY\Sigma_Z=\frac{1}{4\pi}\partial_Z(\Sigma_Z). \tag{2.4.20}$$

Using that p and q are real-valued, the formula for $\operatorname{Re}(p^{\mathrm{Wick}}q^{\mathrm{Wick}})$ becomes

$$\begin{aligned}\operatorname{Re}\big(p^{\mathrm{Wick}}q^{\mathrm{Wick}}\big)&=\int(pq)(Z)\Sigma_Z dZ+\int p'(Z)q(Z)\frac{1}{4\pi}\partial_Z\Sigma_Z dZ+\operatorname{Re}R_0\\ &=\int\Big((pq)(Z)-\frac{1}{4\pi}p'(Z)\cdot q'(Z)\Big)\Sigma_Z dZ-\int\frac{1}{4\pi}\operatorname{trace}p''(Z)q(Z)\Sigma_Z dZ+\operatorname{Re}R_0,\end{aligned}$$

that is the result of the lemma, using (2.4.18) and (2.4.5) for the penultimate term on the line above. □

The next lemma is more involved.

Lemma 2.4.8. *For p measurable real-valued function such that $p'',(p'p'')',(pp'')''\in L^\infty$, we have*

$$p^{Wick}p^{Wick}=\int\Big[p(Z)^2-\frac{1}{4\pi}|\nabla p(Z)|^2\Big]\Sigma_Z dZ+S, \tag{2.4.21}$$

$$\|S\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\leq C(n)\left(\|p''\|_{L^\infty}^2+\|(p''p')'\|_{L^\infty}+\|(pp'')''\|_{L^\infty}\right). \tag{2.4.22}$$

Here p'' stands for the vector (tensor) with components $(\partial_X^\alpha p)_{|\alpha|=2}$, whereas the components of $(p''p')'$ are $\partial_X^\alpha\big(\partial_X^\beta \partial_X^\gamma p\big)_{\substack{|\alpha|=1,|\beta|=2\\|\gamma|=1}}$, and those of $(pp'')''$ are

$$\partial_X^\alpha\big(p\partial_X^\beta p\big)_{|\alpha|=|\beta|=2}.$$

Proof. We have

$$\begin{aligned} p^{\mathrm{Wick}}p^{\mathrm{Wick}} &= \iint p(Y)p(Z)\Sigma_Y\Sigma_Z dYdZ \\ &= \iint p(Y)\big(p(Y)+p'(Y)(Z-Y)\big)\Sigma_Y\Sigma_Z dYdZ \\ &\quad + \iiint_0^1 p(Y)(1-\theta)p''(Y+\theta(Z-Y))d\theta(Z-Y)^2\Sigma_Y\Sigma_Z dYdZd\theta \end{aligned}$$

so that, using (2.4.20) for the terms pp' in the double integral above, we get, noting $\mathrm{trace}(p'') = \Delta p$,

$$p^{\mathrm{Wick}}p^{\mathrm{Wick}} = \big(p^2 - \frac{1}{4\pi}|\nabla p|^2 - \frac{1}{4\pi}p\Delta p\big)^{\mathrm{Wick}} + \mathrm{Re}(\Omega_0+\Omega_1+\Omega_2), \tag{2.4.23}$$

with

$$\Omega_0 = \iiint_0^1 p\big(Y+\theta(Z-Y)\big)p''\big(Y+\theta(Z-Y)\big)(Z-Y)^2\Sigma_Y\Sigma_Z dYdZ(1-\theta)d\theta, \tag{2.4.24}$$

$$\begin{aligned} \Omega_1 = \iiint_0^1 & p'\big(Y+\theta(Z-Y)\big)\theta(Y-Z) \\ & \times p''\big(Y+\theta(Z-Y)\big)(Z-Y)^2\Sigma_Y\Sigma_Z dYdZ(1-\theta)d\theta \end{aligned} \tag{2.4.25}$$

and from Remark 2.4.7,

$$\|\Omega_2\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_1(n)\|p''\|_{L^\infty}^2. \tag{2.4.26}$$

We write now $\Omega_0 = \Omega_{00} + \Omega_{01},\ \Omega_1 = \Omega_{10} + \Omega_{11}$ with

$$\begin{aligned} \Omega_{00} &= \frac{1}{2}\iint p(Y)p''(Y)(Z-Y)^2\Sigma_Y\Sigma_Z dYdZ, \\ \Omega_{01} &= \iiint_0^1 \Big((pp'')\big(Y+\theta(Z-Y)\big) - (pp'')(Y)\Big)(Z-Y)^2\Sigma_Y\Sigma_Z dYdZ(1-\theta)d\theta \\ \Omega_{10} &= -\frac{1}{6}\iiint_0^1 p'(Y)(Z-Y)p''(Y)(Z-Y)^2\Sigma_Y\Sigma_Z dYdZ, \end{aligned}$$

$$\|\Omega_{11}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_2(n)\|(p'p'')'\|_{L^\infty}. \tag{2.4.27}$$

We have also $\Omega_{01} = \Omega_{010} + \Omega_{011}$ with

$$\Omega_{010} = \frac{1}{6}\iint (pp'')'(Y)(Z-Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ,$$

$$\|\Omega_{011}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_3(n)\|(pp'')''\|_{L^\infty}. \tag{2.4.28}$$

From (2.4.23)—(2.4.28), it suffices to check that the following term is a remainder satisfying the estimate (2.4.22) to get the result of Lemma 2.4.8:

$$\begin{aligned}\widetilde{\Omega} = -\frac{1}{4\pi}\int p(Y)\,\mathrm{trace}\, p''(Y)\Sigma_Y dY + \frac{1}{2}\,\mathrm{Re}\iint (pp'')(Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ\\ + \frac{1}{6}\,\mathrm{Re}\iint (pp'')'(Y)(Z-Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ\\ - \frac{1}{6}\,\mathrm{Re}\iiint_0^1 p'(Y)(Z-Y)p''(Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ. \end{aligned}\tag{2.4.29}$$

The real part of the Weyl symbol of $\int (Z_j - Y_j)(Z_k - Y_k)(Z_l - Y_l)\Sigma_Y\Sigma_Z dZ$ is (see (2.4.10))

$$\begin{aligned}&\int (Z_j - Y_j)(Z_k - Y_k)(Z_l - Y_l)e^{-\frac{\pi}{2}|Y-Z|^2}\\ &\qquad\qquad \times \cos(2\pi[X-Y, X-Z])2^n e^{-2\pi|X-\frac{Y+Z}{2}|^2} dZ\\ &= \int T_j T_k T_l e^{-2\pi|T/2|^2}\cos(2\pi[X-Y,T])2^n e^{-2\pi|X-Y-\frac{T}{2}|^2} dT\\ &= \int T_j T_k T_l \cos(2\pi[X-Y,T])e^{-\pi|X-Y-T|^2} dT 2^n e^{-\pi|X-Y|^2} = \nu_{jkl}(X-Y)\end{aligned}$$

with

$$\nu_{jkl}(S) = \int T_j T_k T_l \cos(2\pi[S,T])e^{-\pi|S-T|^2} dT 2^n e^{-\pi|S|^2} \tag{2.4.30}$$

$$= 2^n e^{-\pi|S|^2}\int (T_j+S_j)(T_k+S_k)(T_l+S_l)\cos(2\pi[S,T])e^{-\pi|T|^2} dT \tag{2.4.31}$$

$$= 2^n e^{-\pi|S|^2}\int (T_j T_k S_l + T_k T_l S_j + T_l T_j S_k + S_j S_k S_l)\cos(2\pi[S,T])e^{-\pi|T|^2} dT. \tag{2.4.32}$$

We notice that the function $S \mapsto \int_{\mathbb{R}^{2n}} T_j T_k \exp(2i\pi[S,T])e^{-\pi|T|^2} dT$ is a second-order derivative of $S \mapsto \int_{\mathbb{R}^{2n}} \exp(2i\pi[S,T])e^{-\pi|T|^2} dT = e^{-\pi|S|^2}$ so that

$$2^n e^{-\pi|S|^2} S_l \int_{\mathbb{R}^{2n}} T_j T_k \cos(2\pi[S,T])e^{-\pi|T|^2} dT = e^{-2\pi|S|^2} S_l P_{jk}(S),$$

with P_{jk} even, second-order and real polynomial. The function $S_{l_1}S_{l_2}S_{l_3}e^{-2\pi|S|^2}$ is always a linear combination of derivatives of Schwartz functions on $\mathbb{R}^{2n}$, since

if $l_1 < l_2 \le l_3$ it is the derivative with respect to S_{l_1} of $S_{l_2}S_{l_3}e^{-2\pi|S|^2}(-4\pi)^{-1}$,

if $l_1 = l_2 < l_3$ it is the derivative with respect to S_{l_3} of $S_{l_1}S_{l_2}e^{-2\pi|S|^2}(-4\pi)^{-1}$,

if $l_1 = l_2 = l_3 = l$ it is a linear combination of the third and first derivative with respect to S_l of $e^{-2\pi|S|^2}$, since

$$(e^{t^2})''' = (12t + 8t^3)e^{t^2}, \quad t^3e^{t^2} = \frac{1}{8}(e^{t^2})''' - \frac{3}{4}(e^{t^2})'.$$

As a result the function ν_{jkl} defined by (2.4.32) is a linear combination of derivatives with respect to S_j, S_k or S_l of Schwartz functions on $\mathbb{R}^{2n}$. Integrating by parts in the last two terms of (2.4.29), we see that their $\mathcal{L}(L^2)$ norm is bounded from above by $C_4(n)(\|(pp'')''\|_{L^\infty} + \|(p'p'')'\|_{L^\infty})$. Looking at (2.4.29), we see that we are left with

$$\widetilde{\Omega}_0 = -\frac{1}{4\pi}\int p(Y)\,\mathrm{trace}\,p''(Y)\Sigma_Y dY + \frac{1}{2}\,\mathrm{Re}\iint (pp'')(Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ. \tag{2.4.33}$$

The real part of the operator $\int (Z_j - Y_j)(Z_k - Y_k)\Sigma_Y\Sigma_Z dZ$ has the Weyl symbol (function of X)

$$\int T_jT_k e^{-\pi|X-Y-T|^2}\cos(2\pi[X-Y,T])dT 2^n e^{-\pi|X-Y|^2} \tag{2.4.34}$$

$$= \int \Big((X_j - Y_j)(X_k - Y_k) + T_jT_k\Big)e^{-\pi|T|^2}\cos(2\pi[X-Y,T])dT 2^n e^{-\pi|X-Y|^2} \tag{2.4.35}$$

$$= \int \big(S_jS_k + T_jT_k\big)e^{-\pi|T|^2}\cos(2\pi[S,T])dT 2^n e^{-\pi|S|^2}, \quad S = X - Y. \tag{2.4.36}$$

• If $j \neq k$, both terms in (2.4.36) are second order derivatives with respect to Y of a Schwartz function in $\mathbb{R}^{2n}$. In fact the first term is

$$S_jS_k 2^n e^{-2\pi|S|^2} = \partial_{S_j}\partial_{S_k}\big(2^n e^{-2\pi|S|^2}/16\pi^2\big) = \partial_{Y_j}\partial_{Y_k}\big(2^n e^{-2\pi|S|^2}/16\pi^2\big)$$

and the second term is equal to $-S_{j'}S_{k'}2^n e^{-2\pi|S|^2}$, with $j' \neq k'$, also a second-order derivative. The contribution of these terms in (2.4.33) is then, after integration by parts, an L^2 bounded operator with norm $\le C_5(n)\|(pp'')''\|_{L^\infty}$.

• If $j = k$, with $j' = j \pm n$ (in fact $j' = j + n$ if $1 \le j \le n$ and $j' = j - n$ if $1 + n \le j \le 2n$), we note that (2.4.36) is equal to

$$S_j^2 2^n e^{-2\pi|S|^2} - \frac{1}{4\pi^2}e^{-\pi|S|^2}\partial_{S_{j'}}^2\big(2^n e^{-\pi|S|^2}\big) = 2^n e^{-2\pi|S|^2}\Big(S_j^2 - \frac{1}{4\pi^2}(4\pi^2 S_{j'}^2 - 2\pi)\Big).$$

Taking into account the contribution of all these terms in (2.4.33), we see that we are left with

$$-\frac{1}{4\pi}\int p(Y)\,\mathrm{trace}\,p''(Y)\Sigma_Y dY + \frac{1}{2}\iint \frac{1}{2\pi}\,\mathrm{trace}(pp'')(Y)\Sigma_Y dY = 0.$$

The proof of Lemma 2.4.8 is complete. □

2.5 Basic estimates for pseudodifferential operators

2.5.1 L^2 estimates

Theorem 2.5.1. *Let g be an admissible metric on $\mathbb{R}^{2n}$ (see Definition 2.2.15), $a \in S(1,g)$ (cf. Definition 2.2.6). Then a^w is bounded on $L^2(\mathbb{R}^n)$ and $\|a^w\|_{\mathcal{L}(L^2)}$ can be estimated by a seminorm of the symbol a.*

Before proving this theorem, we need a lemma.

Lemma 2.5.2. *Let g be a positive-definite quadratic form on $\mathbb{R}^{2n}$ such that $g \le g^\sigma$. Let a be a smooth function in $\mathbb{R}^{2n}$ and U be a g-ball with radius ≤ 1 such that a is g-confined in U (see Definition 2.3.1). Then*

$$\|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C(n)\|\!|a|\!\|_{g,U}^{(2n+1)}. \tag{2.5.1}$$

Proof of the lemma. For $k \in \mathbb{N}$, we have

$$\hat{a}(-\Xi) = \int \Big(\big(1 - \langle g^{-1}D_X, \Xi\rangle\big)^k a\Big)(X) e^{2i\pi X\cdot\Xi} dX \big(1 + \langle g^{-1}\Xi, \Xi\rangle\big)^{-k},$$

so that, with $k = 2n+1$,

$$|\hat{a}(-\Xi)| \le \big(1 + \langle g^{-1}\Xi, \Xi\rangle\big)^{-2n-1} \int \sum_{0\le l\le 2n+1} C_{2n+1}^l |a^{(l)}(X)(g^{-1}\Xi)^l| dX,$$

entailing

$$|\hat{a}(-\Xi)| \le \big(1 + \langle g^{-1}\Xi, \Xi\rangle\big)^{-2n-1}$$
$$\times \int \sum_{0\le l\le 2n+1} C_{2n+1}^l \|a\|_{g,U}^{(2n+1,2n+1)} \langle gg^{-1}\Xi, g^{-1}\Xi\rangle^{l/2} \big(1 + g^\sigma(X-U)\big)^{-n-\frac{1}{2}} dX.$$

This gives

$$|\hat{a}(-\Xi)| \le \big(1 + \langle g^{-1}\Xi, \Xi\rangle\big)^{-2n-1} \big(1 + \langle g^{-1}\Xi, \Xi\rangle^{1/2}\big)^{2n+1} \|a\|_{g,U}^{(2n+1,2n+1)}$$
$$\times \int \big(1 + g^\sigma(X-U)\big)^{-n-\frac{1}{2}} dX.$$

With X_0 standing for the center of U, we have, with $X' \in U$,

$$1 + g(X - X_0) \le 1 + 2g(X - X') + 2g(X' - X_0) \le 3 + 2g^\sigma(X - X')$$

and consequently $1 + g(X - X_0) \le 3\big(1 + g^\sigma(X - U)\big)$, so that

$$|\hat{a}(-\Xi)|$$
$$\le \big(1 + \langle g^{-1}\Xi, \Xi\rangle\big)^{-n-\frac{1}{2}} 2^{n+\frac{1}{2}} \|a\|_{g,U}^{(2n+1,2n+1)} \int \big(1 + g(X - X_0)\big)^{-n-\frac{1}{2}} dX 3^{n+\frac{1}{2}}$$
$$\le \big(1 + \langle g^{-1}\Xi, \Xi\rangle\big)^{-n-\frac{1}{2}} \|a\|_{g,U}^{(2n+1,2n+1)} |g|^{-1/2} c(n).$$

As a consequence, the L^1-norm of $\hat{a}$ is bounded above by the rhs of (2.5.1), and using the estimate (2.1.13) we get the result of the lemma. □

Proof of the theorem 2.5.1. It is now a simple matter to obtain the L^2-boundedness of Theorem 2.5.1. We write, using the partition of unity given by Theorem 2.2.7,

$$a^w = \int a_Y^w |g_Y|^{1/2} dY$$

and we note that from the previous lemma, $\sup_Y \|a_Y^w\|_{\mathcal{L}(L^2)} < \infty$. Moreover, the estimates (2.3.4), (2.2.29) imply that, for all $l, k = k_1 + k_2$,

$$\begin{aligned}
&|(\bar{a}_Y \sharp a_Z)^{(l)} T^l| \\
&\leq A_{k,l}(g_Y + g_Z)(T)^{l/2} \Big(1 + (g_Y^\sigma \wedge g_Z^\sigma)(X - U_Y) + (g_Y^\sigma \wedge g_Z^\sigma)(X - U_Z)\Big)^{-k/2} \\
&\leq A_{k,l} g_Y(T)^{l/2} \delta_r(Y,Z)^{N_0 l} \delta_r(Y,Z)^{-k_1} \big(1 + g_Y^\sigma(X - U_Y)\big)^{-k_2/2} \delta_r(Y,Z)^{N_0 k_2} \\
&\leq A_{k,l} g_Y(T)^{l/2} \big(1 + g_Y^\sigma(X - U_Y)\big)^{-k_2/2} \delta_r(Y,Z)^{N_0 k_2 + N_0 l - k_1},
\end{aligned}$$

so that, for l, N, M given, one can choose

$$k_2 = N, \quad k_1 - N_0 N - N_0 l \geq M$$

and get that $\bar{a}_Y \sharp a_Z$ is g_Y-confined in U_Y, so that with the previous lemma, we obtain

$$\max(\|\bar{a}_Y^w a_Z^w\|_{\mathcal{L}(L^2)}^{1/2}, \|a_Y^w \bar{a}_Z^w\|_{\mathcal{L}(L^2)}^{1/2}) \leq C_M \delta(Y,Z)^{-M} \tag{2.5.2}$$

for all M. To conclude, we note that (2.5.2) implies that the assumptions of (Cotlar's) Lemma 4.7.1 are fulfilled. The proof of the theorem 2.5.1 is complete. □

The following lemma will be useful later on.

Lemma 2.5.3. *Let g be an admissible metric on $\mathbb{R}^{2n}$ and $\int \varphi_Y |g_Y|^{1/2} dY = 1$ be a partition of unity related to g (cf. Theorem 2.2.7). There exists a positive constant C such that for all $u \in L^2(\mathbb{R}^n)$*

$$C^{-1} \|u\|_{L^2(\mathbb{R}^n)}^2 \leq \int \|\varphi_Y^w u\|_{L^2(\mathbb{R}^n)}^2 |g_Y|^{1/2} dY \leq C \|u\|_{L^2(\mathbb{R}^n)}^2. \tag{2.5.3}$$

Proof of the lemma. The right inequality is indeed a direct consequence of Cotlar's lemma 4.7.1 so we leave it to the reader and provide a proof of the more involved left inequality. The confinement estimates of Theorem 2.3.3 imply that for all $N \geq 1$, there exists C_N such that

$$\|\varphi_Y^w \varphi_Z^w\|_{\mathcal{L}(L^2)} \leq C_N \delta_r(Y,Z)^{-N}.$$

Now since the φ_Y are real-valued and a partition of unity, we obtain with $L^2(\mathbb{R}^n)$ norms and dot products

$$\begin{aligned}\|u\|^2 &= \iint \langle \varphi_Y^w u, \varphi_Z^w u\rangle |g_Y|^{1/2}|g_Z|^{1/2} dY dZ \\ &\le \iint_{\delta_r(Y,Z)\le\alpha} \Big(\frac{\alpha}{\delta_r(Y,Z)}\Big)^{N_0} \|\varphi_Y^w u\|\|\varphi_Z^w u\||g_Y|^{1/2}|g_Z|^{1/2} dY dZ \\ &\qquad + \underbrace{\iint_{\delta_r(Y,Z)>\alpha} \langle \varphi_Y^w \varphi_Z^w u, u\rangle |g_Y|^{1/2}|g_Z|^{1/2} dY dZ}_{\langle Ru,u\rangle}, \quad (2.5.4)\end{aligned}$$

where α is a positive parameter to be chosen later. We check now the self-adjoint operator R defined above: from Cotlar's lemma, we have

$$\|R\|_{\mathcal{L}(L^2)} \le \sup_{\substack{Y_0,Z_0 \\ \text{with } \delta_r(Y_0,Z_0)>\alpha}} \left[\iint_{\delta_r(Y,Z)>\alpha} \|\varphi_{Y_o}^w \varphi_{Z_o}^w \varphi_Y^w \varphi_Z^w\|_{\mathcal{L}(L^2)}^{1/2} |g_Y|^{1/2}|g_Z|^{1/2} dY dZ\right],$$

and since

$$\begin{aligned}\|\varphi_{Y_o}^w \varphi_{Z_o}^w \varphi_Y^w \varphi_Z^w\|_{\mathcal{L}(L^2)}^2 &\le \|\varphi_{Y_o}^w \varphi_{Z_o}^w\|\ \|\varphi_Y^w \varphi_Z^w\|\ \|\varphi_{Y_o}^w\|\ \|\varphi_{Z_o}^w \varphi_Y^w\|\ \|\varphi_Z^w\| \\ &\le C_N \delta_r(Z_o,Y_o)^{-N} C_N \delta_r(Z,Y)^{-N} C_N \delta_r(Z_o,Y)^{-N} \sup_Z \|\varphi_Z^w\|^2,\end{aligned}$$

we get that for $N/4 = N_1$, with N_1 such that (2.2.29) holds,

$$\begin{aligned}\|R\|_{\mathcal{L}(L^2)} &\le C_N^{3/4} \sup_Y \|\varphi_Y^w\|^{1/2} \\ &\sup_{\substack{Y_o,Z_o \\ \text{with } \delta_r(Y_o,Z_o)>\alpha}} \delta_r(Y_o,Z_o)^{-N/4} \iint \delta_r(Y,Z)^{-N/4} \delta_r(Z_o,Y)^{-N/4} |g_Y|^{1/2}|g_Z|^{1/2} dY dZ \\ &\le C_N^{3/4} \sup_Y \|\varphi_Y^w\|^{1/2} \alpha^{-N/4} \int \delta_r(Z_o,Y)^{-N/4} \int \delta_r(Z,Y)^{-N/4} |g_Z|^{1/2} dZ |g_Y|^{1/2} dY \\ &\le C_{4N_1}^{3/4} \sup_Y \|\varphi_Y^w\|^{1/2}\ \alpha^{-N_1} C_0^2. \quad (2.5.5)\end{aligned}$$

We choose now the parameter α so that $C_{4N_1}^{3/4} \sup_Y \|\varphi_Y^w\|^{1/2}\ \alpha^{-N_1} C_0^2 = 1/2$ and we get from (2.5.4) and (2.5.5) that

$$\|u\|^2 \le \iint_{\delta_r(Y,Z)\le\alpha} \Big(\frac{\alpha}{\delta_r(Y,Z)}\Big)^{N_1} \|\varphi_Y^w u\|\|\varphi_Z^w u\||g_Y|^{1/2}|g_Z|^{1/2} dZ dY + \frac{1}{2}\|u\|^2.$$

The kernel $\delta_r(Y,Z)^{-N_1}$ is of Schur type (Remark 4.7.3) on $L^2(\mathbb{R}^{2n}, |g_Z|^{1/2}dZ)$ from (2.2.29) and thus we obtain

$$\|u\|^2 \le 2\alpha^{N_1} C_0 \int \|\varphi_Y^w u\|^2 |g_Y|^{1/2} dY,$$

which is the statement of the lemma whose proof is now complete. □

2.5.2 The Gårding inequality with gain of one derivative

We want to prove in this section a generalization of Theorems 1.1.26, 1.1.39; it says that a non-negative symbol of order 1, related to an admissible metric, is quantized by an operator which is semi-bounded from below. According to the remark 2.2.17, to be of order 1 for a symbol a means that $a \in S(\lambda_g, g)$. The main point in this generalization is that the non-negativity for the operator as a consequence of the non-negativity of its symbol holds true as well for any admissible metric g.

However, we want also to deal with systems, including infinite-dimensional systems and prove our inequality in that framework. So far we have dealt only with scalar-valued (complex-valued) symbols ; let us first consider a symbol a defined on $\mathbb{R}^{2n}$ but valued in the algebra of $N \times N$ matrices. It means simply that $a = (a_{jk})_{1\leq j,k\leq N}$ where each a_{jk} belongs to $S(m, g)$ for some g-admissible weight m. In the remark 2.1.8, we have already considered that case, including the infinite-dimensional case where the $N \times N$ matrices are replaced by the algebra $\mathscr{B}(\mathbb{H})$ of bounded operator on some Hilbert space. Although many results can be extended without much change to this "matrix-valued" case, it is very important to keep in mind that $\mathscr{B}(\mathbb{H})$ is not commutative as soon as $\dim \mathbb{H} > 1$ and that the composition formula and the Poisson bracket should be given the proper definition, taking into account the position of the various terms. Anyhow, we shall skip checking all the details of that calculus for systems of pseudodifferential operators and take advantage of the very simple proof using the Wick calculus to extend the result to that case.

Theorem 2.5.4. *Let g be an admissible metric on $\mathbb{R}^{2n}$, $\mathbb{H}$ be a Hilbert space, a be a symbol in $S(\lambda_g, g)$ valued in the non-negative symmetric bounded operators on $\mathbb{H}$. Then the operator a^w is semi-bounded from below, and more precisely, there exists $l \in \mathbb{N}$ and C depending only on n such that*

$$\forall u \in \mathscr{S}(\mathbb{R}^n; \mathbb{H}), \quad \langle a^w u, u\rangle + C\|a\|^{(l)}_{S(\lambda_g,g)}\|u\|^2_{L^2(\mathbb{R}^n;\mathbb{H})} \geq 0. \tag{2.5.6}$$

Under the same hypothesis, the same result is true with $\langle a^w u, u\rangle$ replaced by $\mathrm{Re}\langle a(x, D)u, u\rangle$.

Proof. Using Theorem 2.2.7, one can find a family $(\varphi_Y)_{Y\in\mathbb{R}^{2n}}$ of functions uniformly in $S(1, g)$ supported in $U_{Y,r}$, nonnegative, such that $\int \varphi_Y |g_Y|^{1/2} dY = 1$. With $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$ uniformly in $S(1, g)$ and real-valued, supported in $U_{Y,2r}$, equal to 1 on $U_{Y,r}$, we have

$$\psi_Y \sharp \varphi_Y a \sharp \psi_Y = \varphi_Y a + r_Y, \tag{2.5.7}$$

and Proposition 2.3.16 implies that $(r_Y)_{Y\in\mathbb{R}^{2n}}$ is a uniformly confined family of symbols, so that

$$a^w \equiv \int_{\mathbb{R}^{2n}} \psi^w_Y (\varphi_Y a)^w \psi^w_Y |g_Y|^{1/2} dY, \qquad \mathrm{mod}\ \mathscr{L}(L^2(\mathbb{R}^n)). \tag{2.5.8}$$

The symbol $\varphi_Y a$ belongs uniformly to $S(\lambda_g(Y), g_Y) \subset S(\lambda_g(Y), \lambda_g(Y)^{-1} g_Y^\natural)$, thanks to (2.2.22) and $g_Y^\natural = (g_Y^\natural)^\sigma$. Using a linear symplectic mapping and Segal's formula of Theorem 2.1.2, we get that $(\varphi_Y a)^w$ is unitary equivalent to some α^w with $0 \leq \alpha \in S(\mu, \mu^{-1}|dX|^2)$ with seminorms bounded above independently of Y and $\mu = \lambda_g(Y)$. Proposition 2.4.3(1)(2) imply that $\alpha^w + C \geq 0$, where C is a seminorm of α and thus of a, so that $(a\varphi_Y)^w + C \geq 0$. Plugging this in (2.5.8), we get the result since

$$\int \psi_Y^w \psi_Y^w |g_Y|^{1/2} dY \in \mathscr{L}(L^2(\mathbb{R}^n)), \tag{2.5.9}$$

thanks to Cotlar's lemma. □

Comment. The reader may think that we did not pay much attention to the fact that the symbol was valued in $\mathscr{B}(\mathbb{H})$; in fact, since the ψ_Y, χ_Y are scalar-valued, the formulas (2.5.7), (2.5.8) hold without change (except that $L^2(\mathbb{R}^n)$ becomes $L^2(\mathbb{R}^n; \mathbb{H})$) and it is a simple matter to check that the $\mathscr{B}(\mathbb{H})$-valued version of Proposition 2.4.3 holds true, with the non-negativity condition $a(X) \geq 0$ meaning $a(X)$ nonnegative symmetric bounded operator in $\mathbb{H}$.

2.5.3 The Fefferman-Phong inequality

First versions

Let us consider a classical second-order symbol $a(x, \xi)$, i.e. a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that, for all multi-indices α, β

$$|(\partial_\xi^\alpha \partial_x^\beta a)(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{2-|\alpha|}. \tag{2.5.10}$$

The Fefferman-Phong inequality states that, if a satisfies (2.5.10) and is a non-negative function, there exists C such that, for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathrm{Re}\langle a(x, D)u, u\rangle_{L^2(\mathbb{R}^n)} + C\|u\|^2_{L^2(\mathbb{R}^n)} \geq 0, \tag{2.5.11}$$

or equivalently (with an a priori different constant C)

$$a^w + C \geq 0. \tag{2.5.12}$$

The constant C in (2.5.11), (2.5.12) depends only a finite number of $C_{\alpha\beta}$ in (2.5.10). More generally, we shall prove the following extension of Theorem 2.5.4, extension which works only in the scalar case (i.e. for scalar-valued symbols).

Theorem 2.5.5. *Let g be an admissible metric on $\mathbb{R}^{2n}$, a be a nonnegative symbol in $S(\lambda_g^2, g)$. Then the operator a^w is semi-bounded from below, and more precisely, there exists $l \in \mathbb{N}$ and C depending only on n such that*

$$\forall u \in \mathscr{S}(\mathbb{R}^n), \quad \langle a^w u, u\rangle + C\|a\|^{(l)}_{S(\lambda_g^2, g)}\|u\|^2_{L^2(\mathbb{R}^n)} \geq 0. \tag{2.5.13}$$

Under the same hypothesis, the same result is true with $\langle a^w u, u\rangle$ replaced by $\mathrm{Re}\langle a(x, D)u, u\rangle$.

Reduction to the constant-metric case. This theorem is a consequence of the same statement where g is replaced by a constant metric, i.e. a positive-definite quadratic form on $\mathbb{R}^{2n}$ such that $g \leq g^\sigma$. Using the partition of unity $(\varphi_Y)_{Y\in\mathbb{R}^{2n}}$ given by Theorem 2.2.7, of functions uniformly in $S(1,g)$ supported in $U_{Y,r}$, nonnegative, such that $\int \varphi_Y |g_Y|^{1/2} dY = 1$. With $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$ uniformly in $S(1,g)$, supported in $U_{Y,2r}$, equal to 1 on $U_{Y,r}$, we have

$$\psi_Y \sharp \varphi_Y a \sharp \psi_Y = \varphi_Y a + r_Y, \tag{2.5.14}$$

with $(r_Y)_{Y\in\mathbb{R}^{2n}}$ a uniformly confined family of symbols (see Proposition 2.3.16), implying that a^w is equal modulo $\mathscr{L}(L^2(\mathbb{R}^n))$ to $\int \psi_Y^w (\varphi_Y a)^w \psi_Y^w |g_Y|^{1/2} dY$. Now the symbol $\varphi_Y a$ is nonnegative and uniformly in $S(\lambda_g(Y)^2, g_Y)$ and the result for a constant metric g_Y will give the result, applying (2.5.9). *We are left with the proof of Theorem 2.5.5 in the case g constant metric.* □

The papers [25], [118] contain some counterexamples to the generalization of the above statement to systems, and we shall see that the proof is heavily relying on a Calderón-Zygmund decomposition for a nonnegative function. We are also concerned with the number of derivatives necessary to handle (2.5.13), i.e. to give an explicit bound from above for the number l occurring there. We have chosen to prove a more precise result involving an algebra of pseudodifferential operators introduced by J.Sjöstrand in [130] and [131]. To formulate our result, we need first to introduce that algebra.

Sjöstrand algebra of pseudodifferential operators

In [130] and [131], J. Sjöstrand introduced a Wiener-type algebra of pseudodifferential operators as follows. Let $\mathbb{Z}^{2n}$ be the standard lattice in $\mathbb{R}^{2n}_X$ and let $1 = \sum_{j\in\mathbb{Z}^{2n}} \chi_0(X-j), \chi_0 \in C_c^\infty(\mathbb{R}^{2n})$, be a partition of unity. We note $\chi_j(X) = \chi_0(X-j)$.

Proposition 2.5.6. *Let a be a tempered distribution on $\mathbb{R}^{2n}$. We shall say that a belongs to the class $\mathcal{A}$ if $\omega_a \in L^1(\mathbb{R}^{2n})$, with $\omega_a(\Xi) = \sup_{j\in\mathbb{Z}^{2n}} |\mathcal{F}(\chi_j a)(\Xi)|$, where $\mathcal{F}$ is the Fourier transform. Moreover, we have*

$$S^0_{0,0} \subset S^0_{0,0;2n+1} \subset \mathcal{A} \subset C^0(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n}), \tag{2.5.15}$$

where $S_{0,0;2n+1}$ is the set of functions defined on $\mathbb{R}^{2n}$ such that $|(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)| \leq C_{\alpha\beta}$ for $|\alpha| + |\beta| \leq 2n+1$. $\mathcal{A}$ is a Banach algebra for the multiplication with the norm $\|a\|_{\mathcal{A}} = \|\omega_a\|_{L^1(\mathbb{R}^{2n})}$.

Proof. In fact, we have the implications $a \in \mathcal{A} \Longrightarrow \mathcal{F}(\chi_j a) \in L^1(\mathbb{R}^{2n}) \Longrightarrow \chi_j a \in C^0 \cap L^\infty$, and, since the sum is locally finite with a fixed overlap[12], we get $a \in C^0 \cap L^\infty$. Moreover, if $a \in S^0_{0,0;2n+1}$, i.e. is bounded as well as all its derivatives

[12] If $\cap_{j\in J} \operatorname{supp}\chi_j \neq \emptyset$ then $\operatorname{card} J \leq N_0$, where N_0 depends only on the compact set $\operatorname{supp}\chi_0$.

of order $\leq 2n+1$, we have, with $P(\Xi) = (1+\|\Xi\|^2)^n$ the formula $\mathcal{F}(\chi_j a)(\Xi) = P(\Xi)^{-1}\mathcal{F}\big(P(D_X)(\chi_j a)\big)$. We get the identity

$$\mathcal{F}(\chi_j a)(\Xi) = P(\Xi)^{-1}(\Xi_1+i)^{-1}\mathcal{F}\big((D_{X_1}+i)P(D_X)(\chi_j a)\big).$$

This entails, in the cone $\{\Xi \in \mathbb{R}^{2n}, 2n|\Xi_1| \geq \|\Xi\|\}$ and thus everywhere[13]

$$|\mathcal{F}(\chi_j a)(\Xi)| \leq \underbrace{P(\Xi)^{-1}(1+\|\Xi\|)^{-1}}_{\in L^1(\mathbb{R}^{2n})} \operatorname{mes}(\operatorname{supp}\chi_0) \sup_{0\leq k\leq 2n+1} \|a^{(l)}\|_{L^\infty} C_n,$$

yielding the result. □ □

Remark 2.5.7. Since $1 \in \mathcal{A}$, $\mathcal{A}$ is not included in $\mathcal{F}(L^1(\mathbb{R}^{2n}))$. Moreover $\mathcal{A}$ contains $\mathcal{F}(L^1)$: let a be a function in $\mathcal{F}(L^1)$. With the above notations, we have

$$|\mathcal{F}(\chi_j a)(\Xi)| = \left|\int \hat{\chi}_0(\Xi - N)\hat{a}(N)e^{2i\pi j(N-\Xi)}dN\right| \leq \int |\hat{\chi}_0(\Xi-N)||\hat{a}(N)|dN,$$

and thus $\int |\omega_a(\Xi)|d\Xi \leq \|\hat{a}\|_{L^1}\|\widehat{\chi_0}\|_{L^1}$, which gives the inclusion. Moreover, $\mathcal{A}$ is a Banach commutative algebra for the multiplication.

Proposition 2.5.8. *The algebra $\mathcal{A}$ is stable by change of quantization, i.e. for all t real, $a \in \mathcal{A} \Longleftrightarrow J^t a = \exp(2i\pi t D_x \cdot D_\xi)a \in \mathcal{A}$. The bilinear map $a_1, a_2 \mapsto a_1 \sharp a_2$ is defined on $\mathcal{A}\times\mathcal{A}$ and continuous valued in $\mathcal{A}$, which is a (noncommutative) Banach algebra for $\sharp$. The maps $a \mapsto a^w, a(x,D)$ are continuous from $\mathcal{A}$ to $\mathcal{L}(L^2(\mathbb{R}^n))$.*

Proof. The proof is given in [130]. A. Boulkhemair established a lot more results on this algebra in his paper [21]. □

Remark 2.5.9. The standard Wiener's Lemma states that if $a \in \ell^1(\mathbb{Z}^d)$ is such that $u \mapsto a * u = C_a u$ is invertible as an operator on $\ell^2(\mathbb{Z}^d)$, then the inverse operator is of the form C_b for some $b \in \ell^1(\mathbb{Z}^d)$. In [131] the author is proving several types of Wiener lemma for $\mathcal{A}$. First a commutative version, saying that if $a \in \mathcal{A}$ and $1/a$ is a bounded function, then $1/a$ belongs to $\mathcal{A}$. Next, the theorem 4.1 of [131] provides a noncommutative version of the Wiener lemma for the algebra $\mathcal{A}$: if an operator a^w with $a \in \mathcal{A}$ is invertible as a continuous operator on L^2, then the inverse operator is b^w with $b \in \mathcal{A}$. In the paper [51], K. Gröchenig and M. Leinert prove several versions of the noncommutative Wiener lemma, and their definition of the twisted convolution ((1.1) in [51]) is indeed very close to (a discrete version of) the composition formula (1.2.2) above.

Also J. Sjöstrand proved in the proposition 5.1 of [131] the standard Gårding inequality with gain of one derivative for his class, in the semi-classical setting, where h is a small parameter in $(0,1]$:

$$a \geq 0, a'' \in \mathcal{A} \Longrightarrow a(x,h\xi)^w + Ch \geq 0. \tag{2.5.16}$$

[13] $\mathbb{R}^{2n} = \cup_{1\leq k\leq 2n}\{\Xi \in \mathbb{R}^{2n}, 2n|\Xi_k| \geq \|\Xi\|\}$ since the complement of that union is empty: it is not possible to find Ξ so that $\max_{1\leq k\leq 2n} 2n|\Xi_k| < \|\Xi\| \leq 2n \max_{1\leq k\leq 2n}|\Xi_k|$.

A consequence of the result of [21] is that[14]

$$a \geq 0, a^{(4)} \in S^0_{0,0} \Longrightarrow a(x,h\xi)^w + Ch^2 \geq 0. \tag{2.5.17}$$

We shall improve that statement, keeping the conclusion and weakening the hypothesis:

$$a \geq 0, a^{(4)} \in \mathcal{A} \Longrightarrow a(x,h\xi)^w + Ch^2 \geq 0. \tag{2.5.18}$$

From the first two inclusions in (2.5.15), we see that (2.5.18) implies (2.5.17). Moreover the constant C in (2.5.18) will depend only on the dimension and on the norm of $a^{(4)}$ in $\mathcal{A}$, which is much more precise than the dependence of C in (2.5.17), which depends on a finite number of semi-norms of a in $S^0_{0,0}$. Although (2.5.18) looks stronger than (2.5.16) since $h^2 \ll h$, it is not obvious to actually *deduce* (2.5.16) from (2.5.18). Anyhow we shall see that they are both true and that the proof of (2.5.16) is an immediate consequence of the most elementary properties of the so-called Wick quantization exposed in our section 2.4.1. Note also that a version of the Hörmander-Melin inequality with gain of $6/5$ of derivatives was given, in the semi-classical setting, by F. Hérau ([61]): this author used the assumption (6.4) of the theorem 6.2 of [73], but with a limited regularity on the symbol a, which is only such that $a^{(3)} \in \mathcal{A}$.

Fefferman-Phong inequalities

Theorem 2.5.10. *Let n be a positive integer. There exists a constant C_n such that, for all nonnegative functions a defined on $\mathbb{R}^{2n}$ satisfying $a^{(4)} \in \mathcal{A}$, the operator a^w is semi-bounded from below and, more precisely, satisfies*

$$a^w + C_n \|a^{(4)}\|_{\mathcal{A}} \geq 0. \tag{2.5.19}$$

The Banach algebra $\mathcal{A}$ is defined in Proposition 2.5.6. Note that the constant C_n depends only on the dimension n.

Proof. The proof is given in the subsection starting on page 122. □

Corollary 2.5.11. *Let n be a positive integer.*

(i) Let $a(x,\xi)$ be a nonnegative function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that (2.5.10) *is satisfied for $|\alpha| + |\beta| \leq 2n+5$. Then* (2.5.11) *and* (2.5.12) *hold with a constant C depending only on n and on $\max_{|\alpha|+|\beta|\leq 2n+5} C_{\alpha\beta}$.*

(ii) Let $a(x,\xi,h)$ be a nonnegative function defined on $\mathbb{R}^n \times \mathbb{R}^n \times (0,1]$ such that

$$|(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi,h)| \leq h^{|\alpha|} C_{\alpha\beta}, \qquad \text{for } 4 \leq |\alpha| + |\beta| \leq 2n+5.$$

Then $a^w + Ch^2 \geq 0$ and $\operatorname{Re} a(x,D) + Ch^2 \geq 0$ hold with a constant C depending only on n and on $\max_{4\leq|\alpha|+|\beta|\leq 2n+5} C_{\alpha\beta}$.

[14] In fact the operator $h^{-2}a(x,h\xi)^w$ is unitarily equivalent to $h^{-2}a(h^{1/2}x, h^{1/2}\xi)^w$ and the function $b(x,\xi) = h^{-2}a(h^{1/2}x, h^{1/2}\xi)$ is nonnegative and satisfies $b^{(4)}(x,\xi) = a^{(4)}(h^{1/2}x, h^{1/2}\xi)$ which is uniformly in $S^0_{0,0}$ whenever h is bounded and $a^{(4)} \in S^0_{0,0}$.

(iii) *Let $a(x,\xi)$ be a nonnegative function defined on $\mathbb{R}^n\times\mathbb{R}^n$ such that $a^{(4)}$ belongs to $\mathcal{A}$. Then $a(x,h\xi)^w+C\left\|a^{(4)}\right\|_{\mathcal{A}}h^2\geq 0$ and $\operatorname{Re}a(x,hD)+C\left\|a^{(4)}\right\|_{\mathcal{A}}h^2\geq 0$ hold with a constant C depending only on n.*

(iv) *Let $a(x,\xi,h)$ be a nonnegative function defined on $\mathbb{R}^n\times\mathbb{R}^n\times(0,1]$ such that, for $|\alpha|+|\beta|=4$, the functions $(x,\xi)\mapsto(\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha|}$ belong to $\mathcal{A}$ with a norm bounded above by ν_0 for all $h\in(0,1]$. Then $a^w+C\nu_0h^2\geq 0$ and $\operatorname{Re}a(x,D)+C\nu_0h^2\geq 0$ hold with a constant C depending only on n.*

Proof. That corollary is proven in the subsection starting on page 129. □

Remark 2.5.12. The recent paper [23] by A. Boulkhemair cuts significantly the requirement on the number of derivatives down to $n+4+\varepsilon$.

Remark 2.5.13. Theorem 2.5.10 implies Theorem 2.5.5 since it implies the constant metric particular case of the latter statement. In fact repeating the argument before (2.5.9), we see that for a nonnegative in $S(\lambda_g^2,g)$ where g is a constant admissible metric, i.e. a positive-definite quadratic form such that $g\leq g^\sigma$, the operator a^w is unitarily equivalent to α^w where α is nonnegative in $S(\mu^2,\mu^{-1}|dX|^2)$ with the same seminorms and $\mu\geq 1$; in particular we have that $\alpha^{(4)}$ belongs to $S^0_{0,0}$ which is a subset of $\mathcal{A}$. Applying Theorem 2.5.10 gives readily the constant-metric case.

Remark 2.5.14. We have paid attention to counting precisely the number of derivatives only for the standard metric $|dx|^2+\langle\xi\rangle^{-2}|d\xi|^2$ and not for any admissible metric. However, we do not have any necessary condition supporting the sharpness of the requirements of the best results of [23]. It is certainly possible to obtain a rather explicit estimate of l in Theorem 2.5.5, but the reader will see in the proof that to obtain that the operator $\int r_Y^w|g_Y|^{1/2}dY$ is a bounded operator in $L^2(\mathbb{R}^n)$ with r_Y defined by (2.5.14), requires some effort, even in the standard case. In the case of a general admissible metric, it is quite likely that the Corollary 2.5.11(i) can be generalized, and maybe improved as in [23], but we have not pursued these improvements. The number of derivatives required to have the Fefferman-Phong inequality for a general admissible metric will be of course larger than for the constant metric (say 4+n+ϵ as in [23]), but that number seems also to depend on the temperance properties of the metric, namely on the number N_1 occurring in (2.2.29). The notions introduced in our definition 4.9.10 may play a rôle in formulating a statement on this topic.

The following improvement of the Gårding inequality with gain of one derivative is due to J. Sjöstrand in [131].

Theorem 2.5.15. *Let a be a nonnegative function defined on $\mathbb{R}^{2n}$ such that the second derivatives a'' belongs to $\mathcal{A}$. Then we have*

$$a^w+C_n\|a''\|_{\mathcal{A}}\geq 0. \tag{2.5.20}$$

Proof. Although a proof of this result is given in [131]*(proposition 5.1)*, it is a nice and simple introduction to our more complicated argument of the sequel. From

the proposition 2.4.3, we have

$$a^w = a^{\mathrm{Wick}} - r(a)^w \geq -r(a)^w,$$

with $r(a)(X) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta) a''(X+\theta Y) Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta$. Since $\mathcal{A}$ is stable by translation (see Lemma 4.8.1), we see that $r(a) \in \mathcal{A}$ and thus $r(a)^w$ is bounded on $L^2(\mathbb{R}^n)$ from the proposition 2.5.8. □

Remark 2.5.16. This theorem implies as well the following semi-classical version; let a be function satisfying the assumption of Theorem 2.5.15. For $h \in (0,1]$, we define $A_h(x,\xi) = h^{-1} a(xh^{1/2}, \xi h^{1/2})$. The function A_h is nonnegative with a second derivative bounded in $\mathcal{A}$ by $\mathrm{cst} \times \|a''\|_{\mathcal{A}}$ (see Lemma 4.8.1), so that the previous theorem implies, with C depending only on the dimension, that $A_h^w + C\|a''\|_{\mathcal{A}} \geq 0$. Since A_h^w is unitarily equivalent to $h^{-1} a(x, h\xi)^w$, this gives

$$a(x, h\xi)^w + hC\|a''\|_{\mathcal{A}} \geq 0. \tag{2.5.21}$$

Proof of Theorem 2.5.10, Step 1: Sharp estimates for the remainders

Property (2.4.6) falls short of providing a proof for the Fefferman-Phong inequality, which gains two derivatives. However, we shall be able to use some of the properties of that quantization to handle an improved version of the Fefferman-Phong inequality.

Lemma 2.5.17. *Let a be a function defined on $\mathbb{R}^{2n}$ such that the fourth derivatives $a^{(4)}$ belong to $\mathcal{A}$. Then we have*

$$a^w = \Big(a - \frac{1}{8\pi} \operatorname{trace} a''\Big)^{Wick} + \rho_0(a^{(4)})^w,$$

with $\rho_0(a^{(4)}) \in \mathcal{A}$ and more precisely $\|\rho_0(a^{(4)})\|_{\mathcal{A}} \leq C_n \|a^{(4)}\|_{\mathcal{A}}$.

Proof. The Weyl symbol σ_a of a^{Wick} is

$$\begin{aligned}
\sigma_a(X) &= \int a(X+Y) 2^n e^{-2\pi|Y|^2} dY \\
&= a(X) + \int \frac{1}{2} a''(X) Y^2 2^n e^{-2\pi|Y|^2} dY \\
&\qquad + \frac{1}{3!} \iint_0^1 (1-\theta)^3 a^{(4)}(X+\theta Y) Y^4 2^n e^{-2\pi|Y|^2} dY d\theta \\
&= a(X) + \frac{1}{8\pi} \operatorname{trace} a''(X) \\
&\qquad + \frac{1}{3!} \iint_0^1 (1-\theta)^3 a^{(4)}(X+\theta Y) Y^4 2^n e^{-2\pi|Y|^2} dY d\theta.
\end{aligned}$$

Moreover the Weyl symbol θ_a of $(\operatorname{trace} a'')^{\mathrm{Wick}}$ is, from Proposition 2.4.3,

$$\theta_a(X) = \operatorname{trace} a''(X) + \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta)(\operatorname{trace} a'')''(X+\theta Y) Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta.$$

As a result, the Weyl symbol of the operator $\left(a - \frac{1}{8\pi}\operatorname{trace} a''\right)^{\mathrm{Wick}}$ is

$$a + \frac{1}{3!}\iint_0^1 (1-\theta)^3 a^{(4)}(X+\theta Y) Y^4 2^n e^{-2\pi|Y|^2} dY d\theta \\ - \frac{1}{8\pi}\int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta)(\operatorname{trace} a'')''(X+\theta Y) Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta.$$

We get the equality in the lemma with

$$\rho_0(a^{(4)})(X) = \frac{1}{8\pi}\int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta)(\operatorname{trace} a'')''(X+\theta Y) Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta \\ - \frac{1}{3!}\iint_0^1 (1-\theta)^3 a^{(4)}(X+\theta Y) Y^4 2^n e^{-2\pi|Y|^2} dY d\theta. \quad (2.5.22)$$

We note now that ρ_0 depends linearly on $a^{(4)}$ and that

$$\rho_0(a^{(4)})(X) = \iint_0^1 a^{(4)}(X+\theta Y) \underbrace{M(\theta, Y)}_{\substack{\text{polynomial} \\ \text{in } Y, \theta.}} e^{-2\pi|Y|^2} dY d\theta. \quad (2.5.23)$$

Looking now at the formula (2.5.23) and applying the lemma 4.8.1, we get

$$\|\rho_0(a^{(4)})\|_{\mathcal{A}} \leq \iint_0^1 M(\theta, Y) e^{-2\pi|Y|^2} dY d\theta C_0 \|a^{(4)}\|_{\mathcal{A}} = C_1 \|a^{(4)}\|_{\mathcal{A}}.$$

The proof of Lemma 2.5.17 is complete. □

Remark 2.5.18. We note that, from the lemma 2.5.17 and the L^2-boundedness of operators with symbols in $\mathcal{A}$, Theorem 2.5.19 is reduced to proving

$$a \geq 0, a^{(4)} \in \mathcal{A} \Longrightarrow \left(a - \frac{1}{8\pi}\operatorname{trace} a''\right)^{\mathrm{Wick}} \text{is semi-bounded from below.} \quad (2.5.24)$$

Naturally, one should not expect the quantity $a - \frac{1}{8\pi}\operatorname{trace} a''$ to be nonnegative: this quantity will take negative values even in the simplest case $a(x,\xi) = x^2 + \xi^2$, so that the positivity of the quantization expressed by (2.4.6) is far from enough to get our result. We shall prove below a stronger version of (2.5.24). In particular, one may certainly weaken the assumption down to require only the implicit-looking $(a^{(4)})^w \in \mathscr{L}(L^2(\mathbb{R}^n))$.

The proof, Step 2: nonnegative $C^{3,1}$ functions are sum of squares of $C^{1,1}$ functions

The key element in the proof is the following result on the decomposition of a $C^{3,1}$ nonnegative function as a sum of squares of functions in $C^{1,1}$ with controlled bounds .

Theorem 2.5.19. *Let m be a nonnegative integer. There exists an integer N and a positive constant C such that the following property holds. Let a be a nonnegative $C^{3,1}$ function[15] defined on $\mathbb{R}^m$ such that $a^{(4)} \in L^\infty$; then we can write*

$$a = \sum_{1 \le j \le N} b_j^2, \tag{2.5.25}$$

where the b_j are $C^{1,1}$ functions such that $b_j'', (b_j' b_j'')', (b_j b_j'')'' \in L^\infty$. More precisely, we have

$$\|b_j''\|_{L^\infty}^2 + \|(b_j' b_j'')'\|_{L^\infty} + \|(b_j b_j'')''\|_{L^\infty} \le C \|a^{(4)}\|_{L^\infty}. \tag{2.5.26}$$

Note that this implies that each function b_j is such that b_j^2 is $C^{3,1}$ and that N and C depend only on the dimension m.

Comment. Part of this theorem is a consequence of the classical proof of the Fefferman-Phong inequality ([44]) and of the more refined analysis of J.-M. Bony in [16] (see also [53] and [133]). However the control of the L^∞ norm of the quantities $(b_j' b_j'')', (b_j b_j'')''$ is more difficult to achieve. Naturally the inequality (2.5.26) is a key element of our proof, since it is connected with the estimates (2.4.22).

Proof of the theorem. We define

$$\rho(x) = \left(|a(x)| + |a''(x)|^2\right)^{1/4}, \quad \Omega = \{x, \rho(x) > 0\}, \tag{2.5.27}$$

assuming as we may $\|a^{(4)}\|_{L^\infty} \le 1$. Note that, since ρ is continuous, the set Ω is open. The metric $|dx|^2/\rho(x)^2$ is slowly varying in Ω (see the lemma 4.3.13): $\exists r_0 > 0, C_0 \ge 1$ such that

$$x \in \Omega, |y - x| \le r_0 \rho(x) \Longrightarrow y \in \Omega, C_0^{-1} \le \frac{\rho(x)}{\rho(y)} \le C_0. \tag{2.5.28}$$

The constants r_0, C_0 can be chosen as "universal" constants, thanks to the normalization on $a^{(4)}$ above. Moreover, using Lemma 4.3.11, the nonnegativity of a implies with $\gamma_j = 1$ for $j = 0, 2, 4$, $\gamma_1 = 3, \gamma_3 = 4$,

$$|a^{(j)}(x)| \le \gamma_j \rho(x)^{4-j}, \quad 1 \le j \le 4. \tag{2.5.29}$$

Applying Theorem 4.3.7, we get the following lemma.

[15] A $C^{3,1}$ function is a C^3 function whose third-order derivatives are Lipschitz continuous.

Lemma 2.5.20. *Let a, ρ, Ω, r_0 be as above. There exists a positive number $r_0' \leq r_0$, such that for all $r \in]0, r_0']$, there exists a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ of points in Ω and a positive number M_r, such that the following properties are satisfied. We define $U_\nu, U_\nu^*, U_\nu^{**}$ as the closed Euclidean balls with center x_ν and radius $r\rho_\nu, 2r\rho_\nu, 4r\rho_\nu$ with $\rho_\nu = \rho(x_\nu)$. There exist two families of nonnegative smooth functions on $\mathbb{R}^m$, $(\varphi_\nu)_{\nu\in\mathbb{N}}$, $(\psi_\nu)_{\nu\in\mathbb{N}}$ such that*

$$\sum_\nu \varphi_\nu^2(x) = \mathbf{1}_\Omega(x),\ \operatorname{supp}\varphi_\nu \subset U_\nu, \qquad \psi_\nu \equiv 1 \quad \text{on } U_\nu^*,\ \operatorname{supp}\psi_\nu \subset U_\nu^{**} \subset \Omega.$$

Moreover, for all $l \in \mathbb{N}$, we have

$$\sup_{x\in\Omega,\nu\in\mathbb{N}} \|\varphi_\nu^{(l)}(x)\|\rho_\nu^l + \sup_{x\in\Omega,\nu\in\mathbb{N}} \|\psi_\nu^{(l)}(x)\|\rho_\nu^l < \infty.$$

The overlap of the balls U_ν^{**} is bounded, i.e.

$$\bigcap_{\nu\in\mathcal{N}} U_\nu^{**} \neq \emptyset \qquad \Longrightarrow \qquad \#\mathcal{N} \leq M_r.$$

Moreover, $\rho(x) \sim \rho_\nu$ all over U_ν^{**} (i.e. the ratios $\rho(x)/\rho_\nu$ are bounded above and below by a fixed constant, provided that $x \in U_\nu^{**}$).

Since a is vanishing on Ω^c, we obtain

$$a(x) = \sum_{\nu\in\mathbb{N}} a(x)\varphi_\nu^2(x). \tag{2.5.30}$$

Definition 2.5.21. Let a, ρ, Ω be as above. Let θ be a positive number $\leq \theta_0$, where θ_0 is a fixed constant satisfying the requirements of Lemma 4.3.17. A point $x \in \Omega$ is said to be

(i) θ-elliptic whenever $a(x) \geq \theta\rho(x)^4$,

(ii) θ-nondegenerate whenever $a(x) < \theta\rho(x)^4$: we have then $\|a''(x)\|^2 \geq \rho(x)^4/2$.

We go on now with the proof of Theorem 2.5.19. We choose a positive number θ satisfying the condition in Definition 2.5.21. We choose a positive number $r \leq r_0'$ as defined in Lemma 2.5.20 and we consider a sequence (x_ν) as in that lemma. We assume also that $4r \leq \theta/8$, so that Lemma 2.5.20 can be applied on the ball U_ν^{**}.

Let us first consider the "elliptic" indices ν such that x_ν is θ-elliptic. According to the lemma 4.3.15, for $x \in U_\nu^{**}$, we have $a(x) \sim \rho_\nu^4$, so that with

$$b_\nu(x) = a(x)^{1/2}\psi_\nu(x), \quad b_\nu^2 = a\psi_\nu^2, \quad \varphi_\nu^2 b_\nu^2 = a\varphi_\nu^2 \tag{2.5.31}$$

and on $\operatorname{supp}\varphi_\nu$ (where $\psi_\nu \equiv 1$),

$$\begin{cases} b_\nu' &= 2^{-1}a^{-1/2}a', \\ b_\nu'' &= -2^{-2}a^{-3/2}a'^2 + 2^{-1}a^{-1/2}a'', \\ b_\nu''' &= 3\times 2^{-3}a^{-5/2}a'^3 - \frac{3}{4}a^{-3/2}a'a'' + 2^{-1}a^{-1/2}a''', \\ b_\nu^{(4)} &= -\frac{15}{16}a^{-7/2}a'^4 + \frac{9}{4}a^{-5/2}a'^2a'' - \frac{3}{4}a^{-3/2}a''^2 - a^{-3/2}a'a''' + \frac{1}{2}a^{-1/2}a^{(4)}, \end{cases}$$

yielding

$$\begin{cases} |b'_\nu| & \leq 2^{-1}a^{-1/2}|a'| \lesssim a^{-1/2}\rho^3 \lesssim \rho, \\ |b''_\nu| & \lesssim a^{-3/2}\rho^6 + a^{-1/2}\rho^2 \lesssim 1, \\ |b'''_\nu| & \lesssim a^{-5/2}\rho^9 + a^{-3/2}\rho^3\rho^2 + a^{-1/2}\rho \lesssim \rho^{-1}, \\ |b^{(4)}_\nu| & \lesssim a^{-7/2}\rho^{12} + a^{-5/2}\rho^6\rho^2 + a^{-3/2}\rho^4 + a^{-3/2}\rho^3\rho + a^{-1/2} \lesssim \rho^{-2}. \end{cases}$$

Note in particular that

$$|b_\nu b^{(4)}_\nu| + |b^{(1)}_\nu b^{(3)}_\nu| + |b^{(2)}_\nu b^{(2)}_\nu| \leq C(\theta). \tag{2.5.32}$$

The whole difficulty is concentrated on the next case.

The nondegenerate indices ν are those for which x_ν is θ-nondegenerate. Since $4r \leq \theta/8 \leq \theta^{1/2}$, we can apply the remark 4.3.18 on the product[16]

$$Q_\nu = [-\theta^{1/4}\rho_\nu + x_{\nu 1}, \theta^{1/4}\rho_\nu + x_{\nu 1}] \times B_{\mathbb{R}^{m-1}}(x'_\nu, \theta^{1/2}\rho_\nu),$$

where $x_\nu = (x_{\nu 1}, x'_\nu) \in \mathbb{R} \times \mathbb{R}^{m-1}$. There exists

$$\alpha : B_{\mathbb{R}^{m-1}}(x'_\nu, \theta^{1/2}\rho_\nu) \to [x_{\nu 1} - \theta^{1/4}\rho_\nu, x_{\nu 1} + \theta^{1/4}\rho_\nu]$$

such that

$$\partial_1 a(\alpha(x'), x') = 0 \tag{2.5.33}$$

and $\partial_1^2 a(x) \geq \rho_\nu^2/2$ for $|x - x_\nu| \leq R_0\rho_\nu$ where $R_0 = 10^{-2}$ according to the lemma 4.3.16. We have on Q_ν

$$\begin{aligned} a(x) &= a(x_1, x') \\ &= \int_0^1 (1-t)\partial_1^2 a\big(\alpha(x') + t(x_1 - \alpha(x')), x'\big)dt\big(x_1 - \alpha(x')\big)^2 + a(\alpha(x'), x'). \end{aligned} \tag{2.5.34}$$

According to the remark 4.3.18, we recall that we have for $|x' - x'_\nu| \leq \theta^{1/2}\rho_\nu$,

$$\begin{cases} |\alpha(x') - x_{\nu 1}| & \leq \theta^{1/4}\rho_\nu, \\ |\alpha'(x')| & \leq 2\rho_\nu^{-2}\rho(\alpha(x'), x')^2 \leq 2C_0^2 = C_1, \\ |\alpha''(x')| & \leq 2\rho_\nu^{-2}\big(4^2C_0^4 + 4^2C_0^2 + 12\big)\rho(\alpha(x'), x') \leq C_2\rho_\nu^{-1}, \\ |\alpha'''(x')| & \leq C_3\rho_\nu^{-2}, \end{cases} \tag{2.5.35}$$

with universal constants C_j. Let us now compute the derivatives of the function

$$B' = B_{\mathbb{R}^{m-1}}(x'_\nu, \theta^{1/2}\rho_\nu) \ni x' \mapsto a(\alpha(x'), x') = c(x'). \tag{2.5.36}$$

[16]Naturally the choice of the linear coordinates depends on the index ν, according to the remark 4.3.18. Note also that $U_\nu^{**} \subset Q_\nu \subset B(x_\nu, R_0\rho_\nu)$ since $4r \leq \theta^{1/2} \leq \theta^{1/4} \leq R_0$, according to the previous requirements on r and θ and also to the condition on θ in the lemma 4.3.17.

We have, denoting by ∂_2 the partial derivative with respect to x',

$$\begin{aligned}
c' &= \alpha'\partial_1 a + \partial_2 a = \partial_2 a \qquad \textbf{(here we use the identity } \partial_{\mathbf{1}}\mathbf{a}(\alpha(\mathbf{x}'),\mathbf{x}') \equiv \mathbf{0)},\\
c'' &= \alpha'\partial_1\partial_2 a + \partial_2^2 a,\\
c''' &= \alpha''\partial_1\partial_2 a + \alpha'^2\partial_1^2\partial_2 a + 2\alpha'\partial_1\partial_2^2 a + \partial_2^3 a,\\
c'''' &= \alpha'''\partial_1\partial_2 a + 3\alpha''\alpha'\partial_1^2\partial_2 a + 3\alpha''\partial_1\partial_2^2 a + \alpha'^3\partial_1^3\partial_2 a + 3\alpha'^2\partial_1^2\partial_2^2 a\\
&\qquad + 3\alpha'\partial_1\partial_2^3 a + \partial_2^4 a,
\end{aligned}$$

and we obtain

$$|c'| \lesssim \rho^3, \quad |c''| \lesssim \rho^2, \quad |c'''| \lesssim \rho^{-1}\rho^2 + \rho \sim \rho, \quad |c''''| \lesssim \rho^{-2}\rho^2 + \rho^{-1}\rho + 1 \sim 1,$$

so that

$$c \in C^{3,1}(B'), \quad |c^{(j)}| \lesssim \rho_\nu^{4-j}, 0 \le j \le 4. \tag{2.5.37}$$

Since $\partial_1^2 a \gtrsim \rho^2$ on Q_ν, we can define

$$R(x) = \omega(x)^{1/2}, \quad \omega(x) = \int_0^1 (1-t)\partial_1^2 a\big(\alpha(x') + t(x_1 - \alpha(x')), x'\big)dt. \tag{2.5.38}$$

Note also that the identity (on Q_ν), $a = R(x)^2(x_1-\alpha)^2 + a(\alpha(x'),x')$ forces the function

$$B(x) = R(x)^2(x_1-\alpha)^2$$

to be $C^{3,1}(Q_\nu)$ with a j-th derivative bounded above in absolute value by ρ_ν^{4-j} $(0 \le j \le 4)$ since it is the case for a and c (this fact is not obvious since the function R is a priori only $C^{1,1}$). Defining on Q_ν

$$b(x) = R(x)\big(x_1 - \alpha(x')\big), \tag{2.5.39}$$

we see that

$$a = b^2 + c, \quad |(b^2)^{(j)}| = |B^{(j)}| \lesssim \rho_\nu^{4-j}, \ 0 \le j \le 4. \tag{2.5.40}$$

As a consequence with $\beta = x_1 - \alpha(x')$, $b^2 = R^2\beta^2 = B \in C^{3,1}$,

$$\begin{aligned}
R^2\beta^2 &= \overbrace{B(\alpha(x'),x')}^{=0} + \overbrace{\int_0^1 \partial_1 B(\alpha(x') + \theta(x_1-\alpha(x')),x')d\theta}^{\in C^{2,1}}\,\beta,\\
|\beta^{(j)}| &\lesssim \rho^{1-j}, 0 \le j \le 3,
\end{aligned}$$

and since β vanishes on a hypersurface

$$\begin{cases} R^2\beta & = \int_0^1 \partial_1 B(\alpha(x') + \theta(x_1-\alpha(x')),x')d\theta \in C^{2,1},\\ |(R^2\beta)^{(j)}| & \lesssim \rho_\nu^{3-j}, \ 0 \le j \le 3, \quad \text{(from (2.5.40))}. \end{cases} \tag{2.5.41}$$

Also we have $0 < R^2 = \omega \in C^{1,1}, \omega \sim \rho_\nu^2$ and from (2.5.38), (2.5.35),

$$|\omega^{(j)}| \lesssim \rho_\nu^{2-j}, 0 \leq j \leq 2, \tag{2.5.42}$$

entailing that with $R = \omega^{1/2}$,

$$|R' = \frac{1}{2}\omega^{-1/2}\omega'| \lesssim 1, \quad |R'' = -\frac{1}{4}\omega^{-3/2}\omega'^2 + \frac{1}{2}\omega^{-1/2}\omega''| \lesssim \rho_\nu^{-3}\rho_\nu^2 + \rho_\nu^{-1} \sim \rho_\nu^{-1}. \tag{2.5.43}$$

Using Leibniz' formula, we get

$$(R^2\beta)''' = (\omega\beta)''' = \omega'''\beta + 3\omega''\beta' + 3\omega'\beta'' + \omega\beta''',$$

which makes sense since ω''' is a distribution of order 1 and β is $C^{2,1}$ (see (2.5.35)). From (2.5.41), we know that $(\omega\beta)'''$ is L^∞, and since it is also the case of $\omega''\beta'$, $\omega'\beta''$, $\omega\beta'''$ from (2.5.42) and (2.5.35), we get that $\omega'''\beta$ belongs to L^∞ and

$$|\omega'''\beta| \lesssim 1. \tag{2.5.44}$$

On the other hand we have

$$\omega''' = 2(RR')'' = 2(R'^2 + RR'')' = 4R'R'' + 2(RR'')' = 6R'R'' + 2\underbrace{R}_{C^{1,1}}\underbrace{R'''}_{\substack{\text{distribution}\\ \text{of order 1}}}$$

entailing from (2.5.44), that $\beta(6R'R'' + 2RR''')$ is L^∞ and since it is the case of $\beta R'R''$ (from (2.5.35) and (2.5.43)), we get that $\beta RR'''$ is L^∞ and, using the section 4.2.2, we obtain

$$|\beta RR'''| \lesssim 1, \quad \text{i.e. for all multi-indices } \gamma \text{ with length 3, } |\beta R\partial_x^\gamma R| \lesssim 1. \tag{2.5.45}$$

With $b = R\beta$, we get $b'b'' = (R'\beta + R\beta')(R''\beta + 2R'\beta' + R\beta'')$ and to check that $(b'b'')'$ is in L^∞ with

$$|(b'b'')'| \lesssim 1, \tag{2.5.46}$$

it is enough (see (2.5.35), (2.5.43)) to check the derivatives of $R''\beta R'\beta$, $R''\beta R\beta'$ which are, up to bounded terms (see the section 4.3.2 in the appendix for the meaning of the products)

$$R'''\beta R'\beta = R'''\beta RR'\frac{\beta}{R}, \quad R'''\beta R\beta'$$

which are bounded according to (2.5.45)-(2.5.43)-(2.5.35). Note that b'' is bounded from (2.5.43) and (2.5.35). We want also to verify that $(bb'')''$ is bounded. We use that $(b^2)^{(4)}$ is bounded from (2.5.40) and since we have

$$\underbrace{(b^2)''''}_{\substack{\text{bounded}\\ (2.5.40)}} = 2(b'\otimes b' + bb'')'' = 2\underbrace{(b'\otimes b'' + b''\otimes b')'}_{\substack{\text{bounded}\\ (2.5.46)}} + 2(bb'')'', \tag{2.5.47}$$

we obtain[17] the boundedness of $(bb'')''$.

Remark 2.5.22. Before going on, we should note that our functions b, c above are only defined on Q_ν where holds the identity $a(x) = b(x)^2 + c(x')$. We can replace the function c above by

$$\tilde{c}(x') = c(x')\chi\big((x' - x'_\nu)\theta^{-1/2}\rho_\nu^{-1}\big)$$

where $\chi \in C_c^\infty(\mathbb{R}^{m-1})$ supported in the unit ball and equal to 1 in the ball of radius $1/2$, so that $\tilde{c}$ is defined on $\mathbb{R}^{m-1}$ and the identity $a = b^2 + \tilde{c}$ holds on

$$x_\nu + \frac{1}{2}(Q_\nu - x_\nu) \supset U_\nu^* \supset \operatorname{supp}\varphi_\nu.$$

The bounds on the derivatives are unchanged as long as θ is fixed, which is the case.

Taking that remark into account, as well as the above estimates on the derivatives, we have finally, with E_2 standing for the nondegenerate indices,

$$\begin{aligned} a(x) &= \sum_{\nu\in\mathbb{N}} b_\nu(x)^2\varphi_\nu^2(x) + \sum_{\nu\in E_2} a_\nu(x')\varphi_\nu^2(x) \\ &|b_\nu| \lesssim \rho_\nu^2,\ |b'_\nu| \lesssim \rho_\nu,\ |b''_\nu| \lesssim 1,\ |(b_\nu b''_\nu)''| + |(b'_\nu b''_\nu)'| \lesssim 1 \\ &|a_\nu| \lesssim \rho_\nu^4,\ |a'_\nu| \lesssim \rho_\nu^3,\ |a''_\nu| \lesssim \rho_\nu^2,\ |a'''_\nu| \lesssim \rho_\nu,\ |a''''_\nu| \lesssim 1, \\ &a_\nu \text{ is defined on } \mathbb{R}^{m-1}. \end{aligned}$$

Now, we consider the function $\mathbb{R}^{m-1} \ni t \mapsto A(t) = \rho_\nu^{-4} a_\nu\big(\rho_\nu t\big)$ and we have

$$|A_\nu| \lesssim 1,\ |A'_\nu| \lesssim 1,\ |A''_\nu| \lesssim 1,\ |A'''_\nu| \lesssim 1,\ |A''''_\nu| \lesssim 1.$$

Following the main argument in the proof by C. Fefferman and D.H. Phong, we can use an induction on the dimension m to get

$$A(t) = \sum_{1\le j\le N_{m-1}} B_j^2(t), \quad B_j \in C^{1,1}, \quad \text{and } B_j'', (B'_jB''_j)', (B_jB''_j)'' \in L^\infty.$$

Incorporated in the induction hypothesis is that the bounds on B depend only on the bounds on $A^{(4)}$. We obtain

$$a(x) = \sum_{\nu\in\mathbb{N}} b_\nu(x)^2\varphi_\nu^2(x) + \sum_{\nu\in E_2}\sum_{1\le j\le N_{m-1}} \rho_\nu^4 B_{j,\nu}^2\Big(\frac{x'}{\rho_\nu}\Big)\varphi_\nu^2(x)$$

[17]The equality (2.5.47) is an equality between tensors (0,4) and it might look somewhat pedantic to resort to such notations: the reader may check directly the implication

$$\left.\begin{aligned} &\forall\gamma, |\gamma| = 4, \partial_x^\gamma(b^2) \in L^\infty \\ \forall\gamma_j, 1\le j\le 3, |\gamma_1| = 1 = |\gamma_2|, |\gamma_3| = 2, &\partial_x^{\gamma_1}(\partial_x^{\gamma_2} b\partial_x^{\gamma_3} b) \in L^\infty \end{aligned}\right\}$$
$$\Longrightarrow \forall\gamma_3,\gamma_4, |\gamma_3| = 2 = |\gamma_4|,\ \partial_x^{\gamma_3}(b\partial_x^{\gamma_4} b) \in L^\infty.$$

i.e.

$$a(x) = \sum_{1\leq j\leq N_{m-1}+1} \sum_{\nu\in\mathbb{N}} b_{\nu,j}(x)^2 \varphi_\nu^2(x).$$

One needs to pass to a finite sum, which is quite standard since the overlap of the support of the functions φ_ν is bounded; this last argument is given in the appendix 4.3.6. The proof of the theorem 2.5.19 is complete. □

The proof, Step 3: application of the Wick calculus

Let a be a nonnegative function defined on $\mathbb{R}^{2n}$ such that $a^{(4)}$ belongs to $\mathcal{A}$ (defined in Proposition 2.5.6). Applying Lemma 2.5.17 and the L^2-boundedness of the operators with Weyl symbol in $\mathcal{A}$, we see that it suffices to prove that the operator with Wick symbol $a-\frac{1}{8\pi}\operatorname{trace} a''$ is semi-bounded from below. Since $\mathcal{A}\subset L^\infty(\mathbb{R}^{2n})$, it is enough to prove the following lemma.

Lemma 2.5.23. *Let a be a nonnegative function defined on $\mathbb{R}^{2n}$ such that $a^{(4)}$ belongs to $L^\infty(\mathbb{R}^{2n})$. Theorem 2.5.19 is providing a decomposition $a=\sum_{1\leq j\leq N} b_j^2$ along with the estimates* (2.5.26)*. Then we have*

$$\Big(a-\frac{1}{8\pi}\operatorname{trace} a''\Big)^{Wick} = \sum_{1\leq j\leq N}\left[\Big(b_j-\frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{Wick}\right]^2 + R$$

where R is a L^2-bounded operator such that $\|R\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\leq C\|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})}$, C depending only on the dimension n.

Proof. We have

$$a-\frac{1}{8\pi}\operatorname{trace} a'' = a-\frac{\Delta a}{8\pi} = \sum_{1\leq j\leq N} b_j^2 - \frac{1}{4\pi}|\nabla b_j|^2 - \frac{1}{4\pi} b_j\Delta b_j. \tag{2.5.48}$$

Then using the lemma 2.4.8, we get

$$b_j^{\mathrm{Wick}} b_j^{\mathrm{Wick}} = \Big(b_j^2 - \frac{1}{4\pi}|\nabla b_j|^2\Big)^{\mathrm{Wick}} + S_j, \tag{2.5.49}$$

with

$$\|S_j\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\leq C_1\left(\|b_j''\|_{L^\infty}^2 + \|(b_j''b_j')'\|_{L^\infty} + \|(b_jb_j'')''\|_{L^\infty}\right)\leq C_2\|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})},$$

where C_1, C_2 depend only on the dimension. Moreover, we have, from Lemma 2.4.6,

$$\mathrm{Re}\Big(b_j^{\mathrm{Wick}}(\Delta b_j)^{\mathrm{Wick}}\Big) = \Big(b_j\Delta b_j - \frac{1}{4\pi}\nabla\cdot(\nabla b_j\Delta b_j) + \frac{1}{4\pi}(\Delta b_j)^2\Big)^{\mathrm{Wick}} + R_j, \tag{2.5.50}$$

with $\|R_j\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_3\|b_j''\|^2_{L^\infty(\mathbb{R}^{2n})} \leq C_4\|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})}$. As a consequence, from (2.5.49), (2.5.50), we get

$$\begin{aligned}&\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\mathrm{Wick}}\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\mathrm{Wick}}\\ &= \Big(b_j^2 - \frac{1}{4\pi}|\nabla b_j|^2 - \frac{1}{4\pi}b_j\Delta b_j\Big)^{\mathrm{Wick}} + \frac{1}{16\pi^2}\Big(\nabla\cdot(\nabla b_j \Delta b_j)\Big)^{\mathrm{Wick}} - \frac{1}{16\pi^2}\Big((\Delta b_j)^2\Big)^{\mathrm{Wick}}\\ &\qquad + S_j - \frac{1}{4\pi}R_j + \frac{1}{64\pi^2}(\Delta b_j)^{\mathrm{Wick}}(\Delta b_j)^{\mathrm{Wick}}, \quad (2.5.51)\end{aligned}$$

so that from (2.4.5), (2.5.26) and the estimates above for R_j, S_j, we obtain from (2.5.48) that

$$\sum_{1\leq j\leq N}\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\mathrm{Wick}}\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\mathrm{Wick}} = \Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\mathrm{Wick}} + S$$

with $\|S\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_5\|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})}$, with C_5 depending only on the dimension. This is the result of the lemma, completing as well the proof of Theorem 2.5.10. □

Proof of the Corollary 2.5.11.

Let us begin with the statement (iv) in this corollary. Let us define

$$A(x,\xi) = h^{-2}a(xh^{1/2}, \xi h^{-1/2}, h). \quad (2.5.52)$$

The function A satisfies

$$\begin{aligned}(\partial_\xi^\alpha\partial_x^\beta A)(x,\xi) &= h^{-2-\frac{|\alpha|}{2}+\frac{|\beta|}{2}}(\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{-1/2},h)\\ &= h^{\frac{|\alpha|+|\beta|-4}{2}}(\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha|}\end{aligned}$$

so that for $|\alpha|+|\beta| = 4$, we have $(\partial_\xi^\alpha\partial_x^\beta A)(x,\xi) = (\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha|}$. We have supposed that for $|\alpha|+|\beta| = 4$, the functions

$$(x,\xi)\mapsto(\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha|}$$

belong to $\mathcal{A}$ with a norm bounded above independently by ν_0. As a result the function $A^{(4)}(x,\xi)$ belongs to $\mathcal{A}$ with a norm bounded above by ν_0. Since $A(x,\xi)\geq 0$, Theorem 2.5.10 implies that $A^w + C_n\nu_0 \geq 0$, i.e.

$$\big(a(xh^{1/2},\xi h^{-1/2},h)\big)^w + C_n\nu_0 h^2 \geq 0$$

and since there is a unitary mapping U_h such that

$$U_h^* a(x,\xi,h)^w U_h = \big(a(xh^{1/2},\xi h^{-1/2},h)\big)^w,$$

we obtain

$$a(x,\xi,h)^w + C_n\nu_0 h^2 \geq 0. \tag{2.5.53}$$

To get that $\operatorname{Re} a(x,D,h) + Ch^2 \geq 0$, one[18] should note that the symbols $A^{(4)}$ defined above belong to $\mathcal{A}$, which implies that it is also the case for $J^{-1/2}A^{(4)}$ and $J^{1/2}\overline{A^{(4)}}$. Now we have

$$2\operatorname{Re} a(x,D,h) = 2\operatorname{Re}(J^{-1/2}a)^w = (J^{-1/2}a + J^{1/2}\bar{a})^w,$$

so that rescaling[19] the symbol $J^{-1/2}a + J^{1/2}\bar{a}$, we find $J^{-1/2}A + J^{1/2}\bar{A}$. Since we have

$$J^{-1/2}A = e^{-i\pi D_x\cdot D_\xi}A = A - i\pi D_x\cdot D_\xi A - \int_0^1 (1-\theta)e^{-i\pi\theta D_x\cdot D_\xi}d\theta\pi^2(D_x\cdot D_\xi)^2 A,$$

and that A is real-valued, we get

$$\operatorname{Re}(J^{-1/2}A) = A - \int_0^1 (1-\theta)e^{-i\pi\theta D_x\cdot D_\xi}d\theta\pi^2 \underbrace{(D_x\cdot D_\xi)^2 A}_{\in\mathcal{A}}.$$

Now we have from the previous identity, since $\mathcal{A}$ is stable by the group J^t (Theorem 1.1 in [130]), with a uniform constant for t in a compact set,

$$2\operatorname{Re} A(x,D) = \big(2\operatorname{Re}(J^{-1/2}A)\big)^w \in 2A^w + \mathcal{A}^w.$$

We can then apply the result (2.5.53) and the L^2 boundedness of $\mathcal{A}^w$ to conclude. The proof of *(iv)* in the corollary 2.5.11 is complete.

Let us show that (iv) implies (iii). We define $b(x,\xi,h) = a(x,h\xi)$, which is nonnegative; it is enough to check the functions

$$(x,\xi)\mapsto(\partial_1^\beta\partial_2^\alpha b)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha|},$$

for $|\alpha|+|\beta| = 4$. We have in fact

$$(\partial_1^\beta\partial_2^\alpha b)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha|} = (\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{1/2}).$$

Now, from the lemma 4.8.1 in our appendix, for $h\in(0,1]$, the functions

$$(x,\xi)\mapsto(\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{1/2})$$

[18] With the group J^t defined in Lemma 4.1.2, the formula linking the Weyl quantization with the ordinary quantization is $a(x,D) = (J^{-1/2}a)^w$.

[19] We define $B(x,\xi) =$

$$h^{-2}(J^{-1/2}a)(xh^{1/2},\xi h^{-1/2}) + h^{-2}(J^{1/2}a)(xh^{1/2},\xi h^{-1/2}) = (J^{-1/2}A)(x,\xi) + (J^{1/2}A)(x,\xi).$$

belong to $\mathcal{A}$ with a bounded norm since we have supposed that $a^{(4)} \in \mathcal{A}$. We can then apply the already proven result *(iv)* in the corollary to get

$$a(x,\xi h)^w + Ch^2\|a^{(4)}\|_{\mathcal{A}} \geq 0, \quad \operatorname{Re} a(x,hD) + Ch^2\|a^{(4)}\|_{\mathcal{A}} \geq 0.$$

Let us show that (iv) implies (ii). We assume that $a(x,\xi,h)$ is a nonnegative function satisfying the assumptions of *(ii)*. According to the already proven *(iv)*, we need only to check, for $|\alpha'| + |\beta'| = 4$, the norm in $\mathcal{A}$ of

$$(x,\xi) \mapsto (\partial_1^{\beta'}\partial_2^{\alpha'}a)(xh^{1/2}, \xi h^{-1/2}, h)h^{-|\alpha'|} = c_{\alpha'\beta'}(x,\xi).$$

Because of the second inclusion in (2.5.15), it is enough to find an L^∞ bound on the $2n+1$ first derivatives of that function; we have, for $|\alpha''| + |\beta''| \leq 2n+1$

$$(\partial_\xi^{\alpha''}\partial_x^{\beta''}c_{\alpha'\beta'})(x,\xi) = (\partial_1^{\beta'+\beta''}\partial_2^{\alpha'+\alpha''}a)(xh^{1/2}, \xi h^{-1/2}, h)h^{-|\alpha'|}h^{\frac{-|\alpha''|+|\beta''|}{2}} \tag{2.5.54}$$

and from the assumption in *(ii)*, we get, since $4 \leq |\alpha'+\alpha''| + |\beta'+\beta''| \leq 2n+5$,

$$|(\partial_1^{\beta'+\beta''}\partial_2^{\alpha'+\alpha''}a)(xh^{1/2}, \xi h^{-1/2}, h)| \leq C_{\alpha'+\alpha'',\beta'+\beta''}h^{|\alpha'|+|\alpha''|}, \tag{2.5.55}$$

so that (2.5.54)-(2.5.55) imply

$$\begin{aligned}|(\partial_\xi^{\alpha''}\partial_x^{\beta''}c_{\alpha'\beta'})(x,\xi)| &\leq C_{\alpha'+\alpha'',\beta'+\beta''}h^{|\alpha'|+|\alpha''|}h^{-|\alpha'|}h^{\frac{-|\alpha''|+|\beta''|}{2}} \\ &= C_{\alpha'+\alpha'',\beta'+\beta''}h^{\frac{|\alpha''|+|\beta''|}{2}} \leq C_{\alpha'+\alpha'',\beta'+\beta''}.\end{aligned}$$

yielding the sought bound. The proof of *(ii)* is complete.
Proof of (i) *in the corollary 2.5.11.* Using a Littlewood-Paley decomposition, we have

$$1 = \sum_{\nu\geq 0}\varphi_\nu^2(\xi), \quad \varphi_\nu \in C_c^\infty(\mathbb{R}^n),$$
$$\text{for } \nu \geq 1,\ \operatorname{supp}\varphi_\nu \subset \{2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\}, \quad \sup_{\nu,\xi}|\partial_\xi^\alpha\varphi_\nu(\xi)|2^{\nu|\alpha|} < \infty.$$

We introduce also some smooth nonnegative compactly supported functions $\psi_\nu(\xi)$, satisfying the same uniform estimates than φ_ν and supported in $2^{\nu-3} \leq |\xi| \leq 2^{\nu+3}$ for $\nu \geq 1$, identically 1 on $2^{\nu-2} \leq |\xi| \leq 2^{\nu+2}$ (in particular on the support of φ_ν). We consider a nonnegative symbol a satisfying (2.5.10) for $4 \leq |\alpha| + |\beta| \leq 2n+5$. We write

$$a = \sum_{\nu\geq 0}\varphi_\nu^2 a = \sum_{\nu\geq 0}(\psi_\nu\sharp\varphi_\nu^2 a\sharp\psi_\nu + r_\nu). \tag{2.5.56}$$

The proof relies on the following

Claim 2.5.24. *The operator with Weyl symbol* $\sum_\nu r_\nu$ *is bounded on* $L^2(\mathbb{R}^n)$.

As a matter of fact, if this claim is proven, we are left with the operator $\sum_\nu \psi_\nu^w(\varphi_\nu^2 a)^w\psi_\nu^w$ and we can apply the already proven result *(ii)* in this corollary to get that with a uniform C,

$$\sum_\nu \psi_\nu^w(\varphi_\nu^2 a)^w\psi_\nu^w = \sum_\nu \psi_\nu^w \underbrace{\big((\varphi_\nu^2 a)^w + C\big)}_{\geq 0}\psi_\nu^w - C\underbrace{(\sum_\nu \psi_\nu^2)^w}_{L^2 \text{ bounded}}$$

and so this operator is semi-bounded from below as well as a^w. Let us prove the claim. The following composition formula is proven in Theorem 4.6.1:

$$(a_1\sharp a_2\sharp a_3)(X) = 2^{2n}\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} a_1(Y_1)a_2(Y_2)a_3(X-Y_1+Y_2) \\ e^{-4i\pi[X-Y_1,X-Y_2]}dY_1dY_2. \quad (2.5.57)$$

Applying this to $\psi_\nu\sharp a_\nu\sharp\psi_\nu$ with $a_\nu = \varphi_\nu^2 a$, we get

$$r_\nu(x,\xi) = -2^n\iiint_0^1 (1-\theta)e^{-4i\pi y\eta}\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)(\partial_x^2 a_\nu)(x+\theta y,\xi)y^2 dyd\eta d\theta$$
$$= \frac{2^n}{16\pi^2}\iiint_0^1 (1-\theta)\partial_\eta^2(e^{-4i\pi y\eta})\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)(\partial_x^2 a_\nu)(x+\theta y,\xi)dyd\eta d\theta$$
$$= \frac{2^n}{16\pi^2}\iiint_0^1 (1-\theta)e^{-4i\pi y\eta}\partial_\eta^2\big(\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)\big)(\partial_x^2 a_\nu)(x+\theta y,\xi)dyd\eta d\theta.$$

From this formula we see that r_ν is supported where $2^{\nu-1}\leq|\xi|\leq 2^{\nu+1}$ since it is the case for a_ν ($\nu\geq 1$); since the overlap of the rings where $|\xi|\sim 2^\nu$ is bounded, it is enough to check some bounds on the derivatives of r_ν to get similar bounds on the $\sum_\nu r_\nu$. Moreover in the integrand, if the function $\psi_\nu(\xi+\eta)$ is differentiated, we get

$$2^{\nu+2}\leq|\xi+\eta|\leq 2^{\nu+3} \text{ or } 2^{\nu-3}\leq|\xi+\eta|\leq 2^{\nu-2}.$$

As a result, in the first case, we have $|\eta|\geq|\xi+\eta|-|\xi|\geq 2^{\nu+2}-2^{\nu+1}=2^{\nu+1}$, whereas in the second case $|\eta|\geq|\xi|-|\xi+\eta|\geq 2^{\nu-1}-2^{\nu-2}=2^{\nu-2}$, which implies that we always have $|\eta|\geq 2^{\nu-2}$. Since we have also $|\eta|\leq|\xi+\eta|+|\xi|\leq 2^{\nu+3}+2^{\nu+1}$, we obtain (note that the case when the other function $\psi(\xi-\eta)$ is differentiated is similar) on the integrand

$$2^{\nu-2}\leq|\eta|\leq 2^{\nu+4}. \quad (2.5.58)$$

We write now

$$\frac{1}{\alpha!}(\partial_\xi^\alpha\partial_x^\beta r_\nu)(x,\xi) = \sum_{\alpha'+\alpha''=\alpha}\frac{2^n}{\alpha'!\alpha''!16\pi^2}$$
$$\times\iiint_0^1 (1-\theta)e^{-4i\pi y\eta}\partial_\xi^{\alpha'}\partial_\eta^2\big(\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)\big)(\partial_\xi^{\alpha''}\partial_x^\beta\partial_x^2 a_\nu)(x+\theta y,\xi)dyd\eta d\theta$$

and since the integral above is, for N, k even integers, $N > n$,

$$\iiint_0^1 (1-\theta)e^{-4i\pi y\eta}(1+4|\eta|^2)^{-k/2} \\ \times (1+4|y|^2)^{-N/2}(1+D_\eta^2)^{N/2}\partial_\xi^{\alpha'}\partial_\eta^2\big(\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)\big) \\ \times (1+D_y^2)^{k/2}\big((\partial_\xi^{\alpha''}\partial_x^\beta\partial_x^2 a_\nu)(x+\theta y,\xi)\big)dyd\eta d\theta,$$

we get, for $|\alpha|+|\beta|+k \le 2n+3$,

$$|(\partial_\xi^\alpha\partial_x^\beta r_\nu)(x,\xi)| \le C_{\alpha\beta N}(2^\nu)^{-k-|\alpha'|-2+2-|\alpha''|+n} = C_{\alpha\beta N}(2^\nu)^{-|\alpha|+n-k}.$$

For α, β given such that $\max(|\alpha|,|\beta|) \le n+1$, we choose $k = n-|\alpha|$ or $k = n-|\alpha|+1$ so that k is even, and we get, uniformly in ν, $|(\partial_\xi^\alpha\partial_x^\beta r_\nu)(x,\xi)| \lesssim 1$; note that then we have indeed

$$|\alpha|+|\beta|+k \le |\beta|+n+1 \le 2n+2 \le 2n+3.$$

Eventually, from (a mild version of) the theorem 1.2 in [22] we get the claim 2.5.24: we have proven that for $\max(|\alpha|,|\beta|) \le n+1$, $\partial_\xi^\alpha\partial_x^\beta r$ is bounded. The proof of (2.5.12) is complete, under the assumptions of the corollary.

Proof of (1.1.2). To obtain also the result for the ordinary quantization is not a direct consequence of the previous result, because of our limitation on the regularity of a. So we have to revisit our argument above, replacing at each step the Weyl quantization by the standard quantization. It is a bit tedious, but unavoidable. We write

$$a = \sum_{\nu\ge 0}\varphi_\nu^2 a = \sum_{\nu\ge 0}(\psi_\nu \circ \varphi_\nu^2 a \circ \psi_\nu + s_\nu). \tag{2.5.59}$$

The proof relies on the following

Claim 2.5.25. *The operator with standard symbol $\sum_\nu s_\nu$ is bounded on $L^2(\mathbb{R}^n)$.*

As a matter of fact, if this claim is proven, we are left with the operator

$$\mathrm{Re}\sum_\nu \mathrm{op}(\psi_\nu)\mathrm{op}(\varphi_\nu^2 a)\mathrm{op}(\psi_\nu)$$

and we can apply the already proven result (ii) in this corollary to get that with a uniform C,

$$\sum_\nu \mathrm{op}(\psi_\nu)\,\mathrm{Re}\,\mathrm{op}(\varphi_\nu^2 a)\mathrm{op}(\psi_\nu) \\ = \sum_\nu \mathrm{op}(\psi_\nu)\underbrace{\big(\mathrm{Re}\,\mathrm{op}(\varphi_\nu^2 a)+C\big)}_{\ge 0}\mathrm{op}(\psi_\nu) - C\underbrace{\mathrm{op}(\sum_\nu \psi_\nu^2)}_{L^2\text{bounded}},$$

and so this operator is semi-bounded from below as well as $\mathrm{Re}\,\mathrm{op}(a)$. Let us prove the claim. Reminding the ordinary composition formula, we have

$$(a \circ b)(x,\xi) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{-2i\pi y\eta} a(x,\xi+\eta) b(y+x,\xi) dy d\eta. \tag{2.5.60}$$

Applying this to $\psi_\nu \circ a_\nu \circ \psi_\nu$ with $a_\nu = \varphi_\nu^2 a$, we get $\psi_\nu \circ a_\nu \circ \psi_\nu = \psi_\nu \circ a_\nu \psi_\nu = \psi_\nu \circ a_\nu$ and

$$\begin{aligned}(\psi_\nu \circ a_\nu \circ \psi_\nu)(x,\xi) &= \iint e^{-2i\pi y\eta} \psi_\nu(\xi+\eta) a_\nu(y+x,\xi) dy d\eta \\ &= \iint e^{-2i\pi y\eta} \psi_\nu(\xi+\eta) \Big(a_\nu(x,\xi) + \int_0^1 (1-\theta) \partial_x^2 a_\nu(x+\theta y,\xi) y^2 d\theta \Big) dy d\eta \\ &= (a_\nu \psi_\nu)(x,\xi) - s_\nu(x,\xi),\end{aligned}$$

with

$$s_\nu(x,\xi) = -\iiint_0^1 (1-\theta) e^{-2i\pi y\eta} \psi_\nu(\xi+\eta) (\partial_x^2 a_\nu)(x+\theta y,\xi) y^2 dy d\eta d\theta.$$

That formula is so similar to the defining formula of r_ν above that we can resume the discussion and use (a mild version of) the theorem 1.1 in [22] we get the claim 2.5.25: The proof of (2.5.11) is complete, under the assumptions of the corollary.

2.5.4 Analytic functional calculus

Theorem 2.5.26. *Let g be an admissible metric on $\mathbb{R}^{2n}$ (definition 2.2.15). There exists some positive integers κ_0, κ_1, a positive increasing function α defined on $\mathbb{N}$, depending only on the structure constants of g, as given by* (2.2.16) *and* (2.2.3) *such that the following property is true. Let $\nu \geq 2$ be an integer and $a \in S(1,g)$ (see definition 2.2.6). Then $a^{\sharp\nu}$ belongs to $S(1,g)$ and for all $X, T \in \mathbb{R}^{2n}, k \in \mathbb{N}$,*

$$|(a^{\sharp\nu})^{(k)}(X) T^k| \leq g_X(T)^{k/2} \Big(\alpha(k) \|a\|_{S(1,g)}^{(\kappa_0 k + \kappa_1)} \Big)^\nu,$$

which implies

$$\|a^{\sharp\nu}\|_{S(1,g)}^{(k)} \leq \Big(\alpha(k) \|a\|_{S(1,g)}^{(\kappa_0 k + \kappa_1)} \Big)^\nu, \tag{2.5.61}$$

where the semi-norms in $S(1,g)$ are defined in (2.2.7).

N.B. The fact that $a^{\sharp\nu}$ belongs to $S(1,g)$ is clear from the symbolic calculus, as given by the theorem 2.3.7. The point of this lemma is that the k-th semi-norm of $a^{\sharp\nu}$ is bounded above by the ν-th power of the $(\kappa_0 k + \kappa_1)$-th semi-norm of a, a shift which *does not depend on ν*.

Proof. We use the partition of unity of the proposition 2.2.9 to get

$$(a^{\sharp\nu})(X) = \int_{\mathcal{P}^\nu} (a_{Y_1} \sharp \dots \sharp a_{Y_\nu})(X) |g_{Y_1}|^{1/2} \dots |g_{Y_\nu}|^{1/2} dY_1 \dots dY_\nu.$$

We note that with $g_j = g_{Y_j}, U_j = U_{Y_j,r}$, from the first inequality in (2.2.29), the assumption (4.6.12) is satisfied with $\delta_{j,j+1} = \delta_r(Y_j, Y_{j+1})$ where the function δ_r is given by (2.2.27) and N_0, C_0 are depending only on the "structure constants" of the metric g (as given by (2.2.16) and (2.2.3)): this is

$$1 + \frac{g_{Y_{j+1}}(T)}{g_{Y_j}(T)} + \frac{g_{Y_j}(S)}{g_{Y_{j+1}}(S)} \leq C_0 \delta_r(Y_j, Y_{j+1})^{N_0}. \tag{2.5.62}$$

We use now the corollaries 4.6.6, 4.6.7 to obtain for all $N, M \in \mathbb{N}$ to be chosen later[20],

$$\begin{aligned}
|(a^{\sharp\nu})^{(k)}(X)T^k| &\leq \kappa_n^{\nu/2} 2^{\nu N} \nu^k \omega(n, C_0, N_0, M)^{\nu/2} \\
&\int_{\mathcal{P}^\nu} \max_{k_1+\dots+k_\nu=k} \Big(\prod_{1\leq j\leq \nu} \|\!| a_{Y_j} |\!\|_{g_{Y_j}, U_{Y_j,r}}^{(k_j+N+M+(2n+1)(2N_0+2))} \Big) \big(\max_{1\leq j\leq \nu} g_{Y_j}(T) \big)^{k/2} \\
&\times \Big(1 + \nu^{-1} \max_{1\leq j\leq \nu} (g^\sigma_{Y_1} \wedge \dots \wedge g^\sigma_{Y_\nu})(X - U_{Y_j,r}) \Big)^{-N/4} \\
&\prod_{1\leq j<\nu} \delta_r(Y_j, Y_{j+1})^{-M/8} |g_{Y_1}|^{1/2} \dots |g_{Y_\nu}|^{1/2} dY_1 \dots dY_\nu.
\end{aligned}$$

We have from (4.4.37)

$$\begin{aligned}
1 + \nu^{-1} &\max_{1\leq j\leq \nu} (g^\sigma_{Y_1} \wedge \dots \wedge g^\sigma_{Y_\nu})(X - U_{Y_j,r}) \\
&\geq 1 + \nu^{-1} (g^\sigma_{Y_1} \wedge \dots \wedge g^\sigma_{Y_\nu})(X - U_{Y_\nu,r}) \\
&\geq \big(1 + g^\sigma_{Y_\nu}(X - U_{Y_\nu,r})\big) \prod_{1\leq j<\nu} \big(1 + C_0 \delta_r(Y_j, Y_{j+1})^{N_0}\big)^{-1} \\
&\geq \big(1 + g^\sigma_{Y_\nu}(X - U_{Y_\nu,r})\big) (2C_0)^{1-\nu} \prod_{1\leq j<\nu} \delta_r(Y_j, Y_{j+1})^{-N_0}.
\end{aligned}$$

We note also that, with the partition of unity (φ_Y) given in proposition 2.2.9,

$$\|\!| a_Y |\!\|^{(l)}_{g_Y, U_{Y,r}} = \|\!| \varphi_Y a |\!\|^{(l)}_{g_Y, U_{Y,r}} \leq C(l) \|a\|^{(l)}_{S(1,g)}$$

where the semi-norms of the symbol a are given by (2.2.7) and $C(l)$ depends only on the structure constants of the metric g. Thanks to the first inequality in

[20] We have to keep in mind that the constants n, C_0, N_0 are "fixed". We want to track the dependence of the estimates with respect to ν and k. Incidentally, we have assumed, as we may, that N_0 is an integer.

(2.2.29), the ratios $g_{Y_j}(T)/g_{Y_{j+1}}(T)$ are bounded above by $C_3\delta_r(Y_j,Y_{j+1})^{N_0}$. As a consequence, we have

$$\begin{aligned}
&|(a^{\sharp\nu})^{(k)}(X)T^k| \\
&\quad\le \kappa_n^{\nu/2}2^{\nu N}\nu^k\omega(n,C_0,N_0,M)^{\nu/2}C\big(k+N+M+(2n+1)(2N_0+2)\big)^{\nu} \\
&\quad\Big(\max_{k_1+\cdots+k_\nu=k}\prod_{1\le j\le\nu}\|a\|_{S(1,g)}^{(k_j+N+M+(2n+1)(2N_0+2))}\Big) \\
&\quad\int_{\mathcal{P}^\nu} g_{Y_\nu}(T)^{k/2}C_3^{(\nu-1)k/2}\big(1+g^\sigma_{Y_\nu}(X-U_{Y_\nu,r})\big)^{-N/4}(2C_0)^{(\nu-1)N/4} \\
&\qquad\prod_{1\le j<\nu}\delta_r(Y_j,Y_{j+1})^{-\frac{M}{8}+\frac{N_0N}{4}+\frac{N_0k}{2}}|g_{Y_1}|^{1/2}\dots|g_{Y_\nu}|^{1/2}dY_1\dots dY_\nu.
\end{aligned}$$

We note now that from the first inequality in (2.2.29),

$$\frac{g_{Y_\nu}(T)}{g_X(T)}\le C_3\delta_r(X,Y_\nu)^{N_0}\le C_3\big(1+2g^\sigma_{Y_\nu}(X-U_{Y_\nu,r})\big)^{N_0}$$

so that

$$\begin{aligned}
&|(a^{\sharp\nu})^{(k)}(X)T^k| \\
&\quad\le \kappa_n^{\nu/2}2^{\nu N}\nu^k\omega(n,C_0,N_0,M)^{\nu/2}C\big(k+N+M+(2n+1)(2N_0+2)\big)^{\nu} \\
&\qquad\Big(\max_{k_1+\cdots+k_\nu=k}\prod_{1\le j\le\nu}\|a\|_{S(1,g)}^{(k_j+N+M+(2n+1)(2N_0+2))}\Big) \\
&g_X(T)^{k/2}\int_{\mathcal{P}^\nu}(2C_3)^{k/2}C_3^{(\nu-1)k/2}\big(1+g^\sigma_{Y_\nu}(X-U_{Y_\nu,r})\big)^{-\frac{N}{4}+\frac{N_0k}{2}}(2C_0)^{(\nu-1)N/4} \\
&\qquad\prod_{1\le j<\nu}\delta_r(Y_j,Y_{j+1})^{-\frac{M}{8}+\frac{N_0N}{4}+\frac{N_0k}{2}}|g_{Y_1}|^{1/2}\dots|g_{Y_\nu}|^{1/2}dY_1\dots dY_\nu. \quad (2.5.63)
\end{aligned}$$

We know also from (2.2.27) and the second inequality in(4.4.26) that

$$\big(1+g^\sigma_{Y_\nu}(X-U_{Y_\nu,r})\big)\ge\frac{1}{2}\delta_r(X,Y_\nu).$$

This implies that the integral above is, up to some constant terms that we shall make precise,

$$\begin{aligned}
&\int_{\mathcal{P}^\nu}\prod_{1\le j<\nu}\delta_r(Y_j,Y_{j+1})^{-\frac{M}{8}+\frac{N_0N}{4}+\frac{N_0k}{2}}\delta_r(X,Y_\nu)^{-\frac{N}{4}+\frac{N_0k}{2}} \\
&\qquad\qquad|g_{Y_1}|^{1/2}\dots|g_{Y_\nu}|^{1/2}dY_1\dots dY_\nu. \quad (2.5.64)
\end{aligned}$$

We take a look at the second inequality in (2.2.29), and for a given k, we can choose N so that $\frac{N}{4}-\frac{N_0k}{2}\ge N_1$, then choose M so that $\frac{M}{8}-\frac{N_0N}{4}-\frac{N_0k}{2}\ge N_1$ in

such a way that the integral is bounded above by a fixed constant. At this point, we note that N_1, N_0 are depending only on the structure constants of g and that we can choose, assuming as we may that $N_1 \in \mathbb{N}$,

$$N = 4N_1 + 2N_0 k,\ M = 8N_1 + 2N_0 N + 4N_0 k, \tag{2.5.65}$$

so that

$$N = 4N_1 + 2N_0 k, \quad M = 8N_1 + 8N_0 N_1 + (4N_0 + 4N_0^2)k, \tag{2.5.66}$$

giving an affine dependence of N, M on k (N_0, N_1 are fixed constants depending only on the structure constants of g). On the other hand, we have in (2.5.63) a constant

$$B = \kappa_n^{\nu/2} 2^{\nu N} \nu^k \omega(n, C_0, N_0, M)^{\nu/2} C(k + N + M + (2n+1)(2N_0+2))^{\nu} \\ \times (2C_3)^{k/2} C_3^{(\nu-1)k/2} (2C_0)^{(\nu-1)N/4}, \tag{2.5.67}$$

where C_0, C_3 are fixed depending only on the structure constants of g. The constant B can be estimated from above by

$$\beta(k)^{\nu}\nu^k, \quad \text{with } \beta(k) \text{ depending on } k, n \text{ and on the structure constants of } g.$$

Taking into account our choice (2.5.65) of N, M, we see that (2.5.64) is bounded above by C_4^{ν}, where C_4 is a fixed constant depending only on the structure constants of g. We have proven that

$$|(a^{\sharp\nu})^{(k)}(X)T^k| \leq g_X(T)^{k/2} \\ \times \Big(\max_{k_1+\cdots+k_\nu=k} \prod_{1\leq j\leq \nu} \|a\|_{S(1,g)}^{(k_j+N+M+(2n+1)(2N_0+2))} \Big) \nu^k \beta(k)^{\nu} C_4^{\nu}$$

where N, M are given by (2.5.66) and depend in an affine way of k and on the structure constants of g. The constant $\beta(k)$ depends only on k, n and on the structure constants of the metric g. This gives readily (2.5.61), completing the proof of the lemma. □

Corollary 2.5.27. *Let g be an admissible metric on $\mathbb{R}^{2n}$ (definition 2.2.15), $a \in S(1,g)$, and f be an entire function of one complex variable. Then the operator $f(a^w)$ has a symbol in $S(1,g)$.*

Proof. The operator $f(a^w) = (\sum_{\nu\geq 0} c_\nu a^{\sharp\nu})^w = b^w$ and from the estimates (2.5.61) of the Theorem 2.5.26, b is a smooth function and

$$|b^{(k)}(X)T^k| \leq \sum_{\nu\geq 0} |c_\nu| \Big(\alpha(k) \|a\|_{S(1,g)}^{(\kappa_0 k + \kappa_1)} \Big)^{\nu} g_X(T)^{k/2} \leq C_{k,a} g_X(T)^{k/2},$$

since the radius of convergence of the series $\sum_{\nu\geq 0} c_\nu \rho^\nu$ is infinite. □

2.6 Sobolev spaces attached to a pseudodifferential calculus

2.6.1 Introduction

The Weyl quantization is associating an operator a^w to a function a defined on the phase space $\mathbb{R}^{2n}$. When the phase space is equipped with an admissible metric g and M is an admissible weight for g (see Definition 2.2.15), we have obtained an algebra of operators, composition formulas, symbolic calculus and $L^2(\mathbb{R}^n)$ boundedness of the operators with symbols in $S(1,g)$. In the standard case of the metric $|dx|^2+\langle\xi\rangle^{-2}|d\xi|^2$ which is such that

$$S(\langle\xi\rangle^m,|dx|^2+\langle\xi\rangle^{-2}|d\xi|^2)=S^m_{1,0}\quad(m\in\mathbb{R},\text{ cf. Section 1.1.3}),$$

the natural Sobolev spaces (indexed by $s\in\mathbb{R}$) are the standard

$$H^s(\mathbb{R}^n)=\{u\in\mathscr{S}'(\mathbb{R}^n),\langle D\rangle^s u\in L^2(\mathbb{R}^n)\},$$

with the obvious Hilbertian structure; in that case, the operators of order m (i.e. of weight $\langle\xi\rangle^m$) are continuous from $H^{s+m}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$. We wish to generalize this to the case of a general admissible metric on $\mathbb{R}^{2n}$ and construct some Hilbert spaces $H(M,g)$ such that, for an operator a^w with $a\in S(M_1,g)$ (M_1 is another admissible weight for g), a^w will send continuously $H(M,g)$ into $H(MM_1^{-1},g)$. The case of the classes $S^m_{1,0}$ is oversimplified by the fact that the Fourier transform is providing the explicit isomorphism $\langle D\rangle^s$ of $H^s(\mathbb{R}^n)$ with $L^2(\mathbb{R}^n)$, which is the Fourier multiplier $\langle\xi\rangle^s$. With the metric corresponding to the classes (1.1.48),

$$g=(1+|x|^2+|\xi|^2)^{-1}\big(|dx|^2+|d\xi|^2\big)$$

which is proven admissible in Lemma 2.2.18, although it is easy to define the natural Sobolev spaces when $m\in\mathbb{N}/2$,

$$\mathcal{H}((1+|x|^2+|\xi|^2)^m,g)=\{u\in\mathscr{S}'(\mathbb{R}^n),x^\alpha D^\beta u\in L^2(\mathbb{R}^n),|\alpha|+|\beta|\le 2m\},$$

a direct definition for all real m requires some techniques of interpolation and duality[21]. As a result, even in that rather simple case, the construction of the Sobolev spaces requires some effort and displaying an explicit pseudodifferential isomorphism of $L^2(\mathbb{R}^n)$ with $\mathcal{H}((1+|x|^2+|\xi|^2)^m,g)$ is not completely obvious. We shall mainly follow the presentation of J.-M. Bony and J.-Y. Chemin in [19], which generalized the work of A. Unterberger in [145], who dealt with symplectic metrics (see also [26], [88]).

[21] Another way would be to use the spectral decomposition of the harmonic oscillator and Mehler's formula (see e.g. [145] and [76]) to replace the Fourier transform, but it would be very specific of that example.

2.6.2 Definition of the Sobolev spaces

In this section we consider an admissible metric g on $\mathbb{R}^{2n}$, a g-admissible weight m, $(\varphi_Y)_{Y\in\mathbb{R}^{2n}}$ a partition of unity given by Theorem 2.2.7 and $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$, $(\theta_Y)_{Y\in\mathbb{R}^{2n}}$ the uniformly confined families of symbols given by the theorem 2.3.15, i.e. verifying

$$\varphi_Y = \psi_Y \sharp \theta_Y, \quad \int \varphi_Y |g_Y|^{1/2} dY = 1. \tag{2.6.1}$$

Definition 2.6.1. The Sobolev space $H(m,g)$ is the space of $u \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^{2n}} m(Y)^2 \|\theta_Y^w u\|^2_{L^2(\mathbb{R}^n)} |g_Y|^{1/2} dY < \infty. \tag{2.6.2}$$

This is a pre-Hilbert space equipped with the dot-product

$$\langle u, v\rangle_{H(m,g)} = \int_{\mathbb{R}^{2n}} m(Y)^2 \langle \theta_Y^w u, \theta_Y^w v\rangle_{L^2(\mathbb{R}^n)} |g_Y|^{1/2} dY. \tag{2.6.3}$$

We shall prove below (Corollary 2.6.16) that $H(m,g)$ is actually a Hilbert space but some preliminary work has to be done before obtaining this.

Remark 2.6.2. According to Lemma 1.1.12, for $u \in \mathscr{S}'(\mathbb{R}^n)$, we have $\theta_Y^w u \in \mathscr{S}(\mathbb{R}^n)$ so that the definition makes sense; the dot-product (2.6.3) is indeed sesquilinear Hermitian and positive-definite, since $\theta_Y^w u = 0$ for all $Y \in \mathbb{R}^{2n}$ implies $\varphi_Y^w u = 0$ and the partition property yields $u = 0$.

We need first to prove that the space $H(m,g)$ does not depend on the choice of the partition and of the families (ψ_Y), (θ_Y).

Proposition 2.6.3. *A distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ belongs to $H(m,g)$ if and only if for all uniformly confined families of symbols $(\phi_Y)_{Y\in\mathbb{R}^{2n}}$, we have*

$$\int_{\mathbb{R}^{2n}} m(Y)^2 \|\phi_Y^w u\|^2_{L^2(\mathbb{R}^n)} |g_Y|^{1/2} dY < \infty.$$

Proof. The sufficiency follows from the definition of $H(m,g)$. Let us prove that the condition is necessary; if $(\phi_Y)_{Y\in\mathbb{R}^{2n}}$ is a uniformly confined family of symbols and $u \in H(m,g)$, we have (with $L^2(\mathbb{R}^n)$ scalar products)

$$\begin{aligned}
\Omega(u) &= \int m(Y)^2 \langle \phi_Y^w u, \phi_Y^w u\rangle |g_Y|^{1/2} dY \\
&= \iiint m(Y)^2 \langle \phi_Y^w \psi_S^w \theta_S^w u, \phi_Y^w \psi_T^w \theta_T^w u\rangle |g_Y|^{1/2} |g_T|^{1/2} |g_S|^{1/2} dY dS dT \\
&= \iiint m(Y)^2 \langle \bar\psi_T^w \bar\phi_Y^w \phi_Y^w \psi_S^w \theta_S^w u, \theta_T^w u\rangle |g_Y|^{1/2} |g_T|^{1/2} |g_S|^{1/2} dY dS dT.
\end{aligned}$$

According to the estimate (2.5.1) and to the biconfinement estimates (corollary 2.3.3), we have for all N, $\|\psi_T^w \phi_Y^w \phi_Y^w \psi_S^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_N \delta_r(T,Y)^{-N} \delta_r(Y,S)^{-N}$ so that,

$$\Omega(u) \leq C_N \iiint \frac{m(Y)^2}{m(S)m(T)} \delta_r(T,Y)^{-N} \delta_r(Y,S)^{-N} |g_Y|^{1/2} dY \\ \times m(S)\|\theta_S^w u\|_{L^2} m(T) \|\theta_T^w u\|_{L^2} |g_T|^{1/2} |g_S|^{1/2} dS dT.$$

Thanks to the Schur's property (see Remark 4.7.3), we need only to check, using (2.2.31),

$$\sup_T \iint \delta_r(T,Y)^{-N+N_0} \delta_r(Y,S)^{-N+N_0} |g_Y|^{1/2} |g_S|^{1/2} dY dS < \infty,$$

which follows from (2.2.29) if $N \geq N_1 + N_0$. The proof of the proposition is complete. □

Theorem 2.6.4. *Let g be an admissible metric on $\mathbb{R}^{2n}$, let m_1, m_2 be g-admissible weights and let $a \in S(m_2 m_1^{-1}, g)$. Then a^w is continuous from $H(m_2, g)$ into $H(m_1, g)$.*

Proof. We have for $u \in H(m_2, g)$,

$$\|a^w u\|^2_{H(m_1,g)} = \int m_2(Y)^2 \|m_2(Y)^{-1} m_1(Y) \theta_Y^w a^w u\|^2_{L^2} |g_Y|^{1/2} dY,$$

and the family $(m_2(Y)^{-1} m_1(Y) \theta_Y \sharp a)_{Y \in \mathbb{R}^{2n}}$ is a uniformly confined family of symbols from Proposition 2.3.16. Proposition 2.6.3 gives the result. □

Theorem 2.6.5. *Let g be an admissible metric on $\mathbb{R}^{2n}$. The space $H(1,g)$ is equal to $L^2(\mathbb{R}^n)$.*

Proof. The symbol $a = \int_{\mathbb{R}^{2n}} \bar{\theta}_Y \sharp \theta_Y |g_Y|^{1/2} dY$ belongs to $S(1,g)$ and a^w is thus bounded on $L^2(\mathbb{R}^n)$ (cf. Theorem 2.5.1). As a result for $u \in L^2(\mathbb{R}^n)$, we have

$$\|u\|^2_{H(1,g)} = \langle a^w u, u \rangle_{L^2(\mathbb{R}^n)} \leq C \|u\|^2_{L^2(\mathbb{R}^n)}.$$

Conversely, if $u \in H(1,g)$, we have

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \iint \langle \psi_Y^w \theta_Y^w u, \psi_Z^w \theta_Z^w u \rangle_{L^2} |g_Y|^{1/2} |g_Z|^{1/2} dY dZ \\ \leq \iint \|\bar{\psi}_Z^w \psi_Y^w\|_{\mathcal{L}(L^2)} \|\theta_Y^w u\|_{L^2} \|\theta_Z^w u\|_{L^2} |g_Y|^{1/2} |g_Z|^{1/2} dY dZ,$$

and since the kernel $(Y,Z) \mapsto \|\overline{\psi_Z}^w \psi_Y^w\|_{\mathcal{L}(L^2)}$ satisfies the Schur property as in the proof of the theorem above, we get $\|u\|^2_{L^2(\mathbb{R}^n)} \leq C \|u\|^2_{H(1,g)}$, proving the result. □

2.6.3 Characterization of pseudodifferential operators

We start with the result of R. Beals ([7]), characterizing the set of operators with a symbol in $S^0_{0,0}$ (the smooth bounded functions on $\mathbb{R}^{2n}$ as well as all their derivatives, see (2.2.2)). We follow essentially the presentation of [15].

The constant metric

For A a linear operator from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$), L a linear form on $\mathbb{R}^{2n}$, we define $\operatorname{ad} L^w \cdot A = L^w A - AL^w$, which makes sense as an operator from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$.

Theorem 2.6.6. *Let g be a constant metric on $\mathbb{R}^{2n}$ such that $g \leq g^\sigma$ (see (4.4.18)) and let A be a linear operator from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$. The operator A is equal to a^w with $a \in S(1,g)$ (see the definition 2.2.6) if and only if all the iterated commutators*

$$\operatorname{ad} L_1^w \dots \operatorname{ad} L_\nu^w \cdot A \in \mathcal{L}(L^2(\mathbb{R}^n)), \tag{2.6.4}$$

for linear forms L_j. Moreover, for all $l \in \mathbb{N}$, there exists $k(l) \in \mathbb{N}, C_l > 0$ such that, with the notation (2.2.7), we have

$$\|a\|^{(l)}_{S(1,g)} \leq C_l \sup_{\substack{\nu \leq k(l) \\ g(T_j)\leq 1, L_j(X)=[T_j,X]}} \|\operatorname{ad} L_1^w \dots \operatorname{ad} L_\nu^w \cdot A\|_{\mathcal{L}(L^2(\mathbb{R}^n))}. \tag{2.6.5}$$

We shall use later on, for a linear operator A from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$, the notation

$$\|A\|^{(k)}_{\mathrm{op}(g)} = \sup_{\substack{\nu \leq k \\ g(T_j)\leq 1, L_j(X)=[T_j,X]}} \|\operatorname{ad} L_1^w \dots \operatorname{ad} L_\nu^w \cdot A\|_{\mathcal{L}(L^2(\mathbb{R}^n))}. \tag{2.6.6}$$

Proof. The conditions are obviously necessary. To prove their sufficiency, we start with a lemma.

Lemma 2.6.7. *Let $\mathcal{B}$ be a Banach space continuously contained in $\mathscr{S}'(\mathbb{R}^m)$ and translation invariant: for $a \in \mathcal{B}, t \in \mathbb{R}^m, a(\cdot - t) \in \mathcal{B}$ with the same norm as a. There exists C, N such that, for all $a \in \mathcal{B}$ such that $\partial^\alpha a \in \mathcal{B}$ for $|\alpha| \leq N$, a belongs to $L^\infty(\mathbb{R}^m)$ and*

$$\|a\|_{L^\infty(\mathbb{R}^m)} \leq C \sum_{|\alpha|\leq N} \|\partial^\alpha a\|_{\mathcal{B}}.$$

Proof of the lemma. Since the unit ball of $\mathcal{B}$ is a bounded subset of $\mathscr{S}'$, there exists a semi-norm p_0 of $\mathscr{S}$ such that for $a \in \mathcal{B}, \varphi \in \mathscr{S}$,

$$|\langle a, \varphi\rangle_{\mathscr{S}',\mathscr{S}}| \leq \|a\|_{\mathcal{B}}\ p_0(\varphi).$$

On the other hand, with a semi-norm p_1 on $\mathscr{S}$,we have for $\chi \in \mathscr{S}$,

$$(\chi(D)a)(x) = \langle a(\cdot - x), \hat{\chi}\rangle \Longrightarrow \|\chi(D)a)\|_{L^\infty} \leq \|a\|_{\mathcal{B}}\ p_0(\hat{\chi}) = \|a\|_{\mathcal{B}}\ p_1(\chi).$$

Let $\psi \in \mathscr{S}$ such that $\psi \equiv 1$ on the unit ball. For $\epsilon \in (0,1]$ the function $\xi \mapsto \rho_{\epsilon,N}(\xi) = \big(\psi(\epsilon\xi) - \psi(\xi)\big)|\xi|^{-N}$ belongs to $\mathscr{S}$ (note that it vanishes for $|\xi| \leq 1$) and since

$$\begin{aligned}
\xi^\alpha(\partial_\xi^\beta \rho_{\epsilon,N})(\xi) &= \sum_{\beta'+\beta''=\beta} \frac{\beta!}{\beta'!\beta''!}\xi^\alpha \partial_\xi^{\beta'}(|\xi|^{-N})\Big((\partial_\xi^{\beta''}\psi)(\epsilon\xi)\epsilon^{|\beta''|} - (\partial_\xi^{\beta''}\psi)(\xi)\Big) \\
&= \mathbf{1}\{|\xi| \geq 1\} \sum_{\substack{\beta'+\beta''=\beta \\ |\beta''|\geq 1}} O(|\xi|^{-N-|\beta'|-|\beta''|+|\alpha|}) \underbrace{(\partial_\xi^{\beta''}\psi)(\epsilon\xi)(\epsilon|\xi|)^{|\beta|''}}_{\text{bounded}} \\
&\quad + \mathbf{1}\{|\xi| \geq 1\} \sum_{\substack{\beta'+\beta''=\beta \\ |\beta''|\geq 1}} O(|\xi|^{-N-|\beta'|+|\alpha|})(\partial_\xi^{\beta''}\psi)(\xi) \\
&\quad + \mathbf{1}\{|\xi| \geq 1\}\xi^\alpha \partial_\xi^\beta(|\xi|^{-N})\big(\psi(\epsilon\xi) - \psi(\xi)\big),
\end{aligned}$$

we find that for $N \geq |\alpha|$, $\|\xi^\alpha \partial_\xi^\beta \rho_{\epsilon,N}\|_{L^\infty} \leq C_{\psi,\alpha,\beta}$. As a consequence, for N large enough (chosen as an even integer), $p_1(\rho_{\epsilon,N})$ is bounded above by a constant independent of ϵ. We have thus, with C independent of ϵ,

$$\|\psi(\epsilon D)a\|_{L^\infty} \leq \|\psi(D)a\|_{L^\infty} + \|\rho_{\epsilon,N}(D)|D|^N a\|_{L^\infty} \leq C(\|a\|_{\mathcal{B}} + \||D|^N a\|_{\mathcal{B}}) = C_1,$$

and letting ϵ tends to 0 provides the result of the lemma, since the weak limit of $\psi(\epsilon D)a$ in $\mathscr{S}'(\mathbb{R}^m)$ is a and the above estimate gives the bound

$$|\langle a, \varphi\rangle_{\mathscr{S}',\mathscr{S}}| = \lim_{\epsilon\to 0} |\langle \psi(\epsilon D)a, \varphi\rangle| \leq C_1 \|\varphi\|_{L^1(\mathbb{R}^m)},$$

completing the proof. □

We go on with the proof of the theorem. We consider the Banach space $\mathcal{B}$ of $b \in \mathscr{S}'(\mathbb{R}^{2n})$ such that $b^w \in \mathcal{L}(L^2(\mathbb{R}^n))$, equipped with the[22] norm $\|b\|_{\mathcal{B}} = \|b^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$. The invariance by translation of $\mathcal{B}$ is an immediate consequence of (2.1.14) for the phase translation. We need now to prove the embedding in $\mathscr{S}'(\mathbb{R}^{2n})$. For $b \in \mathcal{B}, \Phi \in \mathscr{S}(\mathbb{R}^{2n})$, we have

$$\begin{aligned}
\langle b, \Phi\rangle_{\mathscr{S}',\mathscr{S}} &= \int e^{2i\pi x\xi} b(x,\xi)\langle x\rangle^{-\frac{n+1}{2}}\langle\xi\rangle^{-\frac{n+1}{2}}\Theta(\eta,y)e^{2i\pi(\eta x+y\xi)}dxd\xi dyd\eta \\
&= \int e^{2i\pi x\xi} b(x,\xi) \underbrace{\langle x\rangle^{-\frac{n+1}{2}} e^{2i\pi\eta x}}_{=\overline{u_\eta(x)}} \underbrace{\langle\xi\rangle^{-\frac{n+1}{2}} e^{2i\pi y\xi}}_{=\widehat{v_y}(\xi)} dxd\xi\Theta(\eta,y)dyd\eta,
\end{aligned}$$

[22] It is obviously a norm from the estimate (2.6.7) Moreover the space $\mathcal{B}$ is isomorphic to $\mathcal{L}(L^2(\mathbb{R}^n))$: from the kernel theorem, any linear continuous operator from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ has a distribution kernel in $\mathscr{S}'(\mathbb{R}^{2n})$, and it is the case in particular of $B \in \mathcal{L}(L^2(\mathbb{R}^n))$. As a consequence, the Weyl symbol b of B, linked to the kernel κ_b by the formula (2.1.22) belongs to $\mathscr{S}'(\mathbb{R}^{2n})$ as well.

with $\Theta(\eta,y)=\iint e^{-2i\pi(\eta x'+y\xi')}\langle x'\rangle^{\frac{n+1}{2}}\langle\xi'\rangle^{\frac{n+1}{2}}\Phi(x',\xi')e^{-2i\pi x'\xi'}dx'd\xi'$. As a result, we obtain

$$|\langle b,\Phi\rangle|\le\iint|\Theta(\eta,y)||\langle b(x,D)v_y,u_\eta\rangle|d\eta dy\le\iint|\Theta(\eta,y)|d\eta dy\|b(x,D)\|_{\mathcal{L}(L^2)}$$
$$\le p(\Phi)\|b(x,D)\|_{\mathcal{L}(L^2)},$$

where p is a seminorm on $\mathscr{S}(\mathbb{R}^{2n})$. To get the result with b^w, we use the notation (4.1.14) and, noting that the group J^t is an isomorphism of $\mathscr{S}(\mathbb{R}^{2n})$, $\langle b,\Phi\rangle=\langle J^{1/2}b,J^{-1/2}\Phi\rangle$ so that

$$|\langle b,\Phi\rangle|\le p(J^{-1/2}\Phi)\|b^w\|_{\mathcal{L}(L^2)}=p_0(\Phi)\|b^w\|_{\mathcal{L}(L^2)},\tag{2.6.7}$$

where p_0 is a semi-norm of $\mathscr{S}(\mathbb{R}^{2n})$. Using Lemma 4.4.25, we can find some linear symplectic coordinates y,η such that $g=\sum_{1\le j\le n}h_j(|dy_j|^2+|d\eta_j|^2)$, and the condition $g\le g^\sigma$ means $(0<)h_j\le 1$, so that the vectors $\partial_{y_j},\partial_{\eta_j}$ have a g-norm ≤ 1. Since the assumptions of Lemma 2.6.7 are fulfilled, for a linear operator $A=a^w\in\mathcal{L}(L^2(\mathbb{R}^n))$, satisfying (2.6.4), we have $a\in\mathcal{B}$ and

$$\|a\|_{L^\infty(\mathbb{R}^{2n})}\le C\sum_{|\alpha|+|\beta|\le N}\|(\partial_y^\alpha\partial_\eta^\beta a)^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$$
$$\le C'\sup_{\substack{\nu\le N\\ g(T_j)\le 1,L_j(X)=[T_j,X]}}\|\operatorname{ad}L_1^w\dots\operatorname{ad}L_\nu^w a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))},$$

which is (2.6.5) for $l=0,k=N$. To get the estimates for the derivatives, we consider for linear forms $(L_j=[T_j,\cdot],g(T_j)\le 1)_{1\le j\le l}$, the operator $\operatorname{ad}L_1^w\dots\operatorname{ad}L_l^w a^w$, which belongs to $\mathcal{L}(L^2(\mathbb{R}^n))$ by the hypothesis (2.6.4), with symbol $\partial_{T_1}\dots\partial_{T_l}a$, thus belonging to $\mathcal{B}$. We can apply the previous inequality to that symbol and obtain (2.6.5) with $k(l)=l+N$, completing the proof of the theorem. □

Remark 2.6.8. Let $\rho\in C_c^\infty(\mathbb{R}^{2n})$ with integral 1, supported in the unit ball for the standard Euclidean metric Γ_0, $\epsilon>0$ and $\rho_\epsilon(X)=\epsilon^{-2n}\rho(X/\epsilon)$. For $a\in\mathscr{S}'(\mathbb{R}^{2n})$, the smooth function $a*\rho_\epsilon$ satisfies for any $X\in\mathbb{R}^{2n}$, $(a*\rho_\epsilon)(X)=\langle a(X-\cdot),\rho_\epsilon(\cdot)\rangle$ and thus for N even integer, using (2.6.7) (and (2.1.14) for a phase translation),

$$|(a*\rho_\epsilon)(X)|=|\langle\langle D_Y\rangle^N\big(a(X-Y)\big),\langle D_Y\rangle^{-N}\big(\rho_\epsilon(Y)\big)\rangle|$$
$$\le p_0\big(\langle D\rangle^{-N}\rho_\epsilon\big)\|(\langle D\rangle^N a)^w\|_{\mathcal{L}(L^2)}.$$

Since the seminorm p_0 can be chosen as $p_0(\Phi)=C\|\langle X\rangle^{N_0}\langle D\rangle^{N_0}\Phi\|_{L^1(\mathbb{R}^{2n})}$, we get

$$p_0\big(\langle D\rangle^{-N_0}\rho_\epsilon\big)=C\|\langle X\rangle^{N_0}\rho_\epsilon\|_{L^1(\mathbb{R}^{2n})}\le C_0\quad\text{if }\epsilon\le 1,$$

implying that $\|a\|_{L^\infty(\mathbb{R}^{2n})}\le C_0\|a^w\|^{(N_0)}_{\mathrm{op}(\Gamma_0)}$. We get also

$$\|a\|^{(k)}_{S(1,\Gamma_0)}\le C_k\|a^w\|^{(N_0+k)}_{\mathrm{op}(\Gamma_0)},$$

and the formula (2.1.14) implies as well

$$\|a\|^{(k)}_{S(1,g_0)} \leq C_k \|a^w\|^{(N_0+k)}_{\mathrm{op}(g_0)}, \tag{2.6.8}$$

where g_0 is positive-definite quadratic form such that $g_0 \leq g_0^\sigma$, which provides another proof of Theorem 2.6.6. However Lemma 2.6.7 is of independent interest and useful for the proof displayed above.

Lemma 2.6.9. *For all $k \in \mathbb{N}$, for all $\tau \in [0,1[$, there exists $C_{k,\tau} > 0, l_{k,\tau} \in \mathbb{N}$, such that, for all $a \in C^\infty(\mathbb{R}^{2n})$, for all Euclidean norms g_0 on $\mathbb{R}^{2n}$ such that $g_0 \leq g_0^\sigma$,*

$$\|(a^{(k)})^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_{k,\tau} \|a^w\|^\tau_{\mathcal{L}(L^2)} \|(a^{(l_{k,\tau})})^w\|^{1-\tau}_{\mathcal{L}(L^2)}, \tag{2.6.9}$$

$$\text{where } \|(a^{(k)})^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \sup_{T\in\mathbb{R}^{2n}, g_0(T)=1} \|(a^{(k)}(\cdot)T^k)^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))}. \tag{2.6.10}$$

N.B. It is of key importance to notice that the constants $C_{k,\tau}, l_{k,\tau}$ depend only on k and τ and *not* on g_0 or a.

Proof. For $X, T \in \mathbb{R}^{2n}$, we have

$$a(X+T) = a(X) + a'(X)T + \int_0^1 (1-s)^2 a''(X+sT)dsT^2$$

and thus, with $T = \lambda e_j$ ($\lambda > 0$, e_j is the j-th vector of an orthonormal basis of $\mathbb{R}^{2n}$ for g_0)

$$\forall \lambda > 0, \quad \lambda \|(\partial_j a)^w\|_{\mathcal{L}(L^2)} \leq 2\|a^w\|_{\mathcal{L}(L^2)} + \frac{\lambda^2}{2}\|(\partial_j^2 a)^w\|_{\mathcal{L}(L^2)}$$

so that $\|(\partial_j a)^w\|_{\mathcal{L}(L^2)} \leq 2\|a^w\|^{1/2}_{\mathcal{L}(L^2)}\|(\partial_j^2 a)^w\|^{1/2}_{\mathcal{L}(L^2)}$ and consequently

$$\|(\partial_j\partial_k a)^w\|_{\mathcal{L}(L^2)} \leq 2^{3/2}\|a^w\|^{1/4}_{\mathcal{L}(L^2)}\|(\partial_k^2 a)^w\|^{1/4}_{\mathcal{L}(L^2)}\|(\partial_j^2\partial_k a)^w\|^{1/2}_{\mathcal{L}(L^2)}.$$

Going on with that interpolation argument, using also (4.2.2), we get the result of the lemma. □

Localized commutators

Theorem 2.6.10. *Let A be a linear mapping from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$, g, m be admissible metric and weight on $\mathbb{R}^{2n}$. There exists $a \in S(m,g)$ such that $A = a^w$ if and only if, for all uniformly confined family of symbols $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$ (see def.2.3.14), for all linear forms L_j given by $L_j(X) = [X, T_j]$, with $g_Y(T_j) \leq 1$, for all $k \in \mathbb{N}$,*

$$\sup_{Y\in\mathbb{R}^{2n}} m(Y)^{-1} \| \operatorname{ad} L_1^w \dots \operatorname{ad} L_k^w \cdot \psi_Y^w A\|_{\mathscr{L}(L^2(\mathbb{R}^n))} < +\infty, , \tag{2.6.11}$$

i.e. with the notation (2.6.6), $\sup_{Y\in\mathbb{R}^{2n}} m(Y)^{-1}\|\psi_Y^w A\|^{(k)}_{op(g_Y)} < +\infty$.

Proof. The condition is easily seen to be necessary: if $A = a^w$ with $a \in S(m,g)$, we have with the partition of unity of Theorem 2.2.7, $L(X) = [X,T], g_Y(T) \leq 1$, using (2.1.12), with $a_Z = \varphi_Z a$,

$$\psi_Y^w A = \int_{\mathbb{R}^{2n}} \psi_Y^w a_Z^w |g_Z|^{1/2} dZ,$$
$$2i\pi \operatorname{ad} L^w \cdot \psi_Y^w A = \int_{\mathbb{R}^{2n}} \Big((\psi_Y' \cdot T)^w a_Z^w + \psi_Y^w (a_Z' \cdot T)^w \Big) |g_Z|^{1/2} dZ,$$

and using the estimates in the proof of Theorem 2.3.7 as well as (2.5.1), we get (2.6.11) for $k = 1$ and by iteration for all k. Let us prove the sufficiency. First of all, with $(\varphi_Y)_{Y \in \mathbb{R}^{2n}}$ standing for a partition of unity, as given by Theorem 2.2.7, from Theorem 2.3.15, we can find two uniformly confined families of symbols $(\theta_Y)_{Y \in \mathbb{R}^{2n}}, (\psi_Y)_{Y \in \mathbb{R}^{2n}}$ such that $\varphi_Y = \psi_Y \sharp \theta_Y$. From Theorem 2.6.6 applied to the constant metric g_Y, we get that $m(Y)^{-1} \theta_Y^w A = c_Y^w$, where $c_Y \in S(1, g_Y)$ uniformly (i.e. with seminorms bounded above independently of Y) and thus

$$m(Y)^{-1} \varphi_Y^w A = m(Y)^{-1} \psi_Y^w \theta_Y^w A = \psi_Y^w c_Y^w,$$

which gives

$$A = \int_{\mathbb{R}^{2n}} \varphi_Y^w A |g_Y|^{1/2} dY = \int_{\mathbb{R}^{2n}} m(Y) (\psi_Y \sharp c_Y)^w |g_Y|^{1/2} dY = a^w,$$

with $a \in S(m,g)$ since the family $(\psi_Y \sharp c_Y)_{Y \in \mathbb{R}^{2n}}$ is a uniformly confined family of symbols from the lemma 2.3.6. □

Remark 2.6.11. The previous proof implies that

$$\forall k, \exists C, \exists l, \quad \|a\|_{S(m,g)}^{(k)} \leq C \|a^w\|_{\mathrm{op}(m,g)}^{(l)}, \tag{2.6.12}$$

where $\|a\|_{S(m,g)}^{(k)}$ is given by (2.2.7) and with

$$\|a^w\|_{\mathrm{op}(m,g)}^{(l)} = \sup_{\substack{Y \in \mathbb{R}^{2n}, p \leq l \\ g_Y(L_j) \leq 1}} m(Y)^{-1} \| \operatorname{ad} L_1^w \dots \operatorname{ad} L_p^w \cdot \theta_Y^w A \|_{\mathscr{L}(L^2(\mathbb{R}^n))}, \tag{2.6.13}$$

where the linear form L, given by $LX = [T,X]$ is identified with the vector T and $g_Y(L)$ is defined as $g_Y(T)$. We have also, using the notation (2.6.6),

$$\|a^w\|_{\mathrm{op}(m,g)}^{(l)} = \sup_{Y \in \mathbb{R}^{2n}} m(Y)^{-1} \|\theta_Y^w a^w\|_{\mathrm{op}(g_Y)}^{(l)}. \tag{2.6.14}$$

Theorem 2.6.12. *Let g, m be admissible metric and weight on $\mathbb{R}^{2n}$ and let A be a linear operator from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$. There exists $a \in S(m,g)$ such that $A = a^w$ if and only if the seminorms $\|A\|_{op(m,g)}^{(k)}$ defined in* (2.6.14) *are finite and*

$$\forall k, \exists C, \exists l, \quad \|a\|_{S(m,g)}^{(k)} \leq C \|A\|_{op(m,g)}^{(l)}, \tag{2.6.15}$$
$$\forall k, \forall \tau \in [0,1[, \exists C, \exists l, \quad \|a\|_{S(m,g)}^{(k)} \leq C \big(\|A\|_{op(m,g)}^{(0)} \big)^{\tau} \big(\|A\|_{op(m,g)}^{(l)} \big)^{1-\tau}. \tag{2.6.16}$$

Proof. The equivalence is proven in Theorem 2.6.10 and the first inequality in the above remark. The second inequality follows from the first and from the interpolation result of Lemma 2.6.9. □

2.6.4 One-parameter group of elliptic operators

Lemma 2.6.13. *Let g be an admissible metric on $\mathbb{R}^{2n}$ and m be a g-admissible weight. Then $1+|\operatorname{Log} m|$ is also a g-admissible weight. Assuming $m \in S(m,g)$, we have*[23]

$$\forall k \geq 1,\ \exists C_k \quad |(\operatorname{Log} m)^{(k)}(X)T^k| \leq C_k g_X(T)^{k/2}, \tag{2.6.17}$$

and $\operatorname{Log} m \in S(1+|\operatorname{Log} m|, g)$.

Proof. There exists $r>0, c>0$ such that whenever $g_X(Y-X)\leq r^2$, we have $m(X)m(Y)^{-1}\in[e^{-c},e^c]$, so that $|\operatorname{Log} m(X)-\operatorname{Log} m(Y)|\leq c$, implying

$$\left(\frac{1+|\operatorname{Log} m(Y)|}{1+|\operatorname{Log} m(X)|}\right)^{\pm 1} \leq 1+c,$$

proving the slow variation. Moreover, there exists $C>1, N\geq 0$ such that

$$\begin{aligned}\frac{1+|\operatorname{Log} m(Y)|}{1+|\operatorname{Log} m(X)|} &\leq 1+\frac{|\operatorname{Log} m(Y)-\operatorname{Log} m(X)|}{1+|\operatorname{Log} m(X)|}\\ &\leq 1+\frac{\operatorname{Log} C+N\operatorname{Log}\big(1+(g_X^\sigma\wedge g_Y^\sigma)(Y-X)\big)}{1+|\operatorname{Log} m(X)|},\end{aligned}$$

giving the first result of the lemma. We have for $k\geq 1$ (see (4.3.1)),

$$\frac{1}{k!}(\operatorname{Log} m)^{(k)}T^k = \sum_{k_1+\cdots+k_r=k}(-1)^{r-1}r^{-1}m^{-r}\frac{1}{k_1!}m^{(k_1)}T^{k_1}\cdots\frac{1}{k_r!}m^{(k_r)}T^{k_r},$$

and $|m^{(k_j)}T^{k_j}|\leq Cm(X)g_X(T)^{k_j/2}$ which implies (2.6.17). □

Lemma 2.6.14. *Let g be an admissible metric on $\mathbb{R}^{2n}$, m be a g-admissible weight such that $m\in S(m,g)$. If $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$ is a uniformly confined family of symbols, the family $\big(\psi_Y(\cdot)\sharp\operatorname{Log}(m(\cdot)m(Y)^{-1})\big)_{Y\in\mathbb{R}^{2n}}$ is also uniformly confined.*

Proof. We have with the partition of unity of Theorem 2.2.7,

$$b_Y=\psi_Y\sharp\operatorname{Log}\big(m(\cdot)m(Y)^{-1}\big)=\int\psi_Y\sharp\varphi_Z\operatorname{Log}\big(m(\cdot)m(Y)^{-1}\big)|g_Z|^{1/2}dZ,$$

and $|\varphi_Z(X)\operatorname{Log}\big(m(X)m(Y)^{-1}\big)|\leq C\varphi_Z(X)\Big(1+\operatorname{Log}\big(\delta_r(Z,Y)\big)\Big)$, whereas the X-derivatives of $\operatorname{Log}\big(m(X)m(Y)^{-1}\big)$ satisfy the estimates of $S(1,g)$ (cf. (2.6.17)). We

[23] According to the remark 2.2.8, if m is a g-admissible weight, we may assume $m\in S(m,g)$.

get that $\big(\delta_r(Y,Z)^{-1}\varphi_Z \operatorname{Log}\big(m(\cdot)m(Y)^{-1}\big)\big)_{Z\in\mathbb{R}^{2n}}$ is a uniformly confined family of symbols and the confinement estimates yield

$$|b_Y^{(k)}(X)T^k| \leq C_{N,N'}\int \delta_r(Z,Y)\delta_r(Z,Y)^{-N}\big(1+g_Y^\sigma(X-U_{Y,r})\big)^{-N'}|g_Z|^{1/2}dZ,$$

which gives the result, if $N \geq N_1+1$, with N_1 given by (2.2.29). □

Theorem 2.6.15. *Let g be an admissible metric on $\mathbb{R}^{2n}$, M, m be g-admissible weights such that $M \in S(M,g), m \in S(m,g)$ and let $a \in S(m,g), \alpha \in S(1,g)$. There exists a unique smooth application $\mathbb{R}\times\mathbb{R}^{2n} \ni (t,X) \mapsto a_t(X) \in \mathbb{C}$ such that $a_t \in S(mM^t,g)$ for all t and*

$$\frac{\partial a_t(X)}{\partial t} = \big((\alpha+\operatorname{Log} M)\sharp a_t\big)(X), \quad a_0 = a. \tag{2.6.18}$$

When $m \equiv a_0 \equiv 1$, the unique solution of (2.6.18) satisfies as well $\frac{\partial a_t(X)}{\partial t} = \big(a_t\sharp(\alpha+\operatorname{Log} M)\big)(X)$ and we have

$$a_t\sharp a_s = a_{t+s}, \quad a_t \in S(M^t,g). \tag{2.6.19}$$

First step of the proof. We suppose first the existence of a solution a_t of (2.6.18). We have

$$a_t(X) = a_0(X) + \int_0^t \big(\underbrace{(\alpha+\operatorname{Log} M)}_{S(\langle \operatorname{Log} M\rangle,g)}\sharp \underbrace{a_s}_{S(mM^s,g)}\big)(X)ds$$

and thus $a_t(X) = a_0(X) + \int_0^t b(s,X)$, with $b(s) \in S(m\langle \operatorname{Log} M\rangle M^s, g)$, proving that the mapping $t \mapsto a_t$ is C^1 from $[-\tau,\tau]$ for any positive τ into the Fréchet space

$$S(m\langle \operatorname{Log} M\rangle(M+M^{-1})^\tau, g).$$

We define, using (2.6.1),

$$c_{t,Y} = m(Y)^{-1}M(Y)^{-t}\theta_Y\sharp a_t. \tag{2.6.20}$$

The symbol $c_{t,Y}(X)$ is C^1 in t and such that

$$\frac{\partial c_{t,Y}}{\partial t} = m(Y)^{-1}M(Y)^{-t}\theta_Y\sharp(\alpha+\operatorname{Log} M)\sharp a_t - \operatorname{Log} M(Y)c_{t,Y}$$

and verifying, with

$$\beta_Y(X) = \alpha(X) + \operatorname{Log}\frac{M(X)}{M(Y)}, \tag{2.6.21}$$

the equation $\frac{\partial c_{t,Y}}{\partial t} = m(Y)^{-1}M(Y)^{-t}\int \theta_Y\sharp\beta_Y\sharp\psi_Z\sharp\theta_Z\sharp a_t|g_Z|^{1/2}dZ$, i.e.

$$\frac{\partial c_{t,Y}}{\partial t} = \int \frac{m(Z)M(Z)^t}{m(Y)M(Y)^t}\theta_Y\sharp\beta_Y\sharp\psi_Z\sharp c_{t,Z}|g_Z|^{1/2}dZ.$$

We have thus

$$\frac{\partial c_{t,Y}}{\partial t} = \int K_{t,Y,Z} \sharp c_{t,Z} |g_Z|^{1/2} dZ, \quad c_{0,Y} = m(Y)^{-1} \theta_Y \sharp a, \tag{2.6.22}$$

$$\text{with } K_{t,Y,Z} = \frac{m(Z)M(Z)^t}{m(Y)M(Y)^t} \theta_Y \sharp \beta_Y \sharp \psi_Z. \tag{2.6.23}$$

We shall also consider the analogue

$$\frac{\partial \tilde{c}_{t,Y}}{\partial t} = \int \tilde{K}_{t,Y,Z} \sharp \tilde{c}_{t,Z} |g_Z|^{1/2} dZ, \quad \tilde{c}_{0,Y} = m(Y)^{-1} \theta_Y \sharp a, \tag{2.6.24}$$

$$\text{with } \tilde{K}_{t,Y,Z} = \frac{m(Z)M(Z)^t}{m(Y)M(Y)^t} \theta_Y \sharp \beta_Z \sharp \psi_Z. \tag{2.6.25}$$

Next step: resolution of the equation (2.6.22). For each $k \in \mathbb{N}$, we consider the vector space E_k defined as follows: E_0 is the Banach space $L^\infty\big(\mathbb{R}^{2n}, \mathcal{L}(L^2(\mathbb{R}^n))\big)$, E_1 is the subspace of E_0 such that

$$\|A\|_{E_1} = \|A\|_{E_0} + \sup_{\substack{Y, g_Y(T)=1 \\ L(X)=[X,T]}} \|[L^w, A_Y]\|_{\mathcal{L}(L^2(\mathbb{R}^n))} < \infty.$$

More generally, for $k \geq 1$, the space E_k is defined, using the notation (2.6.6), as the subspace of E_0 such that

$$\|A\|_{E_k} = \|A_Y\|^{(k)}_{\mathrm{op}(g_Y)} < \infty.$$

If (A_j) is a Cauchy sequence in E_1, it converges in E_0 (with limit A) and for $g_Y(T) = 1, L_T(X) = [X, T]$, the sequence $([L^w_T, A_{j,Y}])$ is converging in

$$L^\infty\big(\mathbb{G}_1, \mathcal{L}(L^2(\mathbb{R}^n))\big), \quad \mathbb{G}_1 = \{(Y,T) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}, g_Y(T) \leq 1\},$$

with limit B. Now we have also as operators from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$, the equality $[L^w_T, A_Y] = B_{Y,T}$; this proves that $A \in E_1$ and that the sequence (A_j) is converging in E_1 to A since

$$\sup_{Y, g_Y(T)=1} \|[L^w_T, A_{j,Y} - A_Y]\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$$

$$= \sup_{Y, g_Y(T)=1} \|[L^w_T, A_{j,Y}] - B_{Y,T}\|_{L^\infty(\mathbb{G}_1, \mathcal{L}(L^2(\mathbb{R}^n)))} \underset{j \to +\infty}{\longrightarrow} 0. \tag{2.6.26}$$

The same type of argument works for $k \geq 2$, replacing the bundle $\mathbb{G}_1$ by the suitable $\mathbb{G}_k$. We define the operator $\mathcal{K}_t$ by $(\mathcal{K}_t A)_Y = \int K^w_{t,Y,Z} A_Z |g_Z|^{1/2} dZ$, and we want to prove that it is a bounded operator on E_k, uniformly for t bounded.

Let $Y, T \in \mathbb{R}^{2n}$ such that $g_Y(T) = 1$, L the linear form $LX = [X, T]$. We check

$$\begin{aligned}
&\|(\operatorname{ad} L)^k (\mathcal{K}_t A)_Y\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \\
&\quad \le C \max_{k_1+k_2=k} \int \|(\operatorname{ad} L)^{k_1} K^w_{t,Y,Z}\|_{\mathcal{L}(L^2)} \|(\operatorname{ad} L)^{k_2} A_Z\|_{\mathcal{L}(L^2)} |g_Z|^{1/2} dZ \\
&\quad \le C \max_{k_1+k_2=k} \int \|(\operatorname{ad} L)^{k_1} K^w_{t,Y,Z}\|_{\mathcal{L}(L^2)} \|A\|_{E_k} g_Z(T)^{k_2/2} g_Y(T)^{-k_2/2} |g_Z|^{1/2} dZ.
\end{aligned}$$

According to Lemma 2.6.14, the family $(\theta_Y \sharp \beta_Y)_{Y \in \mathbb{R}^{2n}}$ is uniformly confined and the biconfinement theorem 2.3.2 implies that the families

$$(\delta_r(Y,Z)^N \theta_Y \sharp \beta_Y \sharp \psi_Z)_{Y \in \mathbb{R}^{2n}}$$

are uniformly confined (and this uniformly with respect to Z), so that, using (2.5.1) and (2.2.29) for N large enough,

$$\begin{aligned}
&\sup_{Y, |t| \le T} \|(\operatorname{ad} L)^k (\mathcal{K}_t A)_Y\|_{\mathcal{L}(L^2)} \\
&\quad \le C_N \max_{l \le k} \sup_{Y, |t| \le \tau} \int \delta_r(Y,Z)^{-N} \frac{m(Z) M(Z)^t}{m(Y) M(Y)^t} \|A\|_{E_k} \frac{g_Z(T)^{l/2}}{g_Y(T)^{l/2}} |g_Z|^{1/2} dZ < \infty.
\end{aligned}$$

Handling the same estimate with $(\operatorname{ad} L)^k$ replaced by $(\operatorname{ad} L_1)^{k_1} \dots (\operatorname{ad} L_r)^{k_r}$ is strictly analogous to the previous argument. We have proven that the operator $\mathcal{K}_t$ is bounded on the Banach space E_k, with a norm bounded for t bounded. As a result, since $m(Y)^{-1} \theta^w_Y a^w_0$ belongs to all E_k, the ODE

$$\frac{dC}{dt} = \mathcal{K} C, \quad C(0) = m(Y)^{-1} \theta^w_Y a^w_0, \tag{2.6.27}$$

has a unique solution on $\mathbb{R}$ belonging to all E_k. The same holds for the equation (2.6.24). The characterization of Theorem 2.6.10 implies that $C_Y(t) = c^w_{t,Y}$ with $c_{t,Y} \in S(1, g_Y)$ uniformly in Y and for t bounded.

Third step: proof of the first statement of the theorem. If a_t is a solution of (2.6.18), we have proven that $C(t, Y) = c^w_{t,Y}$ given by (2.6.20) satisfies (2.6.27) and this implies the uniqueness of the solution of (2.6.18), since

$$a^w_t = \int \psi^w_Y \theta^w_Y a^w_t |g_Y|^{1/2} dY = \int m(Y) M(Y)^t \ \psi^w_Y c^w_{t,Y} |g_Y|^{1/2} dY.$$

To prove the existence of a solution of (2.6.18), we consider the solution $\tilde{c}_{t,Y}$ of (2.6.24), and we define

$$a_t = \int m(Y) M(Y)^t \ \psi_Y \sharp \tilde{c}_{t,Y} |g_Y|^{1/2} dY.$$

Thanks to Theorem 2.6.10 and to Proposition 2.3.16, the family $(\psi_Y \sharp \tilde{c}_{t,Y})_{Y \in \mathbb{R}^{2n}}$ is a uniformly confined family of symbols and a_t belongs to $S(mM^t, g)$. On the

other hand, we have

$$\begin{aligned}\frac{\partial a_t}{\partial t} &= \int m(Y)M(Y)^t \operatorname{Log} M(Y)\ \psi_Y \sharp \tilde{c}_{t,Y}|g_Y|^{1/2}dY \\ &\qquad + \iint m(Y)M(Y)^t \psi_Y \sharp \tilde{K}_{t,Y,Z} \sharp \tilde{c}_{t,Z}|g_Z|^{1/2}|g_Y|^{1/2}dYdZ \\ &= \int m(Z)M(Z)^t \operatorname{Log} M(Z)\ \psi_Z \sharp \tilde{c}_{t,Z}|g_Z|^{1/2}dZ \\ &\qquad + \iint m(Z)M(Z)^t \psi_Y \sharp \theta_Y \sharp \beta_Z \sharp \psi_Z \sharp \tilde{c}_{t,Z}|g_Z|^{1/2}|g_Y|^{1/2}dYdZ \\ &= \int m(Z)M(Z)^t \big(\alpha + \operatorname{Log} M\big) \sharp \psi_Z \sharp \tilde{c}_{t,Z}|g_Z|^{1/2}dZ \\ &= \big(\alpha + \operatorname{Log} M\big) \sharp a_t,\end{aligned}$$

along with $a_{t=0} = \int m(Y)\psi_Y \sharp m(Y)^{-1}\theta_Y \sharp a_0 |g_Y|^{1/2}dY = a_0$, providing a solution of (2.6.18). We may consider the unique solution a_t of (2.6.18) for $a_0 \equiv 1$. If moreover $m \equiv 1$, for s given in $\mathbb{R}$, the mappings $t \mapsto a_{t+s}, t \mapsto a_t \sharp a_s$ are both solutions of (2.6.18) with value a_s at $t = 0$ and the already proven uniqueness property (with $M = m^s$) implies

$$a_{t+s} = a_t \sharp a_s.$$

In particular, we have for all t, $a_t \sharp a_{-t} = 1$ and, taking the derivative of that identity, we get $\alpha \sharp a_t \sharp a_{-t} - a_t \sharp \alpha \sharp a_{-t}$, that is $\alpha = a_t \sharp \alpha \sharp a_{-t}$, implying the commutation for $\sharp$ of a_t with α, completing the proof of the theorem. □

Corollary 2.6.16. *Let g be an admissible metric and m be an admissible g-weight. There exists $a \in S(m,g), b \in S(m^{-1},g)$ such that $a\sharp b = b\sharp a = 1$ and for any g-admissible weight M, the operator a^w is an isomorphism from $H(M,g)$ onto $H(Mm^{-1},g)$ with inverse b^w. The space $H(m,g)$ is an Hilbert space with the dot-product* (2.6.3)*. An equivalent Hilbertian structure on $H(m,g)$ is given by the dot-product $\langle a^w u, a^w v\rangle_{L^2(\mathbb{R}^n)}$.*

Proof. We may assume that $m \in S(m,g)$ and solve the equation (2.6.19) with $M = m, \alpha = 0$: we get $a_1 \sharp a_{-1} = 1 = a_{-1}\sharp a_1,\ a_{\pm 1} \in S(m^{\pm 1}, g)$. Moreover, for a g-admissible weight M, thanks to Theorem 2.6.4, the operator a_1^w (resp. a_{-1}^w) sends continuously $H(M,g)$ into $H(Mm^{-1},g)$ (resp. $H(Mm^{-1},g)$ into $H(M,g)$) and since $a_1^w a_{-1}^w = \mathrm{Id}_{H(Mm^{-1},g)}, a_{-1}^w a_1^w = \mathrm{Id}_{H(M,g)}$, we get the result of the theorem. We already know that $H(m,g)$ is a pre-Hilbertian space with the dot product (2.6.3); the completeness of that space follows from the completeness of $L^2(\mathbb{R}^n)$ and the above isomorphisms. Moreover $H(m,g) \ni u \mapsto \|a_1^w u\|_{L^2(\mathbb{R}^n)}$ is a Hilbertian norm on $H(m,g)$, which is equivalent to the norm given by (2.6.3) since a_1^w is an isomorphism from $H(m,g)$ onto $H(1,g) = L^2(\mathbb{R}^n)$. □

Corollary 2.6.17. *Let g be an admissible metric and m be an admissible g-weight.*

(1) *Let $u \in \mathscr{S}'(\mathbb{R}^n)$; the following properties are equivalent,*

(i) $u \in H(m,g)$,

(ii) For all $a \in S(m,g)$, $a^w u \in L^2(\mathbb{R}^n)$,

(iii) There exist $c \in S(m^{-1},g), v \in L^2(\mathbb{R}^n)$, *such that* $u = c^w v$.

(2) *The space $\mathscr{S}(\mathbb{R}^n)$ is dense in $H(m,g)$.*

(3) *The dual space of $H(m,g)$ is canonically identified with $H(m^{-1},g)$.*

Proof. We prove (1): thanks to Theorem 2.6.4, we know $(i) \Longrightarrow (ii)$. Corollary 2.6.16 provides an isomorphism a^w with $a \in S(m,g)$ of $H(m,g)$ with $L^2(\mathbb{R}^n)$, with inverse b^w with $b \in S(m^{-1},g)$; as a result if (ii) holds, we have $u = b^w a^w u$ with $a^w u \in L^2, b \in S(m^{-1},g)$, that is (iii). If (iii) is satisfied and a^w is the isomorphism from $H(m,g)$ onto L^2 (with inverse b^w, $a \in S(m,g), b \in S(m^{-1},g)$) of the corollary 2.6.16, we have

$$u = c^w a^w \underbrace{b^w v}_{\in H(m,g)} = (\underbrace{c \sharp a}_{\in S(1,g)})^w (b^w v),$$

so that Theorem 2.6.4 implies $u \in H(m,g)$, completing the proof of (1). Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the isomorphism b^w sends $L^2(\mathbb{R}^n)$ onto $H(m,g)$ and since we have $\mathscr{S}(\mathbb{R}^n) \supset b^w\big(\mathscr{S}(\mathbb{R}^n)\big)$, the latter is dense in $H(m,g)$, implying (2). To prove (3), using an isomorphism a^w between $H(m,g)$ and $L^2(\mathbb{R}^n)$ with $a \in S(m,g)$, with an inverse b^w between $H(m^{-1},g)$ and $L^2(\mathbb{R}^n)$ with $b \in S(m^{-1},g)$, we consider the sesquilinear mapping

$$\begin{array}{ccc} H(m,g) \times H(m^{-1},g) & \longrightarrow & \mathbb{C} \\ (u,v) & \mapsto & \langle a^w u, b^w v\rangle_{L^2(\mathbb{R}^n)}, \end{array} \tag{2.6.28}$$

and we equip the space $H(m,g)$ with the norm $\|a^w u\|_{L^2}$, and $H(m^{-1},g)$ with the norm $\|b^w v\|_{L^2}$. The anti-linear mapping $\Phi : H(m^{-1},g) \longrightarrow \big(H(m,g)\big)^*$ given by

$$\langle \Phi(v), u\rangle_{H(m,g)^*, H(m,g)} = \langle a^w u, b^w v\rangle_{L^2(\mathbb{R}^n)},$$

is such that

$$\|\Phi(v)\|_{H(m,g)^*} = \sup_{\|a^w u\|_{L^2}=1} |\langle a^w u, b^w v\rangle_{L^2(\mathbb{R}^n)}| = \|b^w v\|_{L^2} = \|v\|_{H(m^{-1},g)},$$

since $a^w\big(\mathscr{S}(\mathbb{R}^n)\big) = \mathscr{S}(\mathbb{R}^n)$ (if $u \in \mathscr{S}$, $u = a^w b^w u$) so that Φ is isometric; it is also onto since the Riesz representation theorem implies that for $\xi_0 \in H(m,g)^*$, we can find $w_0 \in H(m,g)$ so that

$$\forall u \in H(m,g), \quad \xi(u) = \langle u, w_0\rangle_{H(m,g)} = \langle a^w u, a^w w_0\rangle_{L^2} = \langle a^w u, b^w a^w a^w w_0\rangle_{L^2},$$

i.e. $\xi = \Phi(v)$ with $v = a^w a^w w_0$ which belongs to $H(m^{-1},g)$. The open mapping theorem implies that Φ is an isomorphism between $H(m^{-1},g)$ and $H(m,g)^*$ and that the pairing (2.6.28) provides the identification of these spaces. □

Lemma 2.6.18. *Let g_1, g_2 be two admissible metrics on $\mathbb{R}^{2n}$ and let m be an admissible g_j-weight such that $m \in S(m, g_j), j = 1, 2$. Then $H(m, g_1) = H(m, g_2)$.*

Proof. If $a_{j,t} \in S(m^t, g_j)$ verify (2.6.19) with

$$\frac{\partial a_{j,t}(X)}{\partial t} = (\operatorname{Log} m \sharp a_{j,t})(X) = (a_{j,t} \sharp \operatorname{Log} m)(X), \quad a_{j,0}(X) \equiv 1.$$

the operators $a^w_{j,t}$ are continuous endomorphisms of the Fréchet space $\mathscr{S}(\mathbb{R}^n)$ and for all $u \in \mathscr{S}(\mathbb{R}^n)$, the mappings $t \mapsto a^w_{j,t}u \in \mathscr{S}(\mathbb{R}^n)$ are of class C^1. For $u \in \mathscr{S}(\mathbb{R}^n)$, we calculate

$$\partial_t\big(a^w_{2,-t}a^w_{1,t}u\big) = -(a_{2,-t}\sharp \operatorname{Log} m)^w a^w_{1,t}u + a^w_{2,-t}(\operatorname{Log} m \sharp a_{1,t})^w u = 0,\ a^w_{2,0}a^w_{1,0}u = u,$$

so that $a_{2,-t}\sharp a_{1,t} = 1$ and $a_{2,t} = a_{2,t}\sharp a_{1,-t}\sharp a_{1,t} = a_{1,t}$, so that the operators $a^w_{2,t}, a^w_{1,t}$ defining for $t = 1$ the isomorphisms between $H(m, g_j)$ and L^2 are the same, giving the result of the lemma. □

2.6.5 An additional hypothesis for the Wiener lemma: the geodesic temperance

We have seen so far that the pseudodifferential calculus can be developed in great generality using the rather simple notion of admissibility of the metric g, as given by Definition 2.2.15. We have seen in the previous sections that a large part of the theory of the Sobolev spaces attached to a pseudodifferential calculus given by an admissible metric can also be developed under this sole assumptions (see also [19] which we have followed) and that the analytic functional calculus is possible under the only assumption of admissibility. However, we would like to prove a Wiener-type lemma (see e.g. our Remark 2.5.9), as explained below. Let us consider a pseudodifferential operator A with a symbol in the class $S(1, g)$ (A is L^2 bounded if g is admissible) such that A is invertible as an operator in $L^2(\mathbb{R}^n)$: is it true that A^{-1} is a pseudodifferential operator with symbol in $S(1, g)$? A positive answer to that question seems to require an additional assumption, the so-called geodesic temperance. Although no example of an admissible metric which is not geodesically temperate seems to be availaible, we shall study in this section a version of the geodesic temperance which will allow us to prove a Wiener lemma and which is also invariant by conformal change.

Definition 2.6.19. Let g be an admissible metric on $\mathbb{R}^{2n}$ (see the definition 2.2.15) and let $g^\natural$ be the intermediate symplectic metric, as given by the definition 2.2.19. Let d be the geodesic distance on $\mathbb{R}^{2n}$ for the metric $g^\natural$. We shall say that the metric g is geodesically temperate if there exists $C > 0, N \geq 0$ such that, for all $X, Y, T \in \mathbb{R}^{2n}$,

$$\frac{g_X(T)}{g_Y(T)} \leq C\big(1 + d(X, Y)\big)^N. \tag{2.6.29}$$

Lemma 2.6.20. *Let g be an admissible metric on $\mathbb{R}^{2n}$ (def. 2.2.15) so that g is geodesically temperate (def. 2.6.19); then there exist $C, N > 0$ such that*

$$C^{-1}\big(1 + g^\natural_X(X-Y)\big)^{1/N} \leq 1 + d(X,Y) \leq C\big(1 + g^\natural_X(X-Y)\big)^N, \qquad (2.6.30)$$

where d is the geodesic distance for $g^\natural$.

Proof. We have for $X_0, X_1 \in \mathbb{R}^{2n}$, using the admissibility of $g^\natural$,

$$\begin{aligned} d(X_0,X_1) &\leq \int_0^1 \big(g^\natural_{X_0+t(X_1-X_0)}(X_1-X_0)\big)^{1/2}dt \\ &\leq \int_0^1 C'\big(1+g^\natural_{X_0}(t(X_1-X_0))\big)^{N'}dt g^\natural_{X_0}(X_1-X_0)^{1/2} \\ &\leq C'\big(1+g^\natural_{X_0}(X_1-X_0)\big)^{N'} g^\natural_{X_0}(X_1-X_0)^{1/2}, \end{aligned}$$

providing the last inequality of (2.6.30). On the other hand, for a $g^\natural$-geodesic curve between X_0 and X_1, we have from (2.6.29)

$$\begin{aligned} d(X_0,X_1) &= \int_0^1 g^\natural_{X(t)}(\dot X(t))^{1/2}dt \geq \\ &\int_0^1 g^\natural_{X_0}(\dot X(t))^{1/2} C^{-1}\big(1+\max_{t\in[0,1]} d(X_0,X(t))\big)^{-N}dt \geq \\ &C^{-1} g^\natural_{X_0}(X_1-X_0)^{1/2}\big(1+d(X_0,X_1)\big)^{-N}, \end{aligned}$$

since $\int_0^1 g^\natural_{X_0}(\dot X(t))^{1/2}dt \geq g^\natural_{X_0}$-distance from X_0 to X_1 and

$$\max_{t\in[0,1]} d(X_0,X(t)) \leq d(X_0,X_1)$$

since $t \mapsto X(t)$ is a geodesic curve for the Riemannian metric $g^\natural$. We obtain thus the first inequality in (2.6.30), completing the proof of the lemma. □

Remark 2.6.21. With the notations and assumptions of the previous lemma, there exists $C, N > 0$ such that, for all $X, Y \in \mathbb{R}^{2n}$,

$$\frac{1}{C}\big(1+(g^\natural_X \wedge g^\natural_Y)(X-Y)\big)^{1/N} \leq 1+d(X,Y) \leq C\big(1+(g^\natural_X \wedge g^\natural_Y)(X-Y)\big)^N. \quad (2.6.31)$$

The left inequality above follows from the third inequality in (4.4.26) and the first inequality in (2.6.30). Moreover, we have from (4.4.27) (with T depending on X, Y),

$$(g^\natural_X \wedge g^\natural_Y)(X-Y) = 2g^\natural_X(X-T) + 2g^\natural_Y(T-Y)$$

and using the triangle inequality along with the right inequality in (2.6.30), we get

$$1+d(X,Y)\leq 1+d(X,T)+d(Y,T)\leq C\big(1+g_X^\natural(X-T)\big)^N+C\big(1+g_Y^\natural(Y-T)\big)^N$$
$$\leq C\big(2+g_X^\natural(X-T)+g_Y^\natural(T-Y)\big)^N\leq C\big(2+\frac{1}{2}(g_X^\natural\wedge g_Y^\natural)(X-Y)\big)^N,$$

which gives (2.6.31) (with the same N as in (2.6.30)). Defining for $r>0, X, Y\in\mathbb{R}^{2n}$,

$$\delta_r^\natural(X,Y)=1+(g_X^\natural\wedge g_Y^\natural)(U_{X,r}^\natural-U_{Y,r}^\natural), \tag{2.6.32}$$
$$\text{with } U_{X,r}^\natural=\{X'\in\mathbb{R}^{2n}, g_X^\natural(X'-X)\leq r^2\}, \tag{2.6.33}$$

we see that $\delta_r^\natural$ is the main distance function[24] for the admissible metric $g^\natural$ (see (2.2.27) and the fact that $g^\natural$ is symplectic, according to (4.4.29)). We have

$$\delta_r^\natural(X,Y)=1+(g_X^\natural\wedge g_Y^\natural)(X'-Y'),\ X'\in U_{X,r}^\natural,\ Y'\in U_{Y,r}^\natural,$$

so that the ratio $\delta_r^\natural(X,Y)/1+(g_{X'}^\natural\wedge g_{Y'}^\natural)(X'-Y')$ is bounded above and below by some fixed positive constants, implying from (2.6.31) that

$$d(X,Y)\leq d(X,X')+d(X',Y')+d(Y',Y)$$
$$\leq d(X,X')+C\big(1+(g_{X'}^\natural\wedge g_{Y'}^\natural)(X'-Y')\big)^N+d(Y',Y)$$
$$\leq d(X,X')+C'\underbrace{\big(1+(g_X^\natural\wedge g_Y^\natural)(X'-Y')\big)^N}_{\delta_r^\natural(X,Y)^N}+d(Y',Y),$$

and since $g^\natural$ is slowly varying, $X'\in U_{X,r}^\natural, Y'\in U_{Y,r}^\natural$, we get that

$$\exists C>0,\exists N\geq 0,\exists r>0,\quad \forall X,Y\in\mathbb{R}^{2n},\quad 1+d(X,Y)\leq C\delta_r^\natural(X,Y)^N. \tag{2.6.34}$$

Lemma 2.6.22. *Let g be an admissible and geodesically temperate metric on $\mathbb{R}^{2n}$ (Definitions 2.2.15, 2.6.19) and let G be a slowly varying metric on $\mathbb{R}^{2n}$ such that $G=\mu g$ is conformal to g, with $\mu\leq\lambda_g$ (defined in (2.2.13)). Then G is admissible and geodesically temperate on $\mathbb{R}^{2n}$.*

Proof. The fact that G is admissible follows from Lemma 2.2.21. Since G is conformal to g, the definition 2.2.19 implies that $G^\natural=g^\natural$; moreover, from the proposition 2.2.20, the ratio G_X/G_Y is bounded above by

$$C\big(1+(g_X^\natural\wedge g_Y^\natural)(X-Y)\big)^N$$

[24]Note that $\delta_r^\natural(X,Y)$ is larger than $\Delta_r(X,Y)$ as defined by (2.2.32), since in the latter formula $U_{X,r}$ is the g_X-ball with center X, and radius r whereas here in (2.6.32), we are considering the smaller $g_X^\natural$-ball (see (2.2.22)).

and Lemma 2.6.20 entails that this quantity is bounded above by a constant multiplied by a power of $(1+d(X,Y))$, proving the second assertion of the lemma. □

Lemma 2.6.23. *The metrics $\langle\xi\rangle^{2\delta}|dx|^2+\langle\xi\rangle^{-2\rho}|d\xi|^2$, (with $0\le\delta\le\rho\le 1$, $\delta<1$), $|dx|^2+h^2|d\xi|^2$ (with $0<h\le 1$), $(1+|x|^2+|\xi|^2)^{-1}|dx|^2+|d\xi|^2$ are admissible and geodesically temperate. Any slowly varying metric G (see definition 2.2.1) such that $G\le G^\sigma$ (i.e. (2.2.12) is satisfied) and G is conformal to one of the metrics above is admissible and geodesically temperate.*

Proof. The admissibility follows from Lemma 2.2.18, the geodesic temperance for the last two metrics follows from the previous lemma 2.6.22 since they are both conformal to a constant metric (note that all the constants are uniform with respect to $h\in(0,1]$). We see that the first metrics of the lemma are conformal to

$$|dx|^2\langle\xi\rangle^{\delta+\rho}+|d\xi|^2\langle\xi\rangle^{-\rho-\delta}$$

and, using again Lemma 2.6.22, it is enough to prove that the latter is geodesically temperate. Since $(\rho+\delta)/2\in[0,1)$, this amounts to prove that

$$\frac{|\eta|}{1+|\xi|}\le C\big(1+d_0(\xi,\eta)\big)^N \tag{2.6.35}$$

where d_0 is the geodesic distance for $|d\xi|^2\langle\xi\rangle^{-2\theta}$ on $\mathbb{R}^n$, where $\theta\in[0,1)$. We have for a C^1 curve $t\mapsto\xi(t)$ in $\mathbb{R}^n$,

$$(1-\theta)^{-1}\frac{d}{dt}(\langle\xi(t)\rangle^{1-\theta})=\langle\xi(t)\rangle^{-\theta-1}\dot\xi(t)\cdot\xi(t)$$

and consequently

$$\int_0^1|\dot\xi(t)|\langle\xi(t)\rangle^{-\theta}dt\ge(1-\theta)^{-1}\left|\int_0^1\frac{d}{dt}(\langle\xi(t)\rangle^{1-\theta})dt\right|=\frac{\big|\langle\xi(1)\rangle^{1-\theta}-\langle\xi(0)\rangle^{1-\theta}\big|}{(1-\theta)},$$

which proves $\big|\langle\xi\rangle^{1-\theta}-\langle\eta\rangle^{1-\theta}\big|\le(1-\theta)d_0(\xi,\eta)$, implying for $\langle\xi\rangle^{1-\theta}\le\frac12\langle\eta\rangle^{1-\theta}$ that $\langle\eta\rangle^{1-\theta}\le 2(1-\theta)d_0(\xi,\eta)$, which gives (2.6.35) in that case, whereas (2.6.35) is obvious in the other case $\langle\xi\rangle^{1-\theta}>\frac12\langle\eta\rangle^{1-\theta}$. The last statement of the lemma follows from Lemma 2.6.22. The proof of the lemma is complete. □

Remark 2.6.24. The main point of the previous lemma is the stability of the admissible and geodesically temperate metrics by conformal change preserving the uncertainty principle and the slowness. We shall apply this property to some Calderón-Zygmund metrics which are precisely conformal to a constant metric or to the metric defining the $S_{1,0}$ class.

Lemma 2.6.25. *Let g be an admissible metric on $\mathbb{R}^{2n}$. Then $\forall N_0 \geq 0, \exists C_0 > 0, \exists k_0 \in \mathbb{N}, \forall N_1 \geq 0, \exists C_1, \exists k_1 \in \mathbb{N}, \forall \nu \in \mathbb{N}^*, \forall J \subset \{0, \dots, \nu-1\}$, $\forall c_0, \dots, c_\nu \in \mathscr{S}(\mathbb{R}^{2n}), \forall Y_0, \dots, Y_\nu \in \mathbb{R}^{2n}$, we have*

$$\begin{aligned}
&\|c_0^w \dots c_\nu^w\|_{\mathscr{L}(L^2)} \\
&\quad \leq C_0^{\nu - \operatorname{card} J} C_1^{1+\operatorname{card} J} \|\!| c_0 |\!\|^{(k_1)}_{g_{Y_0}, U_{Y_0,r}} \big(\max_{j \notin K, j \neq 0} \|\!| c_j |\!\|^{(k_0)}_{g_{Y_j}, U_{Y_j,r}} \big)^{\nu - \operatorname{card} J} \\
&\quad \times \big(\max_{j \in K, j \neq 0} \|\!| c_j |\!\|^{(k_1)}_{g_{Y_j}, U_{Y_j,r}} \big)^{\operatorname{card} J} \prod_{0 \leq j < \nu} \delta_r(Y_j, Y_{j+1})^{-N_0} \prod_{j \in J} \delta_r(Y_j, Y_{j+1})^{-N_1},
\end{aligned}$$

with $K = J \cup \big((1+J) \cap [0,\nu]\big) = \{j \in \{0,\dots,\nu\}$ such that $j \in J$ or $j-1 \in J\}$.

Proof. We have

$$\begin{aligned}
\|c_0^w \dots c_\nu^w\|_{\mathscr{L}(L^2)} &\leq \|c_0^w c_1^w\|_{\mathscr{L}(L^2)} \|c_2^w c_3^w\|_{\mathscr{L}(L^2)} \dots, \\
\|c_0^w \dots c_\nu^w\|_{\mathscr{L}(L^2)} &\leq \|c_0^w\|_{\mathscr{L}(L^2)} \, \|c_1^w c_2^w\|_{\mathscr{L}(L^2)} \|c_3^w c_4^w\|_{\mathscr{L}(L^2)} \dots,
\end{aligned}$$

and thus

$$\|c_0^w \dots c_\nu^w\|_{\mathscr{L}(L^2)} \leq \|c_0^w\|^{1/2}_{\mathscr{L}(L^2)} \|c_\nu^w\|^{1/2}_{\mathscr{L}(L^2)} \prod_{1 \leq j < \nu} \|c_j^w c_{j+1}^w\|^{1/2}_{\mathscr{L}(L^2)}. \tag{2.6.36}$$

Using (2.5.1), the biconfinement estimates of Corollary 2.3.3 and the first estimates in (2.2.29), we get

$$\begin{aligned}
&\|c_0^w\|_{\mathscr{L}(L^2)} \leq C(n) \|\!| c_0 |\!\|^{(2n+1)}_{g_{Y_0}, U_{Y_0,r}}, \quad \|c_\nu^w\|_{\mathscr{L}(L^2)} \leq C(n) \|\!| c_\nu |\!\|^{(2n+1)}_{g_{Y_\nu}, U_{Y_\nu,r}}, \\
&\text{for } j \notin J, \quad \|c_j^w c_{j+1}^w\|_{\mathscr{L}(L^2)} \leq C_0 \delta_r(Y_j, Y_{j+1})^{-2N_0} \|\!| c_j |\!\|^{(k_0)}_{g_{Y_j}, U_{Y_j,r}} \|\!| c_{j+1} |\!\|^{(k_0)}_{g_{Y_{j+1}}, U_{Y_{j+1},r}},
\end{aligned}$$

and for $j \in J$,

$$\|c_j^w c_{j+1}^w\|_{\mathscr{L}(L^2)} \leq C_1 \delta_r(Y_j, Y_{j+1})^{-2N_0 - 2N_1} \|\!| c_j |\!\|^{(k_1)}_{g_{Y_j}, U_{Y_j,r}} \|\!| c_{j+1} |\!\|^{(k_1)}_{g_{Y_{j+1}}, U_{Y_{j+1},r}},$$

with C_0, k_0 depending only on N_0 and C_1, k_1 on $N_0 + N_1$. We may assume $C_1 \geq C_0, k_1 \geq k_0$. Plugging the above estimates in (2.6.36) yields the result of the lemma. □

Lemma 2.6.26. *Let g be an admissible and geodesically temperate metric such that, for all $X \in \mathbb{R}^{2n}$, $g_X = g_X^\sigma$ (implying $g \equiv g^\flat \equiv g^\sigma$). Then there exist some constants C_0, k_0 depending only on the structure constants[25] of g, and for all $k \in \mathbb{N}$, there exist some constants C_1, k_1, N_1, such that for $a \in S(1,g)$, $\nu \in \mathbb{N}$, with the notations (2.6.13), (2.2.7),*

$$\|(a^w)^\nu\|^{(k)}_{op(1,g)} \leq C_1 (\nu+1)^{N_1} \big(C_0 \|a\|^{(k_0)}_{S(1,g)}\big)^{\nu-k} \big(C_1 \|a\|^{(k_1)}_{S(1,g)}\big)^k.$$

[25] We mean here the constants occurring in (2.2.3), (2.2.16).

Proof. For $Y_0 \in \mathbb{R}^{2n}$, we consider k linear forms on $\mathbb{R}^{2n}$, $L_j(X) = [T_j, X]$, $j = 1, \dots, k$ such that $g_{Y_0}(T_j) = 1$. We have, using the notations (2.6.13), with $a = \int a\varphi_Y dY$, $a_Y = a\varphi_Y$,

$$\omega_k = \| \operatorname{ad} L_1^w \dots \operatorname{ad} L_k^w \cdot \theta_{Y_0}^w (a^w)^\nu \|_{\mathscr{L}(L^2)} \\ \leq \int \| \operatorname{ad} L_1^w \dots \operatorname{ad} L_k^w \cdot \theta_{Y_0}^w a_{Y_1}^w \dots a_{Y_\nu}^w \|_{\mathscr{L}(L^2)} dY_1 \dots dY_\nu.$$

Since $\operatorname{ad} C \cdot AB = [C, A]B + A[C, B]$, we get, with the convention $\prod_{l \in \emptyset} A_l = \mathrm{Id}$ that

$$\operatorname{ad} C \cdot A_1 \dots A_\mu = \sum_{1 \leq j \leq \mu} (\prod_{1 \leq l < j} A_l)[C, A_j](\prod_{j < l \leq \mu} A_l),$$

so that ω_k is bounded above by a sum of $(\nu+1)^k$ terms $\tilde{\omega}_k$, with

$$\tilde{\omega}_k = \int \| b_0^w \dots b_\nu^w \|_{\mathscr{L}(L^2)} dY_1 \dots dY_\nu, \quad b_j = (\prod_{\alpha \in E_j} \partial_{T_\alpha}) a_{Y_j},$$

with the convention $a_{Y_0} = \theta_{Y_0}$ and the $(E_j)_{0 \leq j \leq \nu}$ are (possibly empty) disjoint subsets of $\{1, \dots, k\}$. In particular, defining $J = \{j \in [0, \nu], E_j \neq \emptyset\}$, we have

$$k \geq \sum_{j \in J} \operatorname{card} E_j \geq \sum_{j \in J} 1 = \operatorname{card} J.$$

We define for $j \notin J$, $c_j = b_j (= a_{Y_j})$ and for $j \in J$, $c_j = b_j (\prod_{\alpha \in E_j} g_{Y_j}(T_\alpha)^{-1/2})$. Since $g = g^\natural$ is geodesically temperate and $g_{Y_0}(T_\alpha) = 1$, we have from the triangle inequality for d (and Hölder's inequality)

$$g_{Y_j}(T_\alpha) \leq C(1 + d(Y_0, Y_j))^N \leq C\nu^{N-1} \sum_{0 \leq l < j} (1 + d(Y_l, Y_{l+1}))^N.$$

We reach the estimate

$$\| b_0^w \dots b_\nu^w \|_{\mathscr{L}(L^2)} \leq C_2 \nu^{M_0} \| c_0^w \dots c_\nu^w \|_{\mathscr{L}(L^2)} \prod_{0 \leq j < \nu} \delta_r(Y_j, Y_{j+1})^{m_j M_0},$$

where the $m_j \in \mathbb{N}$ and such that $\sum_{0 \leq j < \nu} m_j \leq k$ and M_0 is depending only on g. We define $\tilde{J} = \{j \in [0, \nu[, m_j > 0\}$ and we have

$$k \geq \sum_{0 \leq j < \nu} m_j \geq \sum_{0 \leq j < \nu} 1 = \operatorname{card} \tilde{J}.$$

We apply now Lemma 2.6.25 to get with $\tilde{K} = \tilde{J} \cup (1 + \tilde{J})$,

$$\begin{aligned}\tilde{\omega}_k &\le C_0^{\nu-\operatorname{card}\tilde{J}} C_1^{1+\operatorname{card}\tilde{J}} C_2 \nu^{M_0} \\ &\quad \times \|\!| c_0 |\!\|^{(k_1)}_{g_{Y_0}, U_{Y_0,r}} \Big(\max_{j\notin\tilde{K}, j\neq 0} \|\!| c_j |\!\|^{(k_0)}_{g_{Y_j}, U_{Y_j,r}} \Big)^{\nu-\operatorname{card}\tilde{J}} \Big(\max_{j\in\tilde{K}, j\neq 0} \|\!| c_j |\!\|^{(k_1)}_{g_{Y_j}, U_{Y_j,r}} \Big)^{\operatorname{card}\tilde{J}} \\ &\qquad \times \int \prod_{0\le j<\nu} \delta_r(Y_j, Y_{j+1})^{-N_0} \prod_{j\in\tilde{J}} \delta_r(Y_j, Y_{j+1})^{-N_1} \\ &\qquad\qquad \times \prod_{j\in\tilde{J}} \delta_r(Y_j, Y_{j+1})^{kM_0} dY_1 \dots dY_\nu.\end{aligned}$$

The seminorms of the c_j for $j \ge 1$ depend only on the seminorms of a in $S(1,g)$, with

$$\begin{cases} \|\!| c_j |\!\|^{(l)}_{g_{Y_j}, U_{Y_j,r}} \le \|\!| a_{Y_j} |\!\|^{(l+k)}_{g_{Y_j}, U_{Y_j,r}} & \text{if } j \in \tilde{K}, \\ \|\!| c_j |\!\|^{(l)}_{g_{Y_j}, U_{Y_j,r}} \le \|\!| a_{Y_j} |\!\|^{(l)}_{g_{Y_j}, U_{Y_j,r}} & \text{if } j \notin \tilde{K}, \end{cases}$$

whereas the seminorms of c_0 are those of θ_{Y_0} and depend only on the structure constants of g. We choose N_0 large enough (depending only on the structure constants of g) to ensure $\sup_Y \int \delta_r(Y,Z)^{-N_0} dZ < +\infty$. For each k, we choose $N_1(k) = kM_0$ and, integrating successively with respect to $Y_\nu, Y_{\nu-1}, \dots, Y_1$, we get with $k_1' = k_1 + k$, $\operatorname{card}\tilde{J} = \tau k$ with $\tau \in [0,1]$,

$$\begin{aligned}\tilde{\omega}_k &\le \big(C_0 \|a\|^{(k_0)}_{S(1,g)}\big)^{\nu-\operatorname{card}\tilde{J}} \big(C_1 \|a\|^{(k_1')}_{S(1,g)}\big)^{\operatorname{card}\tilde{J}} C_3 \nu^{M_0} \\ &= \big(C_0 \|a\|^{(k_0)}_{S(1,g)}\big)^{\nu-k+k-\tau k} \big(C_1 \|a\|^{(k_1')}_{S(1,g)}\big)^{\tau k} C_3 \nu^{M_0} \\ &\qquad \text{and using } k_0 \le k_1, C_0 \le C_1 \\ &\le \big(C_0 \|a\|^{(k_0)}_{S(1,g)}\big)^{\nu-k} \big(C_1 \|a\|^{(k_1')}_{S(1,g)}\big)^{k-\tau k} \big(C_1 \|a\|^{(k_1')}_{S(1,g)}\big)^{\tau k} C_3 \nu^{M_0},\end{aligned}$$

providing the result of the lemma. □

Theorem 2.6.27. *Let g be an admissible and geodesically temperate metric on $\mathbb{R}^{2n}$ and let $a \in S(1,g)$ such that a^w is invertible in $\mathscr{L}(L^2(\mathbb{R}^n))$. Then there exists $b \in S(1,g)$ such that $b^w = (a^w)^{-1}$.*

Proof. (1) We assume first that $g \equiv g^\natural \equiv g^\sigma$. Let us consider $r \in S(1,g)$ such that $\|r^w\|_{\mathscr{L}(L^2)} < 1$. We have $(\mathrm{Id} - r^w)^{-1} = \sum_{\nu\ge 0} (r^w)^\nu$ and for $\tau \in]0,1[$ (to be chosen later), with $R = r^w$, using Theorem 2.6.12, we get for $k \in \mathbb{N}$

$$\|R^\nu\|^{(k)}_{\mathrm{op}(1,g)} \le C_{k,\tau} \big(\|R^\nu\|^{(0)}_{\mathrm{op}(1,g)}\big)^\tau \big(\|R^\nu\|^{(l_{k,\tau})}_{\mathrm{op}(1,g)}\big)^{1-\tau}. \tag{2.6.37}$$

Using Lemma 2.6.26, we have

$$\|R^\nu\|^{(l_{k,\tau})}_{\mathrm{op}(1,g)} \le C'_{k,\tau} (\nu+1)^{N_{k,\tau}} \big(C_0 \|r\|^{(k_0)}_{S(1,g)}\big)^{\nu-l_{k,\tau}} \big(C_1 \|r\|^{(k_1(k,\tau))}_{S(1,g)}\big)^{l_{k,\tau}}, \tag{2.6.38}$$

and noticing that $\|R^\nu\|^{(0)}_{\mathrm{op}(1,g)} = \sup_{Y\in\mathbb{R}^{2n}} \|\theta_Y^w R^\nu\|_{\mathscr{L}(L^2)} \le C_2\|R\|^\nu_{\mathscr{L}(L^2)}$, we get, using (2.6.37), (2.6.38),

$$\begin{aligned}
&\|R^\nu\|^{(k)}_{\mathrm{op}(1,g)}\\
&\le C_{k,\tau}\big(C_2\|R\|^\nu_{\mathscr{L}(L^2)}\big)^\tau\Big(C'_{k,\tau}(\nu+1)^{N_{k,\tau}}\big(C_0\|r\|^{(k_0)}_{S(1,g)}\big)^{\nu-l_{k,\tau}}\big(C_1\|r\|^{(k_1(k,\tau))}_{S(1,g)}\big)^{l_{k,\tau}}\Big)^{1-\tau}\\
&\quad\le \tilde{C}_{k,\tau}(\nu+1)^{N_{k,\tau}}C_0^\nu\|R\|^{\nu\tau}_{\mathscr{L}(L^2)}\big(\|r\|^{(k_0)}_{S(1,g)}\big)^{(\nu-l_{k,\tau})(1-\tau)}\big(\|r\|^{(k_1(k,\tau))}_{S(1,g)}\big)^{(1-\tau)l_{k,\tau}}.
\end{aligned}$$

Choosing now $1-\tau$ small enough, depending only on C_0, k_0 (in particular independent of k), so that

$$\|R\|^\tau_{\mathscr{L}(L^2)}\big(C_0\|r\|^{(k_0)}_{S(1,g)}\big)^{1-\tau} < 1,$$

we obtain for $\nu \ge l_{k,\tau}$

$$\|R^\nu\|^{(k)}_{\mathrm{op}(1,g)} \le C''_{k,\tau}(\nu+1)^{N_{k,\tau}}\Big(\|R\|^\tau_{\mathscr{L}(L^2)}\big(C_0\|r\|^{(k_0)}_{S(1,g)}\big)^{1-\tau}\Big)^\nu$$

which implies the convergence of $\sum_{\nu\ge0} r^{\sharp\nu}$ in $S(1,g)$. Now if $a\in S(1,g)$ satisfies the invertibility assumption of the theorem, we get that the selfadjoint operator $(\bar{a}\sharp a)^w = (a^w)^*a^w$ is positive invertible and thus, with $0<c\le C$,

$$\forall u\in L^2,\quad c\|u\|^2_{L^2} \le \langle (a^w)^*a^w u,u\rangle \le C\|u\|^2_{L^2},$$

so that $C^{-1}(a^w)^*a^w = \mathrm{Id} - b^w$, $\quad \|b^w\|_{\mathscr{L}(L^2)} < 1$. From the previous discussion, we get that $\big((a^w)^*a^w\big)^{-1} = d_1^w$ with $d_1\in S(1,g)$, and also $\big(a^w(a^w)^*\big)^{-1} = d_2^w$ with $d_2\in S(1,g)$, so that

$$\underbrace{d_1\sharp\bar{a}}_{=\alpha_1}\sharp a = 1 = a\sharp\underbrace{\bar{a}\sharp d_2}_{=\alpha_2} \Longrightarrow \alpha_1 = \alpha_1\sharp a\sharp\alpha_2 = \alpha_2,$$

and a^w has in $\mathscr{L}(L^2)$ the inverse α_1^w with $\alpha_1\in S(1,g)$.

(2) We assume now only that g is admissible and geodesically temperate. Then g satisfies (2.6.29), (2.6.31), (2.2.22), and $g^\natural$ is indeed admissible (see Proposition 2.2.20). Let $a\in S(1,g)$ with a^w invertible in $\mathscr{L}(L^2)$; since $S(1,g)\subset S(1,g^\natural)$, we get from the first part of the proof that there exists $b\in S(1,g^\natural)$ so that $a\sharp b = b\sharp a = 1$. For $S,T\in\mathbb{R}^{2n}$ with $g_S(T)=1$, we set $M_{S,T}(X) = g_X(T)^{1/2}$; the functions $M_{S,T}$ are a family of weight functions with uniform structure constants for $g^\natural$ (and for g), thanks to (2.6.29) and (2.6.31): we have

$$\frac{M_{S,T}(X)^2}{M_{S,T}(Y)^2} = \frac{g_X(T)}{g_Y(T)} \le C(1+(g^\natural_X\wedge g^\natural_Y)(X-Y))^N,$$

and if $g^\natural_X(Y-X)\le r^2$, (2.2.22) implies $g_X(Y-X)\le r^2$ so that with r chosen as in (2.2.3), we get $C^{-1}\le M_{S,T}(X)/M_{S,T}(Y)\le C$. We note also that $\partial_T a$ belongs

to $S(M_{S,T}, g) \subset S(M_{S,T}, g^\natural)$ with uniform seminorms with respect to S, T. Using $0 = \partial_T b \sharp a + b \sharp \partial_T a$, we get

$$\partial_T b = -b \sharp \partial_T a \sharp b, \tag{2.6.39}$$

which implies, thanks to Theorem 2.3.7, that $\|\partial_T b\|^{(k)}_{S(M_{S,T},g^\natural)} \leq C_k$ with C_k independent of S, T. In particular, for $k = 0$, we get for $g_S(T) = 1$ that $|(\partial_T b)(S)| \leq C_0$, which implies $\|b\|^{(1)}_{S(1,g)} < \infty$. Moreover, from (2.6.39), we get with $T_1, T_2 \in \mathbb{R}^{2n}$,

$$\begin{aligned}\partial_{T_1}\partial_{T_2} b &= -\partial_{T_1} b \sharp \partial_{T_2} a \sharp b - b \sharp \partial_{T_1}\partial_{T_2} a \sharp b - b \sharp \partial_{T_2} a \sharp \partial_{T_1} b \\ &= b \sharp \partial_{T_1} a \sharp b \sharp \partial_{T_2} a \sharp b - b \sharp \partial_{T_1}\partial_{T_2} a \sharp b + b \sharp \partial_{T_2} a \sharp b \sharp \partial_{T_1} a \sharp b, \end{aligned} \tag{2.6.40}$$

so that with $g_S(T_1) = g_S(T_2) = 1$, using $\partial_{T_j} a \in S(M_{S,T_j}, g^\natural)$ and $\partial_{T_2}\partial_{T_1} a \in S(M_{S,T_1} M_{S,T_2}, g^\natural)$ we get

$$\|\partial_{T_1}\partial_{T_2} b\|^{(k)}_{S(M_{S,T_1}M_{S,T_2},g^\natural)} \leq C'_k$$

with C'_k independent of S, T_1, T_2. In particular, for $k = 0$, this gives

$$|(\partial_{T_1}\partial_{T_2} b)(S)| \leq C'_0, \quad \text{which implies } \|b\|^{(2)}_{S(1,g)} < \infty.$$

We conclude with an induction, based upon the extension of (2.6.39), (2.6.40) to higher order derivatives. □

Corollary 2.6.28. *Let g, m be admissible metric and weight on $\mathbb{R}^{2n}$ with g geodesically temperate and let $a \in S(m, g)$. The existence of $b \in S(m^{-1}, g)$ such that $b \sharp a = a \sharp b = 1$ is equivalent to the invertibility of a^w as an operator from $H(m_1, g)$ onto $H(m_1 m^{-1}, g)$ for some g-admissible weight function m_1.*

Proof. We use the corollary 2.6.16 to find $c_1 \in S(m_1 m^{-1}, g), c_2 \in S(m m_1^{-1}, g)$, $d_1 \in S(m_1, g), d_2 \in S(m_1^{-1}, g)$ with

$$c_1 \sharp c_2 = c_2 \sharp c_1 = d_1 \sharp d_2 = d_2 \sharp d_1 = 1.$$

If a^w is invertible from $H(m_1, g)$ onto $H(m_1 m^{-1}, g)$, we have that $d_2^w a^w c_1^w$ is invertible in $\mathscr{L}(L^2)$. Since $d_2 \sharp a \sharp c_1 \in S(1, g)$ we use the previous theorem, and get that there exists $d \in S(1, g)$ such that

$$d_2 \sharp a \sharp c_1 \sharp d = d \sharp d_2 \sharp a \sharp c_1 = 1 \Longrightarrow c_2 = d \sharp d_2 \sharp a,\ d_1 = a \sharp c_1 \sharp d$$

so that $c_1 \sharp d \sharp d_2 \sharp a = 1 = a \sharp c_1 \sharp d \sharp d_2$ and the symbol $c_1 \sharp d \sharp d_2 \in S(m^{-1}, g)$ is the inverse of a. Conversely, if if for some $b \in S(m^{-1}, g)$, $a \sharp b = b \sharp a = 1$, then the operators $b^w a^w, a^w b^w$ are the identities of $H(m_1, g), H(m_1 m^{-1}, g)$, so that a^w is invertible from $H(m_1, g)$ onto $H(m_1 m^{-1}, g)$ with inverse b^w. This completes the proof of the corollary. □

Theorem 2.6.29. *Let g, m be admissible metric and weight on $\mathbb{R}^{2n}$ with g geodesically temperate. Let $(\varphi_Y)_{Y\in\mathbb{R}^{2n}}$ be a uniformly confined family which is a partition if unity (see Theorem 2.2.7). Then there exists a uniformly confined family of symbols $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$ such that*

$$\int (\psi_Y \sharp \varphi_Y)|g_Y|^{1/2} = 1.$$

Moreover, $H(m,g) = \{u \in \mathscr{S}'(\mathbb{R}^n), \int m(Y)^2 \|\varphi_Y^w u\|^2_{L^2(\mathbb{R}^n)}|g_Y|^{1/2} < \infty\}$.

Proof. Using Lemma 2.5.3 and Theorem 2.6.27, we get that the operator $a^w = A = \int (\varphi_Y^w)^2 |g_Y|^{1/2} dY$ has an inverse b^w with $b \in S(1,g)$. The family $(\psi_Y = b\sharp\varphi_Y)_{Y\in\mathbb{R}^{2n}}$ is uniformly confined, thanks to the proposition 2.3.16. As a result, with the definition given in (2.6.2), thanks to the proposition 2.6.3, we can choose on $H(m,g)$ the norm

$$\|u\|^2_{H(m,g)} = \int \|\varphi_Y^w u\|^2_{L^2} m(Y)^2 |g_Y|^{1/2} dY$$

since $1 = \int (\overbrace{b\sharp\varphi_Y}^{\psi_Y} \sharp \overbrace{\varphi_Y}^{\theta_Y})|g_Y|^{1/2} dY$. This completes the proof. □