# Elements of Graduate Analysis

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# Contents

# Chapter 1

# Fourier Analysis

## 1.1 Preliminaries

The Fourier transform of a function $u \in L^1(\mathbb{R}^n)$ can be defined as

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-2i\pi x\cdot\xi} dx. \tag{1.1.1}$$

**Lemma 1.1.1** (Riemann-Lebesgue Lemma)**.** *Let $u$ be in $L^1(\mathbb{R}^n)$. Then we have*

$$\widehat{u}(\xi) \underset{|\xi|\to\infty}{\longrightarrow} 0.$$

*Moreover the function $\hat{u}$ is uniformly continuous on $\mathbb{R}^n$.*

*Proof.* We note first that (1.1.1) is meaningful as the integral of an $L^1$ function and we have also

$$\sup_{\xi\in\mathbb{R}^n} |\widehat{u}(\xi)| \le \|u\|_{L^1(\mathbb{R}^n)}. \tag{1.1.2}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. With $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we define

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}, \quad D_j = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}, \quad \xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}. \tag{1.1.3}$$

We find the identities

$$\xi_1 \widehat{\varphi}(\xi) = \widehat{D_1\varphi}(\xi), \quad \widehat{D^\alpha\varphi}(\xi) = \xi^\alpha \widehat{\varphi}(\xi), \tag{1.1.4}$$

entailing $\big(1+|\xi|^2\big)\widehat{\varphi}(\xi) = \text{Fourier}\Big(\varphi + \sum_{1\le j\le n} D_j^2\varphi\Big)$. We find thus

$$\big(1+|\xi|^2\big)|\widehat{\varphi}(\xi)| \le \|\varphi + \sum_{1\le j\le n} D_j^2\varphi\|_{L^1(\mathbb{R}^n)},$$

which implies $\lim_{|\xi|\to+\infty} \widehat{\varphi}(\xi) = 0$. For $u \in L^1(\mathbb{R}^n)$, we have

$$|\widehat{u}(\xi)| \le |\widehat{(u-\varphi)}(\xi)| + |\widehat{\varphi}(\xi)| \le \|u-\varphi\|_{L^1(\mathbb{R}^n)} + |\widehat{\varphi}(\xi)|,$$

so that for all $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\limsup_{|\xi|\to\infty} |\widehat{u}(\xi)| \le \|u-\varphi\|_{L^1(\mathbb{R}^n)} \Longrightarrow \limsup_{|\xi|\to\infty} |\widehat{u}(\xi)| \le \inf_{\varphi\in C_c^\infty(\mathbb{R}^n)} \|u-\varphi\|_{L^1(\mathbb{R}^n)} = 0.$$

We have also $\widehat{u}(\xi+\eta)-\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi}\big(e^{-2i\pi x\cdot\eta}-1\big)u(x)dx$, so that

$$|\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| \le \int_{\mathbb{R}^n} |u(x)| \underbrace{|e^{-2i\pi x\cdot\eta}-1|}_{\le 2} dx,$$

and Lebesgue's dominated convergence Theorem shows that, for all $\xi \in \mathbb{R}^n$,

$$\lim_{\eta\to 0} |\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| = 0,$$

proving continuity.We have also for $R>1, |\eta|\le 1$,

$$|\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| \le \sup_{|\xi|\le R} |\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| + 2\sup_{|\xi|\ge R-1} |\hat{u}(\xi)|$$

so that for $0<\varepsilon<1$, if $\omega_\rho$ is a modulus of continuity[1] of the continuous function $\hat{u}$ on the compact set $\{|x|\le\rho\}$

$$\sup_{|\eta|\le\varepsilon, \xi\in\mathbb{R}^m} |\widehat{u}(\xi+\eta)-\widehat{u}(\xi)| \le \omega_{R+1}(\varepsilon) + 2\sup_{|\xi|\ge R-1} |\hat{u}(\xi)|,$$

proving that the $\limsup$ of the lhs when $\varepsilon$ goes to 0 is smaller than

$$2\sup_{|\xi|\ge R-1} |\hat{u}(\xi)|, \qquad \text{for all } R>1.$$

Since that quantity is already proven to go to 0 when $R$ goes to $+\infty$, we obtain the uniform continuity of $\hat{u}$. □

We need to extend this transformation to various other situations and it turns out that L. Schwartz' point of view to define the Fourier transformation on the very large space of tempered distributions is the simplest. However, the cost of the distribution point of view is that we have to define these objects, which is not a completely elementary matter. We have chosen here to limit our presentation to the tempered distributions, topological dual of the so-called Schwartz space of rapidly decreasing functions; this space is a Fréchet space, so its topology is defined by a countable family of semi-norms and is much less difficult to understand than the space of test functions with compact support on an open set. Proving the Fourier inversion formula on the Schwartz space is a truly elementary matter, which yields almost immediately the most general case for tempered distributions, by a duality abstract nonsense argument. This chapter may also serve to the reader as a motivation to the explore the more difficult local theory of distributions.

[1]For a continuous function $v$ defined on a compact subset $K$ of $\mathbb{R}^m$, the modulus of continuity $\omega$ is defined on $\mathbb{R}_+$ by $\omega(\rho) = \sup_{\substack{x,y\in K\\|x-y|\le\rho}} |v(x)-v(y)|$. We have $\lim_{\rho\to 0_+}\omega(\rho)=0$.

## 1.2 Fourier Transform of tempered distributions

### The Fourier transformation on $\mathscr{S}(\mathbb{R}^n)$

**Definition 1.2.1.** Let $n \geq 1$ be an integer. The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is defined as the vector space of $C^\infty$ functions $u$ from $\mathbb{R}^n$ to $\mathbb{C}$ such that, for all multi-indices. $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| < +\infty.$$

Here we have used the multi-index notation: for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we define

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{1\leq j\leq n} \alpha_j. \tag{1.2.1}$$

A simple example of such a function is $e^{-|x|^2}$, ($|x|$ is the Euclidean norm of $x$) and more generally, if $A$ is a symmetric positive definite $n \times n$ matrix, the function

$$v_A(x) = e^{-\pi\langle Ax,x\rangle} \tag{1.2.2}$$

belongs to the Schwartz class. The space $\mathscr{S}(\mathbb{R}^n)$ is a Fréchet space equipped with the countable family of semi-norms $(p_k)_{k\in\mathbb{N}}$

$$p_k(u) = \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|,|\beta|\leq k}} |x^\alpha \partial_x^\beta u(x)|. \tag{1.2.3}$$

**Lemma 1.2.2.** *The Fourier transform sends continuously $\mathscr{S}(\mathbb{R}^n)$ into itself.*

*Proof.* Just notice that

$$\xi^\alpha \partial_\xi^\beta \hat{u}(\xi) = \int e^{-2i\pi x\xi} \partial_x^\alpha (x^\beta u)(x) dx (2i\pi)^{|\beta|-|\alpha|}(-1)^{|\beta|},$$

and since $\sup_{x\in\mathbb{R}^n} (1+|x|)^{n+1} |\partial_x^\alpha (x^\beta u)(x)| < +\infty$, we get the result. □

**Lemma 1.2.3.** *For a symmetric positive definite $n \times n$ matrix $A$, we have*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}, \tag{1.2.4}$$

*where $v_A$ is given by (1.2.2).*

*Proof.* In fact, diagonalizing the symmetric matrix $A$, it is enough to prove the one-dimensional version of (1.2.4), i.e. to check

$$\int e^{-2i\pi x\xi} e^{-\pi x^2} dx = \int e^{-\pi(x+i\xi)^2} dx e^{-\pi\xi^2} = e^{-\pi\xi^2},$$

where the second equality is obtained by taking the $\xi$-derivative of $\int e^{-\pi(x+i\xi)^2} dx$ : we have indeed

$$\frac{d}{d\xi}\Big(\int e^{-\pi(x+i\xi)^2} dx\Big) = \int e^{-\pi(x+i\xi)^2} (-2i\pi)(x+i\xi) dx$$
$$= i \int \frac{d}{dx}\big(e^{-\pi(x+i\xi)^2}\big) dx = 0.$$

For $a > 0$, we obtain $\int_{\mathbb{R}} e^{-2i\pi x\xi} e^{-\pi a x^2} dx = a^{-1/2} e^{-\pi a^{-1}\xi^2}$, which is the sought result in one dimension. If $n \ge 2$, and $A$ is a positive definite symmetric matrix, there exists an orthogonal $n \times n$ matrix $P$ (i.e. ${}^tPP = \mathrm{Id}$) such that

$$D = {}^tPAP, \quad D = \mathrm{diag}(\lambda_1, \dots, \lambda_n), \text{ all } \lambda_j > 0.$$

As a consequence, we have, since $|\det P| = 1$,

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} e^{-\pi\langle Ax,x\rangle} dx &= \int_{\mathbb{R}^n} e^{-2i\pi (Py)\cdot\xi} e^{-\pi\langle APy,Py\rangle} dy \\
&= \int_{\mathbb{R}^n} e^{-2i\pi y\cdot({}^tP\xi)} e^{-\pi\langle Dy,y\rangle} dy \\
(\text{with } \eta = {}^tP\xi) \quad &= \prod_{1\le j\le n} \int_{\mathbb{R}} e^{-2i\pi y_j\eta_j} e^{-\pi\lambda_j y_j^2} dy_j = \prod_{1\le j\le n} \lambda_j^{-1/2} e^{-\pi\lambda_j^{-1}\eta_j^2} \\
&= (\det A)^{-1/2} e^{-\pi\langle D^{-1}\eta,\eta\rangle} = (\det A)^{-1/2} e^{-\pi\langle {}^tPA^{-1}P\ {}^tP\xi, {}^tP\xi\rangle} \\
&= (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}.
\end{aligned}$$

□

**Proposition 1.2.4.** *The Fourier transformation is an isomorphism of the Schwartz class and for $u \in \mathscr{S}(\mathbb{R}^n)$, we have*

$$u(x) = \int e^{2i\pi x\xi} \hat{u}(\xi) d\xi. \tag{1.2.5}$$

*Proof.* Using (1.2.4) we calculate for $u \in \mathscr{S}(\mathbb{R}^n)$ and $\epsilon > 0$, dealing with absolutely converging integrals,

$$\begin{aligned}
u_\epsilon(x) &= \int e^{2i\pi x\xi} \hat{u}(\xi) e^{-\pi\epsilon^2|\xi|^2} d\xi \\
&= \iint e^{2i\pi x\xi} e^{-\pi\epsilon^2|\xi|^2} u(y) e^{-2i\pi y\xi} dy d\xi \\
&= \int u(y) e^{-\pi\epsilon^{-2}|x-y|^2} \epsilon^{-n} dy \\
&= \int \underbrace{\big(u(x+\epsilon y) - u(x)\big)}_{\text{with absolute value} \le \epsilon|y|\|u'\|_{L^\infty}} e^{-\pi|y|^2} dy + u(x).
\end{aligned}$$

Taking the limit when $\epsilon$ goes to zero, we get the Fourier inversion formula

$$u(x) = \int e^{2i\pi x\xi} \hat{u}(\xi) d\xi. \tag{1.2.6}$$

We have also proven for $u \in \mathscr{S}(\mathbb{R}^n)$ and $\check{u}(x) = u(-x)$

$$u = \check{\hat{\hat{u}}}. \tag{1.2.7}$$

Since $u \mapsto \hat{u}$ and $u \mapsto \check{u}$ are continuous homomorphisms of $\mathscr{S}(\mathbb{R}^n)$, this completes the proof of the proposition. □

**Proposition 1.2.5.** *Using the notation*

$$D_{x_j} = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}, \quad D_x^\alpha = \prod_{j=1}^n D_{x_j}^{\alpha_j} \quad \textit{with } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \tag{1.2.8}$$

*we have, for* $u \in \mathscr{S}(\mathbb{R}^n)$

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \qquad (D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha u(x)}(\xi) \tag{1.2.9}$$

*Proof.* We have for $u \in \mathscr{S}(\mathbb{R}^n)$, $\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x)dx$ and thus

$$(D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\int e^{-2i\pi x\cdot\xi} x^\alpha u(x)dx,$$
$$\xi^\alpha \hat{u}(\xi) = \int (-2i\pi)^{-|\alpha|}\partial_x^\alpha\big(e^{-2i\pi x\cdot\xi}\big)u(x)dx = \int e^{-2i\pi x\cdot\xi}(2i\pi)^{-|\alpha|}(\partial_x^\alpha u)(x)dx,$$

proving both formulas. □

*N.B.* The normalization factor $\frac{1}{2i\pi}$ leads to a simplification in Formula (1.2.9), but the most important aspect of these formulas is certainly that the Fourier transformation exchanges the operation of derivation with the operation of multiplication. For instance with

$$P(D) = \sum_{|\alpha|\le m} a_\alpha D_x^\alpha,$$

we have for $u \in \mathscr{S}(\mathbb{R}^n)$, $\widehat{Pu}(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha \hat{u}(\xi) = P(\xi)\hat{u}(\xi)$, and thus

$$(Pu)(x) = \int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi} P(\xi)\hat{u}(\xi)d\xi. \tag{1.2.10}$$

**Proposition 1.2.6.** *Let* $\phi, \psi$ *be functions in* $\mathscr{S}(\mathbb{R}^n)$*. Then the convolution* $\phi * \psi$ *belongs to the Schwartz space and the mapping*

$$\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \ni (\phi, \psi) \mapsto \phi * \psi \in \mathscr{S}(\mathbb{R}^n)$$

*is continuous. Moreover we have*

$$\widehat{\phi * \psi} = \hat{\phi}\hat{\psi}. \tag{1.2.11}$$

*Proof.* The mapping $(x, y) \mapsto F(x, y) = \phi(x - y)\psi(y)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$ since $x, y$ derivatives of the smooth function $F$ are linear combinations of products

$$(\partial^\alpha \phi)(x - y)(\partial^\beta \psi)(y)$$

and moreover

$$(1 + |x| + |y|)^N |(\partial^\alpha \phi)(x - y)(\partial^\beta \psi)(y)|$$
$$\le (1 + |x - y|)^N |(\partial^\alpha \phi)(x - y)|(1 + 2|y|)^N |(\partial^\beta \psi)(y)| \le p(\phi)q(\psi),$$

where $p, q$ are semi-norms on $\mathscr{S}(\mathbb{R}^n)$. This proves that the bilinear mapping $(\phi, \psi) \mapsto F(\phi, \psi)$ is continuous from $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^{2n})$. We have now directly $\partial_x^\alpha(\phi * \psi) = (\partial_x^\alpha \phi) * \psi$ and

$$
\begin{aligned}
(1+|x|)^N |\partial_x^\alpha(\phi * \psi)| &\le \int |F(\partial^\alpha \phi, \psi)(x,y)|(1+|x|)^N dy \\
&\le \int \underbrace{|F(\partial^\alpha \phi, \psi)(x,y)|(1+|x|)^N(1+|y|)^{n+1}}_{\le p(F)}(1+|y|)^{-n-1} dy,
\end{aligned}
$$

where $p$ is a semi-norm of $F$ (thus bounded by a product of semi-norms of $\phi$ and $\psi$), proving the continuity property. Also we obtain from Fubini's Theorem

$$
(\widehat{\phi * \psi})(\xi) = \iint e^{-2i\pi(x-y)\cdot\xi} e^{-2i\pi y\cdot\xi} \phi(x-y)\psi(y) dy dx = \hat{\phi}(\xi)\hat{\psi}(\xi),
$$

completing the proof of the proposition. □

### The Fourier transformation on $\mathscr{S}'(\mathbb{R}^n)$

**Definition 1.2.7.** Let $n$ be an integer $\ge 1$. We define the space $\mathscr{S}'(\mathbb{R}^n)$ as the topological dual of the Fréchet space $\mathscr{S}(\mathbb{R}^n)$: this space is called the space of *tempered distributions* on $\mathbb{R}^n$.

We note that the mapping

$$
\mathscr{S}(\mathbb{R}^n) \ni \phi \mapsto \frac{\partial \phi}{\partial x_j} \in \mathscr{S}(\mathbb{R}^n),
$$

is continuous since for all $k \in \mathbb{N}$, $p_k(\partial\phi/\partial x_j) \le p_{k+1}(\phi)$, where the semi-norms $p_k$ are defined in (1.2.3). This property allows us to define by duality the derivative of a tempered distribution.

**Definition 1.2.8.** Let $u \in \mathscr{S}'(\mathbb{R}^n)$. We define $\partial u/\partial x_j$ as an element of $\mathscr{S}'(\mathbb{R}^n)$ by

$$
\langle \frac{\partial u}{\partial x_j}, \phi \rangle_{\mathscr{S}',\mathscr{S}} = -\langle u, \frac{\partial \phi}{\partial x_j} \rangle_{\mathscr{S}',\mathscr{S}}. \tag{1.2.12}
$$

The mapping $u \mapsto \partial u/\partial x_j$ is a well-defined endomorphism of $\mathscr{S}'(\mathbb{R}^n)$ since the estimates

$$
\forall \phi \in \mathscr{S}(\mathbb{R}^n), \quad |\langle \frac{\partial u}{\partial x_j}, \phi \rangle| \le C_u p_{k_u}(\frac{\partial \phi}{\partial x_j}) \le C_u p_{k_u+1}(\phi),
$$

ensure the continuity on $\mathscr{S}(\mathbb{R}^n)$ of the linear form $\partial u/\partial x_j$.

**Definition 1.2.9.** Let $u \in \mathscr{S}'(\mathbb{R}^n)$ and let $P$ be a polynomial in $n$ variables with complex coefficients. We define the product $Pu$ as an element of $\mathscr{S}'(\mathbb{R}^n)$ by

$$
\langle Pu, \phi \rangle_{\mathscr{S}',\mathscr{S}} = \langle u, P\phi \rangle_{\mathscr{S}',\mathscr{S}}. \tag{1.2.13}
$$

The mapping $u \mapsto Pu$ is a well-defined endomorphism of $\mathscr{S}'(\mathbb{R}^n)$ since the estimates

$$\forall \phi \in \mathscr{S}(\mathbb{R}^n), \quad |\langle Pu, \phi\rangle| \le C_u p_{k_u}(P\phi) \le C_u p_{k_u+D}(\phi),$$

where $D$ is the degree of $P$, ensure the continuity on $\mathscr{S}(\mathbb{R}^n)$ of the linear form $Pu$.

**Lemma 1.2.10.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$, $f \in L^1_{loc}(\Omega)$ such that, for all $\varphi \in C_c^\infty(\Omega)$, $\int f(x)\varphi(x)dx = 0$. Then we have $f = 0$.*

*Proof.* Let $K$ be a compact subset of $\Omega$ and let $\chi \in C_{\rm c}^\infty(\Omega)$ equal to 1 on a neighborhood of $K$ (see e.g. Exercise 2.8.7 in [11]). With $\rho \in C_{\rm c}^\infty(\mathbb{R}^n)$ such that $\int \rho(t)dt = 1$, and for $\epsilon > 0$, $\rho_\epsilon(x) = \rho(x/\epsilon)\epsilon^{-n}$, we get that

$$\lim_{\epsilon\to 0_+} \rho_\epsilon * (\chi f) = \chi f \quad \text{in } L^1(\mathbb{R}^n),$$

since for $w \in L^1(\mathbb{R}^n)$,

$$\begin{aligned}\|\rho_\epsilon * w - w\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\rho_\epsilon(y)\big(w(x-y)-w(x)\big)dy\right|dx\\ &\le \iint |\rho(z)||w(x-\epsilon z)-w(x)|dzdx = \int |\rho(z)|\|\tau_{\epsilon z}w - w\|_{L^1(\mathbb{R}^n)}dz.\end{aligned}$$

We know[2] that $\lim_{h\to 0}\|\tau_h w - w\|_{L^1(\mathbb{R}^n)} = 0$ and $\|\tau_h w - w\|_{L^1(\mathbb{R}^n)} \le 2\|w\|_{L^1(\mathbb{R}^n)}$ so that Lebesgue's dominated convergence theorem provides

$$\lim_{\epsilon\to 0}\|\rho_\epsilon * w - w\|_{L^1(\mathbb{R}^n)} = 0.$$

We have $\big(\rho_\epsilon * (\chi f)\big)(x) = \displaystyle\int f(y)\underbrace{\chi(y)\rho\big((x-y)\epsilon^{-1}\big)\epsilon^{-n}}_{=\varphi_x(y)}dy$, with $\operatorname{supp}\varphi_x \subset \operatorname{supp}\chi$, $\varphi_x \in C_{\rm c}^\infty(\Omega)$, and from the assumption of the lemma, we obtain $\big(\rho_\epsilon * (\chi f)\big)(x) = 0$ for all $x$, implying $\chi f = 0$ from the convergence result and thus $f = 0$, a.e. on $K$; the conclusion of the lemma follows since $\Omega$ is a countable union of compact sets (see e.g. Exercise 2.8.10 in [11]). □

**Definition 1.2.11** (support of a distribution)**.** For $u \in \mathscr{S}'(\mathbb{R}^n)$, we define the support of $u$ and we note $\operatorname{supp} u$ the closed subset of $\mathbb{R}^n$ defined by

$$(\operatorname{supp} u)^c = \{x \in \mathbb{R}^n, \exists V \text{ open} \in \mathscr{V}_x, \quad u_{|V} = 0\}, \tag{1.2.14}$$

where $\mathscr{V}_x$ stands for the set of neighborhoods of $x$ and $u_{|V} = 0$ means that for all $\phi \in C_{\rm c}^\infty(V)$, $\langle u, \phi\rangle = 0$.

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[2] For $\phi \in C_c^0(\mathbb{R}^n)$, we have $\|\tau_h w - w\|_{L^1(\mathbb{R}^n)} \le \|\tau_h w - \tau_h\phi\|_{L^1(\mathbb{R}^n)} + \|\tau_h\phi - \phi\|_{L^1(\mathbb{R}^n)} + \|\phi - w\|_{L^1(\mathbb{R}^n)}$, so that for $|h| \le 1$,

$$\begin{aligned}\|\tau_h w - w\|_{L^1(\mathbb{R}^n)} &\le 2\|\phi - w\|_{L^1(\mathbb{R}^n)} + \int |\phi(x-h)-\phi(x)|dx\\ &\le 2\|\phi - w\|_{L^1(\mathbb{R}^n)} + |\operatorname{supp}\phi + \mathbb{B}^n|\sup|\phi(x-h)-\phi(x)|\end{aligned}$$

which implies that $\limsup_{h\to 0}\|\tau_h w - w\| \le 2\inf_{\phi\in C_c^0(\mathbb{R}^n)}\|\phi - w\|_{L^1(\mathbb{R}^n)} = 0$.

**Proposition 1.2.12.**
(1) *We have* $\mathscr{S}'(\mathbb{R}^n) \supset \cup_{1\le p\le +\infty} L^p(\mathbb{R}^n)$, *with a continuous injection of each* $L^p(\mathbb{R}^n)$ *into* $\mathscr{S}'(\mathbb{R}^n)$. *As a consequence* $\mathscr{S}'(\mathbb{R}^n)$ *contains as well all the derivatives in the sense* (1.2.12) *of all the functions in some* $L^p(\mathbb{R}^n)$.
(2) *For* $u \in C^1(\mathbb{R}^n)$ *such that*

$$\big(|u(x)| + |du(x)|\big)(1+|x|)^{-N} \in L^1(\mathbb{R}^n), \tag{1.2.15}$$

*for some non-negative* $N$, *the derivative in the sense* (1.2.12) *coincides with the ordinary derivative.*

*Proof.* (1) For $u \in L^p(\mathbb{R}^n)$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$, we can define

$$\langle u, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \int_{\mathbb{R}^n} u(x)\phi(x)dx, \tag{1.2.16}$$

which is a continuous linear form on $\mathscr{S}(\mathbb{R}^n)$:

$$|\langle u, \phi\rangle_{\mathscr{S}',\mathscr{S}}| \le \|u\|_{L^p(\mathbb{R}^n)}\|\phi\|_{L^{p'}(\mathbb{R}^n)},$$

$$\|\phi\|_{L^{p'}(\mathbb{R}^n)} \le \sup_{x\in\mathbb{R}^n}\big((1+|x|)^{\frac{n+1}{p'}}|\phi(x)|\big)C_{n,p} \le C_{n,p}p_k(\phi), \text{ for } k \ge k_{n,p} = \frac{n+1}{p'},$$

with $p_k$ given by (1.2.3) (when $p = 1$, we can take $k = 0$). We indeed have a continuous injection of $L^p(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$: in the first place the mapping described by (1.2.16) is well-defined and continuous from the estimate

$$|\langle u, \phi\rangle| \le \|u\|_{L^p} C_{n,p} p_{k_{n,p}}(\phi).$$

Moreover, this mapping is linear and injective from Lemma 1.2.10.

(2) We have for $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\chi_0 \in C_c^\infty(\mathbb{R}^n)$, $\chi_0 = 1$ near the origin,

$$A = \langle \frac{\partial u}{\partial x_j}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = -\langle u, \frac{\partial \phi}{\partial x_j}\rangle_{\mathscr{S}',\mathscr{S}} = -\int_{\mathbb{R}^n} u(x)\frac{\partial \phi}{\partial x_j}(x)dx$$

so that, using Lebesgue's dominated convergence theorem, we find

$$A = -\lim_{\epsilon\to 0_+}\int_{\mathbb{R}^n} u(x)\frac{\partial \phi}{\partial x_j}(x)\chi_0(\epsilon x)dx.$$

Performing an integration by parts on $C^1$ functions with compact support, we get

$$A = \lim_{\epsilon\to 0_+}\Big\{\int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)\chi_0(\epsilon x)dx + \epsilon\int_{\mathbb{R}^n} u(x)\phi(x)(\partial_j\chi_0)(\epsilon x)dx\Big\},$$

with $\partial_j u$ standing for the ordinary derivative. We have also

$$\int_{\mathbb{R}^n}|u(x)\phi(x)(\partial_j\chi_0)(\epsilon x)|dx \le \|\partial_j\chi_0)\|_{L^\infty(\mathbb{R}^n)}\int |u(x)|(1+|x|)^{-N}dx\ p_N(\phi) < +\infty,$$

so that

$$\langle \frac{\partial u}{\partial x_j}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \lim_{\epsilon\to 0_+}\int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)\chi_0(\epsilon x)dx.$$

Since the lhs is a continuous linear form on $\mathscr{S}(\mathbb{R}^n)$ so is the rhs. On the other hand for $\phi \in C_c^\infty(\mathbb{R}^n)$, the rhs is $\int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)dx$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$ (cf. Exercise 1.5.3), we find that

$$\langle \frac{\partial u}{\partial x_j}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)dx,$$

since the mapping $\phi \mapsto \int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)dx$ belongs to $\mathscr{S}'(\mathbb{R}^n)$, thanks to the assumption on $du$ in (1.2.15). This proves that $\frac{\partial u}{\partial x_j} = \partial_j u$. $\square$

The Fourier transformation can be extended to $\mathscr{S}'(\mathbb{R}^n)$. We start with noticing that for $T, \phi$ in the Schwartz class we have, using Fubini Theorem,

$$\int \hat{T}(\xi)\phi(\xi)d\xi = \iint T(x)\phi(\xi)e^{-2i\pi x\cdot\xi}dxd\xi = \int T(x)\hat{\phi}(x)dx,$$

and we can use the latter formula as a definition.

**Definition 1.2.13.** Let $T$ be a tempered distribution ; the Fourier transform $\hat{T}$ of $T$ is the tempered distribution defined by the formula

$$\langle \hat{T}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\varphi}\rangle_{\mathscr{S}',\mathscr{S}}. \tag{1.2.17}$$

The linear form $\hat{T}$ is obviously a tempered distribution since the Fourier transformation is continuous on $\mathscr{S}$. Thanks to Lemma 1.2.10, if $T \in \mathscr{S}$, the present definition of $\hat{T}$ and (1.1.1) coincide.

This definition gives that, with $\delta_0$ standing as the Dirac mass at 0, $\langle \delta_0, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \phi(0)$ (obviously a tempered distribution), we have

$$\widehat{\delta_0} = 1, \tag{1.2.18}$$

since $\langle \widehat{\delta_0}, \varphi\rangle = \langle \delta_0, \widehat{\varphi}\rangle = \widehat{\varphi}(0) = \int \varphi(x)dx = \langle 1, \varphi\rangle$.

**Theorem 1.2.14.** *The Fourier transformation is an isomorphism of $\mathscr{S}'(\mathbb{R}^n)$. Let $T$ be a tempered distribution. Then we have*[3]

$$T = \check{\hat{\hat{T}}}, \quad \check{\hat{T}} = \hat{\check{T}}. \tag{1.2.19}$$

*With obvious notations, we have the following extensions of (1.2.9),*

$$\widehat{D_x^\alpha T}(\xi) = \xi^\alpha \hat{T}(\xi), \qquad (D_\xi^\alpha \hat{T})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha T(x)}(\xi). \tag{1.2.20}$$

*Proof.* We have for $T \in \mathscr{S}'$

$$\langle \check{\hat{\hat{T}}}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{\hat{T}}, \check{\varphi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \hat{\check{\varphi}}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\hat{\check{\varphi}}}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \varphi\rangle_{\mathscr{S}',\mathscr{S}},$$

[3] We define $\check{T}$ as the distribution given by $\langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle$ and if $T \in \mathscr{S}'$, $\check{T}$ is also a tempered distribution since $\varphi \mapsto \check{\varphi}$ is an involutive isomorphism of $\mathscr{S}$.

where the last equality is due to the fact that $\varphi \mapsto \check{\varphi}$ commutes[4] with the Fourier transform and (1.2.6) means

$$\check{\hat{\hat{\varphi}}} = \varphi,$$

a formula also proven true on $\mathscr{S}'$ by the previous line of equality. Formula (1.2.9) is true as well for $T \in \mathscr{S}'$ since, with $\varphi \in \mathscr{S}$ and $\varphi_\alpha(\xi) = \xi^\alpha \varphi(\xi)$, we have

$$\langle \widehat{D^\alpha T}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, (-1)^{|\alpha|} D^\alpha \hat{\varphi} \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \widehat{\varphi_\alpha} \rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \varphi_\alpha \rangle_{\mathscr{S}',\mathscr{S}},$$

and the other part is proven the same way. □

### The Fourier transformation on $L^1(\mathbb{R}^n)$

**Theorem 1.2.15.** *The Fourier transformation is linear continuous from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and for $u \in L^1(\mathbb{R}^n)$, we have*

$$\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x) dx, \quad \|\hat{u}\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}. \tag{1.2.21}$$

*Proof.* Formula (1.1.1) can be used to define directly the Fourier transform of a function in $L^1(\mathbb{R}^n)$ and this gives a $L^\infty(\mathbb{R}^n)$ function which coincides with the Fourier transform: for a test function $\varphi \in \mathscr{S}(\mathbb{R}^n)$, and $u \in L^1(\mathbb{R}^n)$, we have by the definition (1.2.17) above and Fubini theorem

$$\langle \hat{u}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \int u(x)\hat{\varphi}(x)dx = \iint u(x)\varphi(\xi)e^{-2i\pi x\cdot\xi} dx d\xi = \int \widetilde{u}(\xi)\varphi(\xi)d\xi$$

with $\widetilde{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x)dx$ which is thus the Fourier transform of $u$. □

### The Fourier transformation on $L^2(\mathbb{R}^n)$

**Theorem 1.2.16** (Plancherel formula)**.**
*The Fourier transformation can be extended to a unitary operator of $L^2(\mathbb{R}^n)$, i.e. there exists a unique bounded linear operator $F : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, such that for $u \in \mathscr{S}(\mathbb{R}^n)$, $Fu = \hat{u}$ and we have $F^*F = FF^* = \mathrm{Id}_{L^2(\mathbb{R}^n)}$. Moreover*

$$F^* = CF = FC, \quad F^2C = \mathrm{Id}_{L^2(\mathbb{R}^n)}, \tag{1.2.22}$$

*where $C$ is the involutive isomorphism of $L^2(\mathbb{R}^n)$ defined by $(Cu)(x) = u(-x)$. This gives the Plancherel formula: for $u, v \in L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi = \int u(x)\overline{v(x)}dx. \tag{1.2.23}$$

*Proof.* For test functions $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$, using Fubini theorem and (1.2.6), we get[5]

$$(\hat{\psi}, \hat{\varphi})_{L^2(\mathbb{R}^n)} = \int \hat{\psi}(\xi)\overline{\hat{\varphi}(\xi)}d\xi = \iint \hat{\psi}(\xi)e^{2i\pi x\cdot\xi}\overline{\varphi(x)}dx d\xi = (\psi, \varphi)_{L^2(\mathbb{R}^n)}.$$

---

[4] If $\varphi \in \mathscr{S}$, we have $\widehat{\check{\varphi}}(\xi) = \int e^{-2i\pi x\cdot\xi}\varphi(-x)dx = \int e^{2i\pi x\cdot\xi}\varphi(x)dx = \hat{\varphi}(-\xi) = \check{\hat{\varphi}}(\xi)$.

[5] We have to pay attention to the fact that the scalar product $(u, v)_{L^2}$ in the complex Hilbert space $L^2(\mathbb{R}^n)$ is linear with respect to $u$ and antilinear with respect to $v$: for $\lambda, \mu \in \mathbb{C}$, $(\lambda u, \mu v)_{L^2} = \lambda\bar{\mu}(u, v)_{L^2}$.

Next, the density of $\mathscr{S}$ in $L^2$ shows that there is a unique continuous extension $F$ of the Fourier transform to $L^2$ and that extension is an isometric operator (i.e. satisfying for all $u \in L^2(\mathbb{R}^n), \|Fu\|_{L^2} = \|u\|_{L^2}$, i.e. $F^*F = \mathrm{Id}_{L^2}$). We note that the operator $C$ defined by $Cu = \check{u}$ is an involutive isomorphism of $L^2(\mathbb{R}^n)$ and that for $u \in \mathscr{S}(\mathbb{R}^n)$,

$$CF^2u = u = FCFu = F^2Cu.$$

By the density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, the bounded operators

$$CF^2,\ \mathrm{Id}_{L^2(\mathbb{R}^n)},\ FCF,\ F^2C,$$

are all equal. On the other hand for $u, \varphi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$(F^*u, \varphi)_{L^2} = (u, F\varphi)_{L^2} = \int u(x)\overline{\hat{\varphi}(x)}dx$$
$$= \iint u(x)\bar{\varphi}(\xi)e^{2i\pi x\cdot\xi}dxd\xi = (CFu, \varphi)_{L^2},$$

so that $F^*u = CFu$ for all $u \in \mathscr{S}$ and by continuity $F^* = CF$ as bounded operators on $L^2(\mathbb{R}^n)$, thus $FF^* = FCF = \mathrm{Id}$. The proof is complete. □

### Some standard examples of Fourier transform

Let us consider the Heaviside function defined on $\mathbb{R}$ by $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \leq 0$ ; as a bounded measurable function, it is a tempered distribution, so that we can compute its Fourier transform. With the notation of this section, we have, with $\delta_0$ the Dirac mass at 0, $\check{H}(x) = H(-x)$,

$$\widehat{H} + \widehat{\check{H}} = \hat{1} = \delta_0, \quad \widehat{H} - \widehat{\check{H}} = \widehat{\mathrm{sign}}, \qquad \frac{1}{i\pi} = \frac{1}{2i\pi}2\widehat{\delta_0}(\xi) = \widehat{D\,\mathrm{sign}}(\xi) = \xi\widehat{\mathrm{sign}}\xi.$$

We note that $\mathbb{R} \mapsto \ln|x|$ belongs to $\mathscr{S}'(\mathbb{R})$ and[6] we define the so-called principal value of $1/x$ on $\mathbb{R}$ by

$$\mathrm{pv}\big(\frac{1}{x}\big) = \frac{d}{dx}(\ln|x|), \tag{1.2.24}$$

$$\begin{aligned}\text{so that, } \langle \mathrm{pv}\,\frac{1}{x}, \phi\rangle &= -\int \phi'(x)\ln|x|dx = -\lim_{\epsilon\to 0_+}\int_{|x|\geq\epsilon}\phi'(x)\ln|x|dx \\ &= \lim_{\epsilon\to 0_+}\Big(\int_{|x|\geq\epsilon}\phi(x)\frac{1}{x}dx + \underbrace{\big(\phi(\epsilon)-\phi(-\epsilon)\big)\ln\epsilon}_{\to 0}\Big) \\ &= \lim_{\epsilon\to 0_+}\int_{|x|\geq\epsilon}\phi(x)\frac{1}{x}dx.\end{aligned} \tag{1.2.25}$$

This entails $\xi\big(\widehat{\mathrm{sign}}\xi - \frac{1}{i\pi}pv(1/\xi)\big) = 0$ and from Remark 1.2.17 below, we get

$$\widehat{\mathrm{sign}}\xi - \frac{1}{i\pi}pv(1/\xi) = c\delta_0,$$

with $c = 0$ since the lhs is odd[7].

[6] For $\phi \in \mathscr{S}(\mathbb{R})$, we have $\langle \ln|x|, \phi(x)\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = \int_{\mathbb{R}}\phi(x)\ln|x|dx$.

[7] A distribution $T$ on $\mathbb{R}^n$ is said to be odd (resp. even) when $\check{T} = -T$ (resp. $T$).

*Remark* 1.2.17. Let $T \in \mathscr{S}'(\mathbb{R})$ such that $xT = 0$. Then we have $T = c\delta_0$. Let $\phi \in \mathscr{S}(\mathbb{R})$ and let $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ such that $\chi_0(0) = 1$. We have

$$\phi(x) = \chi_0(x)\phi(x) + (1 - \chi_0(x))\phi(x).$$

Applying Taylor's formula with integral remainder, we define the smooth function $\psi$ by

$$\psi(x) = \frac{(1 - \chi_0(x))}{x}\phi(x)$$

and, applying Leibniz' formula, we see also that $\psi$ belongs to $\mathscr{S}(\mathbb{R})$. As a result

$$\langle T, \phi\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = \langle T, \chi_0\phi\rangle = \langle T, \chi_0\big(\phi - \phi(0)\big)\rangle + \phi(0)\langle T, \chi_0\rangle = \phi(0)\langle T, \chi_0\rangle,$$

since the function $x \mapsto \chi_0(x)\big(\phi(x) - \phi(0)\big)/x$ belongs to $C_c^\infty(\mathbb{R})$. As a result $T = \langle T, \chi_0\rangle\delta_0$.

We obtain

$$\widehat{\operatorname{sign}}(\xi) = \frac{1}{i\pi}pv\frac{1}{\xi}, \tag{1.2.26}$$

$$\widehat{pv(\frac{1}{\pi x})} = -i\operatorname{sign}\xi, \tag{1.2.27}$$

$$\hat{H} = \frac{\delta_0}{2} + \frac{1}{2i\pi}pv(\frac{1}{\xi}) = \frac{1}{(x - i0)}\frac{1}{2i\pi}. \tag{1.2.28}$$

Let us consider now for $0 < \alpha < n$ the $L^1_{\text{loc}}(\mathbb{R}^n)$ function $u_\alpha(x) = |x|^{\alpha-n}$ ($|x|$ is the Euclidean norm of $x$); since $u_\alpha$ is also bounded for $|x| \ge 1$, it is a tempered distribution. Let us calculate its Fourier transform $v_\alpha$. Since $u_\alpha$ is homogeneous of degree $\alpha - n$, we get (Exercise 1.5.9) that $v_\alpha$ is a homogeneous distribution of degree $-\alpha$. On the other hand, if $S \in \mathrm{O}(\mathbb{R}^n)$ (the orthogonal group), we have in the distribution sense[8] since $u_\alpha$ is a radial function, i.e. such that

$$v_\alpha(S\xi) = v_\alpha(\xi). \tag{1.2.29}$$

The distribution $|\xi|^\alpha v_\alpha(\xi)$ is homogeneous of degree 0 on $\mathbb{R}^n\backslash\{0\}$ and is also "radial", i.e. satisfies (1.2.29). Moreover on $\mathbb{R}^n\backslash\{0\}$, the distribution $v_\alpha$ is a $C^1$ function which coincides with[9]

$$\int e^{-2i\pi x\cdot\xi}\chi_0(x)|x|^{\alpha-n}dx + |\xi|^{-2N}\int e^{-2i\pi x\cdot\xi}|D_x|^{2N}\big(\chi_1(x)|x|^{\alpha-n}\big)dx,$$

where $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ is 1 near 0 and $\chi_1 = 1 - \chi_0$, $N \in \mathbb{N}, \alpha + 1 < 2N$. As a result $|\xi|^\alpha v_\alpha(\xi) = c_\alpha$ on $\mathbb{R}^n\backslash\{0\}$ and the distribution on $\mathbb{R}^n$ (note that $\alpha < n$)

$$T = v_\alpha(\xi) - c_\alpha|\xi|^{-\alpha}$$

[8] For $M \in Gl(n,\mathbb{R})$, $T \in \mathscr{S}'(\mathbb{R}^n)$, we define $\langle T(Mx), \phi(x)\rangle = \langle T(y), \phi(M^{-1}y)\rangle|\det M|^{-1}$.

[9] We have $\widehat{u_\alpha} = \widehat{\chi_0 u_\alpha} + \widehat{\chi_1 u_\alpha}$ and for $\phi$ supported in $\mathbb{R}^n\backslash\{0\}$ we get,

$$\langle\widehat{\chi_1 u_\alpha}, \phi\rangle = \langle\widehat{\chi_1 u_\alpha}|\xi|^{2N}, \phi(\xi)|\xi|^{-2N}\rangle = \langle\widehat{|D_x|^{2N}\chi_1 u_\alpha}, \phi(\xi)|\xi|^{-2N}\rangle.$$

is supported in $\{0\}$ and homogeneous (on $\mathbb{R}^n$) with degree $-\alpha$. From Exercises 1.5.7(1), 1.5.5, 1.5.8, the condition $0<\alpha<n$ gives $v_\alpha = c_\alpha |\xi|^{-\alpha}$. To find $c_\alpha$, we compute

$$\int_{\mathbb{R}^n} |x|^{\alpha-n} e^{-\pi x^2} dx = c_\alpha \int_{\mathbb{R}^n} |\xi|^{-\alpha} e^{-\pi \xi^2} d\xi$$

which yields

$$2^{-1}\Gamma(\frac{\alpha}{2})\pi^{-\frac{\alpha}{2}} = \int_0^{+\infty} r^{\alpha-1} e^{-\pi r^2} dr = c_\alpha \int_0^{+\infty} r^{n-\alpha-1} e^{-\pi r^2} dr$$
$$= c_\alpha 2^{-1}\Gamma(\frac{n-\alpha}{2})\pi^{-(\frac{n-\alpha}{2})}.$$

We have proven the following lemma.

**Lemma 1.2.18.** *Let $n \in \mathbb{N}^*$ and $\alpha \in (0,n)$. The function $u_\alpha(x) = |x|^{\alpha-n}$ is $L^1_{loc}(\mathbb{R}^n)$ and also a temperate distribution on $\mathbb{R}^n$. Its Fourier transform $v_\alpha$ is also $L^1_{loc}(\mathbb{R}^n)$ and given by*

$$v_\alpha(\xi) = |\xi|^{-\alpha} \pi^{\frac{n}{2}-\alpha} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

### Fourier transform of Gaussian functions

**Proposition 1.2.19.** *Let $A$ be a symmetric nonsingular $n \times n$ matrix with complex entries such that $\operatorname{Re} A \ge 0$. We define the Gaussian function $v_A$ on $\mathbb{R}^n$ by $v_A(x) = e^{-\pi\langle Ax,x\rangle}$. The Fourier transform of $v_A$ is*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}, \tag{1.2.30}$$

*where $(\det A)^{-1/2}$ is defined according to Formula (1.6.8). In particular, when $A = -iB$ with a symmetric real nonsingular matrix $B$, we get*

$$\operatorname{Fourier}(e^{i\pi\langle Bx,x\rangle})(\xi) = \widehat{v_{-iB}}(\xi) = |\det B|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} B} e^{-i\pi\langle B^{-1}\xi,\xi\rangle}. \tag{1.2.31}$$

*Proof.* We use the notations of Section 1.6 (in the subsection *Logarithm of a nonsingular symmetric matrix*). Let us define $\Upsilon_+^*$ as the set of symmetric $n\times n$ complex matrices with a positive definite real part (naturally these matrices are nonsingular since $Ax=0$ for $x \in \mathbb{C}^n$ implies $0 = \operatorname{Re}\langle Ax,\bar{x}\rangle = \langle (\operatorname{Re} A)x, \bar{x}\rangle$, so that $\Upsilon_+^* \subset \Upsilon_+$).

Let us assume first that $A \in \Upsilon_+^*$; then the function $v_A$ is in the Schwartz class (and so is its Fourier transform). The set $\Upsilon_+^*$ is an open convex subset of $\mathbb{C}^{n(n+1)/2}$ and the function $\Upsilon_+^* \ni A \mapsto \widehat{v_A}(\xi)$ is holomorphic and given on $\Upsilon_+^* \cap \mathbb{R}^{n(n+1)/2}$ by (1.2.30). On the other hand the function

$$\Upsilon_+^* \ni A \mapsto e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} e^{-\pi\langle A^{-1}\xi,\xi\rangle},$$

is also holomorphic and coincides with previous one on $\mathbb{R}^{n(n+1)/2}$. By analytic continuation this proves (1.2.30) for $A \in \Upsilon_+^*$.

If $A \in \Upsilon_+$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have $\langle \widehat{v_A}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int v_A(x)\hat{\varphi}(x)dx$ so that $\Upsilon_+ \ni A \mapsto \langle \widehat{v_A}, \varphi\rangle$ is continuous and thus (note that the mapping $A \mapsto A^{-1}$ is an homeomorphism of $\Upsilon_+$), using the previous result on $\Upsilon_+^*$,

$$\langle \widehat{v_A}, \varphi\rangle = \lim_{\epsilon\to 0_+} \langle \widehat{v_{A+\epsilon I}}, \varphi\rangle = \lim_{\epsilon\to 0_+} \int e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log}(A+\epsilon I)} e^{-\pi\langle (A+\epsilon I)^{-1}\xi,\xi\rangle}\varphi(\xi)d\xi,$$

and by continuity of Log on $\Upsilon_+$ and dominated convergence,

$$\langle \widehat{v_A}, \varphi\rangle = \int e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} e^{-\pi\langle A^{-1}\xi,\xi\rangle}\varphi(\xi)d\xi,$$

which is the sought result. □

### Multipliers of $\mathscr{S}'(\mathbb{R}^n)$

**Definition 1.2.20.** The space $\mathscr{O}_M(\mathbb{R}^n)$ of multipliers of $\mathscr{S}(\mathbb{R}^n)$ is the subspace of the functions $f \in C^\infty(\mathbb{R}^n)$ such that,

$$\forall\alpha \in \mathbb{N}^n, \exists C_\alpha > 0, \exists N_\alpha \in \mathbb{N}, \quad \forall x \in \mathbb{R}^n, \quad |(\partial_x^\alpha f)(x)| \le C_\alpha(1+|x|)^{N_\alpha}. \tag{1.2.32}$$

It is easy to check that, for $f \in \mathscr{O}_M(\mathbb{R}^n)$, the operator $u \mapsto fu$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself, and by transposition from $\mathscr{S}'(\mathbb{R}^n)$ into itself: we define for $T \in \mathscr{S}'(\mathbb{R}^n)$, $f \in \mathscr{O}_M(\mathbb{R}^n)$,

$$\langle fT, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, f\varphi\rangle_{\mathscr{S}',\mathscr{S}},$$

and if $p$ is a semi-norm of $\mathscr{S}$, the continuity on $\mathscr{S}$ of the multiplication by $f$ implies that there exists a semi-norm $q$ on $\mathscr{S}$ such that for all $\varphi \in \mathscr{S}$, $p(f\varphi) \le q(\varphi)$. A typical example of a function in $\mathscr{O}_M(\mathbb{R}^n)$ is $e^{iP(x)}$ where $P$ is a real-valued polynomial: in fact the derivatives of $e^{iP(x)}$ are of type $Q(x)e^{iP(x)}$ where $Q$ is a polynomial so that (1.2.32) holds.

**Definition 1.2.21.** Let $T, S$ be tempered distributions on $\mathbb{R}^n$ such that $\hat{T}$ belongs to $\mathscr{O}_M(\mathbb{R}^n)$. We define the convolution $T * S$ by

$$\widehat{T * S} = \hat{T}\hat{S}. \tag{1.2.33}$$

Note that this definition makes sense since $\hat{T}$ is a multiplier so that $\hat{T}\hat{S}$ is indeed a tempered distribution whose inverse Fourier transform is meaningful. We have

$$\langle T * S, \phi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle \widehat{T * S}, \hat{\check{\phi}}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle \hat{S}, \hat{T}\hat{\check{\phi}}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}.$$

**Proposition 1.2.22.** *Let $T$ be a distribution on $\mathbb{R}^n$ such that $T$ is compactly supported. Then $\hat{T}$ is a multiplier which can be extended to an entire function on $\mathbb{C}^n$ such that if* $\operatorname{supp} T \subset \bar{B}(0, R_0)$,

$$\exists C_0, N_0 \ge 0, \forall\zeta \in \mathbb{C}^n, \quad |\hat{T}(\zeta)| \le C_0(1+|\zeta|)^{N_0}e^{2\pi R_0|\operatorname{Im}\zeta|}. \tag{1.2.34}$$

*In particular, for $S \in \mathscr{S}'(\mathbb{R}^n)$, we may define according to (1.2.33) the convolution $T * S$.*

*Proof.* Let us first check the case $R_0 = 0$: then the distribution $T$ is supported at $\{0\}$ and from Exercise 1.5.5 is a linear combination of derivatives of the Dirac mass at 0. Formulas (1.2.18), (1.2.20) imply that $\hat{T}$ is a polynomial, so that the conclusions of Proposition 1.2.22 hold in that case.

Let us assume that $R_0 > 0$ and let us consider a function $\chi$ is equal to 1 in neighborhood of $\operatorname{supp} T$ (this implies $\chi T = T$) and

$$\langle \widehat{T}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \langle \widehat{\chi T}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \chi\hat{\phi}\rangle_{\mathscr{S}',\mathscr{S}}. \tag{1.2.35}$$

On the other hand, defining for $\zeta \in \mathbb{C}^n$ (with $x\cdot\zeta = \sum x_j\zeta_j$ for $x \in \mathbb{R}^n$),

$$F(\zeta) = \langle T(x), \chi(x)e^{-2i\pi x\cdot\zeta}\rangle_{\mathscr{S}',\mathscr{S}}, \tag{1.2.36}$$

we see that $F$ is an entire function (i.e. holomorphic on $\mathbb{C}^n$): calculating

$$\begin{aligned} F(\zeta+h) - F(\zeta) &= \langle T(x), \chi(x)e^{-2i\pi x\cdot\zeta}(e^{-2i\pi x\cdot h} - 1)\rangle \\ &= \langle T(x), \chi(x)e^{-2i\pi x\cdot\zeta}(-2i\pi x\cdot h)\rangle \\ &\quad + \langle T(x), \chi(x)e^{-2i\pi x\cdot\zeta}\int_0^1 (1-\theta)e^{-2i\theta\pi x\cdot h}d\theta(-2i\pi x\cdot h)^2\rangle, \end{aligned}$$

and applying to the last term the continuity properties of the linear form $T$, we obtain that the complex differential of $F$ is

$$\sum_{1\le j\le n} \langle T(x), \chi(x)e^{-2i\pi x\cdot\zeta}(-2i\pi x_j)\rangle d\zeta_j.$$

Moreover the derivatives of (1.2.36) are

$$F^{(k)}(\zeta) = \langle T(x), \chi(x)e^{-2i\pi x\cdot\zeta}(-2i\pi x)^k\rangle_{\mathscr{S}',\mathscr{S}}. \tag{1.2.37}$$

To evaluate the semi-norms of $x \mapsto \chi(x)e^{-2i\pi x\cdot\zeta}(-2i\pi x)^k$ in the Schwartz space, we have to deal with a finite sum of products of type

$$\left|x^\gamma(\partial^\alpha\chi)(x)e^{-2i\pi x\cdot\zeta}(-2i\pi\zeta)^\beta\right| \le (1+|\zeta|)^{|\beta|}\sup_{x\in\mathbb{R}^n}\left|x^\gamma(\partial^\alpha\chi)(x)e^{2\pi|x||\operatorname{Im}\zeta|}\right|.$$

We may now choose a function $\chi_0$ equal to 1 on $B(0,1)$, supported in $B(0, \frac{R_0+2\epsilon}{R_0+\epsilon})$ such that $\|\partial^\beta\chi_0\|_{L^\infty} \le c(\beta)\epsilon^{-|\beta|}$ with $\epsilon = \frac{R_0}{1+|\zeta|}$. We find with

$$\chi(x) = \chi_0(x/(R_0+\epsilon)) \quad \text{(which is 1 on a neighborhood of } B(0,R_0)),$$

$$\begin{aligned} \sup_{x\in\mathbb{R}^n}|x^\gamma(\partial^\alpha\chi)(x)e^{2\pi|x||\operatorname{Im}\zeta|}| &\le (R_0+2\epsilon)^{|\gamma|}\sup_{y\in\mathbb{R}^n}|(\partial^\alpha\chi_0)(y)e^{2\pi(R_0+2\epsilon)|\operatorname{Im}\zeta|}| \\ &\le (R_0+2\epsilon)^{|\gamma|}e^{2\pi(R_0+2\epsilon)|\operatorname{Im}\zeta|}c(\alpha)\epsilon^{-|\alpha|} \\ &= (R_0+2\frac{R_0}{1+|\zeta|})^{|\gamma|}e^{2\pi(R_0+2\frac{R_0}{1+|\zeta|})|\operatorname{Im}\zeta|}c(\alpha)(\frac{1+|\zeta|}{R_0})^{|\alpha|} \\ &\le (3R_0)^{|\gamma|}e^{2\pi R_0|\operatorname{Im}\zeta|}e^{4\pi R_0}c(\alpha)R_0^{-|\alpha|}(1+|\zeta|)^{|\alpha|} \end{aligned}$$

yielding

$$|F^{(k)}(\zeta)| \le e^{2\pi R_0 |\operatorname{Im}\zeta|} C_k (1+|\zeta|)^{N_k},$$

which implies that $\mathbb{R}^n \ni \xi \mapsto F(\xi)$ is indeed a multiplier. We have also

$$\langle T, \chi\hat{\phi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T(x), \chi(x) \int_{\mathbb{R}^n} \phi(\xi) e^{-2i\pi x\xi} d\xi\rangle_{\mathscr{S}',\mathscr{S}}.$$

Since the function $F$ is entire we have for $\phi \in C_c^\infty(\mathbb{R}^n)$, using (1.2.37) and Fubini Theorem on $\ell^1(\mathbb{N}) \times L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} F(\xi)\phi(\xi) d\xi = \sum_{k\ge 0} \langle T(x), \chi(x)(-2i\pi x)^k\rangle \int_{\operatorname{supp}\phi} \frac{\xi^k}{k!}\phi(\xi) d\xi. \tag{1.2.38}$$

On the other hand, since $\hat{\phi}$ is also entire (from the discussion on $F$ or directly from the integral formula for the Fourier transform of $\phi \in C_c^\infty(\mathbb{R}^n)$), we have

$$\begin{aligned}
\langle T, \chi\hat{\phi}\rangle &= \langle T(x), \chi(x) \sum_{k\ge 0} (\hat{\phi})^{(k)}(0) x^k/k!\rangle \\
&= \langle T(x), \underbrace{\chi(x) \lim_{N\to+\infty} \sum_{0\le k\le N} (\hat{\phi})^{(k)}(0) x^k/k!}_{\text{convergence in } C_c^\infty(\mathbb{R}^n)}\rangle \\
&= \lim_{N\to+\infty} \sum_{0\le k\le N} \langle T(x), \chi(x) x^k/k!\rangle \int_{\mathbb{R}^n} \phi(\xi)(-2i\pi\xi)^k d\xi.
\end{aligned}$$

Thanks to (1.2.38), that quantity is equal to $\int_{\mathbb{R}^n} F(\xi)\phi(\xi)d\xi$. As a result, the tempered distributions $\hat{T}$ and $F$ coincide on $C_c^\infty(\mathbb{R}^n)$, which is dense in $\mathscr{S}(\mathbb{R}^n)$ (see Exercise 1.5.3) and so $\hat{T} = F$, concluding the proof. □

## 1.3 The Poisson summation formula

### Wave packets

We define for $x \in \mathbb{R}^n$, $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi(x-y)^2} e^{2i\pi(x-y)\cdot\eta} = 2^{n/4} e^{-\pi(x-y-i\eta)^2} e^{-\pi\eta^2}, \tag{1.3.1}$$

$$\text{where for } \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n, \quad \zeta^2 = \sum_{1\le j\le n} \zeta_j^2. \tag{1.3.2}$$

We note that the function $\varphi_{y,\eta}$ is in $\mathscr{S}(\mathbb{R}^n)$ and with $L^2$ norm 1. In fact, $\varphi_{y,\eta}$ appears as a *phase translation* of a normalized Gaussian. The following lemma introduces the *wave packets transform* as a Gabor wavelet.

**Lemma 1.3.1.** *Let $u$ be a function in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$. We define*

$$(Wu)(y,\eta) = (u, \varphi_{y,\eta})_{L^2(\mathbb{R}^n)} = 2^{n/4} \int u(x) e^{-\pi(x-y)^2} e^{-2i\pi(x-y)\cdot\eta} dx \tag{1.3.3}$$

$$= 2^{n/4} \int u(x) e^{-\pi(y-i\eta-x)^2} dx e^{-\pi\eta^2}. \tag{1.3.4}$$

*For $u \in L^2(\mathbb{R}^n)$, the function $Tu$ defined by*

$$(Tu)(y+i\eta) = e^{\pi\eta^2} Wu(y,-\eta) = 2^{n/4}\int u(x)e^{-\pi(y+i\eta-x)^2}dx \tag{1.3.5}$$

*is an entire function. The mapping $u \mapsto Wu$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. Moreover, we have the reconstruction formula*

$$u(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} (Wu)(y,\eta)\varphi_{y,\eta}(x)dyd\eta. \tag{1.3.6}$$

*Proof.* For $u$ in $\mathscr{S}(\mathbb{R}^n)$, we have

$$(Wu)(y,\eta) = e^{2i\pi y\eta}\widehat{\Omega}^1(\eta,y)$$

where $\widehat{\Omega}^1$ is the Fourier transform with respect to the first variable of the $\mathscr{S}(\mathbb{R}^{2n})$ function $\Omega(x,y) = u(x)e^{-\pi(x-y)^2}2^{n/4}$. Thus the function $Wu$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$. It makes sense to compute

$$2^{-n/2}(Wu,Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+}\int u(x_1)\overline{u}(x_2)e^{-\pi[(x_1-y)^2+(x_2-y)^2+2i(x_1-x_2)\eta+\epsilon^2\eta^2]}dyd\eta dx_1dx_2. \tag{1.3.7}$$

Now the last integral on $\mathbb{R}^{4n}$ converges absolutely and we can use the Fubini theorem. Integrating with respect to $\eta$ involves the Fourier transform of a Gaussian function and we get $\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}$. Since

$$2(x_1-y)^2 + 2(x_2-y)^2 = (x_1+x_2-2y)^2 + (x_1-x_2)^2,$$

integrating with respect to $y$ yields a factor $2^{-n/2}$. We are left with

$$(Wu,Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+}\int u(x_1)\ \overline{u}(x_2)e^{-\pi(x_1-x_2)^2/2}\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}dx_1dx_2. \tag{1.3.8}$$

Changing the variables, the integral is

$$\lim_{\epsilon\to 0_+}\int u(s+\epsilon t/2)\ \overline{u}(s-\epsilon t/2)e^{-\pi\epsilon^2t^2/2}e^{-\pi t^2}dtds = \|u\|^2_{L^2(\mathbb{R}^n)}$$

by Lebesgue's dominated convergence theorem: the triangle inequality and the estimate $|u(x)| \le C(1+|x|)^{-n-1}$ imply, with $v = u/C$,

$$\begin{aligned}|v(s+\epsilon t/2)\ \overline{v}(s-\epsilon t/2)| &\le (1+|s+\epsilon t/2|)^{-n-1}(1+|s+\epsilon t/2|)^{-n-1}\\ &\le (1+|s+\epsilon t/2|+|s-\epsilon t/2|)^{-n-1}\\ &\le (1+2|s|)^{-n-1}.\end{aligned}$$

Eventually, this proves that for $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\|Wu\|^2_{L^2(\mathbb{R}^{2n})} = \|u\|^2_{L^2(\mathbb{R}^n)} \tag{1.3.9}$$

so that by density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$,

$$W : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}) \quad \text{with} \quad W^*W = \mathrm{id}_{L^2(\mathbb{R}^n)}. \tag{1.3.10}$$

Noticing first that $\iint Wu(y,\eta)\varphi_{y,\eta} dy d\eta$ belongs to $L^2(\mathbb{R}^n)$ (with a norm smaller than $\|Wu\|_{L^1(\mathbb{R}^{2n})}$) and applying Fubini's theorem, we get from the polarization of (1.3.9) for $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned}(u,v)_{L^2(\mathbb{R}^n)} = (Wu, Wv)_{L^2(\mathbb{R}^{2n})} &= \iint Wu(y,\eta)(\varphi_{y,\eta}, v)_{L^2(\mathbb{R}^n)} dy d\eta \\ &= (\iint Wu(y,\eta)\varphi_{y,\eta} dy d\eta, v)_{L^2(\mathbb{R}^n)},\end{aligned}$$

yielding $u = \iint Wu(y,\eta)\varphi_{y,\eta} dy d\eta$, which is the result of the lemma. □

### Poisson's formula

The following lemma is in fact the Poisson summation formula for Gaussian functions in one dimension.

**Lemma 1.3.2.** *For all complex numbers $z$, the following series are absolutely converging and*

$$\sum_{m \in \mathbb{Z}} e^{-\pi(z+m)^2} = \sum_{m \in \mathbb{Z}} e^{-\pi m^2} e^{2i\pi m z}. \tag{1.3.11}$$

*Proof.* We set $\omega(z) = \sum_{m \in \mathbb{Z}} e^{-\pi(z+m)^2}$. The function $\omega$ is entire and 1-periodic since for all $m \in \mathbb{Z}$, $z \mapsto e^{-\pi(z+m)^2}$ is entire and for $R > 0$,

$$\sup_{|z| \le R} |e^{-\pi(z+m)^2}| \le \sup_{|z| \le R} |e^{-\pi z^2}| e^{-\pi m^2} e^{2\pi |m| R} \in \ell^1(\mathbb{Z}).$$

Consequently, for $z \in \mathbb{R}$, we obtain, expanding $\omega$ in Fourier series[10],

$$\omega(z) = \sum_{k \in \mathbb{Z}} e^{2i\pi k z} \int_0^1 \omega(x) e^{-2i\pi k x} dx.$$

We also check, using Fubini's theorem on $L^1(0,1) \times \ell^1(\mathbb{Z})$

$$\begin{aligned}\int_0^1 \omega(x) e^{-2i\pi k x} dx &= \sum_{m \in \mathbb{Z}} \int_0^1 e^{-\pi(x+m)^2} e^{-2i\pi k x} dx \\ &= \sum_{m \in \mathbb{Z}} \int_m^{m+1} e^{-\pi t^2} e^{-2i\pi k t} dt \\ &= \int_{\mathbb{R}} e^{-\pi t^2} e^{-2i\pi k t} = e^{-\pi k^2}.\end{aligned}$$

[10] Note that we use this expansion only for a $C^\infty$ 1-periodic function. The proof is simple and requires only to compute $1 + 2\operatorname{Re} \sum_{1 \le k \le N} e^{2i\pi k x} = \frac{\sin \pi(2N+1)x}{\sin \pi x}$. Then one has to show that for a smooth 1-periodic function $\omega$ such that $\omega(0) = 0$,

$$\lim_{\lambda \to +\infty} \int_0^1 \frac{\sin \lambda x}{\sin \pi x} \omega(x) dx = 0,$$

which is obvious since for a smooth $\nu$ (here we take $\nu(x) = \omega(x)/\sin \pi x$), $|\int_0^1 \nu(x) \sin(\lambda x) dx| = O(\lambda^{-1})$ by integration by parts.

So the lemma is proven for real $z$ and since both sides are entire functions, we conclude by analytic continuation. □

It is now straightforward to get the $n$-th dimensional version of the previous lemma: for all $z \in \mathbb{C}^n$, using the notation (1.3.2), we have

$$\sum_{m\in\mathbb{Z}^n} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}^n} e^{-\pi m^2} e^{2i\pi m\cdot z}. \tag{1.3.12}$$

**Theorem 1.3.3** (Poisson summation formula). *Let $n$ be a positive integer and let $u$ be a function in $\mathscr{S}(\mathbb{R}^n)$. Then we have*

$$\sum_{k\in\mathbb{Z}^n} u(k) = \sum_{k\in\mathbb{Z}^n} \hat{u}(k), \tag{1.3.13}$$

*where $\hat{u}$ stands for the Fourier transform of $u$. In other words the tempered distribution $D_0 = \sum_{k\in\mathbb{Z}^n} \delta_k$ is such that $\widehat{D_0} = D_0$.*

*Proof.* We write, according to (1.3.6) and to Fubini's theorem

$$\begin{aligned}\textstyle\sum_{k\in\mathbb{Z}^n} u(k) &= \sum_{k\in\mathbb{Z}^n} \iint Wu(y,\eta)\varphi_{y,\eta}(k)dyd\eta \\ &= \iint Wu(y,\eta) \sum_{k\in\mathbb{Z}^n} \varphi_{y,\eta}(k)dyd\eta.\end{aligned}$$

Now, (1.3.12), (1.3.1) give

$$\sum_{k\in\mathbb{Z}^n} \varphi_{y,\eta}(k) = \sum_{k\in\mathbb{Z}^n} \widehat{\varphi}_{y,\eta}(k),$$

so that (1.3.6) and Fubini's theorem imply the result. □

## 1.4 Periodic distributions

### The Dirichlet kernel

For $N \in \mathbb{N}$, the Dirichlet kernel $D_N$ is defined on $\mathbb{R}$ by

$$\begin{aligned}D_N(x) &= \sum_{-N\le k\le N} e^{2i\pi kx} \\ &= 1 + 2\operatorname{Re}\sum_{1\le k\le N} e^{2i\pi kx} \underbrace{=}_{x\notin\mathbb{Z}} 1 + 2\operatorname{Re}\left(e^{2i\pi x}\frac{e^{2i\pi Nx}-1}{e^{2i\pi x}-1}\right) \\ &= 1 + 2\operatorname{Re}\big(e^{2i\pi x - i\pi x + i\pi Nx}\big)\frac{\sin(\pi Nx)}{\sin(\pi x)} = 1 + 2\cos(\pi(N+1)x)\frac{\sin(\pi Nx)}{\sin(\pi x)} \\ &= 1 + \frac{1}{\sin(\pi x)}\Big(\sin\big(\pi x(2N+1)\big) - \sin(\pi x)\Big) = \frac{\sin\big(\pi x(2N+1)\big)}{\sin(\pi x)},\end{aligned}$$

and extending by continuity at $x \in \mathbb{Z}$ that 1-periodic function, we find that

$$D_N(x) = \frac{\sin\bigl(\pi x(2N+1)\bigr)}{\sin(\pi x)}. \tag{1.4.1}$$

Now, for a 1-periodic $v \in C^1(\mathbb{R})$, with

$$(D_N \star u)(x) = \int_0^1 D_N(x-t)u(t)dt, \tag{1.4.2}$$

we have

$$\lim_{N\to+\infty} \int_0^1 D_N(x-t)v(t)dt = v(x) + \lim_{N\to+\infty} \int_0^1 \sin(\pi t(2N+1)) \frac{\bigl(v(x-t)-v(x)\bigr)}{\sin(\pi t)} dt,$$

and the function $\theta_x$ given by $\theta_x(t) = \frac{v(x-t)-v(x)}{\sin(\pi t)}$ is continuous on $[0,1]$, and from the Riemann-Lebesgue Lemma 1.1.1, we obtain

$$\lim_{N\to+\infty} \sum_{-N\le k\le N} e^{2i\pi kx} \int_0^1 e^{-2i\pi kt} v(t)dt = \lim_{N\to+\infty} \int_0^1 D_N(x-t)v(t)dt = v(x).$$

On the other hand if $v$ is 1-periodic and $C^{1+l}$, the Fourier coefficient

$$c_k(v) = \int_0^1 e^{-2i\pi kt} v(t)dt$$

$$\overbrace{=}^{\text{for } k\neq 0} \frac{1}{2i\pi k}[e^{-2i\pi kt}v(t)]_{t=1}^{t=0} + \int_0^1 \frac{1}{2i\pi k} e^{-2i\pi kt} v'(t)dt,$$

and iterating the integration by parts, we find $c_k(v) = O(k^{-1-l})$ so that for a 1-periodic $C^2$ function $v$, we have

$$\sum_{k\in\mathbb{Z}} e^{2i\pi kx} c_k(v) = v(x). \tag{1.4.3}$$

### Pointwise convergence of Fourier series

**Lemma 1.4.1.** *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a 1-periodic $L^1_{loc}(\mathbb{R})$ function and let $x_0 \in [0,1]$. Let us assume that there exists $w_0 \in \mathbb{R}$ such that the Dini condition is satisfied, i.e.*

$$\int_0^{1/2} \frac{|u(x_0+t)+u(x_0-t)-2w_0|}{t} dt < +\infty. \tag{1.4.4}$$

*Then, $\lim_{N\to+\infty} \sum_{|k|\le N} c_k(u)e^{2i\pi kx_0} = w_0$ with $c_k(u) = \int_0^1 e^{-2i\pi tk}u(t)dt$.*

*Proof.* Using the above calculations, we find

$$\sum_{|k|\le N} c_k(u)e^{2i\pi kx_0} = (D_N \star u)(x_0) = w_0 + \int_0^1 \frac{\sin\bigl(\pi t(2N+1)\bigr)}{\sin(\pi t)} \bigl(u(x_0-t)-w_0\bigr)dt,$$

so that, using the periodicity of $u$ and the fact that $D_N$ is an even function , we get

$$(D_N \star u)(x_0) - w_0 = \int_0^{1/2} \frac{\sin\bigl(\pi t(2N+1)\bigr)}{\sin(\pi t)}\bigl(u(x_0-t)+u(x_0+t)-2w_0\bigr)dt.$$

Thanks to the hypothesis (1.4.4), the function

$$t \mapsto \mathbf{1}_{[0,\frac12]}(t)\frac{u(x_0-t)+u(x_0+t)-2w_0}{\sin(\pi t)}$$

belongs to $L^1(\mathbb{R})$ and Riemann-Lebesgue Lemma 1.1.1 gives the conclusion. □

**Theorem 1.4.2.** *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a 1-periodic $L^1_{loc}$ function.*
*(1) Let $x_0 \in [0,1], w_0 \in \mathbb{R}$. We define $\omega_{x_0,w_0}(t) = |u(x_0+t)+u(x_0-t)-2w_0|$ and we assume that*

$$\int_0^{1/2} \omega_{x_0,w_0}(t)\frac{dt}{t} < +\infty. \tag{1.4.5}$$

*Then the Fourier series $(D_N \star u)(x_0)$ converges with limit $w_0$. In particular, if (1.4.5) is satisfied with $w_0 = u(x_0)$, the Fourier series $(D_N \star u)(x_0)$ converges with limit $u(x_0)$. If $u$ has a left and right limit at $x_0$ and is such that (1.4.5) is satisfied with $w_0 = \frac12\bigl(u(x_0+0)+u(x_0-0)\bigr)$, the Fourier series $(D_N \star u)(x_0)$ converges with limit $\frac12\bigl(u(x_0-0)+u(x_0+0)\bigr)$.*
*(2) If the function $u$ is Hölder-continuous[11], the Fourier series $(D_N\star u)(x)$ converges for all $x \in \mathbb{R}$ with limit $u(x)$.*
*(3) If $u$ has a left and right limit at each point and a left and right derivative at each point, the Fourier series $(D_N \star u)(x)$ converges for all $x \in \mathbb{R}$ with limit*

$$\frac12\bigl(u(x-0)+u(x+0)\bigr).$$

*Proof.* (1) follows from Lemma 1.4.1; to obtain (2), we note that for a Hölder continuous function of index $\theta \in ]0,1]$, we have for $t \in ]0,1/2]$

$$t^{-1}\omega_{x,u(x)}(t) \le Ct^{\theta-1} \in L^1([0,1/2]).$$

(3) If $u$ has a right-derivative at $x_0$, it means that

$$u(x_0+t) = u(x_0+0) + u'_r(x_0)t + t\epsilon_0(t), \quad \lim_{t\to 0_+}\epsilon_0(t) = 0.$$

As a consequence, for $t \in ]0,1/2]$, $t^{-1}|u(x_0+t)-u(x_0+0)| \le |u'_r(x_0)+\epsilon_0(t)|$. Since $\lim_{t\to 0_+}\epsilon_0(t) = 0$, there exists $T_0 \in ]0,1/2]$ such that $|\epsilon_0(t)| \le 1$ for $t \in [0,T_0]$. As a result, we have

$$\int_0^{1/2} t^{-1}|u(x_0+t)-u(x_0+0)|dt$$
$$\le \int_0^{T_0}(|u'_r(x_0)|+1)dt + \int_{T_0}^{1/2}|u(x_0+t)-u(x_0+0)|dtT_0^{-1} < +\infty,$$

since $u$ is also $L^1_{\mathrm{loc}}$. The integral $\int_0^{1/2} t^{-1}|u(x_0-t)-u(x_0-0)|dt$ is also finite and the condition (1.4.5) holds with $w_0 = \frac12\bigl(u(x_0-0)+u(x_0+0)\bigr)$. The proof of the lemma is complete. □

[11] Hölder-continuity of index $\theta \in ]0,1]$ means that $\exists C>0, \forall t,s,\ |u(t)-u(s)| \le C|t-s|^\theta$.

### Periodic distributions

We consider now a distribution $u$ on $\mathbb{R}^n$ which is periodic with periods $\mathbb{Z}^n$. Let $\chi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}_+)$ such that $\chi = 1$ on $[0,1]^n$. Then the function $\chi_1$ defined by

$$\chi_1(x) = \sum_{k\in\mathbb{Z}^n} \chi(x-k)$$

is $C^\infty$ periodic[12] with periods $\mathbb{Z}^n$. Moreover since

$$\mathbb{R}^n \ni x \in \prod_{1\le j\le n} [E(x_j), E(x_j)+1[,$$

the bounded function $\chi_1$ is also bounded from below and such that $1 \le \chi_1(x)$. With $\chi_0 = \chi/\chi_1$, we have

$$\sum_{k\in\mathbb{Z}^n} \chi_0(x-k) = 1, \quad \chi_0 \in C_c^\infty(\mathbb{R}^n).$$

For $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have from the periodicity of $u$

$$\langle u,\varphi\rangle = \sum_{k\in\mathbb{Z}^n} \langle u(x), \varphi(x)\chi_0(x-k)\rangle = \sum_{k\in\mathbb{Z}^n} \langle u(x), \varphi(x+k)\chi_0(x)\rangle,$$

where the sums are finite. Now if $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have, since $\chi_0$ is compactly supported (say in $|x| \le R_0$),

$$\begin{aligned}|\langle u(x), \varphi(x+k)\chi_0(x)\rangle| &\le C_0 \sup_{|\alpha|\le N_0, |x|\le R_0} |\varphi^{(\alpha)}(x+k)| \\ &\le C_0 \sup_{|\alpha|\le N_0, |x|\le R_0} |(1+R_0+|x+k|)^{n+1}\varphi^{(\alpha)}(x+k)|(1+|k|)^{-n-1} \\ &\le p_0(\varphi)(1+|k|)^{-n-1},\end{aligned}$$

where $p_0$ is a semi-norm of $\varphi$ (independent of $k$). As a result $u$ is a tempered distribution and we have for $\varphi \in \mathscr{S}(\mathbb{R}^n)$, using Poisson's summation formula,

$$\langle u,\varphi\rangle = \langle u(x), \sum_{k\in\mathbb{Z}^n} \underbrace{\varphi(x+k)\chi_0(x)}_{\psi_x(k)}\rangle = \langle u(x), \sum_{k\in\mathbb{Z}^n} \widehat{\psi_x}(k)\rangle.$$

Now we see that $\widehat{\psi_x}(k) = \int_{\mathbb{R}^n} \varphi(x+t)\chi_0(x) e^{-2i\pi kt} dt = \chi_0(x) e^{2i\pi kx}\hat\varphi(k)$, so that

$$\langle u,\varphi\rangle = \sum_{k\in\mathbb{Z}^n} \langle u(x), \chi_0(x)e^{2i\pi kx}\rangle \hat\varphi(k),$$

which means

$$u(x) = \sum_{k\in\mathbb{Z}^n} \langle u(t), \chi_0(t)e^{2i\pi kt}\rangle e^{-2i\pi kx} = \sum_{k\in\mathbb{Z}^n} \langle u(t), \chi_0(t)e^{-2i\pi kt}\rangle e^{2i\pi kx}.$$

[12] Note that the sum is locally finite since for $K$ compact subset of $\mathbb{R}^n$, $(K-k)\cap \operatorname{supp}\chi_0 = \emptyset$ except for a finite subset of $k \in \mathbb{Z}^n$.

**Theorem 1.4.3.** *Let $u$ be a periodic distribution on $\mathbb{R}^n$ with periods $\mathbb{Z}^n$. Then $u$ is a tempered distribution and if $\chi_0$ is a $C_c^\infty(\mathbb{R}^n)$ function such that $\sum_{k\in\mathbb{Z}^n}\chi_0(x-k)=1$, we have*

$$u=\sum_{k\in\mathbb{Z}^n} c_k(u)e^{2i\pi kx}, \tag{1.4.6}$$

$$\hat{u}=\sum_{k\in\mathbb{Z}^n} c_k(u)\delta_k, \quad \textit{with} \quad c_k(u)=\langle u(t),\chi_0(t)e^{-2i\pi kt}\rangle, \tag{1.4.7}$$

*and convergence in $\mathscr{S}'(\mathbb{R}^n)$. If $u$ is in $C^m(\mathbb{R}^n)$ with $m>n$, the previous formulas hold with uniform convergence for (1.4.6) and*

$$c_k(u)=\int_{[0,1]^n} u(t)e^{-2i\pi kt}dt. \tag{1.4.8}$$

*Proof.* The first statements are already proven and the calculation of $\hat{u}$ is immediate. If $u$ belongs to $L^1_{\text{loc}}$ we can redo the calculations above choosing $\chi_0=\mathbf{1}_{[0,1]^n}$ and get (1.4.6) with $c_k$ given by (1.4.8). Moreover, if $u$ is in $C^m$ with $m>n$, we get by integration by parts that $c_k(u)$ is $O(|k|^{-m})$ so that the series (1.4.6) is uniformly converging. □

**Theorem 1.4.4.** *Let $u$ be a periodic distribution on $\mathbb{R}^n$ with periods $\mathbb{Z}^n$. If $u\in L^2_{loc}$ (i.e. $u\in L^2(\mathbb{T}^n)$ with $\mathbb{T}^n=(\mathbb{R}/\mathbb{Z})^n$), then*

$$u(x)=\sum_{k\in\mathbb{Z}^n} c_k(u)e^{2i\pi kx}, \quad \textit{with} \quad c_k(u)=\int_{[0,1]^n} u(t)e^{-2i\pi kt}dt, \tag{1.4.9}$$

*and convergence in $L^2(\mathbb{T}^n)$. Moreover $\|u\|^2_{L^2(\mathbb{T}^n)}=\sum_{k\in\mathbb{Z}^n}|c_k(u)|^2$. Conversely, if the coefficients $c_k(u)$ defined by (1.4.7) are in $\ell^2(\mathbb{Z}^n)$, the distribution $u$ is $L^2(\mathbb{T}^n)$*

*Proof.* As said above the formula for the $c_k(u)$ follows from changing the choice of $\chi_0$ to $\mathbf{1}_{[0,1]^n}$ in the discussion preceding Theorem 1.4.3. Formula (1.4.6) gives the convergence in $\mathscr{S}'(\mathbb{R}^n)$ to $u$. Now, since

$$\int_{[0,1]^n} e^{2i\pi(k-l)t}dt=\delta_{k,l}$$

we see from Theorem 1.4.3 that for $u\in C^{n+1}(\mathbb{T}^n)$,

$$\langle u,u\rangle_{L^2(\mathbb{T}^n)}=\sum_{k\in\mathbb{Z}^n}|c_k(u)|^2.$$

As a consequence the mapping $L^2(\mathbb{T}^n)\ni u\mapsto (c_k(u))_{k\in\mathbb{Z}^n}\in\ell^2(\mathbb{Z}^n)$ is isometric with a range containing the dense subset $\ell^1(\mathbb{Z}^n)$ (if $(c_k(u))_{k\in\mathbb{Z}^n}\in\ell^1(\mathbb{Z}^n)$, $u$ is a continuous function); since the range is closed[13], the mapping is onto and is an isometric isomorphism from the open mapping theorem (see e.g. Theorem 2.1.10 in [10]). □

[13]If $A:\mathcal{H}_1\to\mathcal{H}_2$ is an isometric linear mapping between Hilbert spaces and $(Au_k)$ is a converging sequence in $\mathcal{H}_2$, then by linearity and isometry, the sequence $(u_k)$ is a Cauchy sequence in $\mathcal{H}_1$, thus converges. The continuity of $A$ implies that if $u=\lim_k u_k$, we have

$$v=\lim_k Au_k=Au, \quad \text{proving that the range of } A \text{ is closed.}$$

## 1.5 Exercises

**Exercise 1.5.1.** *Let $A$ be a positive definite $n \times n$ symmetric matrix. Prove that the function $\psi_A$ defined by $\psi_A(x) = e^{-\langle Ax,x\rangle}$ belongs to $\mathscr{S}(\mathbb{R}^n)$.*

*Answer.* The function $\psi_A$ is smooth and such that

$$x^\alpha(\partial_x^\beta \psi_A)(x) = P_{\alpha,\beta}(x)\psi_A(x),$$

where $P_{\alpha,\beta}$ is a polynomial (obvious induction). Since $\langle Ax, x\rangle \ge \delta\|x\|^2$ with a positive $\delta$ and $|P_{\alpha,\beta}(x)| \le C(1+\|x\|^2)^{d/2}$, where $d$ is the degree of $P$, we obtain the boundedness of $x^\alpha(\partial_x^\beta\psi_A)(x)$, proving the sought result.

**Exercise 1.5.2.** *The Schwartz class of functions is defined by*

$$\mathscr{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n), \forall \alpha, \beta \in \mathbb{N}^n, \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| = p_{\alpha\beta}(u) < \infty\},$$

*where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$, $\beta \in \mathbb{N}^n, \partial_x^\beta = \partial_{x_1}^{\beta_1}\dots\partial_{x_n}^{\beta_n}$. Show that the $p_{\alpha\beta}$ are semi-norms on $\mathscr{S}(\mathbb{R}^n)$, making this space a Fréchet space.*

*Answer.* The $p_{\alpha\beta}$ are semi-norms, i.e. valued in $\mathbb{R}_+$ such that $p_{\alpha\beta}(\lambda u) = |\lambda| p_{\alpha\beta}(u)$ and they satisfy the triangle inequality. We consider a Cauchy sequence $(u_k)_{k\in\mathbb{N}}$. It means that for all $\alpha, \beta$, for all $\epsilon > 0$, there exists $k_{\alpha\beta\epsilon}$ such that for all $k \ge k_{\alpha\beta\epsilon}, l \ge 0$

$$p_{\alpha\beta}(u_{k+l} - u_k) \le \epsilon.$$

Using the case $\alpha = \beta = 0$, we find a continuous function $u$ uniform limit of $u_k$. Using the uniform convergence of the sequence $(\partial_x^\alpha u_k)_{k\in\mathbb{N}}$, we get that $u$ is $C^\infty$ and that the sequences $(\partial_x^\alpha u_k)_{k\in\mathbb{N}}$ are uniformly converging towards $\partial_x^\alpha u$. We write then

$$|x^\alpha\partial_x^\beta(u_k - u)(x)| = \lim_{l\to+\infty} |x^\alpha\partial_x^\beta(u_k - u_l)(x)| \le \limsup_l p_{\alpha\beta}(u_k - u_l) \le \epsilon$$

for $k \ge k_{\alpha\beta\epsilon}$. We get $p_{\alpha\beta}(u_k - u) \le \epsilon$ for $k \ge k_{\alpha\beta\epsilon}$, proving the convergence in $\mathscr{S}(\mathbb{R}^n)$.

**Exercise 1.5.3.** *Prove that $C_c^\infty(\mathbb{R}^n)$ is dense in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$.*

*Answer.* Let $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on the unit ball. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ and let us define for $k \in \mathbb{N}^*$

$$\phi_k(x) = \chi_0(x/k)\phi(x), \quad \phi_k \in C_c^\infty(\mathbb{R}^n), \quad \phi_k(x) - \phi(x) = \phi(x)\big(\chi_0(x/k) - 1\big),$$

and with the $p_{\alpha\beta}$ defined in Exercise 1.5.2, we have

$$\begin{aligned} p_{\alpha\beta}\big(\phi_k - \phi\big) = \sup_{x\in\mathbb{R}^n} |x^\alpha \sum_{\substack{\beta'+\beta''=\beta\\ |\beta''|\ge 1}} \frac{\beta!}{\beta'!\beta''!} \partial_x^{\beta'}\phi(x)\partial_x^{\beta''}\chi_0(x/k)k^{-|\beta''|}| \\ + \sup_{x\in\mathbb{R}^n, |x|\ge k} |x^\alpha(\partial_x^\beta\phi)(x)(\chi_0(\frac{x}{k}) - 1)|, \\ \le Ck^{-1} p_{\max(|\alpha|,|\beta|)}(\phi) p_{\max(|\alpha|,|\beta|)}(\chi_0) + k^{-1}\sup_{x\in\mathbb{R}^n} ||x|x^\alpha(\partial_x^\beta\phi)(x)|, \end{aligned}$$

with $p_k$ defined in (1.2.3), proving the convergence towards $\phi$ in the Schwartz space of the sequence $(\phi_k)_{k\in\mathbb{N}}$.

**Exercise 1.5.4.** *Let $T \in \mathscr{S}'(\mathbb{R})$ such that $xT = 0$. Prove that $T = c\delta_0$.*

*Answer.* Let $\phi \in \mathscr{S}(\mathbb{R})$ and let $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ such that $\chi_0(0) = 1$. We have

$$\phi(x) = \chi_0(x)\phi(x) + (1 - \chi_0(x))\phi(x).$$

Applying Taylor's formula with integral remainder, we define the smooth function $\psi$ by

$$\psi(x) = \frac{(1 - \chi_0(x))}{x}\phi(x)$$

and, applying Leibniz' formula, we see also that $\psi$ belongs to $\mathscr{S}(\mathbb{R})$. As a result

$$\langle T, \phi\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = \langle T, \chi_0\phi\rangle = \langle T, \chi_0\big(\phi - \phi(0)\big)\rangle + \phi(0)\langle T, \chi_0\rangle = \phi(0)\langle T, \chi_0\rangle,$$

since the function $x \mapsto \chi_0(x)\big(\phi(x) - \phi(0)\big)/x$ belongs to $C_c^\infty(\mathbb{R})$. As a result $T = \langle T, \chi_0\rangle\delta_0$.

**Exercise 1.5.5.** *Prove that a distribution with support $\{0\}$ is a linear combination of derivatives of the Dirac mass at $0$, i.e.*

$$u = \sum_{|\alpha|\le N} c_\alpha \delta_0^{(\alpha)},$$

*where the $c_\alpha$ are some constants.*

*Answer.* Let $N_0 \in \mathbb{N}$ such that $|\langle u, \varphi\rangle| \le C p_{N_0}(\varphi)$, where the semi-norms $p_k$ are given by (1.2.3). For $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\varphi(x) = \sum_{|\alpha|\le N_0} \frac{(\partial_x^\alpha\varphi)(0)}{\alpha!}x^\alpha + \underbrace{\int_0^1 \frac{(1-\theta)^{N_0}}{N_0!}\varphi^{(N_0+1)}(\theta x)d\theta\, x^{N_0+1}}_{\psi(x),\quad \psi\in C^\infty(\mathbb{R}^n)},$$

and thus for $\chi_0 \in C_c^\infty(\mathbb{R}^n), \chi_0 = 1$ near $0$,

$$\langle u, \varphi\rangle = \langle u, \chi_0\varphi\rangle = \sum_{|\alpha|\le N_0} \frac{(\partial_x^\alpha\varphi)(0)}{\alpha!}\langle u, \chi_0(x)x^\alpha\rangle + \langle u, \chi_0(x)\psi(x)x^{N_0+1}\rangle. \tag{1.5.1}$$

We note that

$$|\langle u, \chi_0(x)\psi(x)x^{N_0+1}\rangle| \le C_0 \sup_{|\alpha|\le N_0} |\partial_x^\alpha\big(\chi_0(x)\psi(x)x^{N_0+1}\big)|. \tag{1.5.2}$$

We can take $\chi_0(x) = \rho(x/\epsilon)$, where $\rho \in C_c^\infty(\mathbb{R}^n)$ is supported in the unit ball $B_1$, $\rho = 1$ in $\frac{1}{2}B_1$ and $\epsilon > 0$. We have then

$$\chi_0(x)\psi(x)x^{N_0+1} = \epsilon^{N_0+1}\rho(\frac{x}{\epsilon})\psi\big(\epsilon\frac{x}{\epsilon}\big)\frac{x^{N_0+1}}{\epsilon^{N_0+1}} = \epsilon^{N_0+1}\rho_1(\frac{x}{\epsilon}),$$

with $\rho_1(t) = \rho(t)\psi(\epsilon t)t^{N_0+1}$, so that $\rho_1 \in C_c^\infty(\mathbb{R}^n)$ is supported in the unit ball $B_1$ has all its derivatives bounded independently of $\epsilon$. From (1.5.2), we get for all $\epsilon > 0$,

$$|\langle u, \chi_0(x)\psi(x)x^{N_0+1}\rangle| \le C_0 \sup_{|\alpha|\le N_0} \epsilon^{N_0+1-|\alpha|}|(\partial_t^\alpha \rho_1)(\frac{x}{\epsilon})| \le C_1\epsilon,$$

which implies that the left-hand-side of (1.5.2) is zero.

**Exercise 1.5.6.** *Let $u \in \mathscr{S}'(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. The distribution $u$ is said to be homogeneous with degree $\lambda$ if for all $t > 0$, $u(t\cdot) = t^\lambda u(\cdot)$. Prove that the distribution $u$ is homogeneous of degree $\lambda$ if and only if Euler's equation is satisfied, namely*

$$\sum_{1\le j\le n} x_j \partial_{x_j} u = \lambda u. \tag{1.5.3}$$

*Answer.* A distribution $u$ on $\mathbb{R}^n$ is homogeneous of degree $\lambda$ means:

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n), \forall t > 0, \qquad \langle u(y), \varphi(y/t)t^{-n}\rangle = t^\lambda \langle u(x), \varphi(x)\rangle,$$

which is equivalent to $\forall \varphi \in C_c^\infty(\mathbb{R}^n), \forall s > 0, \langle u(y), \varphi(sy)s^{n+\lambda}\rangle = \langle u(x), \varphi(x)\rangle$, also equivalent to

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n), \qquad \frac{d}{ds}\big(\langle u(y), \varphi(sy)s^{n+\lambda}\rangle\big) = 0 \quad \text{on } s > 0. \tag{1.5.4}$$

The differentiability property is easy to derive[14] and that

$$\langle u(y), \varphi(sy)s^{n+\lambda}\rangle = \langle u(x), \varphi(x)\rangle \quad \text{at } s = 1.$$

As a consequence, we get that the homogeneity of degree $\lambda$ of $u$ is equivalent to

$$\forall s > 0, \quad \langle u(y), s^{n+\lambda-1}\big((n+\lambda)\varphi(sy) + \sum_{1\le j\le n} (\partial_j\varphi)(sy)sy_j\big)\rangle = 0,$$

also equivalent to $0 = \langle u(y), (n + \lambda + \sum_{1\le j\le n} y_j\partial_j)\big(\varphi(sy)\big)\rangle$ and by the definition of the differentiation of a distribution, it is equivalent to

$$(n+\lambda)u - \sum_{1\le j\le n} \partial_j(y_j u) = 0,$$

which is (1.5.3) by Leibniz' rule.

[14] We have for $s > 0$,

$$\varphi((s+h)y) - \varphi(sy) = \varphi'(sy)hy + \int_0^1 (1-\theta)\varphi''\big((s+\theta h)y\big)d\theta h^2y^2.$$

It is enough to prove that for $\sigma$ in a neighborhood $V$ of $s$, the function $y \mapsto \varphi^{(l)}(\sigma y)$ is bounded in $\mathscr{S}(\mathbb{R}^n)$. This is obvious, choosing for instance $V = (s/2, 2s)$.

**Exercise 1.5.7.**
(1) *Prove that the Dirac mass at* 0 *in* $\mathbb{R}^n$ *is homogeneous of degree* $-n$.
(2) *Prove that if* $T$ *is an homogeneous distribution of degree* $\lambda$, *then* $\partial_x^\alpha T$ *is also homogeneous with degree* $\lambda - |\alpha|$.
(3) *Prove that the distribution* $\operatorname{pv}(\frac{1}{x})$ *is homogeneous of degree* $-1$ *as well as*

$$1/(x \pm i0).$$

(4) *For* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > -1$ *we define the* $L^1_{loc}(\mathbb{R})$ *functions*

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases} \qquad \chi_+^\lambda = \frac{x_+^\lambda}{\Gamma(\lambda+1)}. \tag{1.5.5}$$

*Prove that the distributions* $\chi_+^\lambda$ *and* $x_+^\lambda$ *are homogeneous of degree* $\lambda$.

*Answer.* (1) We have for $t > 0$

$$\langle \delta_0(tx), \varphi(x)\rangle = \langle \delta_0(y), \varphi(y/t)t^{-n}\rangle = t^{-n}\varphi(0) = t^{-n}\langle \delta_0, \varphi\rangle.$$

(2) Taking the derivative of the Euler equation (1.5.3), we get

$$\partial_{x_k} u + \sum_{1 \le j \le k} x_j \partial_{x_j}\partial_{x_k} u - \lambda \partial_{x_k} u = 0,$$

proving that $\partial_{x_k} u$ is homogeneous of degree $\lambda - 1$ and the result by iteration.
(3) It follows immediately from the definition (1.2.25) that the distribution $\operatorname{pv}(\frac{1}{x})$ is homogeneous of degree $-1$. The same is true for the distributions $\frac{1}{x \pm i0}$ as it is clear from

$$\frac{1}{x \pm i0} = \frac{d}{dx}\big(\operatorname{Log}(x \pm i0)\big) = \frac{d}{dx}\big(\ln|x| \pm i\pi \check{H}(x)\big) = \operatorname{pv}\frac{1}{x} \mp i\pi\delta_0. \tag{1.5.6}$$

(4) The distributions $\chi_+^\lambda$ and $x_+^\lambda$ are homogeneous of degree $\lambda$ and by an analytic continuation argument, we can prove that $\chi_+^\lambda$ may be defined for any $\lambda \in \mathbb{C}$, is an homogeneous distribution of degree $\lambda$ and satisfies

$$\chi_+^\lambda = (\frac{d}{dx})^k(\chi_+^{\lambda+k}), \quad \chi_+^{-k} = \delta_0^{(k-1)}, \quad k \in \mathbb{N}^*.$$

**Exercise 1.5.8.** *Let* $(u_j)_{1\le j\le m}$ *be non-zero homogeneous distributions on* $\mathbb{R}^n$ *with distinct degrees* $(\lambda_j)_{1\le j\le m}$ ($j \ne k$ *implies* $\lambda_j \ne \lambda_k$). *Prove that they are independent in the complex vector space* $\mathscr{S}'(\mathbb{R}^n)$.

*Answer.* We assume that $m \ge 2$ and that there exists some complex numbers $(c_j)_{1\le j\le m}$ such that $\sum_{1\le j\le m} c_j u_j = 0$. Then applying the (Euler) operator

$$\mathcal{E} = \sum_{1\le j\le m} x_j \partial_{x_j},$$

we get for all $k \in \mathbb{N}$, $0 = \sum_{1\le j\le m} c_j \mathcal{E}^k(u_j) = \sum_{1\le j\le m} c_j \lambda_j^k u_j$. We consider now the Vandermonde matrix $m \times m$

$$V_m = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \dots & & & \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \dots & \lambda_m^{m-1} \end{pmatrix}, \quad \det V_m = \prod_{1\le j<k\le m} (\lambda_k - \lambda_j) \ne 0.$$

We note that for $\varphi \in C_c^\infty(\mathbb{R}^n)$, and $X \in \mathbb{C}^m$ given by

$$X = \begin{pmatrix} c_1\langle u_1, \varphi\rangle \\ c_2\langle u_2, \varphi\rangle \\ \dots\dots\dots \\ c_m\langle u_m, \varphi\rangle \end{pmatrix},$$

we have $V_m X = 0$, so that $X = 0$, i.e. $\forall j, \forall \varphi \in C_c^\infty(\mathbb{R}^n), c_j\langle u_j, \varphi\rangle = 0$, i.e. $c_j u_j = 0$ and since $u_j$ is not the zero distribution, we get the sought conclusion $c_j = 0$ for all $j$.

**Exercise 1.5.9.** *Let $T \in \mathscr{S}'(\mathbb{R}^n)$ be a homogeneous distribution of degree $m$. Prove that its Fourier transform is a homogeneous distribution of degree $-m-n$.*

*Answer.* We check

$$(\xi \cdot D_\xi)\hat{T} = -\xi \cdot \widehat{xT} = -(\widehat{D_x \cdot xT}) = -\frac{n}{2i\pi}\hat{T} - \frac{1}{2i\pi}(\widehat{x \cdot \partial_x T}) = -\frac{(n+m)}{2i\pi}\hat{T},$$

so that Euler's equation $\xi \cdot \partial_\xi \hat{T} = -(n+m)\hat{T}$ is satisfied.

**Exercise 1.5.10.** *Let $u \in \mathscr{S}'(\mathbb{R}^n)$ such that $\nabla u = (\partial_1 u, \dots, \partial_n u) = 0$. Prove that $u$ is a constant.*

*Answer.* For all $j$, we have $\xi_j \hat{u}(\xi) = 0$ and since a polynomial is a multiplier of $\mathscr{S}$, we have also $|\xi|^2\hat{u}(\xi) = 0$, which implies that $\operatorname{supp} \hat{u} \subset \{0\}$. From Exercise 1.5.5, we find that $\hat{u}$ is a linear combination of derivatives of the Dirac mass at 0 and (1.2.18) implies along with (1.2.20) that $u$ is a polynomial. Now a polynomial with a vanishing gradient is a constant (use Taylor's formula).

## 1.6 Appendix: The Complex Logarithm

### Logarithm on $\mathbb{C}\backslash\mathbb{R}_-$

The set $\mathbb{C}\backslash\mathbb{R}_-$ is star-shaped with respect to 1, so that we can define the principal determination of the logarithm for $z \in \mathbb{C}\backslash\mathbb{R}_-$ by the formula

$$\operatorname{Log} z = \oint_{[1,z]} \frac{d\zeta}{\zeta} = \int_0^1 \frac{(z-1)dt}{(1-t)+tz}. \tag{1.6.1}$$

The function Log is holomorphic on $\mathbb{C}\backslash\mathbb{R}_-$ and we have $\operatorname{Log} z = \ln z$ for $z \in \mathbb{R}_+^*$ and by analytic continuation

$$e^{\operatorname{Log} z} = z = e^{\operatorname{Re}\operatorname{Log} z} e^{i \operatorname{Im}\operatorname{Log} z}, \qquad \begin{cases} |z| &= e^{\operatorname{Re}\operatorname{Log} z}, \\ \operatorname{Arg} z &= \operatorname{Im}\operatorname{Log} z, \end{cases}$$

for $z \in \mathbb{C}\backslash\mathbb{R}_-$. For $z = re^{i\theta}, |\theta| < \pi$, we have for $r > 0$,

$$\operatorname{Log}(re^{i\theta}) = \oint_{[1,re^{i\theta}]} \frac{d\zeta}{\zeta} = \ln r + \int_0^\theta \frac{ire^{it}}{re^{it}} dt = \ln r + i\theta, \quad \operatorname{Im}\operatorname{Log} z = \theta.$$

We get also by analytic continuation, that $\operatorname{Log} e^z = z$ for $|\operatorname{Im} z| < \pi$. Note also that for $|z| < 1$, we have

$$\operatorname{Log}(1+z) = z \int_0^1 \frac{dt}{1+tz} = \sum_{k\ge 0} z(-1)^k \frac{z^k}{k+1} = \sum_{l\ge 1} (-1)^{l+1} \frac{z^l}{l}. \tag{1.6.2}$$

Note that we have also for $|z| = 1, z \neq -1$,

$$\operatorname{Log}(1+z) = z \int_0^1 \frac{dt}{1+tz} = z \int_0^1 \lim_N \Big( \sum_{0\le k\le N} (-1)^k t^k z^k \Big) dt.$$

Since with $z = e^{i\theta}, |\theta| < \pi, t \in [0,1]$,

$$\Big| \sum_{0\le k\le N} (-1)^k t^k z^k = \frac{1 + (-1)^N (tz)^{1+N}}{1+tz} \Big| \le \frac{2}{|1+tz|} = \frac{2}{\sqrt{1 + 2t\cos\theta + t^2}}$$
$$\le \frac{2\mathbf{1}\{\cos\theta \ge 0\}}{\sqrt{1+t^2}} + \frac{2\mathbf{1}\{-1 < \cos\theta \le 0\}}{\sqrt{1-\cos^2\theta}} \in L^1([0,1]_t),$$

so that Lebesgue's dominated convergence implies

$$\operatorname{Log}(1+z) = z \lim_N \sum_{0\le k\le N} (-1)^k \frac{z^k}{k+1},$$

implying that (1.6.2) holds as well for $|z| = 1, z \neq -1$. We consider the following open subset of $\mathbb{C}$

$$\{z \in \mathbb{C}, \exp z \notin \mathbb{R}_-^*\} = \{z \in \mathbb{C}, \operatorname{Im} z \not\equiv \pi (2\pi)\}$$
$$= \cup_{k\in\mathbb{Z}} \underbrace{\{z \in \mathbb{C}, (2k-1)\pi < \operatorname{Im} z < (2k+1)\pi\}}_{\omega_k}.$$

Let $k \in \mathbb{Z}$. On the open set $\omega_k$, the function $z \mapsto \operatorname{Log}(\exp z) - z$ is holomorphic with a null derivative. As a result for $z \in \omega_k$,

$$\operatorname{Log}(\exp z) - z = \operatorname{Log}\big(\exp(2ik\pi)\big) - 2ik\pi = \ln(1) - 2ik\pi = -2ik\pi,$$
$$\text{i.e. } \operatorname{Log}(\exp z) = z - 2ik\pi.$$

We sum-up these results as follows.

**Theorem 1.6.1.** *For $z \in \mathbb{C}\backslash\mathbb{R}_-$, we define* $\operatorname{Log} z$ *by* (1.6.1). *This is an holomorphic function on $\mathbb{C}\backslash\mathbb{R}_-$, with derivative $1/z$, and* Log *coincides with* ln *on $\mathbb{R}_+^*$.*

$$\textit{For } z \in \mathbb{C}\backslash\mathbb{R}_-,\ e^{\operatorname{Log} z} = z = re^{i\theta},$$
$$r = |z| = e^{\operatorname{Re}\operatorname{Log} z},\ \theta = \operatorname{Arg} z = \operatorname{Im}\operatorname{Log} z \in (-\pi, \pi). \tag{1.6.3}$$
$$\textit{For } k \in \mathbb{Z}, z \in \mathbb{C},\ (2k-1)\pi < \operatorname{Im} z < (2k+1)\pi, \quad \operatorname{Log}(e^z) = z - 2ik\pi. \tag{1.6.4}$$
$$\textit{For } z \in \mathbb{C}\backslash\{-1\}, |z| \le 1, \quad \operatorname{Log}(1+z) = \sum_{l\ge 1}(-1)^{l+1}\frac{z^l}{l}. \tag{1.6.5}$$

### Logarithm of a nonsingular symmetric matrix

Let $\Upsilon_+$ be the set of symmetric nonsingular $n \times n$ matrices with complex entries and nonnegative real part. The set $\Upsilon_+$ is star-shaped with respect to the Id: for $A \in \Upsilon_+$, the segment $[1, A] = \big((1-t)\operatorname{Id} + tA\big)_{t\in[0,1]}$ is obviously made with symmetric matrices with nonnegative real part which are invertible, since for $0 \le t < 1$, $\operatorname{Re}\big((1-t)\operatorname{Id} + tA\big) \ge (1-t)\operatorname{Id} > 0$ and for $t = 1$, $A$ is assumed to be invertible[15]. We can now define for $A \in \Upsilon_+$

$$\operatorname{Log} A = \int_0^1 (A - I)\big(I + t(A - I)\big)^{-1} dt. \tag{1.6.6}$$

We note that $A$ commutes with $(I + sA)$ (and thus with $\operatorname{Log} A$), so that, for $\theta > 0$,

$$\frac{d}{d\theta}\operatorname{Log}(A + \theta I) = \int_0^1 \big(I + t(A + \theta I - I)\big)^{-1} dt - \int_0^1 (A + \theta I - I)t\big(I + t(A + \theta I - I)\big)^{-2} dt,$$

and since

$$\frac{d}{dt}\Big\{\big(I + t(A + \theta I - I)\big)^{-1}\Big\} = -\big(I + t(A + \theta I - I)\big)^{-2}(A + \theta I - I),$$

we obtain by integration by parts $\frac{d}{d\theta}\operatorname{Log}(A + \theta I) = (A + \theta I)^{-1}$. As a result, we find that for $\theta > 0, A \in \Upsilon_+$, since all the matrices involved are commuting,

$$\frac{d}{d\theta}\left((A + \theta I)^{-1} e^{\operatorname{Log}(A+\theta I)}\right) = 0,$$

[15] If $A$ is a $n \times n$ symmetric matrix with complex entries such that $\operatorname{Re} A$ is positive definite, then $A$ is invertible: if $AX = 0$, then,

$$0 = \langle AX, \bar X\rangle = \langle A \operatorname{Re} X, \operatorname{Re} X\rangle + \langle A \operatorname{Im} X, \operatorname{Im} X\rangle + \overbrace{\langle A \operatorname{Re} X, -i \operatorname{Im} X\rangle + \langle Ai \operatorname{Im} X, \operatorname{Re} X\rangle}^{=0 \text{ since } A \text{ symmetric}}$$

and taking the real part give $\langle \operatorname{Re} A \operatorname{Re} X, \operatorname{Re} X\rangle + \langle \operatorname{Re} A \operatorname{Im} X, \operatorname{Im} X\rangle = 0$, implying $X = 0$ from the positive-definiteness of $\operatorname{Re} A$.

so that, using the limit $\theta \to +\infty$, we get[16] that

$$\forall A \in \Upsilon_+, \forall \theta > 0, \; e^{\operatorname{Log}(A+\theta I)} = (A + \theta I),$$

and by continuity

$$\forall A \in \Upsilon_+, \quad e^{\operatorname{Log} A} = A, \quad \text{which implies} \quad \det A = e^{\operatorname{trace} \operatorname{Log} A}. \tag{1.6.7}$$

Using (1.6.7), we can define for $A \in \Upsilon_+$,

$$(\det A)^{-1/2} = e^{-\frac{1}{2} \operatorname{trace} \operatorname{Log} A} = |\det A|^{-1/2} e^{-\frac{i}{2} \operatorname{Im}(\operatorname{trace} \operatorname{Log} A)}. \tag{1.6.8}$$

- When $A$ is a positive definite matrix, $\operatorname{Log} A$ is real-valued and $(\det A)^{-1/2} = |\det A|^{-1/2}$.
- When $A = -iB$ where $B$ is a real nonsingular symmetric matrix, we note that $B = PD{}^tP$ with $P \in O(n)$ and $D$ diagonal. We see directly on the formulas (1.6.6), (1.6.1) that

  $$\operatorname{Log} A = \operatorname{Log}(-iB) = P(\operatorname{Log}(-iD)){}^tP, \quad \operatorname{trace} \operatorname{Log} A = \operatorname{trace} \operatorname{Log}(-iD),$$

  and thus, with $(\mu_j)$ the (real) eigenvalues of $B$, we have $\operatorname{Im}(\operatorname{trace} \operatorname{Log} A) = \operatorname{Im} \sum_{1 \le j \le n} \operatorname{Log}(-i\mu_j)$, where the last Log is given by (1.6.1). Finally we get,

  $$\operatorname{Im}(\operatorname{trace} \operatorname{Log} A) = -\frac{\pi}{2} \sum_{1 \le j \le n} \operatorname{sign} \mu_j = -\frac{\pi}{2} \operatorname{sign} B,$$

  where $\operatorname{sign} B$ is the signature of $B$. As a result, we have when $A = -iB$, $B$ real symmetric nonsingular matrix

  $$(\det A)^{-1/2} = |\det B|^{-1/2} e^{i\frac{\pi}{4} \operatorname{sign} B}. \tag{1.6.9}$$

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[16] We have $e^{\operatorname{Log}(A+\theta)} = (A + \theta) B_A$ and with $\tau = \theta - 1$,

$$e^{\operatorname{Log}(A+\theta)} e^{-\ln \theta} = e^{C_\theta}, \quad C_\theta = A \int_0^1 (1 + tA + t\tau)^{-1} (1 + t\tau)^{-1} dt.$$

For $t, \tau \in \mathbb{R}_+$, the matrix $1 + tA + t\tau$ is invertible (see the footnote on page 34) and we have

$$\operatorname{Re}\langle (1 + tA + t\tau)X, X \rangle \ge (1 + t\tau)\|X\|^2, \text{ so that this implies } \|(1 + tA + t\tau)X\| \ge (1 + t\tau)\|X\|$$

and thus $\|(1 + tA + t\tau)^{-1}\| \le (1 + t\tau)^{-1}$. We get

$$\|C_\theta\| \le \|A\| \int_0^1 (1 + t\tau)^{-2} dt = \frac{\|A\|}{1 + \tau} \Longrightarrow \lim_{\theta \to +\infty} C_\theta = 0$$
$$\Longrightarrow B_A = \lim_{\theta \to +\infty} (A + \theta) B_A e^{-\ln \theta} = \lim_{\theta \to +\infty} e^{\operatorname{Log}(A+\theta)} e^{-\ln \theta} = \lim_{\theta \to +\infty} e^{C_\theta} = I.$$

# Chapter 2

# Basic Convolution Inequalities: Young, Hardy-Littlewood-Sobolev

## 2.1 The Banach algebra $L^1(\mathbb{R}^n)$

Let $u, v \in C_c(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$, the mapping $y \mapsto u(x-y)v(y)$ is continuous with compact support $\subset \operatorname{supp} v$. We may thus consider

$$(u * v)(x) = \int_{\mathbb{R}^n} u(x-y)v(y)dy. \tag{2.1.1}$$

We shall say that $u * v$ is the convolution of $u$ with $v$. For a given $x$, the change of variables $y' = x - y$ show that $u * v = v * u$. We see easily that $u * v$ is continuous and moreover if $x \notin \operatorname{supp} u + \operatorname{supp} v$, then for all $y \in \operatorname{supp} v$, $x - y \notin \operatorname{supp} u$ (otherwise $x = x - y + y \in \operatorname{supp} u + \operatorname{supp} v$) so that for all $y \in \mathbb{R}^n, u(x-y)v(y) = 0$. As a result, $(\operatorname{supp} u + \operatorname{supp} v)^c \subset \{u * v = 0\}$ and thus $\{u * v \neq 0\} \subset \operatorname{supp} u + \operatorname{supp} v$. Since $\operatorname{supp} u + \operatorname{supp} v$ is compact (as a sum of compact sets), we have

$$\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v = \{x + y\}_{\substack{x \in \operatorname{supp} u \\ y \in \operatorname{supp} v}} \tag{2.1.2}$$

and $u * v \in C_c(\mathbb{R}^n)$. Moreover convolution is associative, since for $u, v, w \in C_c(\mathbb{R}^n)$, we have

$$\big((u * v) * w\big)(x) = \int_{\mathbb{R}^n} (u * v)(x-y)w(y)dy = \iint_{\mathbb{R}^n \times \mathbb{R}^n} u(x-y-z)v(z)w(y)dydz$$
$$= \iint_{\mathbb{R}^n \times \mathbb{R}^n} u(x-z)v(z-y)w(y)dydz = \int_{\mathbb{R}^n} u(x-z)(v*w)(z)dz = \big(u*(v*w)\big)(x).$$

**Proposition 2.1.1.** *The binary operation of $C_c(\mathbb{R}^n)$ given by $(u, v) \mapsto u * v$ is associative, commutative and distributive with respect to addition and such that*

$$\|u * v\|_{L^1(\mathbb{R}^n)} \leq \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^1(\mathbb{R}^n)}. \tag{2.1.3}$$

*Proof.* The estimate is the only point to be proven. For $u, v \in C_c(\mathbb{R}^n)$, we have

$$\|u * v\|_{L^1(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} u(x-y)v(y)dy \right| dx \le \iint_{\mathbb{R}^n \times \mathbb{R}^n} |u(x-y)||v(y)|dydx$$
$$= \|u\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |v(y)|dy = \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^1(\mathbb{R}^n)}.$$

With $u_0(x) = \exp -\pi|x|^2$ we have $\|u_0\|_{L^1(\mathbb{R}^n)} = 1$ and

$$\|u_0 * u_0\|_{L^1(\mathbb{R}^n)} = \int |(u_0 * u_0)(x)|dx = \iint e^{-\pi|x-y|^2 - \pi|y|^2} dydx = 1,$$

proving that the estimate (2.1.3) is optimal. □

**Proposition 2.1.2.** *Let $k \in \mathbb{N}$, $\varphi \in C_c^k(\mathbb{R}^n)$ and let $u \in L^1_{loc}(\mathbb{R}^n)$ (i.e. $\forall K$ compact, $u\mathbf{1}_K \in L^1(\mathbb{R}^n)$). We define*

$$(\varphi * u)(x) = \int_{\mathbb{R}^n} \varphi(x-y)u(y)dy. \tag{2.1.4}$$

*The function $\varphi * u$ belongs to $C^k(\mathbb{R}^n)$ and if $u \in L^1(\mathbb{R}^n)$, then $\varphi * u$ belongs to $L^1(\mathbb{R}^n)$ and is such that $\|\varphi * u\|_{L^1(\mathbb{R}^n)} \le \|\varphi\|_{L^1(\mathbb{R}^n)} \|u\|_{L^1(\mathbb{R}^n)}$. Moreover, we have* $\operatorname{supp}(\varphi * u) \subset \operatorname{supp} \varphi + \operatorname{supp} u$, *where the support of $u$ is defined by*

$$\operatorname{supp} u = \{x \in X, \nexists V \in \mathscr{V}_x, u_{|V} = 0, \text{-}a.e.\}, \tag{2.1.5}$$

*Proof.* Let $x \in \mathbb{R}^n$ be given. The function $y \mapsto u(y)\varphi(x-y)$ is supported in $x - \operatorname{supp} \varphi = \{x - z\}_{z \in \operatorname{supp} \varphi}$, a compact set (since $\operatorname{supp} \varphi$ is compact). Since $\varphi$ is bounded, the function $y \mapsto u(y)\varphi(x-y)$ belongs to $L^1_{\text{comp}}(\mathbb{R}^n)$, so that (2.1.4) makes sense. We see that $\varphi * u$ belongs to $C^k(\mathbb{R}^n)$: indeed, we have

$$|\varphi^{(k)}(x-y)u(y)| \le |u(y)|\mathbf{1}_{\operatorname{supp}\varphi}(x-y) \sup |\varphi^{(k)}|$$

so that for $K$ compact, since $K - \operatorname{supp} \varphi = \{x - z\}_{x \in K, z \in \operatorname{supp} \varphi}$ is also compact, we have

$$\sup_{x \in K} |\varphi^{(k)}(x-y)u(y)| \le |u(y)|\mathbf{1}_{K - \operatorname{supp}\varphi}(y) \sup |\varphi^{(k)}| \in L^1(\mathbb{R}^n_y).$$

Whenever $u \in L^1(\mathbb{R}^n)$, the inequality on $L^1$-norms is proven as (2.1.3).

Let us prove now the inclusion of supports. Since $\operatorname{supp} \varphi$ is compact and $\operatorname{supp} u$ is closed the set $\operatorname{supp} u + \operatorname{supp} \varphi$ is closed: if $\lim_k (y_k + z_k) = x$, with $y_k \in \operatorname{supp} u$, $z_k \in \operatorname{supp} \varphi$, extracting a subsequence, we get $\lim_l z_{k_l} = z \in \operatorname{supp} \varphi$ and $\lim_l (y_{k_l} + z_{k_l}) = x$, so that the sequence $y_{k_l}$ is converging and since $\operatorname{supp} u$ is closed $\operatorname{supp} u \ni \lim_l y_{k_l} = x - z$, proving $x \in \operatorname{supp} u + \operatorname{supp} \varphi$. We consider now the open set $V_0 = (\operatorname{supp} u + \operatorname{supp} \varphi)^c$. For all $y \in \mathbb{R}^n$, we have

$$V_0 - y \subset (\operatorname{supp} \varphi)^c \quad \text{or} \quad y \notin \operatorname{supp} u, \tag{2.1.6}$$

otherwise, we could find $y_0$ such that $V_0 - y_0 \cap (\operatorname{supp} \varphi) \neq \emptyset$ and $y_0 \in \operatorname{supp} u$. This would imply the existence of $x \in V_0$ such that $x - y_0 \in \operatorname{supp} \varphi$ and thus

$$V_0 \ni x = x - y_0 + y_0 \in \operatorname{supp} \varphi + \operatorname{supp} u = V_0^c,$$

which is impossible. As a result (2.1.6) implies that for $x \in V_0$, and $y \in \mathbb{R}^n$, we have $\varphi(x-y) = 0$ or $y \notin \operatorname{supp} u$. Since the domain of integration in (2.1.4) is $\operatorname{supp} u$, this implies $(\varphi * u)(x) = 0$ and $(\operatorname{supp} u + \operatorname{supp} \varphi)^c \subset \big(\operatorname{supp}(\varphi * u)\big)^c$, that is the sought result. □

**Proposition 2.1.3.** *Let $\Omega$ be an open set of $\mathbb{R}^n$, let $u \in L^1_{loc}(\Omega)$ and let $V$ be open $\subset \Omega$. Then*

$$u_{|_V} = 0 \Longleftrightarrow \forall \varphi \in C_c(V), \int u(x)\varphi(x)dx = 0.$$

*N.B.* This result is important for distribution theory: a function in $L^1_{loc}(\Omega)$ can be viewed as a Radon measure on $\Omega$, i.e. a continuous linear form on $C_c(\Omega)$. For $u \in L^1_{loc}(\Omega)$, we define the linear form $l_u$

$$C_c(\Omega) \ni \varphi \mapsto l_u(\varphi) = \int_\Omega \varphi(x)u(x)dx,$$

which is continuous since

$$|\int_\Omega \varphi(x)u(x)dx| \le \sup|\varphi(x)| \int_{\operatorname{supp}\varphi} |u(x)|dx.$$

This proposition proves that the mapping $u \mapsto l_u$ is injective.

*Proof of the proposition.* The condition is obviously necessary. Let us prove that it is sufficient. Let $K$ be a compact set included in $V$ and let $\chi_K \in C_c(V;[0,1])$, $\chi_K = 1$ on $K$. With

$$\rho \in C_c^\infty(\mathbb{R}^n;\mathbb{R}_+), \int \rho(x)dx = 1,\ \operatorname{supp}\rho = \{\|x\| \le 1\},\ \epsilon > 0,\ \rho_\epsilon(\cdot) = \rho(\cdot/\epsilon)\epsilon^{-n},$$

$$\text{we obtain} \quad (\rho_\epsilon * \chi_K u)(x) = \int u(y) \overbrace{\chi_K(y)\rho_\epsilon(x-y)}^{\in C_c(V)} dy = 0.$$

As a consequence, we have

$$\begin{aligned}\|\chi_K u\|_{L^1(\mathbb{R}^n)} \\ &\le \|\chi_K u - \varphi\|_{L^1(\mathbb{R}^n)} + \|\varphi - \varphi * \rho_\epsilon\|_{L^1(\mathbb{R}^n)} + \|\varphi * \rho_\epsilon - \chi_K u * \rho_\epsilon\|_{L^1(\mathbb{R}^n)} \\ &\le 2\|\chi_K u - \varphi\|_{L^1(\mathbb{R}^n)} + \|\varphi - \varphi * \rho_\epsilon\|_{L^1(\mathbb{R}^n)}. \quad (2.1.7)\end{aligned}$$

**Lemma 2.1.4.** *Let $\varphi \in C_c^k(\mathbb{R}^n)$. Then $\varphi * \rho_\epsilon \in C_c^\infty(\mathbb{R}^n)$ and $\varphi * \rho_\epsilon \to \varphi$ in $C_c^k(\mathbb{R}^n)$ when $\epsilon$ goes to 0.*

*Proof of the lemma.* We have indeed $(\varphi * \rho_\epsilon)(x) = \int \varphi(x - \epsilon y)\rho(y)dy$ so that

$$|(\varphi * \rho_\epsilon)(x) - \varphi(x)| \le \int \rho(y)|\varphi(x-\epsilon y) - \varphi(x)|dy \le \sup_{|x_1 - x_2| \le \epsilon} |\varphi(x_1) - \varphi(x_2)|$$

which goes to 0 with $\epsilon$. Similar estimates hold for derivatives of order $\le k$, and moreover we have $\operatorname{supp}(\varphi * \rho_\epsilon) \subset \operatorname{supp}\varphi + \epsilon\mathbb{B}^n \subset \operatorname{supp}\varphi + \epsilon_0\mathbb{B}^n$ for $\epsilon \le \epsilon_0$, yielding the lemma. □

We go on with the proof of Proposition 2.1.3. From (2.1.7) and Lemma 2.1.4, we obtain

$$\|\chi_K u\|_{L^1(\mathbb{R}^n)} \le 2 \inf_{\varphi\in C_c(V)} \|\chi_K u - \varphi\|_{L^1(\mathbb{R}^n)} = 0,$$

since $\chi_K u \in L^1(V)$. Thus we have $\chi_K u = 0$ for all compact sets $K \subset V$, and since $\chi_K = 1$ on $K$, and $V$ is a countable union of compact sets, we find that $u = 0$ a.e. on $V$. □

**Theorem 2.1.5.** *There exists a unique bilinear mapping*

$$\begin{array}{ccc} L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) & \to & L^1(\mathbb{R}^n) \\ (u, v) & \mapsto & u * v \end{array}$$

*such that if $u, v \in C_c(\mathbb{R}^n)$, $u * v$ is the convolution of $u$ and $v$ and*

$$\|u * v\|_{L^1(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}\|v\|_{L^1(\mathbb{R}^n)}.$$

*The space $L^1(\mathbb{R}^n)$ is a commutative Banach algebra*[1] *for addition and convolution.*

*Proof.* Uniqueness: if $\star$ is another mapping with the same properties, $u, v \in L^1(\mathbb{R}^n)$, $\varphi, \psi \in C_c(\mathbb{R}^n)$

$$u \star v - u * v =$$
$$(u - \varphi) \star v + \varphi \star (v - \psi) + \varphi \star \psi - (u - \varphi) * v - \varphi * (v - \psi) - \varphi * \psi,$$

using $\varphi * \psi = \varphi \star \psi$, and with $L^1(\mathbb{R}^n)$ norms,

$$\|u \star v - u * v\| \le 2\|u - \varphi\|\|v\| + 2\|v - \psi\|\|\varphi\|.$$

The density of $C_c(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$ and the above inequality entail $u * v = u \star v$. To prove existence, we consider sequences $(\varphi_k), (\psi_k)$ in $C_c(\mathbb{R}^n)$, converging in $L^1(\mathbb{R}^n)$: it is easily proven that $\varphi_k * \psi_k$ are Cauchy sequences since (with $L^1(\mathbb{R}^n)$ norms),

$$\|\varphi_{k+l} * \psi_{k+l} - \varphi_k * \psi_k\| \le \|\varphi_{k+l} - \varphi_k\|\|\psi_{k+l}\| + \|\psi_{k+l} - \psi_k\|\|\varphi_k\|$$

Moreover, using the same inequality, we prove that the limit does not depend on the choice of the sequences $\varphi_k, \psi_\kappa$ but only on their limits. □

**Proposition 2.1.6.** *Let $u, v \in L^1(\mathbb{R}^n)$. Then for almost all $x$*

$$\int |u(x - y)v(y)|dy \ < +\infty.$$

*Defining $h(x) = \int u(x - y)v(y)dy$, we have $h \in L^1(\mathbb{R}^n)$,*

$$\|h\|_{L^1(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}\|v\|_{L^1(\mathbb{R}^n)} \quad \textit{and } h = u * v\ .$$

[1] A complex Banach space $B$ equipped with a multiplication $*$ which is associative, distributive with respect to the addition, such that for $\lambda \in \mathbb{C}$ and $x, y \in B$, $(\lambda x) * y = \lambda(x * y) = x * (\lambda y)$ and so that $\|x * y\| \le \|x\|\|y\|$ is called a Banach algebra. When the multiplication is commutative the Banach algebra is said to be commutative. When the multiplication has a unit element, the Banach algebra is said to be unital.

*Proof.* We consider the measurable function $F$ on $\mathbb{R}^{2n}$, given by $F(x,y) = u(x-y)v(y)$. We have

$$\int \left( \int |F(x,y)| dx \right) dy = \int \left( \int |u(x-y)| dx \right) |v(y)| dy$$
$$= \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^1(\mathbb{R}^n)} < +\infty.$$

As a result $F \in L^1(\mathbb{R}^{2n})$ and Fubini Theorem implies that

$$h(x) = \int F(x,y) dy$$

is an $L^1$ function of $x$. We have also proven that $\|h\|_{L^1(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^1(\mathbb{R}^n)}$. Since for $u, v \in C_c(\mathbb{R}^n)$, we have $h = u * v$, Theorem 2.1.5 yields the conclusion. □

**Lemma 2.1.7.** *The Banach algebra $L^1(\mathbb{R}^n)$ is not unital.*

*Proof.* Let us assume that $L^1(\mathbb{R}^n)$ has a unit $\nu$. We would have for all $x \in \mathbb{R}^n$, $e^{-\pi|x|^2} = \int e^{-\pi|x-y|^2} \nu(y) dy$ and thus for all $\xi \in \mathbb{R}^n$,

$$(\dagger) \qquad \int e^{-\pi|x|^2} e^{-2i\pi x\cdot\xi} dx = \int e^{-\pi|x|^2} e^{-2i\pi x\cdot\xi} dx \int e^{-2i\pi y\cdot\xi} \nu(y) dy.$$

Claim: for $\tau \in \mathbb{R}$,

$$\int_{\mathbb{R}} e^{-\pi t^2} e^{-2i\pi t\tau} dt = e^{-\pi\tau^2}. \tag{2.1.8}$$

To prove this claim, we note that

$$F(\tau) = \int_{\mathbb{R}} e^{-\pi t^2} e^{-2i\pi t\tau} e^{\pi\tau^2} dt = \int_{\mathbb{R}} e^{-\pi(t+i\tau)^2} dt,$$

so that $F'(\tau) = \int_{\mathbb{R}} \frac{d}{idt}\left(e^{-\pi(t+i\tau)^2}\right) dt = 0$ and $F(\tau) = F(0) = 1$, proving the Claim. Applying this to ($\dagger$), we get $e^{-\pi|\xi|^2} = e^{-\pi|\xi|^2} \int e^{-2i\pi y\cdot\xi} \nu(y) dy$. Thanks to the Riemann-Lebesgue Lemma 1.1.1, $\xi \mapsto \int e^{-2i\pi y\cdot\xi} \nu(y) dy$ is a continuous function with limit 0 at infinity, so we cannot have $\int e^{-2i\pi y\cdot\xi} \nu(y) dy = 1$ which is a consequence of the previous equality. □

## 2.2 Young's inequality

**Lemma 2.2.1.** *Let $(X, \mathcal{M}, \mu)$ be a measure space where $\mu$ is a $\sigma$-finite positive measure. Let $1 \le r \le \infty, 1/r + 1/r' = 1$. For $u \in L^r(\mu), w \in L^{r'}(\mu)$, the product $uw$ belongs to $L^1(\mu)$. Moreover we have*

$$\|u\|_{L^r(\mu)} = \sup_{\|w\|_{L^{r'}(\mu)}=1} |\langle u, w \rangle|, \quad \textit{with} \quad \langle u, w \rangle = \int_X u\bar{w} d\mu.$$

*Proof.* The first statement follows from Hölder's inequality. Also that inequality implies for $\|w\|_{L^{r'}} = 1$

$$\left|\int_X u\bar{w}d\mu\right| \le \|u\|_{L^r(\mu)} \Longrightarrow \|u\|_{L^r(\mu)} \ge \sup_{\|w\|_{L^{r'}(\mu)}=1} \left|\int_X u\bar{w}d\mu\right|.$$

*We assume first that* $1 < r < +\infty$. Taking $w = \alpha|u|^{r-1}$, with $u = \alpha|u|$, $|\alpha| \equiv 1$ (we define $\alpha = u/|u|$ on $\{u \neq 0\}$, $\alpha = 1$ on $\{u = 0\}$: $\alpha$ is easily seen to be a measurable function), we find for $u \neq 0$ in $L^r$,

$$\|w\|_{L^{r'}}^{r'} = \int_X |u|^{(r-1)r'=r} d\mu = \|u\|_{L^r}^r > 0,$$

and $\int_X u\bar{w} = \int_X u\bar{\alpha}|u|^{r-1} = \int_X |u|\alpha\bar{\alpha}|u|^{r-1} = \|u\|_{L^r}^r$. We obtain thus

$$\langle u, w/\|w\|_{L^{r'}}\rangle = \|u\|_{L^r}^{r-\frac{r}{r'}=r(1-\frac{1}{r'})=1},$$

proving the result.
*We assume now* $r = 1$. We take $w = \mathbf{1}_{u\neq 0}\frac{u}{|u|}$, so that we find for $u \neq 0$ in $L^1$,

$$\|w\|_{L^\infty} = 1, \quad \int_X u\bar{w}d\mu = \int |u|d\mu = \|u\|_{L^1}, \quad \text{proving the result in that case.}$$

*We assume* $r = +\infty, \mu(X) < +\infty$. Let $u \in L^\infty(\mu), u \neq 0$, and let $\epsilon > 0$: then we have

$$+\infty > \mu\Bigl(\underbrace{\{x \in X, |u(x)| \ge \|u\|_{L^\infty(\mu)} - \epsilon\}}_{G_\epsilon}\Bigr) > 0.$$

We define for $\epsilon \in (0, \|u\|_{L^\infty(\mu)})$, $w = \frac{\bar{u}\mathbf{1}_{G_\epsilon}}{|u|\mu(G_\epsilon)}$, so that $\|w\|_{L^1(\mu)} = 1$. We have also

$$\langle u, w\rangle = \int_X |u|\frac{\mathbf{1}_{G_\epsilon}}{\mu(G_\epsilon)}d\mu \ge \|u\|_{L^\infty(\mu)} - \epsilon,$$

so that $\sup_{\|w\|_{L^1}=1} |\langle u, w\rangle| \ge \|u\|_{L^\infty(\mu)} - \epsilon$. Since the latter is true for all $\epsilon > 0$, this gives the result.
*We assume* $r = +\infty$, $\mu$ *σ-finite.* Let $X = \cup_{\mathbb{N}} X_N, \mu(X_N) < +\infty$. We may assume that the sequence $(X_N)_{N\in\mathbb{N}}$ is increasing. Let $u \in L^\infty(\mu), u \neq 0$. We define for $\epsilon \in (0, \|u\|_{L^\infty(\mu)})$,

$$G_{\epsilon,N} = \{x \in X_N, |u(x)| \ge \|u\|_{L^\infty(\mu)} - \epsilon\}.$$

Since $G_\epsilon = \cup_{N\in\mathbb{N}} G_{\epsilon,N} = \{x \in X, |u(x)| \ge \|u\|_{L^\infty(\mu)} - \epsilon\}$ which has a positive measure, Proposition 1.4.4 (2) in [11] implies

$$\lim_N \mu(G_{\epsilon,N}) = \mu(G_\epsilon) > 0 \Longrightarrow \exists N_\epsilon, \forall N \ge N_\epsilon, \mu(G_{\epsilon,N}) > 0.$$

We define $w = \frac{\bar{u}\mathbf{1}_{G_{\epsilon,N_\epsilon}}}{|u|\mu(G_{\epsilon,N_\epsilon})}$, so that $\|w\|_{L^1(\mu)} = 1$, and we have

$$\langle u, w\rangle = \int_X |u|\frac{\mathbf{1}_{G_{\epsilon,N_\epsilon}}}{\mu(G_{\epsilon,N_\epsilon})}d\mu \ge \|u\|_{L^\infty(\mu)} - \epsilon,$$

proving the result in that case as well. The proof of the lemma is complete. □

**Theorem 2.2.2** (Young's inequality). *Let $p, q, r \in [1, +\infty]$ such that*

$$1 - \frac{1}{r} = 1 - \frac{1}{p} + 1 - \frac{1}{q}. \tag{2.2.1}$$

*Then for $u, v \in C_c(\mathbb{R}^n)$, we have*

$$\|u * v\|_{L^r(\mathbb{R}^n)} \le \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^q(\mathbb{R}^n)}. \tag{2.2.2}$$

*Moreover the bilinear mapping $C_c(\mathbb{R}^n)^2 \ni (u, v) \mapsto u * v \in L^r(\mathbb{R}^n)$ can be extended to a bilinear mapping from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ satisfying (2.2.2).*

*Proof.* (1) *We note first that if $r = 1$,* then $p = q = 1$ and the inequality is already proven as well as the unique extension property.
(2) *Moreover if $r = +\infty$,* then $1/p + 1/q = 1$, the requested inequality is

$$\|u * v\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^q(\mathbb{R}^n)},$$

which follows immediately from Hölder's inequality. The extension property holds obviously for $1 \le p, q < +\infty$. If $p = +\infty = r$, then $q = 1$ and

$$(u * v)(x) = \int u(x - y) v(y) dy,$$

and $(u, v) \mapsto u * v$ is a bilinear continuous mapping from $L^\infty \times L^1$ into $L^\infty$ satisfying (2.2.2).
(3) *We may thus assume that $r \in ]1, +\infty[$.* If $p = +\infty$ (resp. $q = +\infty$), we have $1 + 1/r = 1/q$ (resp. $1 + 1/r = 1/p$), so that $r = +\infty$, a case now excluded. If $p = 1$ we have $q = r$ ; if $q = r = 1$, the inequality is proven. We thus may assume that $1 \le p < +\infty$, $1 < q, r < +\infty$. Let $w \in C_c(\mathbb{R}^n)$. We consider

$$(u * v * w)(0) = \int (u * v)(y) w(-y) dy = \iint u(y - x) v(x) w(-y) dy dx,$$

we define

$$t = \frac{1}{p}, \quad s = \frac{1}{q}, \quad \sigma = 1 - \frac{1}{r}, \quad u_0 = |u|^p, v_0 = |v|^q, w_0 = |w|^{1/\sigma},$$

and we find

$$(\sharp) \qquad |(u * v * w)(0)| \le \iint u_0^t(y - x) v_0^s(x) w_0^\sigma(-y) dy dx.$$

We note that

$$1 - t + 1 - s = \sigma, \text{ i.e. } 1 - t + 1 - s + 1 - \sigma = 1, \quad 1 - t, 1 - s, 1 - \sigma \ge 0.$$

**Lemma 2.2.3.** *Let $u_0, v_0, w_0$ be nonnegative functions in $L^1(\mathbb{R}^n)$ with norm 1. Let $s, t, \sigma \in [0, 1]$ such that $1 - t + 1 - s + 1 - \sigma = 1$. Then*

$$\iint u_0^t(y - x) v_0^s(x) w_0^\sigma(-y) dy dx \le 1.$$

*Proof of the lemma.* We have for $u_0(y-x), v_0(x), w_0(-y)$ positive,

$$t\,\mathrm{Log}\,u_0(y-x)+s\,\mathrm{Log}\,v_0(x)+\sigma\,\mathrm{Log}\,w_0(-y)$$
$$=\left[(1-t)\underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{a_1}+(1-s)\underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{a_2}+(1-\sigma)\underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{a_3}\right]\cdot\underbrace{\begin{pmatrix}\mathrm{Log}\,u_0(y-x)\\ \mathrm{Log}\,v_0(x)\\ \mathrm{Log}\,w_0(-y)\end{pmatrix}}_{L}.$$

Consequently, we obtain, using the convexity of the exponential function,

$$\begin{aligned}u_0^t(y-x)v_0^s(x)w_0^\sigma(-y)&\\ &=\exp\big[(1-t)(a_1\cdot L)+(1-s)(a_2\cdot L)+(1-\sigma)(a_3\cdot L)\big]\\ &\le(1-t)\exp(a_1\cdot L)+(1-s)\exp(a_2\cdot L)+(1-\sigma)\exp(a_3\cdot L),\end{aligned}$$

so that

$$\begin{aligned}\iint u_0^t(y-x)v_0^s(x)w_0^\sigma(-y)dydx\le&\\ \iint\Big\{(1-t)v_0(x)w_0(-y)&+(1-s)u_0(y-x)w_0(-y)\\ &+(1-\sigma)u_0(y-x)v_0(x)\Big\}dydx=1,\end{aligned}\tag{2.2.3}$$

concluding the proof of the lemma. □

Going back to the proof of the theorem, we note that the previous lemma and (♯) imply

$$\begin{aligned}|(u*v*w)(0)|\le\iint\Big\{(1-t)v_0(x)w_0(-y)+(1-s)u_0(y-x)w_0(-y)&\\ +(1-\sigma)u_0(y-x)v_0(x)\Big\}dydx.&\end{aligned}\tag{2.2.4}$$

We get thus with $1/r+1/r'=1, \check{w}(x)=w(-x)$, $\langle u,v\rangle=\int u\bar{v}$,

$$|\langle u*v,\check{w}\rangle|\le(1-t)\|v\|_{L^q}^q\|w\|_{L^{r'}}^{r'}+(1-s)\|u\|_{L^p}^p\|w\|_{L^{r'}}^{r'}+(1-\sigma)\|u\|_{L^p}^q\|v\|_{L^q}^q.$$

Let us assume $\|u\|_{L^p}=\|v\|_{L^q}=\|w\|_{L^{r'}}=1$. We have then $|\langle u*v,\check{w}\rangle|\le 1$ so that by homogeneity,

$$|\langle u*v,w\rangle|\le\|u\|_{L^p}\|v\|_{L^q}\|w\|_{L^{r'}}.\tag{2.2.5}$$

Since we have assumed that $r\in(1,+\infty]$, we know that $r'\in[1,+\infty)$ and $C_c(\mathbb{R}^n)$ is dense in $L^{r'}(\mathbb{R}^n)$. Inequality (2.2.5) implies for $u,v,w\in C_c(\mathbb{R}^n)$, $W\in L^{r'}(\mathbb{R}^n)$,

$$\begin{aligned}\Big|\int\underbrace{(u*v)}_{\substack{C_c(\mathbb{R}^n)\\ \subset L^r(\mathbb{R}^n)}}(x)\,\underbrace{\overline{W(x)}}_{L^{r'}(\mathbb{R}^n)}\,dx\Big|&\le|\langle u*v,W-w\rangle|+|\langle u*v,w\rangle|\\ &\le\|u*v\|_{L^r}\|W-w\|_{L^{r'}}+\|u\|_{L^p}\|v\|_{L^q}\|w\|_{L^{r'}}.\end{aligned}$$

As a result for $u, v \in C_c(\mathbb{R}^n), W \in L^{r'}(\mathbb{R}^n)$, and $\epsilon > 0$, there exists $w \in C_c(\mathbb{R}^n)$ such that $\|W - w\|_{L^{r'}} \le \epsilon$ and thus

$$|\langle u * v, W\rangle| \le \epsilon \|u * v\|_{L^r} + \|u\|_{L^p}\|v\|_{L^q}(\|W\|_{L^{r'}} + \epsilon),$$

which implies $|\langle u * v, W\rangle| \le \|u\|_{L^p}\|v\|_{L^q}\|W\|_{L^{r'}}$ and from Lemma 2.2.1 this gives $\|u * v\|_{L^r} \le \|u\|_{L^p}\|v\|_{L^q}$.

To prove that the mapping $(u, v) \mapsto u * v$ can be continuously extended from $C_c(\mathbb{R}^n)^2$ into $L^r(\mathbb{R}^n)$ to a continuous mapping from $L^p \times L^q$ into $L^r$, we may assume that $p, q \in [1, +\infty)$. For $(u, v) \in L^p \times L^q$ and $(u_k, v_k)$ sequences in $C_c(\mathbb{R}^n)$ converging towards $u, v$ respectively in $L^p, L^q$, we note that the sequence $(u_k * v_k)$ is a Cauchy sequence in $L^r$ since

$$\begin{aligned}\|u_{k+l} * v_{k+l} - u_k * v_k\|_{L^r} &= \|(u_{k+l} - u_k) * v_{k+l} + u_k * (v_{k+l} - v_k)\|_{L^r} \\ &\le \|u_{k+l} - u_k\|_{L^p}\|v_{k+l}\|_{L^q} + \|v_{k+l} - v_k\|_{L^q}\|u_k\|_{L^p},\end{aligned}$$

and the numerical sequences $(\|v_k\|_{L^q})_k, (\|v_k\|_{L^q})_k$ are bounded. We may define $u * v$ for $(u, v) \in L^p \times L^q$ as the limit in $L^r$ of $u_k * v_k$. That limit does not depend on the approximating sequences, thanks to the same inequality: with $\tilde{u}_k, \tilde{v}_k$ other approximating sequences, we have

$$u_k * v_k - \tilde{u}_k * \tilde{v}_k = (u_k - \tilde{u}_k) * v_k + \tilde{u}_k * (v_k - \tilde{v}_k),$$

and thus $\|u_k * v_k - \tilde{u}_k * \tilde{v}_k\|_{L^r} \le \|u_k - \tilde{u}_k\|_{L^p}\|v_k\|_{L^q} + \|\tilde{u}_k\|_{L^p}\|v_k - \tilde{v}_k\|_{L^q}$, entailing that $\lim_k u_k * v_k = \lim_k \tilde{u}_k * \tilde{v}_k$ in $L^r$. □

There is a more constructive approach to the definition of the convolution product between $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ for $p, q, r$ satisfying (2.2.1). The case $r = +\infty$ is settled directly by Hölder's inequality. We assume in the sequel that $1 \le r < +\infty$.

Let $u \in L^p(\mathbb{R}^n), v \in L^q(\mathbb{R}^n)$, both non-negative functions. Then the function $(x, y) \mapsto u(y - x)v(x)$ is measurable and Tonelli Theorem implies that

$$(u * v)(y) = \int u(y - x)v(x)dx$$

is a measurable non-negative function of $y$. Moreover choosing $w(y) = \mathbf{1}_{\mathbb{B}^n}(y/k)$, inequalities (2.2.4), (2.2.5) entail that $\int_{|y|\le k}(u * v)(y)dy$ is finite for all $k$. As a result the non-negative function $u * v$ is locally integrable (thus almost everywhere finite). We use now Lemma 2.2.1: for $B$ with finite measure and $\lambda > 0$

$$\left(\int_{B\cap\{y,(u*v)(y)\le\lambda\}} \big((u * v)(y)\big)^r dy\right)^{1/r} = \sup_{\substack{w\ge 0\\ \|w\|_{L^{r'}}=1}} \int_{B\cap\{y,(u*v)(y)\le\lambda\}} (u * v)(y)w(y)dy,$$

and inequality (2.2.5) implies

$$\int_{B\cap\{y,(u*v)(y)\le\lambda\}} \big((u*v)(y)\big)^r dy \le \|u\|^r_{L^p(\mathbb{R}^n)}\|v\|^r_{L^q(\mathbb{R}^n)},$$

which proves that for $u, v$ non-negative respectively in $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ for $p, q, r$ satisfying (2.2.1), we find that $u*v$ belongs to $L^r(\mathbb{R}^n)$ and (2.2.2) holds. Now if $u, v$ are respectively in $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$, we may write

$$u = (\operatorname{Re} u)_+ - (\operatorname{Re} u)_- + i(\operatorname{Im} u)_+ - i(\operatorname{Im} u)_-,$$

and define $u * v = (\operatorname{Re} u)_+ * (\operatorname{Re} v)_+ + \dots$ The bilinearity is obvious as well as the continuity $L^p * L^q \subset L^r$. To obtain the inequality (2.2.2) , we use again inequalities (2.2.4), (2.2.5). We sum-up our discussion.

**Definition 2.2.4.** Let $p, q, r \in [1, +\infty]$ satisfying (2.2.1). For $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, we define

$$(u * v)(y) = \int u(y-x)v(x)dx$$

which is a locally integrable function (thus a.e. finite).

**Theorem 2.2.5.** *Let $p, q, r \in [1, +\infty]$ satisfying (2.2.1). The mapping*

$$L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \ni (u, v) \mapsto u * v \in L^r(\mathbb{R}^n),$$

*is continuous and (2.2.2) holds.*

## 2.3 Weak $L^p$ spaces

**Definition 2.3.1.** Let $p \in [1, +\infty)$. We define the *weak-$L^p(\mathbb{R}^n)$ space* $L^p_w(\mathbb{R}^n)$ as the set of measurable functions $u : \mathbb{R}^n \to \mathbb{C}$ such that

$$\sup_{t>0} t^p \lambda_n\big(\{x \in \mathbb{R}^n, |u(x)| > t\}\big) = \Omega_p(u) < +\infty, \tag{2.3.1}$$

where $\lambda_n$ is the Lebesgue measure on $\mathbb{R}^n$.

*Remark* 2.3.2. (1) We have $L^p(\mathbb{R}^n) \subset L^p_w(\mathbb{R}^n)$: let $u \in L^p(\mathbb{R}^n)$. We have for $t > 0$

$$t^p\lambda_n(\{|u| > t\}) = \int_{|u|>t} t^p dx \le \int_{|u|>t} |u(x)|^p dx \le \|u\|^p_{L^p(\mathbb{R}^n)},$$

so that, with $\Omega_p(u)$ defined in (2.3.1), we have

$$\Omega_p(u) \le \|u\|^p_{L^p(\mathbb{R}^n)}. \tag{2.3.2}$$

(2) For $x \in \mathbb{R}^n$, we define $v_p(x) = |x|^{-n/p}$ (a measurable function). For $R > 0$, we have

$$\int_{B(0,R)} v_p(x)^p dx = \int_{B(0,R)} |x|^{-n} dx \ge |\mathbb{S}^{n-1}| \int_0^R dr/r = +\infty,$$

so that $v_p$ is not in $L^p_{\text{loc}}(\mathbb{R}^n)$. On the other hand, we have for $t > 0$

$$t^p \lambda_n\big(\{|x|^{-n/p} > t\}\big) = t^p t^{-\frac{p}{n} n} \lambda_n(\mathbb{B}^n) = \lambda_n(\mathbb{B}^n),$$

so that $v_p$ belongs to $L^p_w(\mathbb{R}^n)$.

**Lemma 2.3.3.** *Let $p \in [1, +\infty)$. Then $L^p_w(\mathbb{R}^n)$ is a $\mathbb{C}$-vector space. For $u, v \in L^p_w(\mathbb{R}^n), \alpha \in \mathbb{C}$, we have*

$$\big(\Omega_p(\alpha u)\big)^{\frac{1}{p}} = |\alpha| \big(\Omega_p(u)\big)^{\frac{1}{p}}, \qquad \big(\Omega_p(u+v)\big)^{\frac{1}{p}} \le 2^{\frac{1}{p}} \big(\Omega_p(u)^{\frac{1}{p}} + \Omega_p(v)^{\frac{1}{p}}\big).$$

*Remark* 2.3.4. The mapping $L^p_w(\mathbb{R}^n) \ni u \mapsto \big(\Omega_p(u)\big)^{\frac{1}{p}}$ is a *quasi-norm*: it satisfies the properties of separation and homogeneity, but fails to satisfy the triangle inequality, although a substitute is available with a constant $2^{1/p} > 1$. We shall see below (Lemma 2.3.5) that when $p \in (1, +\infty)$, we can find a true norm equivalent to this quasi-norm.

*Proof of the lemma.* Let $\alpha, \beta$ be non-zero complex numbers and let $u, v \in L^p_w$. Since for $t > 0$, $|\alpha u| \le t/2$ and $|\beta v| \le t/2$ imply $|\alpha u + \beta v| \le t$, we have

$$\{|\alpha u + \beta v| > t\} \subset \{|\alpha u| > t/2\} \cup \{|\beta v| > t/2\},$$

and thus

$$\begin{aligned} &t^p \lambda_n\big(\{|\alpha u + \beta v| > t\}\big) \\ &\quad\le (2|\alpha|)^p \left(\frac{t}{2|\alpha|}\right)^p \lambda_n\big(\{|\alpha u| > t/2\}\big) + (2|\beta|)^p \left(\frac{t}{2|\beta|}\right)^p \lambda_n(\{|\beta v| > t/2\}) \\ &\qquad\qquad \le (2|\alpha|)^p \Omega_p(u) + (2|\beta|)^p \Omega_p(v), \end{aligned}$$

so that $\Omega_p(\alpha u + \beta v) \le (2|\alpha|)^p \Omega_p(u) + (2|\beta|)^p \Omega_p(v) < +\infty$, proving the vector space property. The first homogeneity equality in the lemma is obvious, let us prove the second one. We may of course assume that both quantities $\Omega_p(u), \Omega_p(v)$ are positive ($\Omega_p(u) = 0$ implies $u = 0$ a.e.). Let $\theta \in (0, 1)$. Since for $t > 0$, $|u| \le (1-\theta)t$ and $|\beta v| \le \theta t$ imply $|u + v| \le t$, we have

$$\{|u+v| > t\} \subset \{|u| > t(1-\theta)\} \cup \{|v| > t\theta\},$$

so that

$$\begin{aligned} &t^p \lambda_n(\{|u+v| > t\}) \\ &\quad\le (1-\theta)^{-p} t^p (1-\theta)^p \lambda_n(\{|u| > t(1-\theta)\}) + \theta^{-p} t^p \theta^p \lambda_n(\{|v| > t\theta\}) \\ &\qquad\qquad \le (1-\theta)^{-p} \Omega_p(u) + \theta^{-p} \Omega_p(v). \quad (*) \end{aligned}$$

We consider now the function $(0, 1) \ni \theta \mapsto (1-\theta)^{-p} a + \theta^{-p} b = \phi_{a,b}(\theta)$, where $a, b$ are positive parameters. We have

$$\phi'_{a,b}(\theta) = p(1-\theta)^{-p-1} a - p\theta^{-p-1} b,$$

and the minimum of $\phi$ is attained at $\theta$ such that $(1-\theta)^{-p-1}a = \theta^{-p-1}b$, i.e

$$\frac{\theta}{1-\theta} = (b/a)^{\frac{1}{p+1}}, \quad \text{i.e.} \quad \theta = \frac{(b/a)^{\frac{1}{p+1}}}{1+(b/a)^{\frac{1}{p+1}}} = \frac{b^{\frac{1}{p+1}}}{a^{\frac{1}{p+1}} + b^{\frac{1}{p+1}}},$$

with $\phi_{a,b} = (1-\theta)^{-p}a + \theta^{-p}b = (a^{\frac{1}{p+1}} + b^{\frac{1}{p+1}})^{p+1}$ at this point. We infer from $(*)$ that

$$\big(\Omega_p(u+v)\big)^{\frac{1}{p}} \le \big(\Omega_p(u)^{\frac{1}{p+1}} + \Omega_p(v)^{\frac{1}{p+1}}\big)^{\frac{p+1}{p}} \le 2^{\frac{1}{p}}\big(\Omega_p(u)^{\frac{1}{p}} + \Omega_p(v)^{\frac{1}{p}}\big),$$

where the last inequality comes from the sharp elementary[2]

$$(a^{\frac{1}{p+1}} + b^{\frac{1}{p+1}})^{\frac{p+1}{p}} \le 2^{\frac{1}{p}}\big(a^{\frac{1}{p}} + b^{\frac{1}{p}}\big).$$

□

**Lemma 2.3.5.** *Let $p \in (1,+\infty)$ and let $p'$ be its conjugate exponent. For $u \in L^p_w(\mathbb{R}^n)$, we define*

$$N_p(u) = \sup_{\substack{A \text{ measurable} \\ \text{with finite positive} \\ \text{measure}}} \lambda_n(A)^{-1/p'} \int_A |u(x)|dx. \tag{2.3.3}$$

*Then $N_p$ is a norm on $L^p_w(\mathbb{R}^n)$ which is equivalent to the quasi-norm $\Omega_p(\cdot)^{1/p}$.*

*Proof.* Tonelli's Theorem gives for a measurable subset $A$ of $\mathbb{R}^n$,

$$\int_A |u(x)|dx = \iint \mathbf{1}_A(x)H(|u(x)|-t)H(t)dtdx, \quad \text{with } H = \mathbf{1}_{\mathbb{R}_+}.$$

As a result, for $T \ge 0$ and $A$ measurable with finite measure, we have

$$\begin{aligned}
\int_A |u(x)|dx &= \int_0^{+\infty} \lambda_n\big(A \cap \{|u| > t\}\big)dt \\
&= \int_0^T \lambda_n\big(A \cap \{|u| > t\}\big)dt + \int_T^{+\infty} \lambda_n\big(A \cap \{|u| > t\}\big)dt \\
&\le T\lambda_n(A) + \int_T^{+\infty} \lambda_n\big(\{|u| > t\}\big)dt. \\
&\le T\lambda_n(A) + \int_T^{+\infty} \Omega_p(u)t^{-p}dt = T\lambda_n(A) + \Omega_p(u)\frac{T^{1-p}}{p-1}.
\end{aligned}$$

[2] We have from Hölder's inequality for $a, b$ positive,

$$a^{\frac{1}{p+1}} + b^{\frac{1}{p+1}} \le \Big((a^{\frac{1}{p+1}})^{\frac{p+1}{p}} + (b^{\frac{1}{p+1}})^{\frac{p+1}{p}}\Big)^{\frac{p}{p+1}} (1^{\frac{p+1}{1}} + 1^{\frac{p+1}{1}})^{\frac{1}{p+1}} = 2^{\frac{1}{p+1}}\big(a^{\frac{1}{p}} + b^{\frac{1}{p}}\big)^{\frac{p}{p+1}}.$$

The constant $2^{\frac{1}{p+1}}$ is easily shown to be sharp by taking $a = b$.

We choose $T = \lambda_n(A)^{-1/p}\Omega_p(u)^{1/p}$ and we find

$$\int_A |u(x)|dx \le \lambda_n(A)^{1/p'}\Omega_p(u)^{1/p} + \frac{1}{p-1}\lambda_n(A)^{-\frac{1}{p}+1}\Omega_p(u)^{1+\frac{1}{p}-1}$$
$$= \lambda_n(A)^{1/p'}\Omega_p(u)^{1/p}\frac{p}{p-1},$$

proving

$$N_p(u) \le \frac{p}{p-1}\Omega_p(u)^{1/p}. \tag{2.3.4}$$

For $t > 0$, and $X_k$ measurable with finite measure, we have

$$t^p\lambda_n\big(\{|u| > t\} \cap X_k\big) = t^p \int_{\{|u|>t\}\cap X_k} dx$$
$$\le t^{p-1}\int_{\{|u|>t\}\cap X_k} |u(x)|dx \le t^{p-1}N_p(u)\lambda_n\big(\{|u| > t\} \cap X_k\big)^{1/p'},$$

so that $t\lambda_n\big(\{|u| > t\} \cap X_k\big)^{1/p} \le N_p(u)$. Since $\lambda_n$ is $\sigma$-finite, this implies

$$\Omega_p(u) \le N_p(u)^p. \tag{2.3.5}$$

We see now that $N_p$ is finite $\ge 0$ on $L^p_w$ from (2.3.4). Moreover $N_p(u) = 0$ implies from (2.3.5) that $\lambda_n(\{|u| > t\}) = 0$ for all $t > 0$ and since

$$\{u \ne 0\} = \cup_{n\ge1}\{|u| > 1/n\},$$

we find $u = 0$, a.e. Moreover, for $\alpha \in \mathbb{C}$ and $u \in L^p_w$, we have

$$N_p(\alpha u) = \sup_{\substack{A \text{ measurable} \\ \text{with finite measure}>0}} \lambda_n(A)^{-1/p'}\int_A |\alpha u(x)|dx = |\alpha|N_p(u).$$

Eventually, for $u, v \in L^p_w$ and $A$ measurable with finite measure, we have

$$\lambda_n(A)^{-1/p'}\int_A |u(x) + v(x)|dx$$
$$\le \lambda_n(A)^{-1/p'}\int_A |u(x)|dx + \lambda_n(A)^{-1/p'}\int_A |v(x)|dx \le N_p(u) + N_p(v),$$

which implies $N_p(u+v) \le N_p(u)+N_p(v)$, proving that $N_p$ is a norm on $L^p_w(\mathbb{R}^n)$ and concluding the proof of the lemma. □

**Proposition 2.3.6.** *Let $p \in (1, +\infty)$. Then $L^p_w(\mathbb{R}^n)$ is a Banach space for the norm* (2.3.3).

*Proof.* Let us consider a Cauchy sequence $(u_k)_{k\in\mathbb{N}}$ in $L^p_w(\mathbb{R}^n)$: in particular for every measurable subset $A$ with finite measure, we find that $(u_{k|A})_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^1(A)$, thus convergent with limit $v_A$. Since the Lebesgue measure on $\mathbb{R}^n$ is $\sigma$-finite, we find a measurable function $u$ such that for every $A$ measurable with finite measure, $\lim_k \|u_k - u\|_{L^1(A)} = 0$. We check now for a measurable subset $A$ with finite measure,

$$\begin{aligned}\lambda_n(A)^{-1/p'} &\int_A |u_k(x) - u(x)|dx \\ &\le \lambda_n(A)^{-1/p'} \int_A |u_k(x) - u_l(x)|dx + \lambda_n(A)^{-1/p'} \int_A |u_l(x) - u(x)|dx \\ &\le N_p(u_k - u_l) + \lambda_n(A)^{-1/p'} \|u_l - u\|_{L^1(A)}.\end{aligned}$$

Let $\epsilon > 0$ be given. There exists $N_\epsilon$ such that for $k, l \ge N_\epsilon$, we have $N_p(u_k - u_l) \le \epsilon/2$. We know also that for $l \ge L_{\epsilon,A}$, we have $\lambda_n(A)^{-1/p'} \|u_l - u\|_{L^1(A)} \le \epsilon/2$. We take $k \ge N_\epsilon$ and we choose $l = \max(N_\epsilon, L_{\epsilon,A})$: we find

$$\lambda_n(A)^{-1/p'} \int_A |u_k(x) - u(x)|dx \le \epsilon.$$

As a result $u$ belongs to $L^p_w(\mathbb{R}^n)$ and $N_p(u_k - u) \le \epsilon$ for $k \ge N_\epsilon$, proving the completeness of $L^p_w(\mathbb{R}^n)$. □

## 2.4 The Hardy-Littlewood-Sobolev inequality

We begin with a lemma, following [13].

**Lemma 2.4.1.** *Let $p, q, r > 1$ be real numbers such that*

$$1 - \frac{1}{p} + 1 - \frac{1}{q} = 1 - \frac{1}{r} = \frac{1}{r'}$$

*and let $f, g$ be non-negative measurable functions such that*

$$\|f\|_{L^p(\mathbb{R}^n)} = 1 = \|g\|_{L^{r'}(\mathbb{R}^n)}.$$

*Setting $\tau = n/q$, we define*

$$T_\tau(f, g) = \iint f(x)|x - y|^{-\tau} g(y)dydx$$

*and we have*

$$\begin{aligned}T_\tau(f, g) = \tau \int_{\mathbb{R}^3_+ \times \mathbb{R}^n \times \mathbb{R}^n} & t_3^{-\tau-1} H(t_3 - |x - y|) \\ & H(f(x) - t_1)H(g(y) - t_2)dt_1dt_2dt_3dxdy. \quad (2.4.1)\end{aligned}$$

*Setting for $t_j \ge 0$,*

$$u_1(t_1) = \int_{\mathbb{R}^n} H(f(x) - t_1)dx, \ u_2(t_2) = \int_{\mathbb{R}^n} H(g(y) - t_2)dy, \ u_3(t_3) = \beta_n t_3^n,$$

*with* $\beta_n = |\mathbb{B}^n|$, *and*

$$m(t_1, t_2, t_3) = \max\big(u_1(t_1), u_2(t_2), u_3(t_3)\big),$$

*we have*

$$T_\tau(f, g) \le \tau \int_{\mathbb{R}^3_+} t_3^{-\tau-1} \frac{u_1(t_1)u_2(t_2)u_3(t_3)}{m(t)} dt_1 dt_2 dt_3, \tag{2.4.2}$$

$$p \int_0^{+\infty} t_1^{p-1} u_1(t_1) dt_1 = r' \int_0^{+\infty} t_2^{r'-1} u_2(t_2) dt_2 = 1. \tag{2.4.3}$$

*Proof.* We have for $\tau > 0$,

$$\tau \int_0^{+\infty} t^{-\tau-1} H(t - |x|) dt = \tau \int_{|x|}^{+\infty} t^{-\tau-1} dt = [t^{-\tau}]_{t=+\infty}^{t=|x|} = |x|^{-\tau}$$

and thus

$$\begin{aligned} T_\tau(f, g) &= \iint f(x)|x-y|^{-\tau} g(y) dy dx \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+} f(x) g(y) \tau t_3^{-\tau-1} H(t_3 - |x-y|) dx dy dt_3 \\ &= \tau \int_{\mathbb{R}^3_+ \times \mathbb{R}^n \times \mathbb{R}^n} t_3^{-\tau-1} H(t_3 - |x-y|) H(f(x) - t_1) H(g(y) - t_2) dt_1 dt_2 dt_3 dx dy, \end{aligned}$$

proving (2.4.1). We have thus

$$\begin{aligned} T_\tau(f, g) \le \tau &\int_{\substack{\mathbb{R}^3_+ \times \mathbb{R}^n \times \mathbb{R}^n \\ m(t) = u_3(t_3)}} t_3^{-\tau-1} H(f(x) - t_1) H(g(y) - t_2) dt_1 dt_2 dt_3 dx dy \\ &+ \tau \int_{\substack{\mathbb{R}^3_+ \times \mathbb{R}^n \times \mathbb{R}^n \\ m(t) = u_2(t_2) \\ m(t) > u_3(t_3)}} t_3^{-\tau-1} H(t_3 - |x-y|) H(f(x) - t_1) dt_1 dt_2 dt_3 dx dy \\ &+ \tau \int_{\substack{\mathbb{R}^3_+ \times \mathbb{R}^n \times \mathbb{R}^n \\ m(t) = u_1(t_1) \\ m(t) > \max(u_2(t_2), u_3(t_3))}} t_3^{-\tau-1} H(t_3 - |x-y|) H(g(y) - t_2) dt_1 dt_2 dt_3 dx dy, \end{aligned}$$

so that with

$$\begin{aligned} A_3 &= \{t \in \mathbb{R}^3_+, m(t) = u_3(t_3)\} \\ A_2 &= \{t \in \mathbb{R}^3_+, m(t) = u_2(t_2), m(t) > u_3(t_3)\}, \\ A_1 &= \{t \in \mathbb{R}^3_+, m(t) = u_1(t_1), m(t) > \max(u_2(t_2), u_3(t_3))\}, \end{aligned}$$

we have

$$\begin{aligned} T_\tau(f, g) \le \tau &\int_{A_3} t_3^{-\tau-1} u_1(t_1) u_2(t_2) dt \\ &+ \tau \int_{A_2} t_3^{-\tau-1} \beta_n t_3^n u_1(t_1) dt \\ &+ \tau \int_{A_1} t_3^{-\tau-1} \beta_n t_3^n u_2(t_2) dt \\ &= \tau \int_{\mathbb{R}^3_+} t_3^{-\tau-1} \frac{u_1(t_1) u_2(t_2) u_3(t_3)}{m(t)} dt_1 dt_2 dt_3. \end{aligned}$$

Moreover, we have

$$p\int_0^{+\infty} t_1^{p-1}u_1(t_1)dt_1 = \int_{\mathbb{R}^n}\int_0^{+\infty} pt_1^{p-1}H(f(x)-t_1)dxdt_1 = \int_{\mathbb{R}^n} f(x)^p dx = 1$$

and

$$r'\int_0^{+\infty} t_2^{r'-1}u_2(t_2)dt_2 = \int_{\mathbb{R}^n}\int_0^{+\infty} r't_2^{r'-1}H(g(y)-t_2)dydt_2 = \int_{\mathbb{R}^n} g(y)^{r'}dx = 1,$$

completing the proof of the lemma. □

**Lemma 2.4.2.** *Let $p, q, r, f, g, \tau, T_\tau, \beta_n, u_1, u_2$ as in the previous lemma. Then we have*

$$T_\tau(f,g) \le \frac{n\beta_n^{\tau/n}}{n-\tau}\int_{\mathbb{R}^2_+} \min\big(u_1(t_1)^{1-\frac{\tau}{n}}u_2(t_2), u_1(t_1)u_2(t_2)^{1-\frac{\tau}{n}}\big)dt_1dt_2. \tag{2.4.4}$$

*Proof.* For $t \in \mathbb{R}^3_+$, we set $V(t) = \frac{u_1(t_1)u_2(t_2)u_3(t_3)}{m(t)}$.
*Let us assume that* $u_1(t_1) \ge u_2(t_2)$. In that case we have

$$\begin{aligned}
&\int_0^{+\infty} t_3^{-\tau-1}V(t_1,t_2,t_3)dt_3 = \int_0^{+\infty} t_3^{-\tau-1}\frac{u_1(t_1)u_2(t_2)u_3(t_3)}{\max(u_1(t_1),u_3(t_3))}dt_3 \\
&= u_1(t_1)u_2(t_2)\Big(\int_{\mathbb{R}_+,\beta_n t_3^n \le u_1(t_1)} t_3^{-\tau-1+n}\beta_n dt_3 u_1(t_1)^{-1} + \int_{\mathbb{R}_+,\beta_n t_3^n > u_1(t_1)} t_3^{-\tau-1}dt_3\Big) \\
&= u_1(t_1)u_2(t_2)\beta_n\left(u_1(t_1)^{-1}\left[\frac{t_3^{n-\tau}}{n-\tau}\right]_{t_3=0}^{t_3=u_1(t_1)^{1/n}\beta_n^{-1/n}} + \beta_n^{-1}\left[\frac{t_3^{-\tau}}{\tau}\right]_{t_3=+\infty}^{t_3=u_1(t_1)^{1/n}\beta_n^{-1/n}}\right) \\
&= u_1(t_1)u_2(t_2)\beta_n\Big(u_1(t_1)^{-1+\frac{n-\tau}{n}}\frac{\beta_n^{-1+\frac{\tau}{n}}}{n-\tau} + \tau^{-1}\beta_n^{-1+\frac{\tau}{n}}u_1(t_1)^{-\tau/n}\Big) \\
&= u_1(t_1)^{1-\frac{\tau}{n}}u_2(t_2)\beta_n^{\tau/n}\frac{n}{\tau(n-\tau)}.
\end{aligned}$$

*If we have instead* $u_1(t_1) \le u_2(t_2)$, we find

$$\int_0^{+\infty} t_3^{-\tau-1}V(t_1,t_2,t_3)dt_3 = u_2(t_2)^{1-\frac{\tau}{n}}u_1(t_1)\beta_n^{\tau/n}\frac{n}{\tau(n-\tau)}.$$

From (2.4.2) and the previous estimates, we obtain

$$\begin{aligned}
T_\tau(f,g) &\le \frac{n\beta_n^{\tau/n}}{n-\tau}\int_{\mathbb{R}^2_+} \mathbf{1}\big(u_1(t_1)\ge u_2(t_2)\big)u_1(t_1)^{1-\frac{\tau}{n}}u_2(t_2)^{1-\frac{\tau}{n}}u_2(t_2)^{\frac{\tau}{n}}dt_1dt_2 \\
&\quad + \frac{n\beta_n^{\tau/n}}{n-\tau}\int_{\mathbb{R}^2_+} \mathbf{1}\big(u_1(t_1)\le u_2(t_2)\big)u_1(t_1)^{1-\frac{\tau}{n}}u_2(t_2)^{1-\frac{\tau}{n}}u_1(t_1)^{\frac{\tau}{n}}dt_1dt_2 \\
&\qquad = \frac{n\beta_n^{\tau/n}}{n-\tau}\int_{\mathbb{R}^2_+} u_1(t_1)^{1-\frac{\tau}{n}}u_2(t_2)^{1-\frac{\tau}{n}}\Big(\min\big(u_1(t_1),u_2(t_2)\big)\Big)^{\tau/n}dt_1dt_2,
\end{aligned}$$

which is (2.4.4). □

**Lemma 2.4.3.** *Let $p, q, r, f, g, \tau, T_\tau, \beta_n, u_1, u_2$ as in the previous lemmas. We define*

$$J = \int_{\mathbb{R}_+^2} \min\big(u_1(t_1)^{1-\frac{\tau}{n}} u_2(t_2), u_1(t_1) u_2(t_2)^{1-\frac{\tau}{n}}\big) dt_1 dt_2. \tag{2.4.5}$$

*Then with*

$$J_1 = \int_0^{+\infty} u_1(t_1) \int_0^{t_1^{p/r'}} u_2(t_2)^{1-\frac{\tau}{n}} dt_2 dt_1,\ J_2 = \int_0^{+\infty} u_2(t_2) \int_0^{t_2^{r'/p}} u_1(t_1)^{1-\frac{\tau}{n}} dt_1 dt_2,$$

*we have $J \le J_1 + J_2$. Moreover, we have*

$$J_1 \le \frac{1}{pr'} \left(\frac{p'\tau}{n}\right)^{\tau/n}, \quad J_2 \le \frac{1}{pr'} \left(\frac{r\tau}{n}\right)^{\tau/n}.$$

*Proof.* We have

$$J \le \iint_{0 \le t_1, 0 \le t_2 \le t_1^{p/r'}} \big(u_1(t_1) u_2(t_2)\big)^{1-\frac{\tau}{n}} \min\big(u_1(t_1), u_2(t_2)\big)^{\tau/n} dt_1 dt_2$$
$$+ \iint_{0 \le t_2, 0 \le t_1 \le t_2^{r'/p}} \big(u_1(t_1) u_2(t_2)\big)^{1-\frac{\tau}{n}} \min\big(u_1(t_1), u_2(t_2)\big)^{\tau/n} dt_1 dt_2$$

and thus

$$J \le \int_0^{+\infty} u_1(t_1) \left(\int_0^{t_1^{p/r'}} u_2(t_2)^{1-\frac{\tau}{n}} dt_2\right) dt_1$$
$$+ \int_0^{+\infty} u_2(t_2) \left(\int_0^{t_2^{r'/p}} u_1(t_1)^{1-\frac{\tau}{n}} dt_1\right) dt_2.$$

From Hölder's inequality, since $1 - \frac{\tau}{n} = 1/q'$, we find, choosing $m = \frac{r'-1}{q'}$

$$\int_0^{t_1^{p/r'}} u_2(t_2)^{1-\frac{\tau}{n}} dt_2 = \int_0^{t_1^{p/r'}} t_2^m u_2(t_2)^{1-\frac{\tau}{n}} t_2^{-m} dt_2$$
$$\le \Big( \underbrace{\int_0^{t_1^{p/r'}} t_2^{mq'} u_2(t_2) dt_2}_{=1/r' \text{ from } (2.4.3)} \Big)^{1/q'} \left(\int_0^{t_1^{p/r'}} t_2^{-mq} dt_2\right)^{1/q}.$$

We note also that

$$mq = \frac{r'-1}{q'} q < 1 \iff \frac{r'-1}{q'} < 1/q \iff r' < q' \text{ which holds since } \frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}.$$

As a result, we have

$$J_1 \le \int_0^{+\infty} u_1(t_1) \big(\frac{1}{r'}\big)^{1/q'} \big((t_1^{p/r'})^{1-mq} (1-mq)^{-1}\big)^{1/q} dt_1.$$

Since

$$\frac{p(1-mq)}{r'q} = \frac{p}{r'q}\big(1-\frac{(r'-1)}{q'}q\big) = \frac{p}{r'}\big(1-\frac{r'}{q'}\big) = p\big(\frac{1}{r'}-\frac{1}{q'}\big) = \frac{p}{p'} = p-1,$$

we obtain, using (2.4.3),

$$\begin{aligned}
J_1 &\le \int_0^{+\infty} u_1(t_1)t_1^{p-1}dt_1\big(\frac{1}{r'}\big)^{1/q'}(1-mq)^{-1/q} \\
&= \frac{1}{p}\big(\frac{1}{r'}\big)^{1/q'}(1-mq)^{-1/q} = \frac{1}{pr'}\big(\frac{1}{r'}-\frac{mq}{r'}\big)^{-1/q} = \frac{1}{pr'}\big(\frac{1}{r'}-\frac{q}{q'r}\big)^{-1/q} \\
&= \frac{1}{pr'}\big(\frac{1}{r'}-\frac{(q-1)}{r}\big)^{-1/q} = \frac{1}{pr'}\big(1-\frac{q}{r}\big)^{-1/q} = \frac{1}{pr'}\big(\frac{1}{q}-\frac{1}{r}\big)^{-1/q}q^{-1/q} \\
&= \frac{1}{pr'}\big(\frac{1}{p'}\big)^{-1/q}q^{-1/q} = \frac{1}{pr'}\big(\frac{p'}{q}\big)^{1/q} = \frac{1}{pr'}\big(\frac{p'\tau}{n}\big)^{\tau/n}.
\end{aligned}$$

To estimate $J_2$ from above is analogous: we have, choosing $\mu = \frac{p-1}{q'}$,

$$\begin{aligned}
\int_0^{t_2^{r'/p}} u_1(t_1)^{1-\frac{\tau}{n}}dt_1 &= \int_0^{t_2^{r'/p}} t_1^{\mu}u_1(t_1)^{1-\frac{\tau}{n}}t_1^{-\mu}dt_1 \\
&\le \Big(\underbrace{\int_0^{t_2^{r'/p}} t_1^{\mu q'}u_1(t_1)dt_1}_{=1/p}\Big)^{1/q'}\Big(\int_0^{t_2^{r'/p}} t_1^{-\mu q}dt_1\Big)^{1/q}.
\end{aligned}$$

We check $\mu q < 1$ by the same calculation, exchanging the roles of $p$ and $r'$: $p'$ is replaced by $r$ and $pr'$ replaced by $r'p$ is unchanged. □

**Theorem 2.4.4** (Hardy-Littlewood-Sobolev inequality)**.** *Let $p, q, r \in (1, +\infty)$ such that $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$. There exists $C > 0$ such that, for all $F \in L^p(\mathbb{R}^n)$,*

$$\|(F * |\cdot|^{-n/q})\|_{L^r(\mathbb{R}^n)} \le C\|F\|_{L^p(\mathbb{R}^n)}.$$

*The constant $C$ can be taken as $q'\beta_n^{1/q}\frac{1}{pr'}\left(\left(\frac{p'}{q}\right)^{1/q} + \left(\frac{r}{q}\right)^{1/q}\right)$.*

*Proof.* For $f = |F|/\|F\|_{L^p}, \|g\|_{L^{r'}} = 1$, we have proven from (2.4.4) and Lemma 2.4.3,

$$T_\tau(f,g) \le \frac{n\beta_n^{\tau/n}}{n-\tau}\frac{1}{pr'}\Big(\big(\frac{p'}{q}\big)^{1/q} + \big(\frac{r}{q}\big)^{1/q}\Big) = \beta_n^{1/q}q'\frac{1}{pr'}\Big(\big(\frac{p'}{q}\big)^{1/q} + \big(\frac{r}{q}\big)^{1/q}\Big),$$

providing the sought result. □

## 2.5 Riesz-Thorin Interpolation Theorem

**Theorem 2.5.1** (Hadamard three-lines theorem). *Let $a < b$ be real numbers, let $\Omega = \{z \in \mathbb{C}, a < \operatorname{Re} z < b\}$ and let $f : \overline{\Omega} \to \mathbb{C}$ be a bounded continuous function which is holomorphic on $\Omega$. We define for $x \in [a, b]$,*

$$M(x) = \sup_{y \in \mathbb{R}} |f(x + iy)|.$$

*Then the function $M$ is log-convex on $[a, b]$, i.e.*

$$M(x) \le M(a)^{\frac{b-x}{b-a}} M(b)^{\frac{x-a}{b-a}}. \tag{2.5.1}$$

*N.B.* We note here that this proposition implies in particular that if $f$ vanishes identically on the vertical line $\{\operatorname{Re} z = a\}$ or on $\{\operatorname{Re} z = b\}$, then it should vanish identically on $\Omega$. If $M(a), M(b)$ are both positive, then (2.5.1) reads

$$(\ln M)\big((1-\theta)a + \theta b\big) \le (1-\theta)\ln M(a) + \theta \ln M(b),$$

which means convexity of $\ln M$ on $[a, b]$, i.e. log-convexity. Defining $\ln 0 = -\infty$, we recover the fact that if $f$ vanishes on one vertical line, it vanishes on $\Omega$.

*Proof.* We may of course assume without loss of generality that $a = 0, b = 1$: given $a < b$ real numbers, and $f$ as in the proposition above, we may consider

$$\tilde{f}(z) = f\big((b-a)z + a\big),$$

which is defined on $\{z \in \mathbb{C}, 0 \le \operatorname{Re} z \le 1\}$. If we get the result for $\tilde{f}$, it will read for $\theta \in [0, 1]$

$$\begin{aligned}
\sup_{\{\operatorname{Re}\zeta = a + \theta(b-a) = x\}} |f(\zeta)| &= \sup_{\{\operatorname{Re} z = \theta\}} |\tilde{f}(z)| \\
&\le \big(\sup_{y\in\mathbb{R}} |\tilde{f}(iy)|\big)^{1-\theta} \big(\sup_{y\in\mathbb{R}} |\tilde{f}(1+iy)|\big)^{\theta} \\
&= \big(\sup_{y\in\mathbb{R}} |f(a + (b-a)iy)|\big)^{1-\theta} \big(\sup_{y\in\mathbb{R}} |f(a + b - a + (b-a)iy)|\big)^{\theta} \\
&= \big(\sup_{\operatorname{Re}\zeta = a} |f(\zeta)|\big)^{\frac{b-x}{b-a}} \big(\sup_{\operatorname{Re}\zeta = b} |f(\zeta)|\big)^{\frac{x-a}{b-a}},
\end{aligned}$$

which is the sought result.
*We assume first that $M(0) = M(1) = 1$.* We define for $\epsilon > 0$ the holomorphic function $h_\epsilon$ on $\operatorname{Re} z > -1/\epsilon$ given by

$$h_\epsilon(z) = \frac{1}{1 + \epsilon z}.$$

We note that $\forall z \in \partial\Omega$, $|f(z)h_\epsilon(z)| \le 1$ (in fact $|f(z)| \le 1$ there as well as $h_\epsilon(z)$) and moreover with $C = \sup_{\overline{\Omega}} |f|$, we have for $0 \le \operatorname{Re} z \le 1$, $|\operatorname{Im} z| \ge C/\epsilon$,

$$|f(z)h_\epsilon(z)| \le C|1 + \epsilon z|^{-1} \le C\epsilon^{-1}|\operatorname{Im} z|^{-1} \le 1. \tag{2.5.2}$$

As a result, considering the rectangle $R_\epsilon = \{0 \le \operatorname{Re} z \le 1, |\operatorname{Im} z| \le C/\epsilon\}$, we see that the continuous function $fh_\epsilon : R_\epsilon \to \mathbb{C}$ is bounded above by 1 on the boundary and is holomorphic in the interior. Applying the maximum principle, we obtain that

$$(\sharp) \qquad \forall z \in R_\epsilon, \quad |f(z)h_\epsilon(z)| \le 1.$$

On the other hand if $z \in \bar{\Omega}$ with $|\operatorname{Im} z| > C/\epsilon$, we get from (2.5.2) the same inequality ($\sharp$). Consequently, we have for all $\epsilon > 0$ and all $z \in \bar{\Omega}$, $|f(z)h_\epsilon(z)| \le 1$, which implies the sought result $|f(z)| \le 1$ for $z \in \bar{\Omega}$.

*We assume now that $M(0), M(1)$ are both positive,* and we introduce the function

$$F(z) = M(0)^{-(1-z)}M(1)^{-z}f(z) = f(z)e^{z(\ln M(0) - \ln M(1))}M(0)^{-1}. \tag{2.5.3}$$

The function $F$ is holomorphic on $\Omega = \{0 < \operatorname{Re} z < 1\}$, is bounded on $\bar{\Omega}$ since

$$\sup_{z\in\bar{\Omega}} |F(z)| \le M(0)^{-1}e^{|\ln M(0) - \ln M(1)|}\sup_{\bar{\Omega}} |f|.$$

Moreover, on the vertical lines $\operatorname{Re} z = 0, 1$, $|F|$ is bounded above respectively by

$$M(0)M(0)^{-1} = 1, \qquad M(1)M(0)M(1)^{-1}M(0)^{-1} = 1,$$

so that we may apply the previous result to obtain

$$\forall z \in \bar{\Omega}, \quad |M(0)^{-(1-z)}M(1)^{-z}f(z)| \le 1,$$

which is precisely the sought result.

*We assume now that $M(0) \ge 0, M(1) \ge 0$.* Let $\epsilon > 0$ be given. We introduce the function

$$F_\epsilon(z) = (M(0) + \epsilon)^{-(1-z)}(M(1) + \epsilon)^{-z}f(z). \tag{2.5.4}$$

Then, using the previous result, we obtain

$$\forall \epsilon > 0, \forall z \in \bar{\Omega}, \quad |f(z)| \le |(M(0) + \epsilon)^{(1-z)}(M(1) + \epsilon)^z|,$$

which implies the result, letting $\epsilon \to 0_+$. The proof of the theorem is complete. □

**Theorem 2.5.2** (Riesz-Thorin Interpolation Theorem).
*Let $(X, \mathcal{M}, \mu)$ be a measure space where $\mu$ is a $\sigma$-finite positive measure. Let $p_0, p_1, q_0, q_1 \in [1, +\infty]$ and let $T : L^{p_j}(\mu) \longrightarrow L^{q_j}(\mu), j = 0, 1$, be a linear map such that*

$$\|Tu\|_{L^{q_j}(\mu)} \le M_j\|u\|_{L^{p_j}(\mu)}, \qquad j = 0, 1.$$

*For $\theta \in [0, 1]$ we define $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then $T$ is a bounded linear map from $L^{p_\theta}(\mu)$ into $L^{q_\theta}(\mu)$ and*

$$\forall u \in L^{p_\theta}(\mu), \quad \|Tu\|_{L^{q_\theta}(\mu)} \le M_0^{1-\theta}M_1^\theta\|u\|_{L^{p_\theta}(\mu)}. \tag{2.5.5}$$

*Proof.* We may of course assume that $\theta \in (0,1)$.
[**1**] *Let us first assume that $p_\theta = +\infty$, so that $p_0 = p_1 = +\infty$.*
Let $u$ be a function in $L^\infty(\mu)$: $Tu$ belongs to $L^{q_0}(\mu) \cap L^{q_1}(\mu)$.
**Claim:** *for $\theta \in (0,1)$, we have $L^{q_0}(\mu) \cap L^{q_1}(\mu) \subset L^{q_\theta}(\mu)$.* This is obvious if $q_\theta = +\infty$ (implying $q_0 = q_1 = +\infty$) and if $q_\theta < +\infty$, assuming that $q_0, q_1$ are both finite (and distinct), we find some $t \in (0,1)$ such that

$$q_\theta = (1-t)q_0 + tq_1, \quad \text{so that with } \frac{1}{r} = 1-t,$$

$$\int_X |v|^{q_\theta} d\mu = \int_X |v|^{q_0(1-t)} |v|^{q_1 t} d\mu$$
$$\le \| |v|^{q_0(1-t)} \|_{L^r} \| |v|^{q_1 t} \|_{L^{r'}} = \|v\|_{L^{q_0}}^{q_0(1-t)} \|v\|_{L^{q_1}}^{q_1 t}. \tag{2.5.6}$$

If $q_0 = +\infty, 1 \le q_1 < +\infty$, we have $q_\theta = q_1/\theta$ and

$$\int_X |v|^{q_\theta} d\mu \le \|v\|_{L^\infty}^{q_1(\frac{1}{\theta}-1)} \int_X |v|^{q_1} d\mu, \tag{2.5.7}$$

proving the claim in that case as well.

We find thus that $Tu \in L^{q_\theta}$ and when $q_0, q_1$ are both finite, applying (2.5.6),

$$\|Tu\|_{q_\theta}^{q_\theta} \le \|Tu\|_{q_0}^{q_0(1-t)} \|Tu\|_{q_1}^{q_1 t} \le M_0^{q_0(1-t)} M_1^{q_1 t} \|u\|_\infty^{q_\theta},$$

and since

$$\frac{tq_1}{q_\theta} = \frac{q_\theta - q_0}{q_1 - q_0}\frac{q_1}{q_\theta} = \frac{1 - \frac{q_0}{q_\theta}}{1 - \frac{q_0}{q_1}} = \frac{q_0^{-1} - q_\theta^{-1}}{q_0^{-1} - q_1^{-1}} = \theta, \qquad \text{so that } \frac{(1-t)q_0}{q_\theta} = 1 - \theta,$$

proving (2.5.5). If $q_0 = +\infty, 1 \le q_1 < +\infty$, we have $q_\theta = q_1/\theta$ and applying (2.5.7)

$$\|Tu\|_{q_\theta}^{q_\theta} \le \|Tu\|_{q_0}^{q_1(\frac{1}{\theta}-1)} \|Tu\|_{q_1}^{q_1} \le M_0^{q_1(\frac{1}{\theta}-1)} M_1^{q_1} \|u\|_\infty^{q_\theta},$$

and since

$$\frac{q_\theta - q_1}{q_\theta} = 1 - \frac{q_\theta^{-1}}{q_1^{-1} - q_0^{-1}} = 1 - \theta, \quad \text{so that } \frac{q_1}{q_\theta} = \theta,$$

this implies (2.5.5) in that case as well.
[**2**] *We assume now that $1 \le p_\theta < +\infty$, $q_\theta > 1$.* Let $u$ be a function in $S$, defined by

$$S = \{s : X \to \mathbb{C}, \text{measurable}, s(X) \text{ finite with } \mu\big(\{s \ne 0\}\big) < +\infty\}. \tag{2.5.8}$$

We have

$$u = \sum_{1 \le j \le m} \alpha_j e^{i\phi_j} \mathbf{1}_{A_j}, \quad \alpha_j > 0, \phi_j \in \mathbb{R}, \quad \mu(A_j) < +\infty, \tag{2.5.9}$$

where the $A_j$ are pairwise disjoint elements of $\mathcal{M}$. Then $Tu$ makes sense, belongs to $L^{q_\theta}(\mu)$ and since $S$ is dense in $L^{p_\theta}(\mu)$ (Proposition 3.2.11 in [11]), it is enough to prove that

$$\forall v \in L^{(q_\theta)'}, \quad \left| \int (Tu) v d\mu \right| \le M_0^{1-\theta} M_1^\theta \|u\|_{p_\theta} \|v\|_{(q_\theta)'}. \tag{2.5.10}$$

In fact, if we prove the above inequality, thanks to Lemma 2.2.1, this will imply that $\|Tu\|_{q_\theta} \le M_0^{1-\theta} M_1^\theta \|u\|_{p_\theta}$. Now since $T$ is a linear operator, and $S$ is dense in $L^{p_\theta}(\mu)$, there is a unique extension of $T$ to a bounded linear operator from $L^{p_\theta}(\mu)$ into $L^{q_\theta}(\mu)$ with operator-norm bounded above by $M_0^{1-\theta} M_1^\theta$. To obtain (2.5.10), it is enough to prove that

$$\forall v \in S, \qquad \left|\int (Tu) v d\mu\right| \le M_0^{1-\theta} M_1^\theta \|u\|_{p_\theta} \|v\|_{(q_\theta)'}, \tag{2.5.11}$$

since $q_\theta > 1$ ($S$ is dense in $L^{(q_\theta)'}$). We may thus assume that

$$v = \sum_{1\le k\le N} \beta_k e^{i\psi_k} \mathbf{1}_{B_k}, \quad \beta_k > 0, \psi_k \in \mathbb{R}, \quad \mu(B_k) < +\infty, \tag{2.5.12}$$

where the $B_k$ are pairwise disjoint elements of $\mathcal{M}$. We define the entire functions

$$u(z) = \sum_{1\le j\le m} \alpha_j^{a(z)/a(\theta)} e^{i\phi_j} \mathbf{1}_{A_j}, \quad a(z) = \frac{1-z}{p_0} + \frac{z}{p_1}, \tag{2.5.13}$$

$$v(z) = \sum_{1\le k\le N} \beta_k^{(1-b(z))/(1-b(\theta))} e^{i\psi_k} \mathbf{1}_{B_k}, \quad b(z) = \frac{1-z}{q_0} + \frac{z}{q_1}, \tag{2.5.14}$$

$$F(z) = \int_X \big(Tu(z)\big) v(z) d\mu, \tag{2.5.15}$$

and we note that $a(\theta) = 1/p(\theta), b(\theta) = 1/q(\theta) \in (0,1)$ since $\theta \in (0,1)$. The function $F$ is bounded on $\{z \in \mathbb{C}, 0 \le \operatorname{Re} z \le 1\}$: we have to deal with a finite sum and

$$\operatorname{Re} a(z) \in [0,1], \quad \operatorname{Re}(1-b(z)) \in [0,1].$$

Moreover, for $y \in \mathbb{R}$, we have

$$F(iy) = \int_X T\Big(\sum_{1\le j\le m} \alpha_j^{\frac{a(iy)}{a(\theta)}} e^{i\phi_j} \mathbf{1}_{A_j}\Big)\Big(\sum_{1\le k\le m} \mathbf{1}_{B_k} \beta_k^{\frac{(1-b(iy))}{(1-b(\theta))}} e^{i\psi_k}\Big) d\mu,$$

and thus

$$|F(iy)| \le M_0 \| \sum_{1\le j\le m} \alpha_j^{\frac{a(iy)}{a(\theta)}} e^{i\phi_j} \mathbf{1}_{A_j} \|_{p_0} \| \sum_{1\le k\le m} \mathbf{1}_{B_k} \beta_k^{\frac{(1-b(iy))}{(1-b(\theta))}} e^{i\psi_k} \|_{q_0'}.$$

Since the $(A_j)_{1\le j\le m}$ (and the $(B_k)_{1\le k\le N}$) are pairwise disjoint, we have

$$\begin{aligned} \| \sum_{1\le j\le m} \alpha_j^{\frac{a(iy)}{a(\theta)}} e^{i\phi_j} \mathbf{1}_{A_j} \|_{p_0} &= \| \sum_{1\le j\le m} \alpha_j^{\frac{\operatorname{Re} a(iy)}{a(\theta)}} \mathbf{1}_{A_j} \|_{p_0} = \| \sum_{1\le j\le m} \alpha_j^{\frac{p(\theta)}{p_0}} \mathbf{1}_{A_j} \|_{p_0} \\ &= \Big( \int_X \Big( \sum_{1\le j\le m} \alpha_j^{p(\theta)} \mathbf{1}_{A_j} \Big) d\mu \Big)^{1/p_0} = \Big( \int_X |u(\theta)|^{p(\theta)} d\mu \Big)^{1/p_0} = \|u(\theta)\|_{p(\theta)}^{p_\theta/p_0}, \end{aligned}$$

and

$$\|\sum_{1\le k\le N}\mathbf{1}_{B_k}\beta_k^{\frac{(1-b(iy))}{(1-b(\theta))}}e^{i\psi_k}\|_{q_0'}=\|\sum_{1\le k\le N}\beta_k^{\frac{1-\operatorname{Re}b(iy)}{1-b(\theta)}}\mathbf{1}_{B_k}\|_{q_0'}=\|\sum_{1\le k\le N}\beta_k^{\frac{q'(\theta)}{q_0'}}\mathbf{1}_{B_k}\|_{q_0'}$$

$$=\Big(\int_X\Big(\sum_{1\le k\le N}\beta_k^{q'(\theta)}\mathbf{1}_{B_k}\Big)d\mu\Big)^{1/q_0'}=\Big(\int_X|v(\theta)|^{q'(\theta)}d\mu\Big)^{1/q_0'}=\|v(\theta)\|_{q'(\theta)}^{q_\theta'/q_0'},$$

so that, for $y\in\mathbb{R}$, $|F(iy)|\le M_0\|u(\theta)\|_{p(\theta)}^{p_\theta/p_0}\|v(\theta)\|_{q'(\theta)}^{q_\theta'/q_0'}$. We obtain similarly that

$$|F(1+iy)|\le M_1\|u(\theta)\|_{p(\theta)}^{p_\theta/p_1}\|v(\theta)\|_{q'(\theta)}^{q_\theta'/q_1'}.$$

The last two inequalities and Theorem 2.5.1 imply for $\operatorname{Re}z\in[0,1]$

$$|F(z)|\le\Big(M_0\|u(\theta)\|_{p(\theta)}^{\frac{p_\theta}{p_0}}\|v(\theta)\|_{q'(\theta)}^{\frac{q_\theta'}{q_0'}}\Big)^{1-\operatorname{Re}z}\Big(M_1\|u(\theta)\|_{p(\theta)}^{p_\theta/p_1}\|v(\theta)\|_{q'(\theta)}^{q_\theta'/q_1'}\Big)^{\operatorname{Re}z},$$

so that for $\operatorname{Re}z=\theta$, since

$$\frac{p_\theta}{p_0}(1-\theta)+\frac{p_\theta}{p_1}\theta=1=\frac{q_\theta'}{q_0'}(1-\theta)+\frac{q_\theta'}{q_1'}\theta,$$

we get

$$\Big|\int(Tu)vd\mu\Big|=|F(\theta)|\le M_0^{1-\theta}M_1^\theta\|u\|_{p(\theta)}\|v\|_{q'(\theta)},$$

which is indeed (2.5.11), concluding the proof in this case.
[**3**] *We assume now that* $1\le p_\theta<+\infty$, $q_\theta=1$ *(and thus* $q_0=q_1=1, q_0'=q_1'=+\infty$*).* It is enough to prove (2.5.10)(from Proposition 3.2.11 in [11]), and to get it, (2.5.11) should be modified so that $S$ is replaced by $S_\infty$ (see Proposition 3.2.13 in [11]), meaning that (2.5.12) must be modified so that $\mu(B_k)$ could be $+\infty$. We modify (2.5.14) and take $v(z)=v$. The rest of the proof is unchanged, following case [**2**]. The proof of Theorem 2.5.2 is complete. □

The Riesz-Thorin interpolation theorem appears as a direct consequence of Hadamard's three-lines theorem and is a typical example of a complex interpolation method based on a version of the maximum principle for holomorphic functions on unbounded domains. Of course holomorphic functions in an unbounded domain $\Omega$, continuous in $\bar\Omega$, may fail to satisfy the maximum principle[3]. However Phragmén-Lindelöf principle's is asserting that a maximum principle result holds true, provided we impose some restriction on the growth of the class of functions: Hadamard's three lines theorem, in which we have assumed boundedness for the holomorphic function, is a good example of this technique. We give below some classical consequences of Theorem 2.5.2.

[3] The function $e^z$ on $\Omega=\{z\in\mathbb{C},\operatorname{Re}z>0\}$ is unbounded on $\Omega$ although it has modulus 1 on $\partial\Omega$.

**Theorem 2.5.3** (Generalized Young's inequality). *Let $p, q, r \in [1, +\infty]$ such that (2.2.1) holds. Let $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ be measure spaces where each $\mu_j$ is a $\sigma$-finite positive measure and let $k : X_1 \times X_2 \to \mathbb{C}$ be a measurable mapping (the product $X_1 \times X_2$ is equipped with the $\sigma$-algebra $\mathcal{M}_1 \otimes \mathcal{M}_2$) such that there exists $M \geq 0$ with*

$$\sup_{x_1 \in X_1} \left( \int_{X_2} |k(x_1, x_2)|^p d\mu_2(x_2) \right)^{1/p} \leq M, \tag{2.5.16}$$

$$\sup_{x_2 \in X_2} \left( \int_{X_1} |k(x_1, x_2)|^p d\mu_1(x_1) \right)^{1/p} \leq M. \tag{2.5.17}$$

*The linear operator $L$ defined by*

$$(Lu_2)(x_1) = \int_{X_2} k(x_1, x_2) u_2(x_2) d\mu_2(x_2) \tag{2.5.18}$$

*can be extended to a bounded linear operator from $L^q(\mu_2)$ into $L^r(\mu_1)$ with operator-norm less than $M$.*

*Remark* 2.5.4. The first (resp. second) supremum can be replaced by an esssup in the $\mu_1$ (resp. $\mu_2$) sense. If $p = +\infty$ (which implies $q = 1, r = +\infty$), the hypothesis reads as

$$\text{esssup}_{(x_1,x_2) \in X_1 \times X_2} |k(x_1, x_2)| \leq M = M,$$

and the result in that case is trivial since

$$|(Lu_2)(x_1)| \leq M \|u_2\|_{L^1(\mu_2)} \Longrightarrow \|Lu_2\|_{L^\infty(\mu_1)} \leq M \|u_2\|_{L^1(\mu_2)}.$$

We may thus assume that $1 \leq p < +\infty$. If $q = +\infty$ (which implies $p = 1, r = +\infty$), we get also trivially

$$|(Lu_2)(x_1)| \leq \int_{X_2} |k(x_1, x_2)||u_2(x_2)| d\mu_2(x_2) \leq M \|u_2\|_{L^\infty(\mu_2)}$$
$$\Longrightarrow \|Lu_2\|_{L^\infty(\mu_1)} \leq M \|u_2\|_{L^\infty(\mu_2)}.$$

We may thus assume that $p$ and $q$ are finite. We may define (2.5.18) for $u_2 = \mathbf{1}_{A_2}$, where $A_2 \in \mathcal{M}$, with $\mu_2(A_2) < +\infty$. Then we have

$$\int_{A_2} |k(x_1, x_2)| d\mu_2(x_2) \leq M \|\mathbf{1}_{A_2}\|_{L^{p'}(\mu_2)} \leq M \mu_2(A_2)^{1/p'} < +\infty.$$

As a result for $u_2 \in S_q(\mu_2)$ (the space $S_p(\mu)$ is defined by (2.5.8)), we may define $Lu_2$ as an $L^\infty(\mu_1)$ function. Since for $1 \leq q < +\infty$, $S_q(\mu_2)$ is dense in $L^q(\mu_2)$ (Proposition 3.2.11 in [11]), the statement of Theorem 2.5.3 can be rephrased as follows: the linear operator $L$ defined from $S_q(\mu_2)$ into $L^\infty(\mu_1)$ can be uniquely extended as a bounded linear operator from $L^q(\mu_2)$ into $L^r(\mu_1)$ with operator-norm less than $M$.

*N.B.* Young's inequality (Theorem 2.2.2) is indeed a consequence of the above result, taking $k(x_1, x_2) = a(x_1 - x_2)$ with $x_j \in \mathbb{R}^n$, $\mu_j$ equal to the Lebesgue measure on $\mathbb{R}^n$, $M = \|a\|_{L^p(\mathbb{R}^n)}$.

*Proof of the theorem.* As noted in the above remark, we may assume that $p, q$ are both finite. For $u_2 \in S_q(\mu_2)$ (also if $p' = +\infty$ for $u_2 \in S_\infty(\mu_2)$, where $S_\infty(\mu)$ is defined in Proposition 3.2.13 in [11]), we have

$$\|Lu_2\|_{L^\infty(\mu_1)} \leq M\|u_2\|_{L^{p'}(\mu_2)}. \tag{2.5.19}$$

This implies that $L$ can be extended uniquely as a bounded linear operator from $L^{p'}(\mu_2)$ into $L^\infty(\mu_1)$ so that (2.5.19) holds true. Moreover, for $u_2 \in S_q(\mu_2)$, we have if $p > 1$ (thus $p' < +\infty$),

$$\begin{aligned}\|Lu_2\|_{L^p(\mu_1)} &\underbrace{=}_{\text{Lemma 2.2.1}} \sup_{\substack{\|w\|_{L^{p'}(\mu_1)}=1\\ w\in S_{p'}(\mu_1)}} \left|\int_{X_1}(Lu_2)(x_1)w(x_1)d\mu(x_1)\right| \\ &\leq \sup_{\substack{\|w\|_{L^{p'}(\mu_1)}=1\\ w\in S_{p'}(\mu_1)}} \iint_{X_1\times X_2} |k(x_1,x_2)||u_2(x_2)||w(x_1)|d\mu_1(x_1)d\mu_2(x_2) \\ &\leq M \sup_{\substack{\|w\|_{L^{p'}(\mu_1)}=1\\ w\in S_{p'}(\mu_1)}} \|w\|_{L^{p'}(\mu_1)} \int_{X_2}|u_2(x_2)|d\mu_2(x_2) = M\|u_2\|_{L^1(\mu_2)}.\end{aligned}$$

This implies that if $p > 1$, $L$ can be extended uniquely as a bounded linear operator from $L^1(\mu_2)$ into $L^p(\mu_1)$ so that

$$\|Lu_2\|_{L^p(\mu_1)} \leq M\|u_2\|_{L^1(\mu_2)}. \tag{2.5.20}$$

Applying the Riesz-Thorin interpolation Theorem 2.5.2 to the inequalities (2.5.19)-(2.5.20), we find that the linear operator $L$ sends continuously $L^{\tilde q}(\mu_2)$ into $L^{\tilde r}(\mu_2)$ (with operator norm $M$) with

$$\frac{1}{\tilde q} = \frac{1-\theta}{1} + \frac{\theta}{p'}, \quad \frac{1}{\tilde r} = \frac{1-\theta}{p} + \frac{\theta}{\infty},$$

for all $\theta \in [0,1]$. From (2.2.1), we have $1/p' + 1/q' = 1/r'$ so that $p' \geq r'$ and $1 \leq p \leq r$: thus we may choose

$$[0,1] \ni \theta = 1 - \frac{p}{r} \Longrightarrow \frac{1-\theta}{p} = \frac{1}{r}, \tilde r = r, \quad \frac{1-\theta}{1} + \frac{\theta}{p'} = 1 - \frac{1}{p} + \frac{1}{r} = \frac{1}{q}, \ \tilde q = q.$$

This completes the proof for $p > 1$. Note that if $p = 1$ then $r = q$ (which can be assumed finite from Remark 2.5.4), we have directly

$$\begin{aligned}&\int_{X_1}\left(\int_{X_2}|k(x_1,x_2)||u_2(x_2)|d\mu_2(x_2)\right)^q d\mu_1(x_1) \\ &\leq \int_{X_1}\left(\int_{X_2}\left(|k(x_1,x_2)|^{\frac{1}{q}}|u_2(x_2)|\right)^q d\mu_2(x_2)\right)\left(\int_{X_2}|k(x_1,x_2)|^{\frac{q'}{q'}}d\mu_2(x_2)\right)^{\frac{q}{q'}} d\mu_1(x_1) \\ &\qquad \leq M^{q/q'}\iint_{X_1\times X_2}|k(x_1,x_2)||u_2(x_2)|^q d\mu_2(x_2)d\mu_1(x_1) \\ &\qquad\qquad \leq M^{\frac{q}{q'}+1}\|u_2\|^q_{L^q(\mu_2)},\end{aligned}$$

so that in this case as well, we find that

$$\|Lu_2\|_{L^q(\mu_1)} \le M\|u_2\|_{L^q(\mu_2)}. \tag{2.5.21}$$

The proof of Theorem 2.5.3 is complete. □

## 2.6 Hausdorff-Young Inequality

**Theorem 2.6.1** (Hausdorff-Young). *Let $n \ge 1$ be an integer. The Fourier transform $F$ maps injectively and continuously $L^p(\mathbb{R}^n)$ into $L^{p'}(\mathbb{R}^n)$ for $1 \le p \le 2$ and*

$$\forall u \in L^p(\mathbb{R}^n), \quad \|\hat{u}\|_{L^{p'}(\mathbb{R}^n)} \le \|u\|_{L^p(\mathbb{R}^n)}. \tag{2.6.1}$$

*Proof.* Note first that we have defined the Fourier transformation on the space of tempered distributions (see Definition 1.2.13), so that Proposition 1.2.12(1) provides a definition of the Fourier transform for any function in $L^p(\mathbb{R}^n)$ and that this transformation is injective on $\mathscr{S}'(\mathbb{R}^n)$, since it is an isomorphism (see Theorem 1.2.14). We have seen as well in Theorem 1.2.15 that the Fourier transformation on $L^1(\mathbb{R}^n)$ is given by the explicit formula (1.2.21) and satisfies the inequality

$$\forall u \in L^1(\mathbb{R}^n), \text{we have } \hat{u} \in L^\infty(\mathbb{R}^n) \text{ and } \quad \|\hat{u}\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}.$$

Moreover, Theorem 1.2.16 shows that the Fourier transformation is a unitary transformation of $L^2(\mathbb{R}^n)$ so that

$$\forall u \in L^2(\mathbb{R}^n), \text{we have } \hat{u} \in L^2(\mathbb{R}^n) \text{ and } \quad \|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}.$$

Applying the Riesz-Thorin interpolation Theorem 2.5.2 yields readily that the Fourier transformation is a bounded linear map from $L^p(\mathbb{R}^n)$ into $L^{p'}(\mathbb{R}^n)$ for $1 \le p \le 2$ since for $\theta$ ranging in $[0,1]$, we have

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2} \Longrightarrow \frac{1}{p'} = \frac{\theta}{2}.$$

□

*N.B.* The constant 1 in (2.6.1) is not sharp. The best constant can be found in a paper by E. Lieb [12] who proved that for $1 < p < 2$,

$$\sup_{\|u\|_{L^p(\mathbb{R}^n)}=1} \|\hat{u}\|_{L^{p'}(\mathbb{R}^n)} = \left(p^{1/p} p'^{-1/p'}\right)^{n/2}. \tag{2.6.2}$$

*Remark* 2.6.2. The mapping $L^1(\mathbb{R}^n) \ni u \mapsto \hat{u} \in L^\infty(\mathbb{R}^n)$ is one-to-one and **not onto**: if it were onto it would be a bijective continuous mapping from $L^1(\mathbb{R}^n)$ onto $L^\infty(\mathbb{R}^n)$ and thus, from the Open Mapping Theorem (a direct consequence of Baire Theorem, see e.g Theorem 2.1.10 in [10]), it would be an isomorphism. Since

$$\hat{\hat{\check{v}}} = v \quad \text{for a tempered distribution } v,$$

the inverse isomorphism from $L^\infty(\mathbb{R}^n)$ onto $L^1(\mathbb{R}^n)$ would be the inverse Fourier transform $\check{\ }$ and this would imply that the Fourier transform of an $L^\infty(\mathbb{R}^n)$ function belongs to $L^1(\mathbb{R}^n)$. However the latter is not true since the Fourier transform of $\mathbf{1}_{[-1,1]}$(a function in $L^\infty \cap L^1$) is

$$\int_{-1}^{1} e^{-2i\pi x\xi} dx = \left[\frac{e^{-2i\pi x\xi}}{-2i\pi\xi}\right]_{x=-1}^{x=1} = \frac{e^{2i\pi\xi} - e^{-2i\pi\xi}}{2i\pi\xi} = \frac{\sin(2\pi\xi)}{\pi\xi},$$

which does not belong to $L^1(\mathbb{R})$.

# Chapter 3

# Sobolev Injections Theorems

## 3.1 Marcinkiewicz Interpolation Theorem

**Definition 3.1.1.** Let $p, q \in [1, +\infty]$. A (not necessarily linear) mapping

$$T : L^p(\mathbb{R}^n) \longrightarrow L^q_w(\mathbb{R}^n) = L^{q,\infty}(\mathbb{R}^n),$$

$$\text{such that} \quad \exists C \geq 0, \forall u \in L^p(\mathbb{R}^n), \quad \|Tu\|_{L^{q,\infty}(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)},$$

where the Lorentz space $L^{q,\infty}(\mathbb{R}^n)$ is defined in Definition 2.3.1 is said to be of **weak-type** $(p, q)$.

*N.B.* When $q = +\infty$, this means:

$$\exists C \geq 0, \forall u \in L^p(\mathbb{R}^n), \quad \|Tu\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)}. \tag{3.1.1}$$

For $1 \leq q < +\infty$ this means: $\exists C \geq 0, \forall u \in L^p(\mathbb{R}^n), \forall t > 0$,

$$\lambda_n\Big(\big\{x \in \mathbb{R}^n, |(Tu)(x)| > t\big\}\Big) \leq \big(C\|u\|_{L^p(\mathbb{R}^n)} t^{-1}\big)^q, \tag{3.1.2}$$

where $\lambda_n$ stands for the Lebesgue measure on $\mathbb{R}^n$.

**Definition 3.1.2.** A bounded mapping $T : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)$, i.e. such that

$$\exists C \geq 0, \forall u \in L^p(\mathbb{R}^n), \quad \|Tu\|_{L^q(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)}, \tag{3.1.3}$$

will be said of **strong-type** $(p, q)$.

Of course, a strong-type $(p, q)$ mapping is also of weak-type $(p, q)$, since the notions are identical for $q = +\infty$ and if $1 \leq q < +\infty$, this follows from Inequality (2.3.2) (and the related inclusion $L^q \subset L^q_w$).

**Theorem 3.1.3** (Marcinkiewicz Interpolation Theorem)**.** *Let $r \in (1, +\infty]$ and let $T : L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n) \longrightarrow \{mesurable\ functions\}$ be a mapping such that*

$$|T(u + v)| \leq |Tu| + |Tv|. \tag{3.1.4}$$

*We assume that $T$ is of weak-type $(1, 1)$ and $(r, r)$ (see Definition 3.1.1). Then $T$ is of strong-type $(p, p)$ for all $p \in (1, r)$ (see (3.1.3)).*

**Lemma 3.1.4.** *Let $(X, \mathcal{M}, \mu)$ be a measure space where $\mu$ is a positive measure and let*
$$1 \le p_1 \le p \le p_2 \le +\infty.$$
*Then we have $L^p(\mu) \subset L^{p_1}(\mu) + L^{p_2}(\mu)$.*

*Proof.* We may of course assume that $p_1 < p < p_2$. We note then that, for $u \in L^p(\mu)$,
$$\mu\big(\{x \in X, |u(x)| > 1\}\big) = \int_{\{|u(x)|>1\}} d\mu \le \int_{\{|u(x)|>1\}} |u(x)|^p d\mu \le \|u\|^p_{L^p(\mu)} < +\infty.$$
We have $u = u\mathbf{1}_{\{|u|>1\}} + u\mathbf{1}_{\{|u|\le 1\}}$ and $u\mathbf{1}_{\{|u|\le 1\}} \in L^\infty(\mu)$. We have also with $q = p/p_1 \ge 1$, $1/q' = 1 - p_1/p$,
$$\begin{aligned}\int_X |u\mathbf{1}_{\{|u|>1\}}|^{p_1} d\mu &\le \Big(\int_X |u|^{p_1 q} d\mu\Big)^{1/q}\Big(\int_X \mathbf{1}_{\{|u|>1\}}{}^{p_1 q'} d\mu\Big)^{1/q'} \\ &= \|u\|^{p_1}_{L^p(\mu)} \mu\big(\{|u| > 1\}\big)^{1-\frac{p_1}{p}} \le \|u\|_{L^p(\mu)}^{p_1 + (1-\frac{p_1}{p})p} = \|u\|^p_{L^p(\mu)} < +\infty,\end{aligned}$$
so that we have proven that $u\mathbf{1}_{\{|u|>1\}} \in L^{p_1}(\mu)$. If $p_2 = +\infty$, we use $u\mathbf{1}_{\{|u|\le 1\}} \in L^\infty(\mu)$ to conclude. If $p_2 < +\infty$, we estimate
$$\int_X |u\mathbf{1}_{\{|u|\le 1\}}|^{p_2} d\mu = \int_X |u\mathbf{1}_{\{|u|\le 1\}}|^{p_2 - p}|u|^p d\mu \le \int_X |u|^p d\mu = \|u\|^p_{L^p(\mu)} < +\infty.$$
Finally we have proven more precisely that for $u \in L^p(\mu)$,
$$u = \underbrace{u\mathbf{1}_{\{|u|>1\}}}_{u_1} + \underbrace{u\mathbf{1}_{\{|u|\le 1\}}}_{u_2}, \|u_1\|_{L^{p_1}(\mu)} \le \|u\|^{p/p_1}_{L^p(\mu)}, \|u_2\|_{L^{p_2}(\mu)} \le \|u\|^{p/p_2}_{L^p(\mu)}. \tag{3.1.5}$$
□

*N.B.* From the inclusion $L^p \subset L^1 + L^r$ (see Lemma 3.1.4 above), we see that $T$ is indeed defined on $L^p$. This very useful theorem (see [15] for the 1939 original paper and [14] for a historical perspective) is also very remarkable by the fact that it is providing a strong-type information from a weak-type assumption.

*Notation.* Let $(X, \mathcal{M}, \mu)$ be a measure space where $\mu$ is a positive measure; we shall use the following notation, for a measurable function $u$ and $t > 0$:
$$\omega(t, u) = \mu\big(\{x \in \mathbb{R}^n, |u(x)| > t\}\big). \tag{3.1.6}$$
With $\Omega_p(u)$ given by Definition 2.3.1, we find that $\Omega_p(u) = \sup_{t>0} t^p \omega(t, u)$. For $p \in [1, +\infty)$ and $u \in L^p(\mu)$, we have, using Fubini Theorem,
$$\begin{aligned}\int_0^{+\infty} pt^{p-1}\omega(t, u)dt &= \int_0^{+\infty} pt^{p-1}\Big(\int_{\{x, |u(x)|>t\}} d\mu\Big)dt \\ &= \iint_{\mathbb{R}_+ \times X} pt^{p-1} H(|u(x)| - t) d\mu(x) dt \\ &= \int_X \int_0^{|u(x)|} pt^{p-1} dt d\mu(x) = \int_X |u(x)|^p d\mu(x),\end{aligned}$$

$$\text{so that} \quad \|u\|_{L^p(\mu)} = p^{1/p} \|t\omega(t,u)^{1/p}\|_{L^p(\mathbb{R}_+, \frac{dt}{t})}. \tag{3.1.7}$$

On the other hand for $u \in L^\infty(\mu)$ we have

$$\|u\|_{L^\infty(\mu)} = \inf\{t > 0, \omega(t,u) = 0\}.$$

*Proof of Theorem 3.1.3.* We use the above notations with $\mu = \lambda_n$, the Lebesgue measure on $\mathbb{R}^n$. **Let us assume first $\mathbf{r} = +\infty$.** The weak type $(\infty,\infty)$ hypothesis means $\|Tu\|_{L^\infty} \le C\|u\|_{L^\infty}$ and we may assume that $C = 1$. We write for $u \in L^1 + L^\infty$, $t > 0$,

$$u = \underbrace{u\mathbf{1}_{\{|u|>t/2\}}}_{u_1} + \underbrace{u\mathbf{1}_{\{|u|\le t/2\}}}_{u_2} \text{ and this gives}$$

$$|(Tu)(x)| \le |(Tu_1)(x)| + |(Tu_2)(x)| \le |(Tu_1)(x)| + \|u_2\|_{L^\infty} \le |(Tu_1)(x)| + \frac{t}{2},$$

so that we find the inclusion

$$(\sharp) \qquad \{x, |(Tu)(x)| > t\} \subset \{x, |(Tu_1)(x)| > t/2\}.$$

The weak-type $(1,1)$ assumption reads $t\omega(t,Tv) \le c_{11}\|v\|_{L^1}$ so that

$$(\flat) \qquad \frac{t}{2}\lambda_n\bigl(\{x, |(Tu_1)(x)| > \frac{t}{2}\}\bigr) \le c_{11}\|u_1\|_{L^1} \Longrightarrow \omega(\frac{t}{2}, Tu_1) \le \frac{2c_{11}}{t}\int_{|u|>t/2} |u| dx.$$

Applying Formula (3.1.7) to $Tu$, we find, using Tonelli Theorem and $1 < p < +\infty$,

$$\|Tu\|_{L^p}^p = p\int_0^{+\infty} t^{p-1}\omega(t,Tu)dt$$

$$\begin{aligned}
\text{(from } (\sharp)) \quad &\le p\int_0^{+\infty} t^{p-1}\omega(\frac{t}{2}, Tu_1)dt \\
\text{(from } (\flat)) \quad &\le p\int_0^{+\infty} t^{p-1}\frac{2c_{11}}{t}\int_{|u|>t/2} |u| dx dt \\
&= 2pc_{11}\iint_{\mathbb{R}_+\times\mathbb{R}^n} t^{p-2}H(2|u(x)| - t)|u(x)| dt dx \\
&= \frac{2pc_{11}}{p-1}\int_{\mathbb{R}^n} (2|u(x)|)^{p-1}|u(x)| dx = \frac{2^p pc_{11}}{p-1}\|u\|_{L^p}^p,
\end{aligned}$$

which gives the strong-type $(p,p)$ for $T$.
**We assume now $\mathbf{1} < \mathbf{r} < +\infty$.** Let $u \in L^p$, let $t > 0$ and let $u_1, u_2$ be defined as above. Since $|(Tu)(x)| \le |(Tu_1)(x)| + |(Tu_2)(x)|$, we find

$$\{x, |(Tu)(x)| > t\} \subset \{x, |(Tu_1)(x)| > t/2\} \cup \{x, |(Tu_2)(x)| > t/2\},$$

and thus $\omega(t,Tu) \le \omega(\frac{t}{2}, Tu_1) + \omega(\frac{t}{2}, Tu_2)$. Following (3.1.5), we see that $u_1 \in L^1, u_2 \in L^r$. The weak-type assumptions imply with fixed positive constants $c_1, c_r$

$$\frac{t}{2}\omega(\frac{t}{2}, Tu_1) \le c_1\|u_1\|_{L^1}, \quad (\frac{t}{2})^r\omega(\frac{t}{2}, Tu_2) \le c_r^r\|u_2\|_{L^r}^r.$$

We obtain thus

$$(\natural) \qquad \omega(t,Tu) \le \frac{2c_1}{t}\int |u(x)|H(2|u(x)|-t)dx + \frac{2^r c_r^r}{t^r}\int_{0<|u(x)|\le t/2} |u(x)|^r dx.$$

Tonelli's theorem implies

$$\begin{aligned}
\int_0^{+\infty} pt^{p-1}\omega(t,Tu)dt &\le \iint_{\mathbb{R}_+\times\mathbb{R}^n} pt^{p-1}\frac{2c_1}{t}|u(x)|H(2|u(x)|-t)dtdx \\
&\quad + \iint_{\mathbb{R}_+\times\mathbb{R}^n} pt^{p-1}\frac{2^r c_r^r}{t^r}\mathbf{1}_{\{0<|u|\le t/2\}}|u(x)|^r dtdx \\
&= \frac{2pc_1}{p-1}\int |u(x)|(2|u(x)|)^{p-1}dx + 2^r c_r^r p\int_{|u(x)|>0}|u(x)|^r \underbrace{\int_{2|u(x)|}^{+\infty} t^{p-1-r}dt\,dx}_{\text{note that } p-r<0} \\
&= \frac{2^p pc_1}{p-1}\int |u(x)|^p dx + 2^r c_r^r p\int |u(x)|^r\frac{(2|u(x)|)^{p-r}}{r-p}dx \\
&= \|u\|_{L^p}^p\Big(\frac{2^p pc_1}{p-1}+\frac{2^p c_r^r p}{r-p}\Big),
\end{aligned}$$

so that $\|Tu\|_{L^p} \le \|u\|_{L^p} 2p^{1/p}\left(\frac{c_1}{p-1}+\frac{c_r^r}{r-p}\right)^{1/p}$, concluding the proof. □

## 3.2 Maximal Function

**Definition 3.2.1.** Let $f$ be a function in $L^1_{\text{loc}}(\mathbb{R}^n)$. The maximal function of $f$, denoted by $\mathcal{M}_f$, is defined on $\mathbb{R}^n$ by

$$\mathcal{M}_f(x) = \sup_{t>0}\frac{1}{|B(x,t)|}\int_{B(x,t)}|f(y)|dy, \tag{3.2.1}$$

where $|B(x,t)|$ is the Lebesgue measure of the ball with center $x$ and radius $t$.

Using the notation

$$⨍_A f d\mu = \int_A f d\mu/\mu(A),$$

we find

$$\mathcal{M}_f(x) = \sup_{t>0} ⨍_{B(x,t)}|f(y)|dy = \sup_{t>0} ⨍_{\mathbb{B}^n}|f(x+tz)|dz.$$

We note also that the maximal function (of a measurable function) is measurable (see Exercise 3.6.3).

*Remark* 3.2.2. Let us evaluate $\mathcal{M}_{\mathbf{1}_{\mathbb{B}^n}}$. Let $x\in\mathbb{R}^n$. For $t\ge 1+|x|$, we have

$$|y|\le 1 \Longrightarrow |y-x|\le 1+|x| \Longrightarrow y\in\bar{B}(x,t).$$

We have thus for $t\ge 1+|x|$, $t^{-n}|\mathbb{B}^n|^{-1}\int_{B(x,t)}\mathbf{1}_{\mathbb{B}^n}(y)dy = t^{-n}$, implying

$$\mathcal{M}_{\mathbf{1}_{\mathbb{B}^n}}(x) \ge (1+|x|)^{-n} \Longrightarrow \mathcal{M}_{\mathbf{1}_{\mathbb{B}^n}} \notin L^1(\mathbb{R}^n),$$

proving that the mapping $f \mapsto \mathcal{M}_f$ **does not** send $L^1$ into itself. We shall see below that the maximal function of an $L^1(\mathbb{R}^n)$ function is nevertheless in $L^1_w(\mathbb{R}^n)$, proving that the mapping $f \mapsto \mathcal{M}_f$ is of weak-type $(1,1)$.

**Theorem 3.2.3** (Hardy-Littlewood maximal inequality)**.** *The mapping $f \mapsto \mathcal{M}_f$ is of weak-type $(1,1)$ and of strong-type $(p,p)$ for all $p \in (1+\infty]$ (see Definitions 3.1.1, 3.1.2).*

*Proof.* Since the mapping $f \mapsto \mathcal{M}_f$ is obviously of strong-type $(\infty,\infty)$ (since $\|\mathcal{M}_f\|_{L^\infty} \le \|f\|_{L^\infty}$), according[1] to the Marcinkiewicz interpolation Theorem 3.1.3, it is enough to prove the weak-type $(1,1)$ property:

$$\exists C_n, \forall f \in L^1(\mathbb{R}^n), \quad \sup_{t>0} t\big|\{x \in \mathbb{R}^n, \mathcal{M}_f(x) > t\}\big| \le C_n \|f\|_{L^1(\mathbb{R}^n)}. \tag{3.2.2}$$

Note that from Remark 3.2.2, Riesz-Thorin Theorem 2.5.2 cannot be used since the mapping fails to be of strong-type $(1,1)$. We start with a lemma.

**Lemma 3.2.4** (Wiener covering lemma)**.** *Let $E$ be measurable subset of $\mathbb{R}^n$ such that $E \subset \cup_{j\in J} B_j$ where $(B_j)_{j\in J}$ is a family of open balls such that*

$$2\rho_0 = \sup_{j\in J} \operatorname{diam} B_j < +\infty.$$

*Then there exists a countable subfamily $(B_j)_{j\in D}$ of pairwise disjoint balls such that*

$$\lambda_n(E) \le 5^n \sum_{j\in D} \lambda_n(B_j).$$

*Proof of the lemma.* Let $B_{j_0} = B(x_0, r_0)$ be a ball[2] such that $\operatorname{diam} B_{j_0} = 2r_0 > \rho_0$. Next, we define

$$J_0 = J, \quad J_1 = \{j \in J_0, B_j \cap B_{j_0} = \emptyset\}.$$

If $j \notin J_1$, then $B_j \cap B_{j_0} \ne \emptyset$, so that $\exists y_0 \in B_j \cap B_{j_0}$ and

$$x \in B_j \Longrightarrow |x - x_0| \le \underbrace{|x - y_0|}_{x, y_0 \in B_j} + \underbrace{|y_0 - x_0|}_{y_0 \in B(x_0, r_0)} \le 2\rho_0 + r_0 < 5r_0,$$

entailing $j \notin J_1 \Longrightarrow B_j \subset B^*_{j_0}$ which is defined as the ball with same center as $B_{j_0}$ and a diameter equal to five times the diameter of $B_{j_0}$.

• For the family $(B_j)_{j\in J_0}$ of open balls with bounded diameters,

$$\exists j_0 \in J_0, \text{ with } J_1 = \{j \in J_0, B_j \cap B_{j_0} = \emptyset\}, \quad \begin{cases} j \in J_1 & \Longrightarrow B_j \cap B_{j_0} = \emptyset, \\ j \notin J_1 & \Longrightarrow B_j \subset B^*_{j_0}. \end{cases}$$

[1] Note that the subadditivity property is fulfilled since

$$0 \le (\mathcal{M}_{f+g})(x) = \sup_{t>0} \fint_{\mathbb{B}^n} |(f+g)(x+tz)| dz \le \sup_{t>0} \fint_{\mathbb{B}^n} |f(x+tz)| dz + \sup_{t>0} \fint_{\mathbb{B}^n} |g(x+tz)| dz.$$

[2] We may of course assume that $E$ has positive measure, which implies that $J$ is not empty and $\rho_0 > 0$.

• Let us assume that we have found $J_0 \supset J_1 \supset \cdots \supset J_k, k \ge 1,\ j_0 \in J_0, \dots, j_k \in J_k$ such that

$$\text{(1) } \operatorname{diam} B_{j_0} > \frac{1}{2} \sup_{j\in J_0} \operatorname{diam} B_j,\ \dots\dots\dots\dots,\ \operatorname{diam} B_{j_k} > \frac{1}{2} \sup_{j\in J_k} \operatorname{diam} B_j,$$

$$\text{(2)} \big\{ j \in J_0, j \notin J_1 \Longrightarrow B_j \subset B^*_{j_0} \big\}, \qquad \text{(3)} \big\{ j \in J_1 \Longrightarrow B_j \cap B_{j_0} = \emptyset \big\},$$

$$\dots\dots\dots$$

$$\text{(2)} \big\{ j \in J_{k-1}, j \notin J_k \Longrightarrow B_j \subset B^*_{j_{k-1}} \big\}, \qquad \text{(3)} \big\{ j \in J_k \Longrightarrow B_j \cap B_{j_{k-1}} = \emptyset \big\}.$$

We define then $J_{k+1} = \{j \in J_k, B_j \cap B_{j_k} = \emptyset\}$ and if $J_{k+1} \ne \emptyset$ we find $j_{k+1} \in J_{k+1}$ such that

$$\operatorname{diam} B_{j_{k+1}} > \frac{1}{2} \sup_{j\in J_{k+1}} \operatorname{diam} B_j,$$

fulfilling (1) for $k+1$ as well. Moreover (3) holds true for $k+1$ by construction and if $j \in J_k \backslash J_{k+1}$, we have $B_j \cap B_{j_k} \ne \emptyset$, so that $\exists y_k \in B_j \cap B_{j_k}$, $B_{j_k} = B(x_k, r_k)$, and

$$x \in B_j \Longrightarrow |x - x_k| \le \underbrace{|x - y_k|}_{x, y_k \in B_j} + \underbrace{|y_k - x_k|}_{y_k \in B(x_k, r_k)}$$
$$\le \underbrace{\operatorname{diam} B_j}_{j \in J_k} + r_k < 2 \operatorname{diam} B_{j_k} + r_k = 5 r_k,$$

entailing $B_j \subset B^*_{j_k}$, proving (2) for $k+1$.

• As a result, assuming that all the $J_k$ are non-empty, we find

$$J_0 \supset J_1 \supset \cdots \supset J_k \supset \dots,\ j_k \in J_k,$$

$$\text{such that} \quad \begin{cases} k \ge 1 : j \in J_{k-1} \backslash J_k \Longrightarrow B_j \subset B^*_{j_{k-1}}, \\ k \ge 1 : j \in J_k \Longrightarrow B_j \cap B_{j_{k-1}} = \emptyset. \end{cases}$$

The family $(B_{j_k})_{k\ge0}$ is pairwise disjoint: we consider $k' \ge k'' + 1$. We have $j_{k'} \in J_{k'} \subset J_{k''+1}$ and $j_{k''} \in J_{k''}$ so that

$$\underbrace{B_{j_{k'}}}_{j_{k'} \in J_{k''+1}} \cap B_{j_{k''}} = \emptyset.$$

**Claim.** If $\sum_{k\ge0} |B_{j_k}| < +\infty$ we have for all $j \in J_0$, $B_j \subset \cup_{k\ge1} B^*_{j_{k-1}}$.
The Claim is obvious if $j \in \cup_{k\ge1}(J_{k-1} \backslash J_k)$. Otherwise we have

$$j \in \cap_{k\ge1}(J^c_{k-1} \cup J_k), \text{ which means } j \in \cap_{k\ge1} J_k:$$

in fact, we have $\cap_{k\ge1}(J^c_{k-1} \cup J_k) = \cap_{k\ge1} J_k$ since

$$\{\forall k \ge 1,\ j \in J_k \cup J^c_{k-1}\} \text{ and } \{\exists k_0 \ge 1, j \notin J_{k_0}\}$$
$$\Longrightarrow j \in J^c_{k_0-1}, k_0 \ge 2, \text{ since } J^c_0 = \emptyset,$$
$$\Longrightarrow j \in J^c_{k_0-2}, k_0 \ge 3 \dots \Longrightarrow j \in J^c_1 \Longrightarrow j \in J^c_0 = \emptyset,$$

which is impossible. If $j \in \cap_{k\ge 1} J_k$, we have $\forall k \ge 1,\ 2\operatorname{diam} B_{j_k} > \operatorname{diam} B_j$ and since the series $\sum_{k\ge 0} |B_{j_k}|$ converges, this implies $\lim_k \operatorname{diam} B_{j_k} = 0$, and $\operatorname{diam} B_j = 0$ so that the open ball $B_j$ is empty. The Claim is proven.

• Finally, we have either $\sum_{k\ge 0} |B_{j_k}| = +\infty$ (a case where the conclusion of the lemma is reached trivially) or $\sum_{k\ge 0} |B_{j_k}| < +\infty$ and the above Claim is implying that

$$E \subset \cup_{k\ge 1} B^*_{j_{k-1}},$$

providing the sought answer.

• When $J_{k_0} = \emptyset$ for some $k_0 \ge 1$, we find that $J_0 = \cup_{1\le k\le k_0}(J_{k-1}\backslash J_k)$ and we have obviously $\forall j \in J_0,\ B_j \subset \cup_{k\ge 1} B^*_{j_{k-1}}$, obtaining the conclusion as well in that case. The proof of the Wiener covering lemma is complete. □

Let us go back to the proof of Theorem 3.2.3. Let $s > 0$ be given. If $x \in \mathbb{R}^n$ is such that $\mathcal{M}_f(x) > s$, we can find $t_{s,x} > 0$ such that

$$\frac{1}{|B(x,t_{s,x})|} \int_{B(x,t_{s,x})} |f(y)|dy > s \Longrightarrow |B(x,t_{s,x})| \le s^{-1}\|f\|_{L^1(\mathbb{R}^n)} < +\infty.$$

We consider the measurable set

$$E_s = \{x \in \mathbb{R}^n, \mathcal{M}_f(x) > s\} \subset \cup_{x\in E_s} B(x,t_{s,x})$$

and we note that $t^n_{s,x}|\mathbb{B}^n| \le s^{-1}\|f\|_{L^1(\mathbb{R}^n)}$ so that we may apply Wiener covering Lemma 3.2.4. We find a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that the balls $B(x_k, t_{s,x_k})$ are pairwise disjoint and

$$\begin{aligned} |E_s = \{x \in \mathbb{R}^n, \mathcal{M}_f(x) > s\}| &\le 5^n \sum_{k\in\mathbb{N}} |B(x_k, t_{s,x_k})| \\ &\le s^{-1} 5^n \sum_{k\in\mathbb{N}} \int_{B(x_k,t_{s,x_k})} |f(y)|dy \\ &\le s^{-1} 5^n \int_{\mathbb{R}^n} |f(y)|dy, \end{aligned}$$

proving $s|E_s| \le 5^n \|f\|_{L^1(\mathbb{R}^n)}$ and the weak-type $(1,1)$ property. □

*Remark* 3.2.5. Note that with the result of Exercise 3.6.2, this implies

$$\text{for } 1 < p \le +\infty, \quad \|\mathcal{M}_f\|_{L^p(\mathbb{R}^n)} \le \frac{p^{1+\frac{1}{p}}}{p-1} 5^{\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.2.3}$$

A result due to E.M. Stein and J.-O. Stromberg [17] shows that the $L^p$ to $L^p$ norm of $\mathcal{M}$ can be chosen independently of the dimension $n$.

## 3.3 Lebesgue differentiation theorem

**Theorem 3.3.1** (Lebesgue Differentiation Theorem)**.**
*Let $f$ be a function in $L^1(\mathbb{R}^n)$. Then, there exists a Borel set $L_f$ such that $\lambda_n(L_f^c) = 0$ such that*

$$\forall x \in L_f, \quad \lim_{r\to 0_+} \frac{1}{\lambda_n(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dy = 0. \tag{3.3.1}$$

*The set $L_f$ is called the set of Lebesgue points of $f$.*

*Remark* 3.3.2. Note that this implies that for $f \in L^1(\mathbb{R}^n)$, for all $x \in L_f$,

$$\lim_{r\to 0} \fint_{B(x,r)} f(y)dy = f(x).$$

*Proof.* For $\rho > 0$ we define the measurable set

$$E_\rho = \{x \in \mathbb{R}^n, \limsup_{t\to 0_+} \underbrace{\frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y) - f(x)| dy}_{\mathcal{N}_f(t,x)} > \rho\}. \tag{3.3.2}$$

Let $\phi \in C_c^0(\mathbb{R}^n)$. We have

$$\begin{aligned}\mathcal{N}_f(t,x) &\le \fint_{B(x,t)} |f(y) - \phi(y)|dy + \fint_{B(x,t)} |\phi(y) - \phi(x)|dy + |\phi(x) - f(x)| \\ &\le \mathcal{M}_{\phi - f}(x) + \fint_{B(x,t)} |\phi(y) - \phi(x)|dy + |\phi(x) - f(x)|.\end{aligned}$$

Since $\phi$ is uniformly continuous, we get

$$\limsup_{t\to 0} \mathcal{N}_f(t,x) \le \mathcal{M}_{\phi-f}(x) + |f(x) - \phi(x)|.$$

As a result the set $E_\rho$ defined by (3.3.2) is such that

$$E_\rho \subset \{x, |f(x) - \phi(x)| > \rho/2\} \cup \{x, \mathcal{M}_{\phi-f}(x) > \rho/2\},$$

and this implies $|E_\rho| \le |\{x, |f(x) - \phi(x)| > \rho/2\}| + |\{x, \mathcal{M}_{\phi-f}(x) > \rho/2\}|$. Using now Theorem 3.2.3, we obtain for any $\phi \in C_c^0(\mathbb{R}^n)$,

$$|E_\rho| \le \frac{2}{\rho} \int_{\mathbb{R}^n} |f(x) - \phi(x)|dx + C_n \frac{2}{\rho} \|f - \phi\|_{L^1(\mathbb{R}^n)} = \frac{2(1 + C_n)}{\rho} \|f - \phi\|_{L^1(\mathbb{R}^n)}.$$

The density of $C_c^0(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$ implies that $|E_\rho| = 0$ for all $\rho > 0$ and since

$$\{x \in \mathbb{R}^n, \limsup_{t\to 0_+} \mathcal{N}_f(t,x) > 0\} = \cup_{k\ge 1} E_{1/k},$$

this gives as well that $|E_0| = 0$. We define $L_f = E_0^c$ and we have for $x \in L_f$, $\lim_{t\to 0} \mathcal{N}_f(t,x) = 0$, which is the sought result. □

**Theorem 3.3.3.** *Let $f \in L^1_{loc}(\mathbb{R})$. We define for $x \in \mathbb{R}$, $F(x) = \int_0^x f(y)dy$.*
(1) *Then the function $F$ is continuous on $\mathbb{R}$, differentiable almost everywhere with derivative $f(x)$.*
(2) *The weak derivative of $F$ is $f$.*

*Proof.* (1) The continuity of $F$ is obvious since for $h \geq 0$,

$$F(x+h) - F(x) = \int_{[x,x+h]} f(y)dy,$$

and for $h \leq 0$, $F(x+h) - F(x) = -\int_{[x+h,x]} f(y)dy$. Proposition 3.3.4 below implies $\lim_{h\to 0}(F(x+h) - F(x)) = 0$. We consider now for $h \neq 0$,

$$\begin{aligned}\Big|\frac{F(x+h) - F(x)}{h} - f(x)\Big| &\leq \frac{1}{|h|}\int_{[x,x+h]\cup[x+h,x]} |f(y) - f(x)|dy \\ &\leq \frac{2}{2|h|}\int_{[x-|h|,x+|h|]} |f(y) - f(x)|dy.\end{aligned}$$

Applying the previous theorem if $f \in L^1(\mathbb{R})$, we find that $F$ is differentiable at the Lebesgue points of $f$, with derivative $f$.
(2) We have for $\phi \in C^\infty_c(\mathbb{R}^n)$, using Fubini Theorem,

$$\begin{aligned}\langle F', \phi\rangle &= -\int F(x)\phi'(x)dx \\ &= -\int \phi'(x)\int \big(H(x)\mathbf{1}_{[0,x]}(y) - H(-x)\mathbf{1}_{[x,0]}(y)\big)f(y)dydx \\ &= \int f(y)\Big(-\int_{0\leq y\leq x}\phi'(x)dx + \int_{x\leq y\leq 0}\phi'(x)dx\Big)dy \\ &= \int f(y)\big(H(y)\phi(y) + H(-y)\phi(y)\big)dy = \langle f, \phi\rangle,\end{aligned}$$

proving the result. □

**Proposition 3.3.4.** *Let $(X, \mathcal{M}, \mu)$ be a measure space where $\mu$ is a positive measure. Let $f : X \longrightarrow \overline{\mathbb{R}}_+$ be a measurable mapping such that $\int_X f d\mu < \infty$.*
*(1) The set $N = \{x \in X, f(x) = +\infty\} \in \mathcal{M}$ and $\mu(N) = 0$.*
*(2) For any $\epsilon > 0$, there exists $\alpha > 0$ such that for all $E \in \mathcal{M}$ satisfying $\mu(E) \leq \alpha$, we have $\int_E f d\mu < \epsilon$. In other words, $\lim_{\substack{\mu(E)\to 0\\ E\in\mathcal{M}}} \int_E f d\mu = 0$.*
*In particular, for $u \in L^1(\mu)$, we have*

$$\lim_{\substack{\mu(E)\to 0\\ E\in\mathcal{M}}} \int_E |u|d\mu = 0. \tag{3.3.3}$$

*Proof.* (1) The set $N = \{x \in X, f(x) = +\infty\}$ belongs to $\mathcal{M}$ as the inverse image of the closed set $\{+\infty\}$ by the measurable $f$. For all integers $k$, $k\mathbf{1}_N \leq f$, so that $k\mu(N) \leq \int_X f d\mu < +\infty$. The nonnegative sequence $(k\mu(N))_{k\in\mathbb{N}}$ is bounded so that $\mu(N) = 0$.

(2) Let $E \in \mathcal{M}$ and $n \in \mathbb{N}$: since $\mu(N) = 0$, we have

$$(\natural) \quad \int_E f d\mu = \int_{E\cap N^c} f d\mu = \int_{E\cap N^c \cap\{f\le n\}} f d\mu + \int_{E\cap N^c\cap\{f>n\}} f d\mu$$
$$\le n\mu(E) + \int f\mathbf{1}_{E\cap N^c\cap\{f>n\}} d\mu \le n\mu(E) + \int f\mathbf{1}_{n<f<+\infty} d\mu.$$

The sequence $g_n = f\mathbf{1}_{\{n<f<+\infty\}}$ is such that $g_n(x) = 0$ for $n \ge f(x)$, which is verified for $x \in N^c$ if $n$ is large enough. Since $g_n(x) = 0$ for $x \in N$, we find

$$(\flat) \qquad \forall x \in X, \quad g_n(x) \to 0.$$

Moreover

$$(\sharp) \qquad 0 \le g_n \le f\mathbf{1}_{N^c} \quad \text{and} \quad f\mathbf{1}_{N^c} \in L^1(\mu).$$

Lebesgue dominated convergence Theorem shows that ($\flat$) and ($\sharp$) imply the convergence of $g_n$towards 0 in $L^1(\mu)$. From ($\natural$),we get

$$0 \le \int_E f d\mu \le n\mu(E) + \theta_n, \quad \text{with} \quad \theta_n \underset{n\to+\infty}{\longrightarrow} 0_+.$$

Let $\epsilon > 0$ be given : $\exists N \in \mathbb{N}$ such that $\theta_N < \epsilon/2$. Defining $\alpha = \frac{\epsilon}{2N+1}$ (we have indeed $\alpha > 0$), we get for $\mu(E) \le \alpha$

$$0 \le \int_E f d\mu \le \frac{N\epsilon}{2N+1} + \theta_N < \epsilon/2 + \epsilon/2 = \epsilon, \quad \text{qed}.$$

A slightly shorter reasoning from ($\natural$) would be

$$\forall n \in \mathbb{N}, \quad 0 \le \limsup_{\mu(E)\to 0} \int_E f d\mu \le \theta_n \Longrightarrow 0 \le \limsup_{\mu(E)\to 0} \int_E f d\mu \le \lim_n \theta_n = 0.$$

□

*Remark* 3.3.5. *Almost everywhere differentiability is a very weak piece of information.* Almost everywhere differentiability of a function $F$ is a very weak property that does not tell much about the function $F$: in the first place the trivial example of the Heaviside function shows that a bounded function can be differentiable almost everywhere in $\mathbb{R}$ with a zero derivative without being a constant. The much more elaborate example of the Cantor function shows that a continuous function can be differentiable almost everywhere with a null derivative without being a constant, so is not the integral of its a.e. derivative.

*Remark* 3.3.6. It may also happen that a continuous function is differentiable everywhere but with a derivative which is not integrable in the Lebesgue sense (see Exercise 3.6.4).

The distribution (or weak) derivative does not miss jumps and singularities as the notion of everywhere differentiability. Here the reader may consider only tempered distributions as in Chapter 8, but the statements are true as well for general distributions defined as local objects.

**Lemma 3.3.7.** *Let $T$ be a distribution on $\mathbb{R}$ such that $T' = 0$. Then $T$ is a constant.*

*Proof.* Let $\phi \in C_c^\infty(\mathbb{R})$ and let $\chi_0 \in C_c^\infty(\mathbb{R})$ with integral 1. Denoting $I(\phi) = \int_{\mathbb{R}} \phi(y)dy$, the function $\psi$ defined by

$$\psi(x) = \phi(x) - I(\phi)\chi_0(x),$$

belongs to $C_c^\infty(\mathbb{R})$ and is the derivative of $\Psi(x) = \int_{-\infty}^x \psi(y)dy$. Note that $\Psi$ is $C^\infty$ and with compact support, since for $x$ large enough

$$\Psi(x) = \int_{\mathbb{R}} \phi(y)dy - I(\phi)\int_{\mathbb{R}} \chi_0(y)dy = 0.$$

As a result, we find

$$\langle T, \phi\rangle = \langle T, \psi\rangle + I(\phi)\langle T, \chi_0\rangle = \langle T, \Psi'\rangle + I(\phi)\langle T, \chi_0\rangle = -\langle T', \Psi\rangle + I(\phi)\langle T, \chi_0\rangle,$$

so that $T = \langle T, \chi_0\rangle$ □

**Theorem 3.3.8.** *Let $F$ be a locally integrable function in $\mathbb{R}$ such that its distribution derivative $F'$ is locally integrable. Then the function $F$ is bounded continuous and for all $a \in \mathbb{R}$*

$$F(x) = F(a) + \int_a^x F'(y)dy. \tag{3.3.4}$$

*The function $F$ is also a.e. differentiable with (ordinary) derivative $F'(x)$.*

*Proof.* We define $G(x) = \int_a^x F'(y)dy$ and from Theorem 3.3.3, we find that the distribution derivative $G'$ of $G$ is equal to $F'$ (and that $G$ is continuous). Thus the distribution derivative of $F - G$ is zero, so that $F - G$ is the constant $F(a) - G(a) = F(a)$. The last statement follows from Theorem 3.3.3. □

## 3.4 Gagliardo-Nirenberg Inequality

**Proposition 3.4.1.** *For all $\phi \in C_c^1(\mathbb{R}^n)$, we have*

$$\|\phi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \frac{1}{2}\prod_{1\le j\le n} \|\frac{\partial\phi}{\partial x_j}\|^{1/n}_{L^1(\mathbb{R}^n)}. \tag{3.4.1}$$

*Proof.* The cases $n = 1, 2$ are very easy: for $n = 1$, we have

$$2\phi(x) = \int_{-\infty}^x \phi'(t)dt + \int_{+\infty}^x \phi'(t)dt \Longrightarrow 2\|\phi\|_{L^\infty(\mathbb{R})} \le \|\phi'\|_{L^1(\mathbb{R})}.$$

For $n = 2$, we have, using the previous result,

$$|\phi(x_1, x_2)| \le \frac{1}{2}\int_{\mathbb{R}} |\partial_1\phi(t_1, x_2)|dt_1, \quad |\phi(x_1, x_2)| \le \frac{1}{2}\int_{\mathbb{R}} |\partial_2\phi(x_1, t_2)|dt_2,$$

so that

$$4\|\phi\|^2_{L^2(\mathbb{R}^2)} \le \int_{\mathbb{R}^4} |\partial_1\phi(t_1, x_2)||\partial_2\phi(x_1, t_2)|dt_1dt_2dx_1dx_2 = \|\partial_1\phi\|_{L^1(\mathbb{R}^2)}\|\partial_2\phi\|_{L^1(\mathbb{R}^2)}.$$

The cases $n \ge 3$ are more complicated and we need to start with a couple of lemmas.

**Lemma 3.4.2.** *Let $(X, \mathcal{M}, \mu)$ be a measure space where $\mu$ is a positive measure. Let $f_1, \dots, f_N$ be non-negative measurable functions and let $p_1, \dots, p_N \in [1, +\infty]$ such that*

$$\sum_{1\le j\le N} \frac{1}{p_j} = 1.$$

*Then we have*

$$\int_X f_1 \dots f_N d\mu \le \prod_{1\le j\le N} \|f_j\|_{L^{p_j}(\mu)}.$$

*Proof.* When $N = 2$, this is Hölder's inequality. We may assume that all $f_j$ are not vanishing $\mu$-a.e. (otherwise the lhs is 0) and that each $f_j$ belongs to $L^{p_j}(\mu)$ (otherwise the rhs is $+\infty$ as the product of positive quantities in $\overline{\mathbb{R}}_+$ with one of them $+\infty$). Induction on $N$: let $N \ge 2$ and $p_1, \dots, p_{N+1} \in [1, +\infty]$ with $\sum_{1\le j\le N+1} \frac{1}{p_j} = 1$. Applying Hölder's inequality we find

$$(\sharp) \qquad \int_X f_1 \dots f_N f_{N+1} d\mu \le \|\prod_{1\le j\le N} f_j\|_{L^{p'_{N+1}}(\mu)} \|f_{N+1}\|_{L^{p_{N+1}}(\mu)}.$$

Since $\sum_{1\le j\le N} \frac{p'_{N+1}}{p_j} = 1$ (ensuring that $p_j/p'_{N+1} \ge 1$) and

$$\|\prod_{1\le j\le N} f_j\|_{L^{p'_{N+1}}(\mu)} = \|\prod_{1\le j\le N} f_j^{p'_{N+1}}\|_{L^1(\mu)}^{\frac{1}{p'_{N+1}}},$$

we may use the induction hypothesis to obtain

$$\|\prod_{1\le j\le N} f_j\|_{L^{p'_{N+1}}(\mu)} \le \Big(\prod_{1\le j\le N} \|f_j^{p'_{N+1}}\|_{L^{p_j/p'_{N+1}}}\Big)^{\frac{1}{p'_{N+1}}}.$$

The rhs of that inequality equals $\prod_{1\le j\le N} \|f_j\|_{L^{p_j}}$, and with $(\sharp)$ this provides the answer. □

**Lemma 3.4.3.** *Let $n \ge 2$ be an integer and let $\omega_1, \dots, \omega_n$ be non-negative measurable functions on $\mathbb{R}^{n-1}$ so that $\omega_j$ is a function of $(x_k)_{1\le k\le n, k\ne j}$. Then, we have*

$$\int_{\mathbb{R}^n} \omega_1^{\frac{1}{n-1}} \dots \omega_n^{\frac{1}{n-1}} dx_1 \dots dx_n \le \prod_{j=1}^n \Big(\int_{\mathbb{R}^{n-1}} \omega_j d\widehat{x_j}\Big)^{\frac{1}{n-1}},$$

*where $d\widehat{x_j} = \prod_{\substack{1\le k\le n \\ k\ne j}} dx_k$.*

*Proof of the lemma.* For $n = 2$ we have indeed

$$\int_{\mathbb{R}^2} \omega_1(x_2)\omega_2(x_1)dx_1dx_2 = \|\omega_1\|_{L^1(\mathbb{R})}\|\omega_2\|_{L^1(\mathbb{R})}.$$

Let us assume now that $n \ge 3$: we have

$$I_n = \int_{\mathbb{R}^n} \omega_1^{\frac{1}{n-1}} \dots \omega_n^{\frac{1}{n-1}} dx_1 \dots dx_n = \int_{\mathbb{R}^{n-1}} \overbrace{\omega_1^{\frac{1}{n-1}}}^{\substack{\text{does not}\\ \text{depend on } x_1}} \Big(\int_{\mathbb{R}} \prod_{2\le j\le n} \omega_j^{\frac{1}{n-1}} dx_1\Big) d\widehat{x_1},$$

and since $\frac{1}{n-1} + \frac{n-2}{n-1} = 1$, Hölder's inequality implies

$$I_n \le \|\omega_1\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}} \left\{\int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} \prod_{2\le j\le n} \omega_j^{\frac{1}{n-1}} dx_1\Big)^{\frac{n-1}{n-2}} d\widehat{x_1}\right\}^{\frac{n-2}{n-1}}.$$

We have, using the generalized Hölder's inequality of Lemma 3.4.2,

$$\int_{\mathbb{R}} \prod_{2\le j\le n} \omega_j^{\frac{1}{n-1}} dx_1 \le \prod_{2\le j\le n} \Big(\int_{\mathbb{R}} (\omega_j^{\frac{1}{n-1}})^{n-1} dx_1\Big)^{\frac{1}{n-1}} = \Big(\prod_{2\le j\le n} \int_{\mathbb{R}} \omega_j dx_1\Big)^{\frac{1}{n-1}}.$$

This gives

$$I_n \le \|\omega_1\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}} \left\{\int_{\mathbb{R}^{n-1}} \prod_{2\le j\le n} \underbrace{\Big(\int_{\mathbb{R}} \omega_j dx_1\Big)^{\frac{1}{n-2}}}_{=\Omega_j^{\frac{1}{n-2}}} d\widehat{x_1}\right\}^{\frac{n-2}{n-1}},$$

with $\Omega_j$ independent of $x_1, x_j$ (here $1 \ne j$ since $j \ge 2$). We may apply the induction hypothesis to obtain

$$I_n \le \|\omega_1\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}} \left\{\prod_{2\le j\le n} \|\Omega_j\|_{L^1(\mathbb{R}^{n-2})}^{\frac{1}{n-2}}\right\}^{\frac{n-2}{n-1}}$$
$$= \|\omega_1\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}} \left\{\prod_{2\le j\le n} \|\Omega_j\|_{L^1(\mathbb{R}^{n-2})}\right\}^{\frac{1}{n-1}},$$

and since for $2 \le j \le n$,

$$\|\Omega_j\|_{L^1(\mathbb{R}^{n-2})} = \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} \omega_j dx_1 \prod_{2\le k\le n, k\ne j} dx_k = \|\omega_j\|_{L^1(\mathbb{R}^{n-1})},$$

this proves the lemma. □

Let us go back to the proof of Proposition 3.4.1. We have

$$2|\phi(x)| \le \int_{\mathbb{R}} |\partial_j\phi(x_1, \dots, x_{j-1}, t_j, x_{j+1}, \dots, x_n)|dt_j = \omega_j(x),$$

where $\omega_j$ does not depend on $x_j$. This implies that

$$2^{\frac{n}{n-1}}|\phi(x)|^{\frac{n}{n-1}} \le \prod_{1\le j\le n} \omega_j(x)^{\frac{1}{n-1}},$$

and from Lemma 3.4.3, this implies

$$2^{\frac{n}{n-1}} \int |\phi(x)|^{\frac{n}{n-1}} dx \le \Bigl(\prod_{1\le j\le n} \|\omega_j\|_{L^1(\mathbb{R}^{n-1})}\Bigr)^{\frac{1}{n-1}} = \Bigl(\prod_{1\le j\le n} \|\partial_j\phi\|_{L^1(\mathbb{R}^n)}\Bigr)^{\frac{1}{n-1}},$$

which is (3.4.1), concluding the proof. □

**Proposition 3.4.4.** *The space $W^{1,1}(\mathbb{R}^n)$ is defined as the set of functions $u \in L^1(\mathbb{R}^n)$ such that the distribution $\nabla u$ belongs as well to $L^1(\mathbb{R}^n)$. This space is a Banach space for the norm*

$$\|u\|_{W^{1,1}(\mathbb{R}^n)} = \|u\|_{L^1(\mathbb{R}^n)} + \|\nabla u\|_{L^1(\mathbb{R}^n)}.$$

*Proof.* Let $(u_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $W^{1,1}(\mathbb{R}^n)$. Then, we find $u, V \in L^1$ such that $\lim_k u_k = u, \lim \nabla u_k = V$ in the space $L^1(\mathbb{R}^n)$. Now for $\phi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\int V\phi dx = \lim_k \int \phi \nabla u_k dx = \lim_k \langle \nabla u_k, \phi\rangle = -\lim_k \langle u_k, \nabla\phi\rangle$$
$$= -\lim_k \int u_k \nabla\phi dx = -\int u\nabla\phi dx = \langle \nabla u, \phi\rangle,$$

proving $V = \nabla u$. □

**Theorem 3.4.5** (Gagliardo-Nirenberg inequality)**.** *Let $u \in W^{1,1}(\mathbb{R}^n)$. Then $u$ belongs to $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ and is such that*

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \frac12 \prod_{1\le j\le n} \|\partial_j u\|^{1/n}_{L^1(\mathbb{R}^n)}. \tag{3.4.2}$$

*Proof.* Let $\rho \in C_c^\infty(\mathbb{R}^n;\mathbb{R}_+)$ such that $\int \rho(x)dx = 1$. For $\epsilon > 0$, we define $\rho_\epsilon(x) = \rho(x/\epsilon)\epsilon^{-n}$. The function $(u*\rho_\epsilon)(x) = \int u(y)\rho_\epsilon(x-y)dy$ is smooth, belongs to $L^1(\mathbb{R}^n)$ (Proposition 2.1.1) and converges to $u$ in $L^1(\mathbb{R}^n)$: for $\phi \in C_c^0(\mathbb{R}^n)$, we have

$$u * \rho_\epsilon - u = (u-\phi) * \rho_\epsilon + \phi * \rho_\epsilon - \phi + \phi - u,$$

so that with $L^1$ norms, using (2.1.3), for $\epsilon \le 1$,

$$\|u * \rho_\epsilon - u\| \le 2\|u-\phi\| + \int_K |(\phi * \rho_\epsilon)(x) - \phi(x)|dx,$$

where $K$ is the compact set $\operatorname{supp}\phi + \operatorname{supp}\rho$. From Lemma 2.1.4, we find uniform convergence of the sequence of continuous functions $\phi * \rho_\epsilon$ and this implies

$$\forall \phi \in C_c^0(\mathbb{R}^n), \quad \limsup_{\epsilon\to 0} \|u * \rho_\epsilon - u\| \le 2\|u-\phi\|.$$

The density of $C_c^0(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$ entails that $\lim_\epsilon \|u * \rho_\epsilon - u\| = 0$. We have also

$$\rho_\epsilon * \nabla u = \nabla(\rho_\epsilon * u) \tag{3.4.3}$$

since for $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (\rho_\epsilon * \nabla u)(x)\phi(x)dx = \iint \rho_\epsilon(x-y)(\nabla u)(y)\phi(x)dxdy$$
$$= \langle \nabla u, \check{\rho}_\epsilon * \phi\rangle = -\langle u, \check{\rho}_\epsilon * \nabla\phi\rangle = -\int (\rho_\epsilon * u)(x)\nabla\phi(x)dx = \langle \nabla(\rho_\epsilon * u), \phi\rangle,$$

proving (3.4.3).

*Let us assume first that $u$ belongs to $W^{1,1}(\mathbb{R}^n)$ and is compactly supported.* We may apply (3.4.1) to the smooth compactly supported $\rho_\epsilon * u$. We note that the sequence $\partial_j(\rho_\epsilon * u) = \rho_\epsilon * \partial_j u$ converges in $L^1(\mathbb{R}^n)$ towards $\partial_j u$. Moreover the inequality (3.4.1) applied to $\rho_{\epsilon_1} * u - \rho_{\epsilon_2} * u$ implies that $\rho_\epsilon * u$ is a Cauchy sequence in $L^{n/n-1}(\mathbb{R}^n)$ thus converges with a limit $v$; since that sequence is converging towards $u$ in $L^1(\mathbb{R}^n)$, and for $\phi \in C_c^0(\mathbb{R}^n)$, we have

$$\int v(x)\phi(x)dx = \lim_\epsilon \int (\rho_\epsilon * u)(x)\phi(x)dx = \int u(x)\phi(x)dx,$$

Lemma 1.2.10 implies $u = v$ which belongs to $L^{n/n-1}$. Inequality (3.4.2) holds true by taking the limits in (3.4.1).

*Let us assume now that $u$ belongs to $W^{1,1}(\mathbb{R}^n)$.* Let $\chi$ be in $C_c^\infty(\mathbb{R}^n; [0,1])$, equal to 1 on $B(0,1)$ and supported in $B(0,2)$. For $\epsilon > 0$ we have obviously (dominated convergence)

$$\lim_{\epsilon\to 0} \chi(\epsilon x)u(x) = u(x) \quad \text{in } L^1(\mathbb{R}^n).$$

Let us calculate for $\chi_\epsilon(x) = \chi(\epsilon x)$, $\nabla(u\chi_\epsilon) = \chi_\epsilon \nabla u + u\nabla\chi_\epsilon$. We have

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^n} |u(x)\chi'(\epsilon x)|dx\epsilon = 0 = \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} |u(x)|(1-\chi(\epsilon x))dx,$$

where the first equality is obvious (domination by $\|u\|_{L^1}\epsilon\|\chi'\|_{L^\infty}$) as well as the next one since

$$\int_{\mathbb{R}^n} |u(x)|(1-\chi(\epsilon x))dx \le \int_{|x|\ge 1/\epsilon} |u(x)|dx.$$

We have thus

$$\lim_{\epsilon\to 0}\chi_\epsilon u = u, \quad \lim_{\epsilon\to 0}\nabla(\chi_\epsilon u) = \nabla u \quad \text{in } L^1.$$

Since $u_\epsilon = \chi_\epsilon u$ is compactly supported in $W^{1,1}$, we may apply the previous result to get Inequality (3.4.2) for $u_\epsilon$. That inequality implies as well that $u_\epsilon$ is a Cauchy sequence in $L^{n/n-1}$ and thus converges in that space towards a function $v$. Since the sequence $u_\epsilon$ converges in $L^1$ towards $u$, the same reasoning as above shows $v = u$ and the result.

*Remark* 3.4.6. Gagliardo-Nirenberg inequality (3.4.2) has some interesting properties, beyond the most remarkable of being true. In the first place, this inequality has a scaling invariance: take $u \in W^{1,1}(\mathbb{R}^n)$ and $A \in Gl(n, \mathbb{R})$, and consider the function

$$u_A(x) = u(Ax)|\det A|^{\frac{n-1}{n}}, \text{ so that } (\nabla u_A)(x) = (\nabla u)(Ax)A|\det A|^{\frac{n-1}{n}}.$$

We have

$$\|u_A\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} = \Big(\int |u(Ax)|^{\frac{n}{n-1}}|\det A| dx\Big)^{\frac{n-1}{n}} = \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)},$$

and

$$\|\nabla u_A\|_{L^1(\mathbb{R}^n)} = \int |(\nabla u)(Ax)A||\det A|^{\frac{n-1}{n}} dx = \int |(\nabla u)(y)A||\det A|^{-\frac{1}{n}} dx.$$

Considering $(\nabla u)(x)$ as a linear form on $\mathbb{R}^n$, and $A$ as a linear endomorphism of $\mathbb{R}^n$, we have

$$\|(\nabla u)(x)A\| = \sup_{|T|=1} \|(\nabla u)(x)AT\|.$$

Let us assume now that $A = \alpha\Omega$, where $\alpha \in \mathbb{R}^*, \Omega \in O(n)$. We get then

$$\|(\nabla u)(x)A\| = |\alpha|\|(\nabla u)(x)\|, \quad |\det A| = |\alpha|^n,$$

so that $\|\nabla u_A\|_{L^1(\mathbb{R}^n)} = \|\nabla u\|_{L^1(\mathbb{R}^n)}$. Inequality (3.4.2) implies

$$\begin{aligned}\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \frac{1}{2n} \sum_{1\le j\le n} \int |(\partial_j u)(x)| dx \le \frac{1}{2\sqrt{n}} \int \big( \sum_{1\le j\le n} |(\partial_j u)(x)|^2\big)^{1/2} dx \\ = \frac{1}{2\sqrt{n}} \int \underbrace{\|\nabla u(x)\|}_{\substack{\text{Euclidean}\\ \text{norm on } \mathbb{R}^n}} dx = \frac{1}{2\sqrt{n}} \|\nabla u\|_{L^1(\mathbb{R}^n)}, \quad (3.4.4)\end{aligned}$$

and the latter is invariant by affine similarities (generated by homothetic transformations $x \mapsto x_0 + \alpha x$, $\alpha \in \mathbb{R}^*$, and linear isometries $x \mapsto \Omega x$, $\Omega \in O(n)$).

On the other hand, we shall use Theorem 3.4.5 to prove the so-called Sobolev inequalities of next section. Although these inequalities can be handled via some Fourier analysis methods, this is not the case of Gagliardo-Nirenberg inequality above which involves the $L^1$-norm of the gradient ($L^1$ is not so friendly to Fourier analysis). It is thus an interesting reminder that a clever but elementary combinatorial argument such as Lemma 3.4.3 can find its way into proving a statement which is not accessible to Fourier analysis. □

## 3.5 Sobolev spaces, Sobolev injection theorems

We begin with a lemma.

**Lemma 3.5.1.** *Let $n \geq 1$ be an integer and let $p, q \in [1, +\infty)$ such that $\frac{1}{q} = \frac{1}{n} + \frac{1}{p}$. Then there exists a constant $C(p, n)$ such that for all $v \in C_c^1(\mathbb{R}^n)$,*

$$\|v\|_{L^p(\mathbb{R}^n)} \leq C(p, n)\|\nabla v\|_{L^q(\mathbb{R}^n)}.$$

*Proof.* When $n = 1$, we find that the sought estimate is true as well for $p = +\infty, q = 1$ (this is (3.4.1)) and for $1 \leq p < +\infty$, we cannot have $q \geq 1$. We may thus assume that $n \geq 2$.

*Let us first suppose that* $v \geq 0$. We define $u = v^{\frac{p(n-1)}{n}}$: we note that

$$\frac{1}{p} + \frac{1}{n} \leq 1 \Longrightarrow \frac{1}{p} \leq \frac{n-1}{n} \Longrightarrow \frac{p(n-1)}{n} \geq 1,$$

so that we have with ordinary differentiation, $\partial_j u = \frac{p(n-1)}{n} v^{\frac{p(n-1)}{n}-1} \partial_j v$, and the function $u$ is also $C_c^1$. On the other hand we have, using (3.4.1),

$$\begin{aligned} \|v\|_{L^p}^p = \|u\|_{L^{\frac{n}{n-1}}}^{\frac{n}{n-1}} &\leq 2^{-\frac{n}{n-1}} \prod_{1\leq j\leq n} \|\partial_j u\|_{L^1}^{\frac{1}{n-1}} \\ &\leq 2^{-\frac{n}{n-1}} \Big(\frac{p(n-1)}{n}\Big)^{\frac{n}{n-1}} \underbrace{\Big(\prod_{1\leq j\leq n} \int |\partial_j v||v|^{p-\frac{p}{n}-1} dx\Big)^{\frac{1}{n-1}}}_{\text{term } I}, \end{aligned} \tag{3.5.1}$$

and this implies

$$\|v\|_{L^p}^{p(n-1)} \leq 2^{-n}\Big(\frac{p(n-1)}{n}\Big)^n \prod_{1\leq j\leq n} \big(\|\partial_j v\|_{L^q} \|v^{\frac{np-p-n}{n}}\|_{L^{q'}}\big).$$

We note that $\frac{(np-p-n)}{n} = p(1 - \frac{1}{n} - \frac{1}{p}) = \frac{p}{q'}$, so that if $q > 1$ we have proven

$$\|v\|_{L^p}^{p(n-1)} \leq 2^{-n}\Big(\frac{p(n-1)}{n}\Big)^n \Big(\prod_{1\leq j\leq n} \|\partial_j v\|_{L^q}\Big) \|v^p\|_{L^1}^{\frac{n}{q'}},$$

which gives (the result) for $v \not\equiv 0$,

$$\|v\|_{L^p}^n = \|v\|_{L^p}^{p(n-1)-\frac{np}{q'}} \leq 2^{-n}\Big(\frac{p(n-1)}{n}\Big)^n \prod_{1\leq j\leq n} \|\partial_j v\|_{L^q},$$

since $p(n-1) - \frac{np}{q'} = pn(1 - \frac{1}{n} - \frac{1}{q'}) = pn(\frac{1}{q} - \frac{1}{n}) = n$. If $q = 1$, we have in term $I$ above, $p - \frac{p}{n} - 1 = p(1 - \frac{1}{n} - \frac{1}{p}) = 0$, so that (3.5.1) gives the answer in the case $q = 1$.

*We drop now the non-negativity assumption on* $v$.

For $\epsilon > 0$, and $\chi \in C_c^\infty(\mathbb{R}^n; [0,1])$ equal to 1 near the support of $v$, we define the $C_c^1$ function $u_\epsilon$ by

$$u_\epsilon(x) = \big(v(x)^2 + \epsilon^2\big)^{\frac{1}{2}\frac{p(n-1)}{n}} \chi(x).$$

We have $\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^{\frac{n}{n-1}}}^{\frac{n}{n-1}} = \lim_{\epsilon\to 0} \int \big(v(x)^2 + \epsilon^2\big)^{\frac{p}{2}} \chi(x)^{\frac{n}{n-1}} dx = \|v\|_{L^p}^p$, and calculating

$$\nabla u_\epsilon = (\nabla\chi)\big(v^2 + \epsilon^2\big)^{\frac{1}{2}\frac{p(n-1)}{n}} + \chi\frac{p(n-1)}{2n}(v^2+\epsilon^2)^{\frac{p(n-1)}{2n}-1} 2v\nabla v,$$

using $p(n-1)/n \ge 1$, we get that

$$\lim_{\epsilon\to 0}(\nabla u_\epsilon)(x) = \chi(x)\frac{p(n-1)}{2n}|v(x)|^{\frac{p(n-1)}{n}-2} 2v(x)(\nabla v)(x),$$

so that with dominated convergence, we obtain

$$\lim_{\epsilon\to 0}\|\nabla u_\epsilon\|_{L^1} = \frac{p(n-1)}{n}\int |v|^{\frac{p(n-1)-n}{n}}|\nabla v| dx.$$

Applying Gagliardo-Nirenberg (3.4.4) to $u_\epsilon$ we find

$$\|v\|_{L^p}^{\frac{p(n-1)}{n}} = \lim_\epsilon \|u_\epsilon\|_{L^{\frac{n}{n-1}}} \le \frac{1}{2\sqrt{n}}\lim_\epsilon \|\nabla u_\epsilon\|_{L^1} = \frac{p(n-1)}{2n^{3/2}}\int |v|^{\frac{p(n-1)-n}{n}}|\nabla v| dx.$$

If $q = 1$, we have $p(n-1) - n = pn(1 - \frac{1}{n} - \frac{1}{p}) = \frac{pn}{q'} = 0$, $p(n-1) = n$ and the previous inequality gives the answer. If $q > 1$, we have $p(n-1) - n = \frac{pn}{q'}$ and Hölder's inequality implies

$$\|v\|_{L^p}^{\frac{p(n-1)}{n}} \le \frac{p(n-1)}{2n^{3/2}}\|v\|_{L^p}^{\frac{p}{q'}}\|\nabla v\|_{L^q}.$$

Since $\frac{p(n-1)}{n} - \frac{p}{q'} = p(1 - \frac{1}{n} - \frac{1}{q'}) = p(\frac{1}{q} - \frac{1}{n}) = 1$, this completes the proof of Lemma 3.5.1. □

**Proposition 3.5.2.** *Let $p \in [1, +\infty]$ and $s \in \mathbb{N}$. We define the Sobolev space $W^{s,p}(\mathbb{R}^n)$ as the set of functions $u \in L^p(\mathbb{R}^n)$ such that the distribution derivatives $\partial^\alpha u$ belong to $L^p(\mathbb{R}^n)$ when the multi-index $\alpha \in \mathbb{N}^n$ is such that $|\alpha| \le s$. This space is a Banach space for the norm*

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} = \sum_{|\alpha|\le s}\|\partial^\alpha u\|_{L^p(\mathbb{R}^n)}.$$

*When $p = 2$, it is a Hilbert space with dot-product*

$$(u, v)_{W^{s,2}(\mathbb{R}^n)} = \sum_{|\alpha|\le s}(\partial^\alpha u, \partial^\alpha v)_{L^2(\mathbb{R}^n)}.$$

*Proof.* This set is obviously a vector space. Let $(u_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $W^{s,p}(\mathbb{R}^n)$. Then, we find $u, v_\alpha \in L^p$ such that $\lim_k u_k = u, \lim_k \partial^\alpha u_k = v_\alpha$ in the Banach space $L^p(\mathbb{R}^n)$. Now for $\phi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\int v_\alpha \phi dx = \lim_k \int \phi\partial^\alpha u_k dx = \lim_k \langle\partial^\alpha u_k, \phi\rangle = (-1)^{|\alpha|}\lim_k\langle u_k, \partial^\alpha\phi\rangle$$
$$= (-1)^{|\alpha|}\lim_k \int u_k\partial^\alpha\phi dx = (-1)^{|\alpha|}\int u\partial^\alpha\phi dx = \langle\partial^\alpha u, \phi\rangle,$$

proving $v_\alpha = \partial^\alpha u$. □

**Lemma 3.5.3.** *Let $p \in [1, +\infty)$ and $k \in \mathbb{N}$. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$. More precisely, defining for $\epsilon > 0$, $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\int \rho(t)dt = 1$, $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on a neighborhood of 0, $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$, $\chi_\epsilon(x) = \chi(\epsilon x)$ and*

$$R_\epsilon u = \rho_\epsilon * \chi_\epsilon u, \tag{3.5.2}$$

*we have $\lim_{\epsilon \to 0} R_\epsilon u = u$ with convergence in $W^{k,p}(\mathbb{R}^n)$.*

*Proof.* Let $u \in W^{k,p}(\mathbb{R}^n)$. The sequence of compactly supported functions $(\chi_\epsilon u)$ converges in $L^p(\mathbb{R}^n)$ towards $u$. We have also

$$R_\epsilon u - u = \rho_\epsilon * (\chi_\epsilon u - u) + \rho_\epsilon * u - u,$$

so that $\|R_\epsilon u - u\|_{L^p} \le \|\chi_\epsilon u - u\|_{L^p} + \|\rho_\epsilon * u - u\|_{L^p}$ and the result for $k = 0$. For $|\alpha| \le k$, we have

$$\partial^\alpha R_\epsilon u - \partial^\alpha u = \rho_\epsilon * \partial^\alpha(\chi_\epsilon u) - \partial^\alpha u = \rho_\epsilon * ([\partial^\alpha, \chi_\epsilon]u) + \rho_\epsilon * (\chi_\epsilon \partial^\alpha u) - \partial^\alpha u,$$

entailing

$$\|\partial^\alpha R_\epsilon u - \partial^\alpha u\|_{L^p} \le \|R_\epsilon \partial^\alpha u - \partial^\alpha u\|_{L^p} + \sum_{\substack{\beta \le \alpha \\ |\beta| \ge 1}} \frac{\alpha!}{\beta!} \epsilon^{|\beta|} \|\rho_\epsilon * ((\partial^\beta \chi)_\epsilon \partial^{\alpha-\beta} u\|_{L^p},$$

which implies convergence in $W^{k,p}(\mathbb{R}^n)$ of $R_\epsilon u$. □

**Theorem 3.5.4.** *Let $n \ge 2$ be an integer and let $p, q \in [1, +\infty)$ such that $\frac{1}{p} = \frac{1}{n} + \frac{1}{q}$. Then we have the continuous embedding*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) = W^{0,q}(\mathbb{R}^n),$$

*and there exists $C(p, n) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^n)$,*

$$\|u\|_{L^q(\mathbb{R}^n)} \le C(p, n)\|\nabla u\|_{L^p(\mathbb{R}^n)}. \tag{3.5.3}$$

*Remark* 3.5.5. Note that when $p$ ranges in the interval $[1, n)$, we have $q = \frac{np}{n-p}$ ranging in $[\frac{n}{n-1}, +\infty)$. We shall use the notation

$$p^*(n) = \frac{np}{n-p} \qquad \text{for the } \textit{Sobolev conjugate exponent.} \tag{3.5.4}$$

We may note here that in the limiting case $p = n, q = +\infty$, the above inclusion does not hold for $n \ge 2$ (however Remark 3.5.6 shows that it is true for $n = 1$). Let $\beta \in (\frac{1}{n}, 1)$ and $w(x) = \chi(x)(\ln|x|)^{1-\beta}/(1-\beta)$, where $\chi \in C_c^\infty(\mathbb{R}^n)$ is equal to 1 on $B(0, 1/4)$ and is supported in $B(0, 1/2)$. We have

$$(\nabla w)(x) = (\ln|x|)^{-\beta}|x|^{-1}\frac{x}{|x|}\chi(x) + C_c^\infty(\mathbb{R}^n) \Longrightarrow$$
$$\|\nabla w\|_{L^n}^n \le C + C\int_0^{1/2} r^{n-1}r^{-n}|\ln r|^{-\beta n}dr = C + \int_2^{+\infty} \frac{dR}{R|\ln R|^{\beta n}} < +\infty,$$

since $n\beta > 1$. The function $w$ is also in $L^n(\mathbb{R}^n)$ since

$$\|w\|_{L^n}^n \le C_1 \int_0^{1/2} r^{n-1}|\ln r|^{(1-\beta)n}dr = C_1\int_2^{+\infty} \frac{(\ln R)^{(1-\beta)n}dR}{R^{n+1}} < +\infty.$$

However $w$ does not belong to $L^\infty$ since $\beta < 1$.

*Remark* 3.5.6. In the case $n = 1$, we have then $p = 1, q = +\infty$ and it is indeed true that $W^{1,1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Let $u \in W^{1,1}(\mathbb{R})$. In the proof of Theorem 3.4.5, we have shown the density of $C_c^1(\mathbb{R})$ in $W^{1,1}(\mathbb{R})$: let $(\phi_k)$ be a sequence of functions of $C_c^1(\mathbb{R})$ converging in $W^{1,1}(\mathbb{R})$. We have

$$u(x) = u(x) - \phi_k(x) + \int_{-\infty}^{x} \phi_k'(t)dt \Longrightarrow |u(x)| \le |u(x) - \phi_k(x)| + \|\phi_k'\|_{L^1(\mathbb{R})},$$

and thus $|u(x)| \le |u(x) - \phi_k(x)| + \|\phi_k' - u'\|_{L^1(\mathbb{R})} + \|u'\|_{L^1(\mathbb{R})}$. We may find a subsequence of $(\phi_k)$ converging almost everywhere to $u$ so that we have a.e.

$$|u(x)| \le \|u'\|_{L^1(\mathbb{R})} \Longrightarrow u \in L^\infty(\mathbb{R}),\ \|u\|_{L^\infty(\mathbb{R})} \le \|u'\|_{L^1(\mathbb{R})}.$$

*Proof of Theorem 3.5.4.* Let $u \in W^{1,p}(\mathbb{R}^n)$. Then from Lemma 3.5.3, we have $\lim_\epsilon R_\epsilon u = u$ in $W^{1,p}(\mathbb{R}^n)$. Moreover from Lemma 3.5.1, we find that

$$\|R_\epsilon u\|_{L^q(\mathbb{R}^n)} \le C(p,n)\|\nabla R_\epsilon u\|_{L^p(\mathbb{R}^n)}.$$

This inequality proves that $(R_\epsilon u)$ is a Cauchy sequence in $L^q(\mathbb{R}^n)$, thus converging towards some $v \in L^q(\mathbb{R}^n)$. Since $(R_\epsilon u)$ converges towards $u$ in $W^{1,p}(\mathbb{R}^n)$, we find for $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle v, \phi\rangle = \lim_\epsilon \int (R_\epsilon u)\phi dx = \langle u, \phi\rangle \Longrightarrow v = u, \quad u \in L^q(\mathbb{R}^n).$$

Passing to the limit with respect to $\epsilon$ in the inequality above gives (3.5.3). □

**Theorem 3.5.7.** *Let $0 \le l < k$ be integers, and let $1 \le p < q < +\infty$ be real numbers such that*

$$\frac{k-l}{n} = \frac{1}{p} - \frac{1}{q}. \qquad \text{Then } W^{k,p}(\mathbb{R}^n) \hookrightarrow W^{l,q}(\mathbb{R}^n).$$

*Proof.* If $n = 1$, we should have $p = 1, q = +\infty, k = l + 1$, and we have already seen that $W^{1,1}(\mathbb{R}) \hookrightarrow W^{0,\infty}(\mathbb{R})$, with

$$\|u\|_{L^\infty} \le \frac{1}{2}\|u'\|_{L^1} \text{ for } u, u' \in L^1$$
$$\Longrightarrow \text{ for } l \in \mathbb{N} \text{ and } u^{(l)}, u^{(l+1)} \in L^1(\mathbb{R}),\ \|u^{(l)}\|_{L^\infty} \le \frac{1}{2}\|u^{(l+1)}\|_{L^1},$$

which implies for $l \in \mathbb{N}$, $W^{1+l,1}(\mathbb{R}) \hookrightarrow W^{l,\infty}(\mathbb{R})$. We assume now $n \ge 2$ and we note that Theorem 3.5.4 tackles the case $k = 1, l = 0$ with the estimate

$$\forall u \in W^{1,p}(\mathbb{R}^n), \quad \|u\|_{L^q(\mathbb{R}^n)} \le C(p,n)\|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{n}.$$

We note that this implies

$$\forall u \in W^{1+l,p}(\mathbb{R}^n), \quad \|\nabla^l u\|_{L^q(\mathbb{R}^n)} \le C(p,n)\|\nabla^{l+1} u\|_{L^p(\mathbb{R}^n)}, \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{n},$$

which deals with the case $k = l + 1$. Let us assume that for $k - l = \nu \geq 1$, we have proven

$$\forall u \in W^{\nu+l,p}(\mathbb{R}^n), \quad \|\nabla^l u\|_{L^q(\mathbb{R}^n)} \leq C(p,n)\|\nabla^{l+\nu} u\|_{L^p(\mathbb{R}^n)}, \quad \frac{1}{p} - \frac{1}{q} = \frac{\nu}{n}.$$

This implies that for

$$\frac{1}{p_{\nu+1}} - \frac{1}{q_{\nu+1}} = \frac{\nu+1}{n}, \quad \frac{1}{p_{\nu+1}} - \frac{1}{q_{\nu+1}} - \frac{1}{n} = \frac{\nu}{n},$$

$$\forall u \in W^{\nu+l+1,p_{\nu+1}}(\mathbb{R}^n), \ \|\nabla^{l+1} u\|_{L^{q_\nu}(\mathbb{R}^n)} \leq C(p_{\nu+1},n)\|\nabla^{l+1+\nu} u\|_{L^{p_{\nu+1}}(\mathbb{R}^n)},$$

with $\frac{1}{p_{\nu+1}} - \frac{1}{q_\nu} = \frac{\nu}{n}$, $q_\nu = \frac{nq_{\nu+1}}{n+q_{\nu+1}}$. But we have

$$\|\nabla^l u\|_{L^r(\mathbb{R}^n)} \leq C(q_\nu,n)\|\nabla^{l+1} u\|_{L^{q_\nu}(\mathbb{R}^n)}, \quad \frac{1}{q_\nu} - \frac{1}{r} = \frac{1}{n},$$

so that $\frac{1}{r} = \frac{1}{q_{\nu+1}} + \frac{1}{n} - \frac{1}{n}$, i.e. $r = q_{\nu+1}$. We have thus proven by induction on $\nu$ that

$$\forall u \in W^{\nu+l,p}(\mathbb{R}^n), \quad \|\nabla^l u\|_{L^q(\mathbb{R}^n)} \leq C(p,n)\|\nabla^{l+\nu} u\|_{L^p(\mathbb{R}^n)}, \quad \frac{1}{p} - \frac{1}{q} = \frac{\nu}{n},$$

proving the sought result. □

*Remark* 3.5.8. We have proven above that

$$W^{k,p}(\mathbb{R}^n) \hookrightarrow W^{l,q}(\mathbb{R}^n), \quad \text{for} \quad \frac{k-l}{n} = \frac{1}{p} - \frac{1}{q}, \quad 1 \leq p < q < +\infty.$$

Note that in this formula, we have $k > l$ but $p < q$ so that the functions in $W^{k,p}$ have more derivatives but less Lebesgue regularity than the functions in $W^{l,q}$. This means that we can somehow trade some regularity in terms of derivatives (first index $k > l$) to buy some $L^q$ regularity according to the fixed exchange rate given by $\frac{k-l}{n} = \frac{1}{p} - \frac{1}{q}$. We see also that Lebesgue regularity is a non-convertible currency which cannot buy a derivative regularity.

## 3.6 Exercises

**Exercise 3.6.1.** *Let $p, q, r \in [1,2]$ such that (2.2.1) holds. Let $u \in L^p(\mathbb{R}^n), v \in L^q(\mathbb{R}^n)$. Prove that $\hat{u} \in L^{p'}(\mathbb{R}^n), v \in L^{q'}(\mathbb{R}^n)$ and that the product $\hat{u}\hat{v}$ belongs to $L^{r'}(\mathbb{R}^n)$. Show that*

$$u * v \in L^r(\mathbb{R}^n) \text{ and } \quad \widehat{u * v} = \hat{u}\hat{v}.$$

*Answer.* The fact that $u * v$ belongs to $L^r$ follows from Young's inequality and we have $\hat{u} \in L^{p'}, \hat{v} \in L^{q'}$ from Hausdorff-Young Theorem. This implies from Hölder's inequality that the product $\hat{u}\hat{v}$ belongs to $L^{r'}$ since

$$\int |\hat{u}|^{r'}|\hat{v}|^{r'} d\xi \leq \left(\int |\hat{u}|^{sr'} d\xi\right)^{1/s} \left(\int |\hat{v}|^{s'r'} d\xi\right)^{1/s'},$$

where we may choose

$$s = \frac{p'}{r'} \Longrightarrow \frac{1}{s'} = 1 - \frac{r'}{p'} = r'\left(\frac{1}{r'} - \frac{1}{p'}\right) = \frac{r'}{q'} \Longrightarrow r's' = q'.$$

The above argument extends when $r' = +\infty$ (which implies $p' = q' = +\infty$ so that $p = q = r = 1$ and $\hat{u}, \hat{v}$ belong to $L^\infty$). We have thus

$$\|\widehat{u * v}\|_{L^{r'}(\mathbb{R}^n)} \le \|\hat{u}\|_{L^{p'}(\mathbb{R}^n)} \|\hat{v}\|_{L^{q'}(\mathbb{R}^n)} \le \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^q(\mathbb{R}^n)}. \tag{3.6.1}$$

To get that $\widehat{u * v} = \hat{u}\hat{v}$, it is enough to prove it for $u, v$ in the Schwartz space since then we shall obtain with $\varphi_k, \psi_k \in \mathscr{S}(\mathbb{R}^n)$ such that $\lim_k \varphi_k = u$ in $L^p$, $\lim_k \psi_k = v$ in $L^q$, thanks to (3.6.1),

$$\hat{u}\hat{v} = \underbrace{\lim_k \widehat{\varphi_k}}_{\substack{\text{limit}\\ \text{in } L^{p'}}} \underbrace{\lim_l \widehat{\psi_l}}_{\substack{\text{limit}\\ \text{in } L^{q'}}} = \underbrace{\lim_k \widehat{\varphi_k * \psi_k}}_{\substack{\text{limit}\\ \text{in } L^{r'}}} = \widehat{u * v}.$$

Formula (1.2.11) gives the result.

**Exercise 3.6.2.** *Show that if $T$ satisfies the assumptions of Theorem 3.1.3 with $r = +\infty$ and*

$$t\omega(t, Tu) \le c_1 \|u\|_{L^1}, \qquad \|Tu\|_{L^\infty} \le c_\infty \|u\|_{L^\infty},$$

*then for $1 < p < +\infty$, we have*

$$\|Tu\|_{L^p} \le \frac{p^{1+\frac{1}{p}}}{p-1} c_1^{1/p} c_\infty^{1/p'} \|u\|_{L^p}.$$

*Answer.* We have only to revisit the proof of Theorem 3.1.3 with paying more attention to the choice of the various constants. We write for $u \in L^1 + L^\infty$, $t > 0$, $\alpha > c_\infty$,

$$u = \underbrace{u\mathbf{1}_{\{|u| > t/\alpha\}}}_{u_1} + \underbrace{u\mathbf{1}_{\{|u| \le t/\alpha\}}}_{u_2}, \tag{3.6.2}$$

and this gives

$$|(Tu)(x)| \le |(Tu_1)(x)| + |(Tu_2)(x)| \le |(Tu_1)(x)| + \|u_2\|_{L^\infty} \le |(Tu_1)(x)| + \frac{c_\infty t}{\alpha},$$

so that we find the inclusion

$$(\sharp) \qquad \{x, |(Tu)(x)| > t\} \subset \{x, |(Tu_1)(x)| > t(1 - c_\infty \alpha^{-1})\}.$$

The weak-type $(1,1)$ assumption reads $t\omega(t, Tv) \le c_1 \|v\|_{L^1}$ so that

$$(\flat) \qquad \omega(t(1 - c_\infty \alpha^{-1}), Tu_1) \le \frac{c_1}{t(1 - c_\infty \alpha^{-1})} \int_{|u| > t/\alpha} |u| dx.$$

Applying Formula (3.1.7) to $Tu$, we find, using Tonelli Theorem and $1 < p < +\infty$,

$$\begin{aligned}
\|Tu\|_{L^p}^p &= p\int_0^{+\infty} t^{p-1}\omega(t, Tu)dt \\
\text{(from } (\sharp)) \quad &\leq p\int_0^{+\infty} t^{p-1}\omega(t(1-c_\infty\alpha^{-1}), Tu_1)dt \\
\text{(from } (\flat)) \quad &\leq p\int_0^{+\infty} t^{p-1}\frac{c_1}{t(1-c_\infty\alpha^{-1})}\int_{|u|>t/\alpha}|u|dxdt \\
&= \frac{pc_1}{1-c_\infty\alpha^{-1}}\iint_{\mathbb{R}_+\times\mathbb{R}^n} t^{p-2}H(\alpha|u(x)|-t)|u(x)|dtdx \\
&= \frac{pc_1}{(1-c_\infty\alpha^{-1})(p-1)}\int_{\mathbb{R}^n}(\alpha|u(x)|)^{p-1}|u(x)|dx \\
&= \frac{\alpha^{p-1}pc_1}{(1-c_\infty\alpha^{-1})(p-1)}\|u\|_{L^p}^p.
\end{aligned}$$

We check now for $\alpha = \lambda c_\infty$ with $\lambda > 1$ (assuming of course $c_\infty > 0$),

$$\frac{\alpha^{p-1}pc_1}{(1-c_\infty\alpha^{-1})(p-1)} = p'c_1\frac{\lambda^p c_\infty^{p-1}}{\lambda-1}.$$

We have proven that for any $\lambda > 1$,

$$\sup_{\|u\|_{L^p}=1}\|Tu\|_{L^p} \leq (p'c_1)^{1/p}\frac{\lambda}{(\lambda-1)^{1/p}}c_\infty^{1/p'},$$

so that choosing $\lambda = p/(p-1)$ gives the sought answer.

**Exercise 3.6.3.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ be an $L^1_{loc}$ function. Prove that $\mathcal{M}_f$ is a measurable function (see Definition 3.2.1).*

*Answer.* For each $t > 0$ the function $\mathbb{R}^n \times \mathbb{R}^n \ni (x, z) \mapsto f(x + tz)$ is measurable and Proposition 4.1.3 in [11] implies that

$$x \mapsto |B(x,t)|^{-1}\int_{B(x,t)}|f(y)|dy = |\mathbb{B}^n|^{-1}\int_{\mathbb{B}^n}|f(x+tz)|dz,$$

is measurable. Proposition 1.3.1 in [11] proves that

$$\widetilde{\mathcal{M}}_f(x) = \sup_{t\in\mathbb{Q}_+^*} \fint_{\mathbb{B}^n}|f(x+tz)|dz$$

is measurable. Let $\epsilon > 0$ be given. Let us consider $t > 0$ and $0 < s \in \mathbb{Q}$ such that $t \leq s \leq t(1+\epsilon)$; we have

$$\frac{1}{t^n|\mathbb{B}^n|}\int_{B(x,t)}|f(y)|dy \leq \frac{1}{t^n|\mathbb{B}^n|}\int_{B(x,s)}|f(y)|dy \leq (\frac{s}{t})^n\widetilde{\mathcal{M}}_f(x) \leq (1+\epsilon)^n\widetilde{\mathcal{M}}_f(x),$$

which implies $\mathcal{M}_f(x) \leq (1+\epsilon)^n\widetilde{\mathcal{M}}_f(x)$. Since $\widetilde{\mathcal{M}}_f(x) \leq \mathcal{M}_f(x)$, we find that for any $\epsilon > 0$, $\mathcal{M}_f(x) \leq (1+\epsilon)^n\widetilde{\mathcal{M}}_f(x) \leq (1+\epsilon)^n\mathcal{M}_f(x)$, proving that $\mathcal{M}_f$ is equal to the measurable $\widetilde{\mathcal{M}}_f$ (this works in particular when $\mathcal{M}_f(x) = +\infty$).

**Exercise 3.6.4.** *Let $F$ be defined on $\mathbb{R}$ by $F(0) = 0$ and for $x \neq 0, F(x) = x^2 \sin(x^{-2})$.*
(1) *Prove that $F$ is differentiable everywhere and calculate its derivative $F'$.*
(2) *Prove that $F'$ is not locally integrable.*
(3) *Prove that the weak derivative of $F$ is not a Radon measure.*

*Answer.*(1) Differentiability outside 0 is obvious with

$$x \neq 0, \quad F'(x) = 2x\sin(x^{-2}) - 2x^{-1}\cos(x^{-2}), \qquad F'(0) = \lim_{x\to 0} x\sin(x^{-2}) = 0.$$

We note in particular that $F'$ is not continuous since $F'(\frac{1}{\sqrt{2k\pi}}) = -2\sqrt{2k\pi}$ for $k \in \mathbb{N}^*$.
(2) Since $2x\sin(x^{-2})$ is locally bounded, we have to prove that $x^{-1}\cos(x^{-2})$ is not locally integrable:

$$\int_0^1 |\cos(x^{-2})|x^{-1}dx = \frac{1}{2}\int_1^{+\infty} |\cos t|\frac{dt}{t} = +\infty.$$

(3) The weak derivative $f$ of $F$ is defined as a linear form on $C_c^\infty(\mathbb{R})$ functions (or as a tempered distribution, cf. Chapter 8 with Definition 1.2.7), with

$$\langle F', \varphi\rangle = -\int_{\mathbb{R}} F(x)\varphi'(x)dx.$$

Let us assume that $\varphi$ is supported in $(0, +\infty)$: we have then

$$\langle F', \varphi\rangle = \int \big(2x\sin(x^{-2}) - 2x^{-1}\cos(x^{-2})\big)\varphi(x)dx.$$

We choose now $\varphi_k \in C_c^\infty((a_k, b_k); [0, 1])$ with $k \in \mathbb{N}^*$,

$$a_k = (2\pi k + \frac{\pi}{4})^{-1/2}, \quad b_k = (2\pi k - \frac{\pi}{4})^{-1/2}$$

so that $x \in (a_k, b_k) \implies x^{-2} \in (2\pi k - \frac{\pi}{4}, 2\pi k + \frac{\pi}{4}) \implies \cos(x^{-2}) \in (2^{-1/2}, 1]$. As a result, we have

$$\int_{a_k}^{b_k} x^{-1}\cos(x^{-2})\varphi_k(x)dx \geq 2^{-1/2}(2\pi k - \frac{\pi}{4})^{1/2}\int_{a_k}^{b_k} \varphi_k(x)dx.$$

We may also assume that $\varphi_k$ equals 1 on $\big[(2\pi k + \frac{\pi}{6})^{-1/2}, (2\pi k - \frac{\pi}{6})^{-1/2}\big]$, implying

$$\int_{a_k}^{b_k} x^{-1}\cos(x^{-2})\varphi_k(x)dx \geq 2^{-1/2}(2\pi k - \frac{\pi}{4})^{1/2}\frac{\pi}{3}\frac{1}{2}(2\pi k + \frac{\pi}{6})^{-3/2} \geq c_0 k^{-1}.$$

Since the intervals $(a_k, b_k)$ are pairwise disjoint, the function

$$\Phi_N(x) = \sum_{1\leq k\leq N} \varphi_k(x),$$

is such that $\Phi_N \in C_c^\infty((0, +\infty); [0, 1])$ and

$$\langle F', \Phi_N\rangle \leq -c_0 \sum_{1\leq k\leq N} \frac{1}{k} + \int_0^1 2x dx \underset{N\to+\infty}{\longrightarrow} -\infty.$$

# Chapter 4

# Introduction to pseudodifferential operators

## 4.1 Prolegomena

A differential operator of order $m$ on $\mathbb{R}^n$ can be written as

$$a(x,D)=\sum_{|\alpha|\le m} a_\alpha(x)D_x^\alpha,$$

where we have used the notation (1.2.8) for the multi-indices. Its *symbol* is a polynomial in the variable $\xi$ and is defined as

$$a(x,\xi)=\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha, \qquad \xi^\alpha=\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}.$$

We have the formula

$$(a(x,D)u)(x)=\int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi}a(x,\xi)\hat u(\xi)d\xi, \tag{4.1.1}$$

where $\hat u$ is the Fourier transform. It is possible to generalize the previous formula to the case where $a$ is a tempered distribution on $\mathbb{R}^{2n}$.

Let $u,v$ be in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$. Then the function

$$\mathbb{R}^n\times\mathbb{R}^n \ni (x,\xi)\mapsto \hat u(\xi)\bar v(x)e^{2i\pi x\cdot\xi}=\Omega_{u,v}(x,\xi) \tag{4.1.2}$$

belongs to $\mathscr{S}(\mathbb{R}^{2n})$ and the mapping $(u,v)\mapsto\Omega_{u,v}$ is sesquilinear continuous. Using these notations, we can provide the following definition.

**Definition 4.1.1.** Let $a\in\mathscr{S}'(\mathbb{R}^{2n})$ be a tempered distribution. We define the operator $a(x,D):\mathscr{S}(\mathbb{R}^n)\longrightarrow\mathscr{S}^*(\mathbb{R}^n)$ by the formula

$$\langle a(x,D)u,v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}=\prec a,\Omega_{u,v}\succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

where $\mathscr{S}^*(\mathbb{R}^n)$ is the antidual of $\mathscr{S}(\mathbb{R}^n)$ (continuous antilinear forms). The distribution $a$ is called the symbol of the operator $a(x,D)$.

*N.B.* The duality product $\langle u, v\rangle_{\mathscr{S}^*(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$, is linear in the variable $u$ and antilinear in the variable $v$. We shall use the same notation for the dot product in the *complex* Hilbert space $L^2$ with the notation

$$\langle u, v\rangle_{L^2} = \int u(x)\overline{v(x)}dx.$$

The general rule that we shall follow is to always use the sesquilinear duality as above, except if specified otherwise. For the real duality, as in the left-hand-side of the formula in Definition 4.1.1, we shall use the notation $\prec u, v \succ = \int u(x)v(x)dx$, e.g. for $u, v \in \mathscr{S}(\mathbb{R}^n)$.

Although the previous formula is quite general, since it allows us to *quantize*[1] any tempered distribution on $\mathbb{R}^{2n}$, it is not very useful, since we cannot compose this type of operators. We are in fact looking for an algebra of operators and the following theorem is providing a simple example.

In the sequel we shall denote by $C_b^\infty(\mathbb{R}^{2n})$ the (Fréchet) space of $C^\infty$ functions on $\mathbb{R}^{2n}$ which are bounded as well as all their derivatives.

**Theorem 4.1.2.** *Let $a \in C_b^\infty(\mathbb{R}^{2n})$. Then the operator $a(x, D)$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself.*

*Proof.* Using Definition 4.1.1, we have for $u, v \in \mathscr{S}(\mathbb{R}^n), a \in C_b^\infty(\mathbb{R}^{2n})$,

$$\langle a(x, D)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \iint e^{2i\pi x\cdot\xi} a(x, \xi)\hat{u}(\xi)\bar{v}(x)dxd\xi.$$

On the other hand the function $U(x) = \int e^{2i\pi x\cdot\xi} a(x, \xi)\hat{u}(\xi)d\xi$ is smooth and such that, for any multi-indices $\alpha, \beta$,

$$\begin{aligned} x^\beta D_x^\alpha U(x) &= (-1)^{|\beta|} \sum_{\alpha'+\alpha''=\alpha} \frac{\alpha!}{\alpha'!\alpha''!} \int e^{2i\pi x\cdot\xi} D_\xi^\beta\big(\xi^{\alpha'}(D_x^{\alpha''}a)(x, \xi)\hat{u}(\xi)\big)d\xi \\ &= (-1)^{|\beta|} \sum_{\alpha'+\alpha''=\alpha} \frac{\alpha!}{\alpha'!\alpha''!} \int e^{2i\pi x\cdot\xi} D_\xi^\beta\big((D_x^{\alpha''}a)(x, \xi)\widehat{D^{\alpha'}u}(\xi)\big)d\xi \end{aligned}$$

and thus

$$\sup_{x\in\mathbb{R}^n} |x^\beta D_x^\alpha U(x)| \le \sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} \frac{\alpha!}{\alpha'!\alpha''!}\frac{\beta!}{\beta'!\beta''!} \|D_\xi^{\beta'} D_x^{\alpha''} a\|_{L^\infty(\mathbb{R}^{2n})} \|D^{\beta''}\widehat{D^\alpha u}\|_{L^1(\mathbb{R}^n)}.$$

Since the Fourier transform and $\partial_{x_j}$ are continuous on $\mathscr{S}(\mathbb{R}^n)$, we get that the mapping $u \mapsto U$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself. The above defining formula for $a(x, D)$ ensures that $a(x, D)u = U$. □

[1] We mean simply here that we are able to define a linear mapping from $\mathscr{S}'(\mathbb{R}^{2n})$ to the set of continuous operators from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$.

The Schwartz space $\mathscr{S}(\mathbb{R}^{2n})$ is not dense in the Fréchet space $C_b^\infty(\mathbb{R}^{2n})$ (e.g. $\forall \varphi \in \mathscr{S}(\mathbb{R}^{2n}), \sup_{x\in\mathbb{R}^{2n}} |1-\varphi(x)| \ge 1$) but, in somewhat pedantic terms, one may say that this density is true for the bornology on $C_b^\infty(\mathbb{R}^{2n})$; in simpler terms, let $a$ be a function in $C_b^\infty(\mathbb{R}^{2n})$ and take for instance

$$a_k(x,\xi) = a(x,\xi)e^{-(|x|^2+|\xi|^2)k^{-2}}.$$

It is easy to see that each $a_k$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$, that the sequence $(a_k)$ is bounded in $C_b^\infty(\mathbb{R}^{2n})$ and converges in $C^\infty(\mathbb{R}^{2n})$ to $a$. This type of density will be enough for the next lemma.

**Lemma 4.1.3.** *Let $(a_k)$ be a sequence in $\mathscr{S}(\mathbb{R}^{2n})$ such that $(a_k)$ is bounded in the Fréchet space $C_b^\infty(\mathbb{R}^{2n})$ and $(a_k)$ is converging in $C^\infty(\mathbb{R}^{2n})$ to a function $a$. Then $a$ belongs to $C_b^\infty(\mathbb{R}^{2n})$ and for any $u \in \mathscr{S}(\mathbb{R}^n)$, the sequence $(a_k(x,D)u)$ converges to $a(x,D)u$ in $\mathscr{S}(\mathbb{R}^n)$.*

*Proof.* The fact that $a$ belongs to $C_b^\infty(\mathbb{R}^{2n})$ is obvious. Using the identities in the proof of Theorem 4.1.2 we see that

$$\begin{aligned}
&x^\beta D_x^\alpha\big(a_k(x,D)u - a(x,D)u\big) = x^\beta D_x^\alpha\big((a_k-a)(x,D)u\big)\\
&= (-1)^{|\beta|}\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} \frac{\alpha!}{\alpha'!\alpha''!}\frac{\beta!}{\beta'!\beta''!}\int e^{2i\pi x\cdot\xi}\big(D_\xi^{\beta'}D_x^{\alpha''}(a_k-a)\big)(x,\xi)D_\xi^{\beta''}\widehat{D^{\alpha'}u}(\xi)d\xi\\
&= \sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} \frac{\alpha!}{\alpha'!\alpha''!}\frac{\beta!}{\beta'!\beta''!}(1+|x|^2)^{-1}\\
&\qquad\times\int (1+|D_\xi|^2)\big(e^{2i\pi x\cdot\xi}\big)\big(D_\xi^{\beta'}D_x^{\alpha''}(a_k-a)\big)(x,\xi)D_\xi^{\beta''}\widehat{D^{\alpha'}u}(\xi)d\xi,
\end{aligned}$$

that is a (finite) sum of terms of type $V_k(x) = (1+|x|^2)^{-1}\int e^{2i\pi x\cdot\xi}b_k(x,\xi)w_u(\xi)d\xi$ with the sequence $(b_k)$ bounded in $C_b^\infty(\mathbb{R}^{2n})$ and converging to 0 in $C^\infty(\mathbb{R}^{2n})$, $u\mapsto w_u$ linear continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself. As a consequence we get that, with $R_1, R_2$ positive parameters,

$$\begin{aligned}
|V_k(x)| \le &\sup_{\substack{|x|\le R_1\\ |\xi|\le R_2}} |b_k(x,\xi)|\int_{|\xi|\le R_2}|w_u(\xi)|d\xi\mathbf{1}_{|x|\le R_1}\\
&+\int_{|\xi|\ge R_2}|w_u(\xi)|d\xi\sup_{k\in\mathbb{N}}\|b_k\|_{L^\infty(\mathbb{R}^{2n})}\mathbf{1}_{|x|\le R_1}\\
&+R_1^{-2}\mathbf{1}_{|x|\ge R_1}\sup_{k\in\mathbb{N}}\|b_k\|_{L^\infty(\mathbb{R}^{2n})}\int|w_u(\xi)|d\xi,
\end{aligned}$$

implying

$$\begin{aligned}
|V_k(x)| \le \varepsilon_k(R_1,R_2)&\int|w_u(\xi)|d\xi + \eta(R_2)\sup_{k\in\mathbb{N}}\|b_k\|_{L^\infty(\mathbb{R}^{2n})}\\
&+\theta(R_1)\sup_{k\in\mathbb{N}}\|b_k\|_{L^\infty(\mathbb{R}^{2n})}\int|w_u(\xi)|d\xi,
\end{aligned}$$

with $\lim_{k\to+\infty} \varepsilon_k(R_1, R_2) = 0, \lim_{R\to+\infty} \eta(R) = \lim_{R\to+\infty} \theta(R) = 0$. Thus we have for all positive $R_1, R_2$,

$$\limsup_{k\to+\infty} \|V_k\|_{L^\infty} \le \eta(R_2) \sup_{k\in\mathbb{N}} \|b_k\|_{L^\infty(\mathbb{R}^{2n})} + \theta(R_1) \sup_{k\in\mathbb{N}} \|b_k\|_{L^\infty(\mathbb{R}^{2n})} \int |w_u(\xi)| d\xi,$$

entailing (by taking the limit when $R_1, R_2$ go to infinity) that $\lim_{k\to+\infty} \|V_k\|_{L^\infty} = 0$ which gives the result of the lemma. □

**Theorem 4.1.4.** *Let $a \in C_b^\infty(\mathbb{R}^{2n})$: the operator $a(x, D)$ is bounded on $L^2(\mathbb{R}^n)$.*

*Proof.* Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, it is enough to prove that there exists a constant $C$ such that for all $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$|\langle a(x, D)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}| \le C\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)}.$$

We introduce the polynomial on $\mathbb{R}^n$ defined by $P_k(t) = (1 + |t|^2)^{k/2}$, where $k \in 2\mathbb{N}$, and the function

$$W_u(x, \xi) = \int u(y)P_k(x - y)^{-1}e^{-2i\pi y\cdot\xi}dy.$$

The function $W_u$ is the partial Fourier transform of the function $\mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto u(y)P_k(x - y)^{-1}$ and if $k > n/2$ (we assume this in the sequel), we obtain that $\|W_u\|_{L^2(\mathbb{R}^{2n})} = c_k\|u\|_{L^2(\mathbb{R}^n)}$. Moreover, since $u \in \mathscr{S}(\mathbb{R}^n)$, the function $W_u$ belongs to $C^\infty(\mathbb{R}^{2n})$ and satisfies for all multi-indices $\alpha, \beta, \gamma$

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} P_k(x)\xi^\gamma|(\partial_x^\alpha\partial_\xi^\beta W_u)(x, \xi)| < \infty.$$

In fact we have

$$\xi^\gamma(\partial_x^\alpha\partial_\xi^\beta W_u)(x, \xi) = \int \overbrace{u(y)(-2i\pi y)^\beta}^{\in\mathscr{S}(\mathbb{R}^n)} \partial^\alpha(1/P_k)(x - y)(-1)^{|\gamma|}D_y^\gamma(e^{-2i\pi y\cdot\xi})dy$$
$$= \sum_{\gamma'+\gamma''=\gamma} \frac{\gamma!}{\gamma'!\gamma''!} \int D_y^{\gamma'}\big(u(y)(-2i\pi y)^\beta\big)(-2i\pi)^{-|\gamma''|}$$
$$\partial^{\gamma''+\alpha}(1/P_k)(x - y)(e^{-2i\pi y\cdot\xi})dy$$

and

$$|\partial^\alpha(1/P_k)(x - y)| \le C_{\alpha,k}(1 + |x - y|)^{-k} \le C_{\alpha,k}(1 + |x|)^{-k}(1 + |y|)^k.$$

From Definition 4.1.1, we have

$$\langle a(x, D)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{2i\pi x\cdot\xi}a(x, \xi)\hat{u}(\xi)\bar{v}(x)dxd\xi,$$

and we obtain, using an integration by parts justified by the regularity and decay of the functions $W$ above,

$$\langle a(x, D)u, v\rangle$$
$$= \iint a(x, \xi)P_k(D_\xi)\left(\int u(y)P_k(x - y)^{-1}e^{2i\pi(x-y)\cdot\xi}dy\right)\bar{v}(x)dxd\xi$$
$$= \iint a(x, \xi)P_k(D_\xi)\big(\underbrace{e^{2i\pi x\cdot\xi}W_u(x, \xi)\bar{v}(x)}_{\in\mathscr{S}(\mathbb{R}^{2n})}\big)dxd\xi$$

$$= \iint (P_k(D_\xi)a)(x,\xi)W_u(x,\xi)P_k(D_x)\left(\int e^{2i\pi x\cdot(\xi-\eta)}P_k(\xi-\eta)^{-1}\overline{\hat{v}(\eta)}d\eta\right)dxd\xi$$
$$= \iint (P_k(D_\xi)a)(x,\xi)W_u(x,\xi)P_k(D_x)\left(W_{\bar{\hat{v}}}(\xi,x)e^{2i\pi x\cdot\xi}\right)dxd\xi$$
$$= \sum_{0\le l\le k/2} C^l_{k/2}\iint |D_x|^{2l}\Big((P_k(D_\xi)a)(x,\xi)W_u(x,\xi)\Big)W_{\bar{\hat{v}}}(\xi,x)e^{2i\pi x\cdot\xi}dxd\xi$$
$$= \sum_{\substack{|\alpha|\le k\\ |\beta|+|\gamma|\le k}} c_{\alpha\beta\gamma}\iint \underbrace{(D^\alpha_\xi D^\beta_x a)(x,\xi)}_{\text{bounded}}D^\gamma_x(W_u)(x,\xi)\underbrace{W_{\bar{\hat{v}}}(\xi,x)}_{\substack{\in L^2(\mathbb{R}^{2n})\text{ with norm}\\ c_k\|v\|_{L^2(\mathbb{R}^n)}}}e^{2i\pi x\cdot\xi}dxd\xi.$$

Checking now the $x$-derivatives of $W_u$, we see that

$$D^\gamma_x(W_u)(x,\xi) = \int u(y)D^\gamma(1/P_k)(x-y)e^{-2i\pi y\cdot\xi}dy,$$

and since $D^\gamma(1/P_k)$ belongs to $L^2(\mathbb{R}^n)$ (since $k > n/2$), we get that the $L^2(\mathbb{R}^{2n})$ norm of $D^\gamma_x(W_u)$ is bounded above by $c_\gamma\|u\|_{L^2(\mathbb{R}^n)}$. Using the Cauchy-Schwarz inequality, we obtain that

$$|\langle a(x,D)u,v\rangle| \le \sum_{\substack{|\alpha|\le k\\ |\beta|+|\gamma|\le k}} c_{\alpha\beta\gamma}\|\partial^\alpha_\xi\partial^\beta_x a\|_{L^\infty(\mathbb{R}^{2n})}\|D^\gamma_x W_u\|_{L^2(\mathbb{R}^{2n})}\|W_{\bar{\hat{v}}}\|_{L^2(\mathbb{R}^{2n})}$$
$$\le C_n\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)}\sup_{\substack{|\alpha|\le k\\ |\beta|\le k}}\|\partial^\alpha_\xi\partial^\beta_x a\|_{L^\infty(\mathbb{R}^{2n})},$$

where $C_n$ depends only on $n$ and $2\mathbb{N}\ni k > n/2$, which is the sought result. □

The next theorem gives us our first algebra of pseudodifferential operators.

**Theorem 4.1.5.** *Let $a,b$ be in $C^\infty_b(\mathbb{R}^{2n})$. Then the composition $a(x,D)b(x,D)$ makes sense as a bounded operator on $L^2(\mathbb{R}^n)$ (also as a continuous operator from $\mathscr{S}(\mathbb{R}^n)$ into itself), and $a(x,D)b(x,D) = (a\diamond b)(x,D)$ where $a\diamond b$ belongs to $C^\infty_b(\mathbb{R}^{2n})$ and is given by the formula*

$$(a\diamond b)(x,\xi) = (\exp 2i\pi D_y\cdot D_\eta)(a(x,\xi+\eta)b(y+x,\xi))_{|y=0,\eta=0}, \tag{4.1.3}$$
$$(a\diamond b)(x,\xi) = \iint e^{-2i\pi y\cdot\eta}a(x,\xi+\eta)b(y+x,\xi)dyd\eta, \tag{4.1.4}$$

*when $a$ and $b$ belong to $\mathscr{S}(\mathbb{R}^{2n})$. The mapping $a,b\mapsto a\diamond b$ is continuous for the topology of Fréchet space of $C^\infty_b(\mathbb{R}^{2n})$. Also if $(a_k),(b_k)$ are sequences of functions in $\mathscr{S}(\mathbb{R}^{2n})$, bounded in $C^\infty_b(\mathbb{R}^{2n})$, converging in $C^\infty(\mathbb{R}^{2n})$ respectively to $a,b$, then $a$ and $b$ belong to $C^\infty_b(\mathbb{R}^{2n})$, the sequence $(a_k\diamond b_k)$ is bounded in $C^\infty_b(\mathbb{R}^{2n})$ and converges in $C^\infty(\mathbb{R}^{2n})$ to $a\diamond b$.*

*Remark* 4.1.6. From Lemma 4.1.2 in [9], we know that the operator $e^{2i\pi D_y\cdot D_\eta}$ is an isomorphism of $C^\infty_b(\mathbb{R}^{2n})$, which gives a meaning to the formula (4.1.3), since for $a,b\in C^\infty_b(\mathbb{R}^{2n})$, $(x,\xi)$ given in $\mathbb{R}^{2n}$, the function $(y,\eta)\mapsto a(x,\xi+\eta)b(y+x,\xi) = C_{x,\xi}(y,\eta)$ belongs to $C^\infty_b(\mathbb{R}^{2n})$ as well as $JC_{x,\xi}$ and we can take the value of the latter at $(y,\eta)=(0,0)$.

*Proof.* Let us first assume that $a, b \in \mathscr{S}(\mathbb{R}^{2n})$. The kernels $k_a, k_b$ of the operators $a(x,D), b(x,D)$ belong also to $\mathscr{S}(\mathbb{R}^{2n})$ and the kernel $k_c$ of $a(x,D)b(x,D)$ is given by (we use Fubini's theorem)

$$k(x,y) = \int k_a(x,z)k_b(z,y)dz = \iiint a(x,\xi)e^{2i\pi(x-z)\cdot\xi}b(z,\zeta)e^{2i\pi(z-y)\cdot\zeta}d\zeta d\xi dz.$$

The function $k$ belongs also to $\mathscr{S}(\mathbb{R}^{2n})$ and we get, for $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned}
&\langle a(x,D)b(x,D)u, v\rangle_{L^2(\mathbb{R}^n)} \\
&\quad = \iiiint\!\!\int a(x,\xi)e^{2i\pi(x-z)\cdot\xi}b(z,\zeta)e^{2i\pi(z-y)\cdot\zeta}u(y)\bar{v}(x)d\zeta d\xi dz dy dx. \\
&\quad = \iiiint a(x,\xi)e^{2i\pi(x-z)\cdot\xi}b(z,\zeta)e^{2i\pi z\cdot\zeta}\hat{u}(\zeta)d\zeta d\xi dz\bar{v}(x)dx. \\
&\quad = \iiiint a(x,\xi)e^{2i\pi(x-z)\cdot\xi}b(z,\zeta)e^{2i\pi(z-x)\cdot\zeta}d\xi dz e^{2i\pi x\cdot\zeta}\hat{u}(\zeta)d\zeta\bar{v}(x)dx. \\
&\quad = \iint c(x,\zeta)e^{2i\pi x\cdot\zeta}\hat{u}(\zeta)d\zeta\bar{v}(x)dx,
\end{aligned}$$

with

$$\begin{aligned}
c(x,\zeta) &= \iint a(x,\xi)e^{2i\pi(x-z)\cdot(\xi-\zeta)}b(z,\zeta)d\xi dz \\
&= \iint a(x,\xi+\zeta)e^{-2i\pi z\cdot\xi}b(z+x,\zeta)d\xi dz, \qquad (4.1.5)
\end{aligned}$$

which is indeed (4.1.4). With $c = a \diamond b$ given by (4.1.4), using that $a, b \in \mathscr{S}(\mathbb{R}^{2n})$ we get, using the notation (1.2.8) and $P_k(t) = (1+|t|^2)^{1/2}, k \in 2\mathbb{N}$,

$$\begin{aligned}
c(x,\xi) &= \iint P_k(D_\eta)\Big(e^{-2i\pi y\cdot\eta}\Big)P_k(y)^{-1}a(x,\xi+\eta)b(y+x,\xi)dyd\eta \\
&= \iint e^{-2i\pi y\cdot\eta}P_k(y)^{-1}(P_k(D_2)a)(x,\xi+\eta)b(y+x,\xi)dyd\eta \\
&= \iint P_k(D_y)\Big(e^{-2i\pi y\cdot\eta}\Big)P_k(\eta)^{-1}P_k(y)^{-1}(P_k(D_2)a)(x,\xi+\eta)b(y+x,\xi)dyd\eta \\
&= \sum_{0\le l\le k/2} C^l_{k/2}\iint e^{-2i\pi y\cdot\eta}|D_y|^{2l}\Big(P_k(y)^{-1}b(y+x,\xi)\Big) \\
&\qquad\qquad P_k(\eta)^{-1}(P_k(D_2)a)(x,\xi+\eta)dyd\eta. \qquad (4.1.6)
\end{aligned}$$

We denote by $a\tilde{\diamond}b$ the right-hand-side of the previous formula and we note that, when $k > n$, it makes sense as well for $a, b \in C_b^\infty(\mathbb{R}^{2n})$, since $|\partial_t^\alpha(1/P_k)(t)| \le C_{\alpha,k}(1+|t|)^{-k}$. We already know that $a \diamond b = a\tilde{\diamond}b$ for $a, b$ in the Schwartz class and we want to prove that it is also true for $a, b \in C_b^\infty(\mathbb{R}^{2n})$. Choosing an even $k > n$ (take $k = n+1$ or $n+2$), we also get

$$\|a\tilde{\diamond}b\|_{L^\infty(\mathbb{R}^{2n})} \le C_n \sup_{|\alpha|\le n+2}\|\partial_\xi^\alpha a\|_{L^\infty(\mathbb{R}^{2n})} \sup_{|\beta|\le n+2}\|\partial_x^\beta b\|_{L^\infty(\mathbb{R}^{2n})}.$$

Moreover, we note from (4.1.6) that

$$\partial_{\xi_j}(a\tilde{\diamond}b) = (\partial_{\xi_j}a)\tilde{\diamond}b + a\tilde{\diamond}(\partial_{\xi_j}b), \partial_{x_j}(a\tilde{\diamond}b) = (\partial_{x_j}a)\tilde{\diamond}b + a\tilde{\diamond}(\partial_{x_j}b)$$

and as a result

$$\|\partial_\xi^\alpha\partial_x^\beta(a\tilde{\diamond}b)\|_{L^\infty(\mathbb{R}^{2n})} \le C_{n,\alpha,\beta} \sup_{\substack{|\alpha'|\le n+2, |\beta'|\le n+2\\ \alpha''+\alpha'''=\alpha,\ \beta''+\beta'''=\beta}} \|\partial_\xi^{\alpha'+\alpha''}\partial_x^{\beta''}a\|_{L^\infty(\mathbb{R}^{2n})}\|\partial_x^{\beta'+\beta'''}\partial_\xi^{\alpha'''}b\|_{L^\infty(\mathbb{R}^{2n})}, \quad (4.1.7)$$

which gives also the continuity of the bilinear mapping $C_b^\infty(\mathbb{R}^{2n}) \times C_b^\infty(\mathbb{R}^{2n}) \ni (a, b) \mapsto a\tilde{\diamond}b \in C_b^\infty(\mathbb{R}^{2n})$. We have for $u, v \in \mathscr{S}(\mathbb{R}^n)$, $a, b \in C_b^\infty(\mathbb{R}^{2n})$,

$$a_k(x,\xi) = e^{-(|x|^2+|\xi|^2)/k^2}a(x,\xi), \quad b_k(x,\xi) = e^{-(|x|^2+|\xi|^2)/k^2}b(x,\xi),$$

from Lemma 4.1.3 and Theorem 4.1.2, with limits in $\mathscr{S}(\mathbb{R}^n)$,

$$a(x,D)b(x,D)u = \lim_k a_k(x,D)b(x,D)u = \lim_k\Big(\lim_l a_k(x,D)b_l(x,D)u\Big),$$

and thus, with $\Omega_{u,v}(x,\xi) = e^{2i\pi x\cdot\xi}\hat{u}(\xi)\bar{v}(x)$ (which belongs to $\mathscr{S}(\mathbb{R}^{2n})$),

$$\begin{aligned}\langle a(x,D)b(x,D)u, v\rangle_{L^2} &= \lim_k\Big(\lim_l\langle(a_k\diamond b_l)(x,D)u, v\rangle\Big)\\ &= \lim_k\Big(\lim_l\iint(a_k\diamond b_l)(x,\xi)\Omega_{u,v}(x,\xi)dxd\xi\Big) = \iint(a\tilde{\diamond}b)(x,\xi)\Omega_{u,v}(x,\xi)dxd\xi,\end{aligned}$$

which gives indeed $a(x,D)b(x,D) = (a\tilde{\diamond}b)(x,D)$. This property gives at once the continuity properties stated at the end of the theorem, since the weak continuity property follows immediately from (4.1.6) and the Lebesgue dominated convergence theorem, whereas the Fréchet continuity follows from (4.1.7). Moreover, with the same notations as above, we have with

$$C_{x,\xi}^{(a,b)}(y,\eta) = a(x,\xi+\eta)b(y+x,\xi)$$

(see Remark 4.1.6) for each $(x,\xi) \in \mathbb{R}^{2n}$,

$$(JC_{x,\xi}^{(a,b)})(0,0) = \lim_k(JC_{x,\xi}^{(a_k,b_k)})(0,0) = \lim_k\big((a_k\diamond b_k)(x,\xi)\big) = (a\tilde{\diamond}b)(x,\xi)$$

which proves (4.1.3). The proof of the theorem is complete. □

**Definition 4.1.7.** Let $A : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n)$ be a linear operator. The adjoint operator $A^* : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n)$ is defined by

$$\langle A^*u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \overline{\langle Av, u\rangle}_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)},$$

where $\mathscr{S}^*(\mathbb{R}^n)$ is the antidual of $\mathscr{S}(\mathbb{R}^n)$ (continuous antilinear forms).

**Lemma 4.1.8.** *Let $n \geq 1$ be an integer and $t \in \mathbb{R}^*$. We define the operator*

$$J^t = \exp 2i\pi t D_x \cdot D_\xi \tag{4.1.8}$$

*on $\mathscr{S}'(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ by $(FJ^t a)(\xi, x) = e^{2i\pi t \xi \cdot x} \hat{a}(\xi, x)$, where $F$ stands here for the Fourier transform in $2n$ dimensions. The operator $J^t$ sends also $\mathscr{S}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ into itself continuously, satisfies (for $s, t \in \mathbb{R}$) $J^{s+t} = J^s J^t$ and is given by*

$$(J^t a)(x, \xi) = |t|^{-n} \iint e^{-2i\pi t^{-1} y \cdot \eta} a(x + y, \xi + \eta) dy d\eta. \tag{4.1.9}$$

*We have*

$$J^t a = e^{i\pi t \langle BD, D\rangle} a = |t|^{-n} e^{-i\pi t^{-1} \langle B\cdot,\cdot\rangle} * a, \tag{4.1.10}$$

*with the $2n \times 2n$ matrix $B = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. The operator $J^t$ sends continuously $C_b^\infty(\mathbb{R}^{2n})$ into itself.*

*Proof.* We have indeed $(FJ^t a)(\xi, x) = e^{2i\pi t \xi \cdot x} \hat{a}(\xi, x) = e^{i\pi t \langle B\Xi, \Xi\rangle} \hat{a}(\Xi)$. Note that $B$ is a $2n \times 2n$ symmetric matrix with null signature, determinant $(-1)^n$ and that $B^{-1} = B$. According to the proposition 1.2.19, the inverse Fourier transform of $e^{i\pi t \langle B\Xi, \Xi\rangle}$ is $|t|^{-n} e^{-i\pi t^{-1} \langle BX, X\rangle}$ so that $J^t a = |t|^{-n} e^{-i\pi t^{-1} \langle B\cdot,\cdot\rangle} * a$. Since the Fourier multiplier $e^{i\pi t \langle B\Xi, \Xi\rangle}$ is smooth bounded with derivatives polynomially bounded, it defines a continuous operator from $\mathscr{S}(\mathbb{R}^{2n})$ into itself.

In the sequel of the proof, we take $t = 1$, which will simplify the notations without corrupting the arguments. Let us consider $a \in \mathscr{S}(\mathbb{R}^{2n})$: we have with $k \in 2\mathbb{N}$ and the polynomial on $\mathbb{R}^n$ defined by $P_k(y) = (1 + |y|^2)^{k/2}$

$$(Ja)(x, \xi) = \iint e^{-2i\pi y \cdot \eta} P_k(y)^{-1} P_k(D_\eta) \Big( P_k(\eta)^{-1} (P_k(D_y) a)(x + y, \xi + \eta) \Big) dy d\eta,$$

so that, with $|T_{\alpha\beta}(\eta)| \leq P_k(\eta)^{-1}$ and constants $c_{\alpha\beta}$, we obtain

$$(Ja)(x, \xi) = \sum_{\substack{|\beta| \leq k \\ |\alpha| \leq k}} c_{\alpha\beta} \iint e^{-2i\pi y \cdot \eta} P_k(y)^{-1} T_{\alpha\beta}(\eta) (D_\xi^\alpha D_x^\beta a)(x + y, \xi + \eta) dy d\eta. \tag{4.1.11}$$

Let us denote by $\tilde{J}a$ the right-hand-side of (4.1.11). We already know that $\tilde{J}a = Ja$ for $a \in \mathscr{S}(\mathbb{R}^{2n})$. We also note that, using an even integer $k > n$, the previous integral converges absolutely whenever $a \in C_b^\infty(\mathbb{R}^{2n})$; moreover we have

$$\|\tilde{J}a\|_{L^\infty} \leq C_n \sup_{\substack{|\alpha| \leq n+2 \\ |\beta| \leq n+2}} \|D_\xi^\alpha D_x^\beta a\|_{L^\infty},$$

and since the derivations are commuting with $J$ and $\tilde{J}$, we also get that

$$\|\partial^\gamma \tilde{J}a\|_{L^\infty} \leq C_n \sup_{\substack{|\alpha| \leq n+2 \\ |\beta| \leq n+2}} \|D_\xi^\alpha D_x^\beta \partial^\gamma a\|_{L^\infty}. \tag{4.1.12}$$

It implies that $\tilde{J}$ is continuous from $C_b^\infty(\mathbb{R}^{2n})$ to itself. Let us now consider $a \in C_b^\infty(\mathbb{R}^{2n} \times \mathbb{R}^m)$; we define the sequence $(a_k)$ in $\mathscr{S}(\mathbb{R}^{2n})$ by

$$a_k(x, \xi) = e^{-(|x|^2+|\xi|^2)/k^2} a(x, \xi).$$

We have $\langle Ja, \Phi\rangle_{\mathscr{S}^*(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} =$

$$\iint a(x, \xi)\overline{(J^{-1}\Phi)}(x, \xi)dxd\xi = \lim_{k\to+\infty} \iint a_k(x, \xi)\overline{(J^{-1}\Phi)}(x, \xi)dxd\xi$$
$$= \lim_{k\to+\infty} \iint (Ja_k)(x, \xi)\bar{\Phi}(x, \xi)dxd\xi = \iint (\tilde{J}a)(x, \xi)\bar{\Phi}(x, \xi)dxd\xi,$$

so that we indeed have $\tilde{J}a = Ja$ and from (4.1.12) the continuity property of the lemma whose proof is now complete. □

**Theorem 4.1.9.** *Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ and $A = a(x, D)$ be given by Definition 4.1.1. Then the operator $A^*$ is equal to $a^*(x, D)$, where $a^* = J\bar{a}$ ($J$ is given in Lemma 4.1.8 above). If $a$ belongs to $C_b^\infty(\mathbb{R}^{2n})$, $a^* = J\bar{a} \in C_b^\infty(\mathbb{R}^{2n})$ and the mapping $a \mapsto a^*$ is continuous from $C_b^\infty(\mathbb{R}^{2n})$ into itself.*

*Proof.* According to the definitions 4.1.7 and 4.1.1, we have for $u, v \in \mathscr{S}(\mathbb{R}^n)$, with $\Omega_{v,u}(x, \xi) = e^{2i\pi x\cdot\xi}\hat{v}(\xi)\bar{u}(x)$,

$$\langle A^*u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \overline{\langle Av, u\rangle}_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \overline{\prec a, \Omega_{v,u} \succ}_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$$
$$= \prec \bar{a}, \overline{\Omega}_{v,u} \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}.$$

On the other hand, we have

$$\big(J^{-1}(\overline{\Omega}_{v,u})\big)(x, \xi) = \iint e^{2i\pi(x-y)\cdot(\xi-\eta)}e^{-2i\pi y\cdot\eta}\bar{\hat{v}}(\eta)u(y)dyd\eta$$
$$= \bar{v}(x)e^{2i\pi x\cdot\xi}\hat{u}(\xi) = \Omega_{u,v}(x, \xi),$$

so that, using (4.1.10), we get

$$\langle A^*u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \prec \bar{a}, J\Omega_{u,v} \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} = \prec J\bar{a}, \Omega_{u,v} \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$$

and finally $A^* = (J\bar{a})(x, D)$. The last statement in the theorem follows from Lemma 4.1.8. □

*N.B.* In this introductory section, we have seen a very general definition of quantization (Definition 4.1.1), an easy $\mathscr{S}$ continuity theorem (Theorem 4.1.2), a trickier $L^2$-boundedness result (Theorem 4.1.4), a composition formula (Theorem 4.1.5) and an expression for the adjoint (Theorem 4.1.7). These five steps are somewhat typical of the construction of a pseudodifferential calculus and we shall see many different examples of this situation. The above prolegomena provide a quite explicit and elementary approach to the construction of an algebra of pseudodifferential operators in a rather difficult framework, since we did not use any asymptotic calculus and did not have at our disposal a "small parameter". The proofs and simple methods that we used here will be useful later as well as many of the results.

## 4.2 Quantization formulas

We have already seen in Definition 4.1.1 and in the formula (4.1.1) a way to associate to a tempered distribution $a \in \mathscr{S}'(\mathbb{R}^{2n})$ an operator from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$. This question of quantization has of course many links with quantum mechanics and we want here to study some properties of various quantizations formulas, such as the Weyl quantization and the Feynman formula along with several variations around these examples. We are given a function $a$ defined on the phase space $\mathbb{R}^n \times \mathbb{R}^n$ ($a$ is a "Hamiltonian") and we wish to associate to this function an operator. For instance, we may introduce the one-parameter formulas, for $t \in \mathbb{R}$,

$$(\operatorname{op}_t a)u(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\big((1-t)x + ty, \xi\big)u(y)dyd\xi. \tag{4.2.1}$$

When $t = 0$, we recognize the standard quantization introduced in Definition 4.1.1, quantizing $a(x)\xi_j$ in $a(x)D_{x_j}$ (see (1.2.8)). However, one may wish to multiply first and take the derivatives afterwards: this is what the choice $t = 1$ does, quantizing $a(x)\xi_j$ in $D_{x_j}a(x)$. The more symmetrical choice $t = 1/2$ was done by Hermann Weyl [21]: we have

$$(\operatorname{op}_{\frac{1}{2}} a)u(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\big(\frac{x+y}{2}, \xi\big)u(y)dyd\xi, \tag{4.2.2}$$

and thus $\operatorname{op}_{\frac{1}{2}}(a(x)\xi_j) = \frac{1}{2}\big(a(x)D_{x_j} + D_{x_j}a(x)\big)$. This quantization is widely used in quantum mechanics, because a real-valued Hamiltonian gets quantized by a (formally) selfadjoint operator. We shall see that the most important property of that quantization remains its symplectic invariance, which will be studied in details in Chapter 2; a different symmetrical choice was made by Richard Feynman who used the formula

$$\iint e^{2i\pi(x-y)\cdot\xi} \,\left(a(x,\xi) + a(y,\xi)\right)\frac{1}{2}u(y)dyd\xi, \tag{4.2.3}$$

keeping the selfadjointness of real Hamiltonians, but loosing the symplectic invariance. The reader may be embarrassed by the fact that we did not bother about the convergence of the integrals above. Before providing a definition, we may assume that $a \in \mathscr{S}(\mathbb{R}^{2n}), u, v \in \mathscr{S}(\mathbb{R}^n), t \in \mathbb{R}$ and compute

$$\begin{aligned}
\langle(\operatorname{op}_t a)u, v\rangle &= \iiint a\big((1-t)x + ty, \xi\big)e^{2i\pi(x-y)\cdot\xi}u(y)\bar v(x)dyd\xi dx \\
&= \iiint a(z,\xi)e^{-2i\pi s\cdot\xi}u(z + (1-t)s)\bar v(z - ts)dzd\xi ds \\
&= \iiint a(x,\xi)e^{-2i\pi z\cdot\xi}u(x + (1-t)z)\bar v(x - tz)dxd\xi dz,
\end{aligned}$$

so that with

$$\Omega_{u,v}(t)(x,\xi) = \int e^{-2i\pi z\cdot\xi}u(x + (1-t)z)\bar v(x - tz)dz, \tag{4.2.4}$$

which is easily seen[2] to be in $\mathscr{S}(\mathbb{R}^{2n})$ when $u, v \in \mathscr{S}(\mathbb{R}^n)$, we can give the following definition.[3]

**Definition 4.2.1.** Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ be a tempered distribution and $t \in \mathbb{R}$. We define the operator $\operatorname{op}_t a : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}^*(\mathbb{R}^n)$ by the formula

$$\langle (\operatorname{op}_t a)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \prec a, \Omega_{u,v}(t) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

where $\mathscr{S}^*(\mathbb{R}^n)$ is the antidual of $\mathscr{S}(\mathbb{R}^n)$ (continuous antilinear forms).

**Proposition 4.2.2.** *Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ be a tempered distribution and $t \in \mathbb{R}$. We have*

$$\operatorname{op}_t a = \operatorname{op}_0(J^t a) = (J^t a)(x, D),$$

*with $J^t$ defined in Lemma 4.1.8.*

*Proof.* Let $u, v \in \mathscr{S}(\mathbb{R}^n)$. With the $\mathscr{S}(\mathbb{R}^{2n})$ function $\Omega_{u,v}(t)$ given above, we have for $t \neq 0$,

$$\begin{aligned}
\big(J^t\Omega_{u,v}(0)\big)(x,\xi) &= |t|^{-n} \iint e^{-2i\pi t^{-1}(x-y)\cdot(\xi-\eta)}\Omega_{u,v}(0)(y,\eta)dyd\eta \\
&= |t|^{-n} \iint e^{-2i\pi t^{-1}(x-y)\cdot(\xi-\eta)}\hat{u}(\eta)\bar{v}(y)e^{2i\pi y\cdot\eta}dyd\eta \\
&= \iint e^{-2i\pi z\cdot(\xi-\eta)}\hat{u}(\eta)\bar{v}(x-tz)e^{2i\pi(x-tz)\cdot\eta}dzd\eta \\
&= \int e^{-2i\pi z\cdot\xi}u(x+(1-t)z)\bar{v}(x-tz)dz = \Omega_{u,v}(t)(x,\xi),
\end{aligned} \tag{4.2.5}$$

so that

$$\begin{aligned}
\langle (\operatorname{op}_t a)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} &= \prec a, \Omega_{u,v}(t) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} && \text{(definition 4.2.1)} \\
&= \prec a, J^t\Omega_{u,v}(0) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} && \text{(property (4.2.5))} \\
&= \prec J^t a, \Omega_{u,v}(0) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} && \text{(easy identity for } J^t) \\
&= \langle (J^t a)(x,D)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} && \text{(definition 4.1.1)},
\end{aligned}$$

completing the proof. □

*Remark* 4.2.3. The theorem 4.1.9 and the previous proposition give in particular that $a(x, D)^* = \operatorname{op}_1(\bar{a}) = (J\bar{a})(x, D)$, a formula which in fact motivates the study of the group $J^t$. On the other hand, using the Weyl quantization simplifies somewhat the matter of taking adjoints since we have

$$\big(\operatorname{op}_{1/2}(a)\big)^* = \big(\operatorname{op}_0(J^{1/2}a)\big)^* = \operatorname{op}_0(J(\overline{J^{1/2}a})) = \operatorname{op}_0(J^{1/2}\bar{a}) = \operatorname{op}_{1/2}(\bar{a})$$

[2]In fact the linear mapping $\mathbb{R}^n \times \mathbb{R}^n \ni (x, z) \mapsto (x-tz, x+(1-t)z)$ has determinant 1 and $\Omega_{u,v}(t)$ appears as the partial Fourier transform of the function $\mathbb{R}^n \times \mathbb{R}^n \ni (x, z) \mapsto \bar{v}(x-tz)u(x+(1-t)z)$, which is in the Schwartz class.

[3]The reader can check that this is consistent with Definition 4.1.1.

and in particular if $a$ is real-valued, $\mathrm{op}_{1/2}(a)$ is formally selfadjoint. The Feynman formula as displayed in (4.2.3) amounts to quantize the Hamiltonian $a$ by

$$\frac{1}{2}\mathrm{op}_0(a + Ja)$$

and we see that $\big(\mathrm{op}_0(a+Ja)\big)^* = \mathrm{op}_0(J\bar{a}+J(\overline{Ja})) = \mathrm{op}_0(J\bar{a}+\bar{a})$, which also provides selfadjointness for real-valued Hamiltonians.

**Lemma 4.2.4.** *Let $a \in \mathscr{S}(\mathbb{R}^{2n})$. Then for all $t \in \mathbb{R}$, $op_t(a)$ is a continuous mapping from $\mathscr{S}'(\mathbb{R}^n)$ in $\mathscr{S}(\mathbb{R}^n)$.*

*Proof.* Let $a \in \mathscr{S}(\mathbb{R}^{2n})$: we have for $u \in \mathscr{S}'(\mathbb{R}^n)$, $A = a(x, D)$,

$$x^\beta(D_x^\alpha Au)(x) = \sum_{\alpha'+\alpha''=\alpha} \frac{1}{\alpha'!\alpha''!}\langle \hat{u}(\xi), e^{2i\pi x\cdot\xi}\xi^{\alpha'}x^\beta(D_x^{\alpha''}a)(x,\xi)\rangle_{\mathscr{S}'(\mathbb{R}^n_\xi),\mathscr{S}(\mathbb{R}^n_\xi)},$$

so that $Au \in \mathscr{S}(\mathbb{R}^n)$ and the same property holds for $\mathrm{op}_t(a)$ since $J^t$ is an isomorphism of $\mathscr{S}'(\mathbb{R}^{2n})$. □

## 4.3 The $S^m_{1,0}$class of symbols

Differential operators on $\mathbb{R}^n$ with smooth coefficients are given by a formula (see (4.1.1))

$$a(x, D)u = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha$$

where the $a_\alpha$ are smooth functions. Assuming some behaviour at infinity for the $a_\alpha$, we may require that they are $C_b^\infty(\mathbb{R}^n)$ (see page 90) and a natural generalization is to consider operators $a(x, D)$ with a symbol $a$ of type $S^m_{1,0}$, i.e. smooth functions on $\mathbb{R}^{2n}$ satisfying

$$|(\partial_\xi^\alpha\partial_x^\beta a)(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}, \quad \langle\xi\rangle = (1+|\xi|^2)^{1/2}. \tag{4.3.1}$$

The best constants $C_{\alpha\beta}$ in (4.3.1) are the semi-norms of $a$ in the Fréchet space $S^m_{1,0}$. We can define, for $a \in S^m_{1,0}$, $k \in \mathbb{N}$,

$$\gamma_{k,m}(a) = \sup_{(x,\xi)\in\mathbb{R}^{2n}, |\alpha|+|\beta|\le k} |(\partial_\xi^\alpha\partial_x^\beta a)(x,\xi)|\langle\xi\rangle^{-m+|\alpha|}. \tag{4.3.2}$$

*Example.* The function $\langle\xi\rangle^m$ belongs to $S^m_{1,0}$: the function

$$\mathbb{R}\times\mathbb{R}^n \ni (\tau,\xi) \mapsto (\tau^2+|\xi|^2)^{m/2}$$

is (positively) homogeneous of degree $m$ on $\mathbb{R}^{n+1}\backslash\{0\}$, and thus $\partial_\xi^\alpha\big((\tau^2+|\xi|^2)^{m/2}\big)$ is homogeneous of degree $m-|\alpha|$ and bounded above by

$$C_\alpha(\tau^2+|\xi|^2)^{\frac{m-|\alpha|}{2}}.$$

Since the restriction to $\tau = 1$ and the derivation with respect to $\xi$ commute, it gives the answer.

We shall see that the class of operators $\mathrm{Op}(S^m_{1,0})$ is suitable ($\mathrm{Op}(b)$ is $\mathrm{op}_0 b$, see Proposition 4.2.2) to invert elliptic operators, and useful for the study of singularities of solutions of PDE. We see that the elements of $S^m_{1,0}$ are temperate distributions, so that the operator $a(x,D)$ makes sense, according to Definition 4.1.1. We have also the following result.

**Theorem 4.3.1.** *Let $m \in \mathbb{R}$ and $a \in S^m_{1,0}$. Then the operator $a(x,D)$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself.*

*Proof.* With $\langle D\rangle = \mathrm{Op}(\langle\xi\rangle)$, we have $a(x,D) = \mathrm{Op}(a(x,\xi)\langle\xi\rangle^{-m})\langle D\rangle^m$. The function $a(x,\xi)\langle\xi\rangle^{-m}$ belongs to $C^\infty_b(\mathbb{R}^{2n})$ so that we can use Theorem 4.1.2 and the fact that $\langle D\rangle^m$ is continuous on $\mathscr{S}(\mathbb{R}^n)$ to get the result. □

**Theorem 4.3.2.** *Let $a \in S^0_{1,0}$. Then the operator $a(x,D)$ is bounded on $L^2(\mathbb{R}^n)$.*

*Proof.* Since $S^0_{1,0} \subset C^\infty_b(\mathbb{R}^{2n})$, it follows from Theorem 4.1.4. □

**Theorem 4.3.3.** *Let $m_1, m_2$ be real numbers and $a_1 \in S^{m_1}_{1,0}, a_2 \in S^{m_2}_{1,0}$. Then the composition $a_1(x,D)a_2(x,D)$ makes sense as a continuous operator from $\mathscr{S}(\mathbb{R}^n)$ into itself and $a_1(x,D)a_2(x,D) = (a_1 \diamond a_2)(x,D)$ where $a_1 \diamond a_2$ belongs to $S^{m_1+m_2}_{1,0}$ and is given by the formula*

$$(a_1 \diamond a_2)(x,\xi) = (\exp 2i\pi D_y \cdot D_\eta)\Big(a_1(x,\xi+\eta)a_2(y+x,\xi)\Big)_{|y=0,\eta=0}. \tag{4.3.3}$$

*N.B.* From Lemma 4.1.5 in [9], we know that the operator $e^{2i\pi D_y\cdot D_\eta}$ is an isomorphism of $S^m_{1,0}(\mathbb{R}^{2n})$, which gives a meaning to the formula (4.3.3), since for $a_j \in S^{m_j}_{1,0}(\mathbb{R}^{2n})$, $(x,\xi)$ given in $\mathbb{R}^{2n}$, the function $(y,\eta) \mapsto a_1(x,\xi+\eta)a_2(y+x,\xi) = C_{x,\xi}(y,\eta)$ belongs to $S^{m_1}_{1,0}(\mathbb{R}^{2n})$ as well as $JC_{x,\xi}$ and we can take the value of the latter at $(y,\eta) = (0,0)$.

*Proof.* We assume first that both $a_j$ belong to $\mathscr{S}(\mathbb{R}^{2n})$. The formula (4.1.4) provides the answer. Now, rewriting the formula (4.1.6) for an even integer $k$, we get

$$(a_1 \diamond a_2)(x,\xi) = \sum_{0\le l\le k/2} C^l_{k/2} \iint e^{-2i\pi y\cdot\eta}|D_y|^{2l}\Big(\langle y\rangle^{-k}a_2(y+x,\xi)\Big) \\ \langle\eta\rangle^{-k}(\langle D_\eta\rangle^k a_1)(x,\xi+\eta)dyd\eta. \tag{4.3.4}$$

We denote by $a_1\tilde{\diamond}a_2$ the right-hand-side of (4.3.4) and we note that, when $k > n+|m_1|$, it makes sense (and it does not depend on $k$) as well for $a_j \in S^{m_j}_{1,0}$, since

$$|\partial^\alpha_y\langle y\rangle^{-k}| \le C_{\alpha,k}\langle y\rangle^{-k}, \ |\partial^\beta_y a_2(y+x,\xi)| \le C_\beta\langle\xi\rangle^{m_2}, \ |\partial^\gamma_\eta a_1(x,\xi+\eta)| \le C_\gamma\langle\xi+\eta\rangle^{m_1}$$

so that the absolute value of the integrand above is[4] [5]

$$\lesssim \langle y\rangle^{-k}\langle\eta\rangle^{-k}\langle\xi\rangle^{m_2}\langle\xi+\eta\rangle^{m_1} \lesssim \langle y\rangle^{-k}\langle\eta\rangle^{-k+|m_1|}\langle\xi\rangle^{m_1+m_2}.$$

[4] We use $\langle\xi+\eta\rangle \le 2^{1/2}\langle\xi\rangle\langle\eta\rangle$ so that,

$$\forall s \in \mathbb{R}, \forall \xi,\eta \in \mathbb{R}^n, \quad \langle\xi+\eta\rangle^s \le 2^{|s|/2}\langle\xi\rangle^s\langle\eta\rangle^{|s|}, \tag{4.3.5}$$

a convenient inequality (to get it for $s \ge 0$, raise the first inequality to the power $s$, and for $s < 0$, replace $\xi$ by $-\xi-\eta$) a.k.a. Peetre's inequality.

[5] We use here the notation $a \lesssim b$ for the inequality $a \le Cb$, where $C$ is a "controlled" constant (here $C$ depends only on $k, m_1, m_2$).

*Remark* 4.3.4. Note that this proves that the mapping

$$S^{m_1}_{1,0} \times S^{m_2}_{1,0} \ni (a_1, a_2) \mapsto a_1 \tilde{\diamond} a_2 \in S^{m_1+m_2}_{1,0}$$

is bilinear continuous. In fact, we have already proven that

$$|(a_1 \tilde{\diamond} a_2)(x, \xi)| \leq C\langle\xi\rangle^{m_1+m_2},$$

and we can check directly that $a_1 \tilde{\diamond} a_2$ is smooth and satisfies

$$\partial_{\xi_j}\big(a_1 \tilde{\diamond} a_2\big) = (\partial_{\xi_j} a_1) \tilde{\diamond} a_2 + a_1 \tilde{\diamond} (\partial_{\xi_j} a_2)$$

so that $|\partial_{\xi_j}(a_1 \tilde{\diamond} a_2)(x, \xi)| \leq C\langle\xi\rangle^{m_1+m_2-1}$, and similar formulas for higher order derivatives.

*Remark* 4.3.5. Let $(c_k)$ be a bounded sequence in the Fréchet space $S^m_{1,0}$ converging in $C^\infty(\mathbb{R}^{2n})$ to $c$. Then $c$ belongs to $S^m_{1,0}$ and for all $u \in \mathscr{S}(\mathbb{R}^n)$, the sequence $(c_k(x, D)u)$ converges to $c(x, D)u$ in $\mathscr{S}(\mathbb{R}^n)$. In fact, the sequence of functions $(c_k(x, \xi)\langle\xi\rangle^{-m})$ is bounded in $C^\infty_b(\mathbb{R}^{2n})$ and we can apply Lemma 4.1.3 to get that $\lim_k \operatorname{Op}(c_k(x, \xi)\langle\xi\rangle^{-m})\langle D\rangle^m u = \operatorname{Op}(c(x, \xi)\langle\xi\rangle^{-m})\langle D\rangle^m u = \operatorname{Op}(c)u$ in $\mathscr{S}(\mathbb{R}^n)$.

The remaining part of the argument is the same than in the proof of Theorem 4.1.5, after (4.1.7). □

**Theorem 4.3.6.** *Let $s, m$ be real numbers and $a \in S^m_{1,0}$. Then the operator $a(x, D)$ is bounded from $H^{s+m}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$.*

*Proof.* Let us recall that $H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n), \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}$. From the theorem 4.3.3, the operator $\langle D\rangle^s a(x, D)\langle D\rangle^{-m-s}$ can be written as $b(x, D)$ with $b \in S^0_{1,0}$ and so from the theorem 4.3.2, it is a bounded operator on $L^2(\mathbb{R}^n)$. Since $\langle D\rangle^\sigma$ is an isomorphism of $H^\sigma(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$ with inverse $\langle D\rangle^{-\sigma}$, it gives the result. □

**Corollary 4.3.7.** *Let $r$ be a symbol in $S^{-\infty}_{1,0} = \cap_m S^m_{1,0}$. Then $r(x, D)$ sends $\mathscr{E}'(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$.*

*Proof.* We have for $v \in \mathscr{E}'$ and $\psi \in C^\infty_c(\mathbb{R}^n)$ equal to 1 on a neighborhood of the support of $v$, iterating

$$x_j D^\beta r(x, D)v = [x_j, D^\beta r(x, D)]\psi v + D^\beta r(x, D)\psi x_j v = r_j(x, D)v, \quad r_j \in S^{-\infty}_{1,0},$$

that $x^\alpha D^\beta r(x, D)v = r_{\alpha\beta}(x, D)v, r_{\alpha\beta} \in S^{-\infty}_{1,0}$, and thus

$$x^\alpha D^\beta r(x, D)v \in \cap_s H^s(\mathbb{R}^n) \subset C^\infty_b(\mathbb{R}^n),$$

completing the proof. □

**Theorem 4.3.8.** *Let $m_1, m_2$ be real numbers and $a_1 \in S^{m_1}_{1,0}, a_2 \in S^{m_2}_{1,0}$. Then $a_1(x,D)a_2(x,D) = (a_1 \diamond a_2)(x,D)$, the symbol $a_1 \diamond a_2$ belongs to $S^{m_1+m_2}_{1,0}$ and we have the asymptotic expansion, for all $N \in \mathbb{N}$,*

$$a_1 \diamond a_2 = \sum_{|\alpha|<N} \frac{1}{\alpha!} D^\alpha_\xi a_1 \partial^\alpha_x a_2 + r_N(a_1, a_2), \tag{4.3.6}$$

*with $r_N(a_1,a_2) \in S^{m_1+m_2-N}_{1,0}$. Note that $D^\alpha_\xi a_1 \partial^\alpha_x a_2$ belong to $S^{m_1+m_2-|\alpha|}_{1,0}$.*

*Proof.* We can use the formula (4.3.3) and apply that lemma to get the desired formula with

$$r_N(a_1,a_2)(x,\xi) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} e^{2i\pi\theta D_z\cdot D_\zeta}(2i\pi D_z \cdot D_\zeta)^N \bigl(a_1(x,\zeta)a_2(z,\xi)\bigr) d\theta_{|z=x,\zeta=\xi}. \tag{4.3.7}$$

The function $(z,\zeta) \mapsto b_{x,\xi}(z,\zeta) = \langle\xi\rangle^{-m_2}(2i\pi D_z \cdot D_\zeta)^N a_1(x,\zeta)a_2(z,\xi)$ belongs to $S^{m_1-N}_{1,0}(\mathbb{R}^{2n}_{z,\zeta})$ uniformly with respect to the parameters $(x,\xi) \in \mathbb{R}^{2n}$: it satisfies, using the notation (4.3.2), for $\max(|\alpha|,|\beta|) \le k$,

$$|\partial^\alpha_\zeta \partial^\beta_z b_{x,\xi}(z,\zeta)| \le \gamma_{k,m_1}(a_1)\gamma_{k,m_2}(a_2)\langle\zeta\rangle^{m_1-N-|\alpha|}.$$

**Lemma 4.3.9.** *Let $n \ge 1$ be an integer and $m,t \in \mathbb{R}$. The operator $J^t$ sends continuously $S^m_{1,0}(\mathbb{R}^{2n})$ into itself and for all integers $N \ge 0$,*

$$(J^t a)(x,\xi) = \sum_{|\alpha|<N} \frac{t^{|\alpha|}}{\alpha!}(D^\alpha_\xi \partial^\alpha_x a)(x,\xi) + r_N(t)(x,\xi), \quad r_N(t) \in S^{m-N}_{1,0},$$

$$r_N(t)(x,\xi) = t^N \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!}\bigl(J^{\theta t}(D_\xi \cdot \partial_x)^N a\bigr)(x,\xi) d\theta.$$

*Proof.* We apply Taylor's formula on $J^t = \exp 2i\pi t D_x \cdot D_\xi$ to get for operators on $\mathscr{S}'(\mathbb{R}^{2n})$,

$$J^t = \sum_{0\le k<N} \frac{t^k}{k!}(D_\xi \cdot \partial_x)^k + \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} J^{\theta t}(tD_\xi \cdot \partial_x)^N d\theta, \tag{4.3.8}$$

and since

$$\frac{1}{k!}(D_\xi \cdot \partial_x)^k = \sum_{\substack{\alpha_1+\dots+\alpha_n=k \\ \alpha_j \in \mathbb{N}}} \frac{(D_{\xi_1}\partial_{x_1})^{\alpha_1}}{\alpha_1!} \dots \frac{(D_{\xi_n}\partial_{x_n})^{\alpha_n}}{\alpha_n!},$$

we obtain the above formulas for $a \in \mathscr{S}'(\mathbb{R}^{2n})$. On the other hand, we get from (4.3.1) that the term $D^\alpha_\xi \partial^\alpha_x a$ belongs to $S^{m-|\alpha|}_{1,0}$. It is thus enough that we show that $J^t$ sends continuously $S^m_{1,0}$ into itself. For that purpose, we can use the formula (4.1.11) (and assume that $t=1$) in the proof of the lemma 4.1.8; also the same reasoning as in the proof of this lemma shows that the right-hand-side of (4.1.11) is meaningful for $a \in S^m_{1,0}$ if $k > n + |m|$ and is indeed the expression of $Ja$. We get, for all $k \in \mathbb{N}$,

$$|Ja(x,\xi)| \le C_{k,n} \iint \langle y\rangle^{-k} \langle \eta\rangle^{-k} \langle \xi+\eta\rangle^m d\xi d\eta$$

so that Peetre's inequality (4.3.5) yields, for $k > n + |m|$, $|Ja(x,\xi)| \le C'_{k,n}\langle \xi\rangle^m$. The estimates for the derivatives are obtained similarly since they commute with $J$. The terms involving integrals of $J^t$ can be handled via Remark 4.1.4 in [9], which provides a polynomial control with respect to $t$. □

Applying Lemma 4.3.9, we obtain that the function

$$\rho_{x,\xi}(z,\zeta) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} (J^\theta b_{x,\xi})(z,\zeta) d\theta$$

belongs to $S^{m_1-N}_{1,0}(\mathbb{R}^{2n}_{z,\zeta})$ uniformly with respect to $x, \xi$, so that in particular

$$\sup_{(x,\xi,z,\zeta)\in\mathbb{R}^{4n}} |\rho_{x,\xi}(z,\zeta)\langle \zeta\rangle^{-m_1+N}| = C_0 < +\infty.$$

Since $r_N(a_1,a_2)(x,\xi)\langle\xi\rangle^{-m_2} = \rho_{x,\xi}(x,\xi)$, we obtain

$$|r_N(a_1,a_2)(x,\xi)| \le C_0\langle\xi\rangle^{m_1+m_2-N}. \tag{4.3.9}$$

Using the formula (4.3.7) above gives as well the smoothness of $r_N(a_1,a_2)$ and with the identities (consequences of $\partial_{x_j}(a_1 \diamond a_2) = (\partial_{x_j} a_1) \diamond a_2 + a_1 \diamond (\partial_{x_j} a_2)$)

$$\begin{aligned} \partial_{x_j}\big(r_N(a_1,a_2)\big) &= r_N(\partial_{x_j} a_1, a_2) + r_N(a_1, \partial_{x_j} a_2) \\ \partial_{\xi_j}\big(r_N(a_1,a_2)\big) &= r_N(\partial_{\xi_j} a_1, a_2) + r_N(a_1, \partial_{\xi_j} a_2), \end{aligned}$$

it is enough to reapply (4.3.9) to get the result $r_N \in S^{m_1+m_2-N}_{1,0}$. □

We have already seen in Theorem 4.1.9 that the adjoint (in the sense of Definition 4.1.7) of the operator $a(x,D)$ is equal to $a^*(x,D)$, where $a^* = J\bar{a}$ ($J$ is given in Lemma 4.1.8). Lemma 4.3.9 gives the following result.

**Theorem 4.3.10.** *Let $a \in S^m_{1,0}$. Then $a^* = J\bar{a}$ and the mapping $a \mapsto a^*$ is continuous from $S^m_{1,0}$ into itself. Moreover, for all integers $N$, we have*

$$a^* = \sum_{|\alpha|<N} \frac{1}{\alpha!} D^\alpha_\xi \partial^\alpha_x \bar{a} + r_N(a), \quad r_N(a) \in S^{m-N}_{1,0}.$$

A consequence of the above results is the following.

**Corollary 4.3.11.** *Let $a_j \in S^{m_j}_{1,0}$, $j = 1,2$. Then we have*

$$a_1 \diamond a_2 \equiv a_1 a_2 \mod S^{m_1+m_2-1}_{1,0}, \tag{4.3.10}$$

$$a_1 \diamond a_2 - a_2 \diamond a_1 \equiv \frac{1}{2i\pi}\{a_1,a_2\} \mod S^{m_1+m_2-2}_{1,0}, \tag{4.3.11}$$

*where the* **Poisson bracket**

$$\{a_1,a_2\} = \sum_{1\le j\le n} \frac{\partial a_1}{\partial \xi_j}\frac{\partial a_2}{\partial x_j} - \frac{\partial a_1}{\partial x_j}\frac{\partial a_2}{\partial \xi_j}. \tag{4.3.12}$$

$$\textit{For } a \in S^m_{1,0}, \quad a^* \equiv \bar{a} \mod S^{m-1}_{1,0}. \tag{4.3.13}$$

**Theorem 4.3.12.** *Let $a$ be a symbol in $S^m_{1,0}$ such that $\inf_{(x,\xi)\in\mathbb{R}^{2n}} |a(x,\xi)|\langle\xi\rangle^{-m} > 0$. Then there exists $b \in S^{-m}_{1,0}$ such that*

$$\begin{array}{rcl} b(x,D)a(x,D) & = & \operatorname{Id} + l(x,D), \\ a(x,D)b(x,D) & = & \operatorname{Id} + r(x,D), \end{array} \quad r, l \in S^{-\infty}_{1,0} = \cap_\nu S^\nu_{1,0}.$$

*Proof.* We remark first that the smooth function $1/a$ belongs to $S^{-m}_{1,0}$: it follows from the Faà de Bruno formula or more elementarily, from the fact that, for $|\alpha|+|\beta| \ge 1$, $\partial_\xi^\alpha \partial_x^\beta (\frac{1}{a} a) = 0$, entailing with the Leibniz formula

$$a\partial_\xi^\alpha\partial_x^\beta(1/a) = \sum_{\substack{\alpha'+\alpha''=\alpha,\beta'+\beta''=\beta\\ |\alpha'|+|\beta'|<|\alpha|+|\beta|}} \partial_\xi^{\alpha'}\partial_x^{\beta'}(1/a)\partial_\xi^{\alpha''}\partial_x^{\beta''}(a)c(\alpha',\beta'),$$

with constants $c(\alpha',\beta')$. Arguing by induction on $|\alpha|+|\beta|$, we get

$$|a\partial_\xi^\alpha\partial_x^\beta(1/a)| \lesssim \sum_{\alpha'+\alpha''=\alpha} \langle\xi\rangle^{-m-|\alpha'|}\langle\xi\rangle^{m-|\alpha''|} \lesssim \langle\xi\rangle^{-|\alpha|}$$

and from $|a| \gtrsim \langle\xi\rangle^m$, we get $1/a \in S^{-m}_{1,0}$. Now, we can compute, using Theorem 4.3.8,

$$\frac{1}{a} \diamond a = 1 + l_1, \quad l_1 \in S^{-1}_{1,0}.$$

Inductively, we can assume that there exist $(b_0,\cdots,b_N)$ with $b_j \in S^{-m-j}$ such that

$$(b_0+\cdots+b_N)\diamond a = 1 + l_{N+1}, \quad l_{N+1} \in S^{-N-1}_{1,0}. \tag{4.3.14}$$

We can now take $b_{N+1} = -l_{N+1}/a$ which belongs to $S^{-m-N-1}$ and this gives

$$(b_0+\cdots+b_N+b_{N+1})\diamond a = 1 + l_{N+1} - l_{N+1} + l_{N+2}, \quad l_{N+2} \in S^{-N-2}_{1,0}.$$

**Lemma 4.3.13.** *Let $\mu \in \mathbb{R}$ and $(c_j)_{j\in\mathbb{N}}$ be a sequence of symbols such that $c_j \in S^{\mu-j}_{1,0}$. Then there exists $c \in S^\mu_{1,0}$ such that*

$$c \sim \sum_j c_j, \qquad i.e. \quad \forall N \in \mathbb{N}, \quad c - \sum_{0\le j<N} c_j \in S^{\mu-N}_{1,0}.$$

*Proof.* The proof is based on a Borel-type argument similar to the one used to construct a $C^\infty$ function with an arbitrary Taylor expansion. Let $\omega \in C^\infty_b(\mathbb{R}^n)$ such that $\omega(\xi) = 0$ for $|\xi| \le 1$ and $\omega(\xi) = 1$ for $|\xi| \ge 2$. Let $(\lambda_j)_{j\in\mathbb{N}}$ be a sequence of numbers $\ge 1$. We want to define

$$c(x,\xi) = \sum_{j\ge 0} c_j(x,\xi)\omega(\xi\lambda_j^{-1}), \tag{4.3.15}$$

and we shall show that a suitable choice of $\lambda_j$ will provide the answer. We note that, since $\lambda_j \ge 1$, the functions $\xi \mapsto \omega(\xi\lambda_j^{-1})$ make a bounded set in the Fréchet space $S^0_{1,0}$. Multiplying the $c_j$ by $\langle\xi\rangle^{-\mu}$, we may assume that $\mu = 0$. We have then, using the notation (4.3.2) (in which we drop the second index),

$$|c_j(x,\xi)|\omega(\xi\lambda_j^{-1}) \le \gamma_0(c_j)\langle\xi\rangle^{-j}\mathbf{1}_{|\xi|\ge\lambda_j} \le \gamma_0(c_j)\lambda_j^{-j/2}\langle\xi\rangle^{-j/2},$$

so that,

$$\forall j \geq 1,\ \lambda_j \geq 2^2 \gamma_0(c_j)^{\frac{2}{j}} = \mu_j^{(0)} \Longrightarrow \forall j \geq 1,\ |c_j(x,\xi)| \omega(\xi \lambda_j^{-1}) \leq 2^{-j} \langle \xi \rangle^{-j/2},$$

showing that the function $c$ can be defined as above in (4.3.15) and is a continuous bounded function. Let $1 \leq k \in \mathbb{N}$ be given. Calculating (with $\omega_j(\xi) = \omega(\xi \lambda_j^{-1})$) the derivatives $\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)$ for $|\alpha| + |\beta| = k$, we get

$$|\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)| \leq \gamma_k(c_j \omega_j) \langle \xi \rangle^{-j-|\alpha|} \mathbf{1}_{|\xi| \geq \lambda_j} \leq \tilde{\gamma}_k(c_j) \lambda_j^{-j/2} \langle \xi \rangle^{-|\alpha| - \frac{j}{2}},$$

so that

$$\forall j \geq k,\ \lambda_j \geq 2^2 \big(\tilde{\gamma}_k(c_j)\big)^{\frac{2}{j}} = \mu_j^{(k)} \Longrightarrow \forall j \geq k,\ |\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)| \leq 2^{-j} \langle \xi \rangle^{-|\alpha| - \frac{j}{2}}, \quad (4.3.16)$$

showing that the function $c$ can be defined as above in (4.3.15) and is a $C^k$ function such that

$$|(\partial_\xi^\alpha \partial_x^\beta c)(x,\xi)| \leq \sum_{0 \leq j < k} \tilde{\gamma}_k(c_j) \langle \xi \rangle^{-j-|\alpha|} + \sum_{j \geq k} 2^{-j} \langle \xi \rangle^{-|\alpha|} \leq C_k \langle \xi \rangle^{-|\alpha|}.$$

It is possible to fulfill the conditions on the $\lambda_j$ above for all $k \in \mathbb{N}$: just take

$$\lambda_j \geq \sup_{0 \leq k \leq j} \mu_j^{(k)}.$$

The function $c$ belongs to $S^0_{1,0}$ and

$$r_N = c - \sum_{0 \leq j < N} c_j = \sum_{0 \leq j < N} \underbrace{(\omega_j - 1) c_j}_{\in S^{-\infty}_{1,0}} + \sum_{j \geq N} c_j \omega_j,$$

and for $|\alpha| + |\beta| = k$, using the estimates (4.3.16), we obtain

$$\sum_{j \geq N} |\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)(x,\xi)| \leq \sum_{N \leq j < \max(2N,k)} \overbrace{|\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)(x,\xi)|}^{\lesssim \langle \xi \rangle^{-|\alpha|-j} \lesssim \langle \xi \rangle^{-|\alpha|-N}}$$
$$+ \sum_{j \geq \max(2N,k)} \underbrace{|\partial_\xi^\alpha \partial_x^\beta (c_j \omega_j)(x,\xi)|}_{\lesssim 2^{-j} \langle \xi \rangle^{-|\alpha| - \frac{j}{2}} \lesssim 2^{-j} \langle \xi \rangle^{-|\alpha|-N}},$$

proving that $r_N \in S^{-N}_{1,0}$. The proof of the lemma is complete. □

Going back to the proof of the theorem, we can take, using Lemma 4.3.13, $S^{-m}_{1,0} \ni b \sim \sum_{j \geq 0} b_j$, and for all $N \in \mathbb{N}$,

$$b \diamond a \in \sum_{0 \leq j < N} b_j \diamond a + S^{-N-m}_{1,0} \diamond a = 1 + S^{-N}_{1,0},$$

providing the first equality in Theorem 4.3.12. To construct a right approximate inverse, i.e. to obtain the second equality in this theorem with an a priori different $b$

follows the same lines (or can be seen as a direct consequence of the previous identity by applying it to the adjoint $a^*$); however we are left with the proof that the right and the left approximate inverse could be taken as the same. We have proven that there exists $b^{(1)}, b^{(2)} \in S^{-m}_{1,0}$ such that

$$b^{(1)} \diamond a \in 1 + S^{-\infty}_{1,0}, \quad a \diamond b^{(2)} \in 1 + S^{-\infty}_{1,0}.$$

Now we calculate, using[6] the theorem 4.1.5, $(b^{(1)} \diamond a) \diamond b^{(2)} = b^{(2)} \mod S^{-\infty}_{1,0}$ which is also $b^{(1)} \diamond (a \diamond b^{(2)}) = b^{(1)} \mod S^{-\infty}_{1,0}$ so that $b^{(1)} - b^{(2)} \in S^{-\infty}_{1,0}$, providing the result and completing the proof of the theorem. □

*Remark* 4.3.14. The mapping $\mathscr{S}'(\mathbb{R}^{2n}) \ni a \mapsto a(x, D)$ is (obviously) linear and one-to-one: if $a(x, D) = 0$, choosing $v(x) = e^{-\pi|x-x_0|^2}$, $\hat{u}(\xi) = e^{-\pi|\xi-\xi_0|^2}$, we get that the convolution of the distribution $\tilde{a}(x, \xi) = a(x, \xi)e^{2i\pi x\cdot\xi}$ with the Gaussian function $e^{-\pi(|x|^2+|\xi|^2)}$ is zero, so that, taking the Fourier transform shows that the product of the same Gaussian function with $\widehat{\tilde{a}}$ is zero, implying that $\tilde{a}$ and thus $a$ is zero. It is a consequence of a version of the Schwartz kernel theorem that the same mapping $\mathscr{S}'(\mathbb{R}^{2n}) \ni a \mapsto a(x, D) \in$ continuous linear operators from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ is indeed onto. However the "onto" part of our statement is highly non trivial and a version of this theorem can be found in the theorem 5.2.1 of [5].

An important consequence of the proof of the previous theorem is the possible microlocalization of this result.

**Theorem 4.3.15.** *Let $\chi$ be a symbol in $S^0_{1,0}$ and let $a$ be a symbol in $S^m_{1,0}$ such that $\inf_{(x,\xi)\in \operatorname{supp}\chi} |a(x, \xi)|\langle\xi\rangle^{-m} > 0$. Let $\psi$ be a symbol in $S^0_{1,0}$ such that $\operatorname{supp}\psi \subset \{\chi = 1\}$. Then there exists $b \in S^{-m}_{1,0}$ such that*

$$b(x, D)a(x, D) \quad = \quad \psi(x, D) + l(x, D), \qquad l \in S^{-\infty}_{1,0}.$$

*Proof.* We consider the symbol $b_0 = \chi/a$, which belongs obviously to $S^{-m}_{1,0}$. We have

$$b_0 \diamond a = \chi + l_1, \quad l_1 \in S^{-1}_{1,0}, \quad \big(-\frac{\chi l_1}{a} + \frac{\chi}{a}\big) \diamond a = \chi + l_1(1-\chi) + l_2, \ l_2 \in S^{-2}_{1,0}.$$

Inductively, we may assume that there exists $(b_0, \dots, b_N)$ with $b_j \in S^{-m-j}$ such that

$$(b_0 + b_1 + \cdots + b_N) \diamond a = \chi + \sum_{1\le j\le N} l_j(1-\chi) + l_{N+1}, \quad l_{N+1} \in S^{-1-N}_{1,0}.$$

Choosing $b_{N+1} = -\chi l_{N+1}/a$, we get

$$(b_0 + b_1 + \cdots + b_N + b_{N+1}) \diamond a = \chi + \sum_{1\le j\le N+1} l_j(1-\chi) + l_{N+2}, \quad l_{N+2} \in S^{-2-N}_{1,0}.$$

[6] A consequence of Theorem 4.1.5 is the associativity of the "law" $\diamond$ since

$$\operatorname{Op}(a \diamond (b \diamond c)) = \operatorname{Op}(a)\big(\operatorname{Op}(b)\operatorname{Op}(c)\big) = \big(\operatorname{Op}(a)\operatorname{Op}(b)\big)\operatorname{Op}(c) = \operatorname{Op}((a \diamond b) \diamond c)$$

so that the injectivity property of Remark 4.3.14 gives the answer.

Taking now a symbol $\psi \in S^0_{1,0}$ such that $\operatorname{supp}\psi \subset \chi^{-1}(\{1\})$, we obtain for all $N \in \mathbb{N}$, the existence of symbols $b_0, \cdots, b_N$ with $b_j \in S^{-m-j}$ such that

$$\psi \diamond (b_0 + b_1 + \cdots + b_N) \diamond a = \psi \diamond \chi + \psi \diamond \sum_{1 \le j \le N} l_j(1-\chi) + \psi \diamond l_{N+1} \qquad (l_{N+1} \in S^{-1-N}_{1,0})$$
$$= \psi + r_{N+1}, \qquad r_{N+1} \in S^{-1-N}_{1,0}.$$

Using now Lemma 4.3.13, we find a symbol $b \in S^{-m}_{1,0}$ such that, for all $N \in \mathbb{N}$, $\psi \diamond b \diamond a \in \psi + S^{-1-N}_{1,0}$, i.e. we find $\tilde b \in S^{-m}_{1,0}$ such that $\tilde b \diamond a \equiv \psi \pmod{S^{-\infty}_{1,0}}$. □

## 4.4 Gårding's inequality

We end this introduction with the so-called *Sharp Gårding inequality*, a result proven in 1966 by L. Hörmander [2] and extended to systems the same year by P. Lax and L. Nirenberg [8].

**Theorem 4.4.1.** *Let $a$ be a nonnegative symbol in $S^m_{1,0}$. Then there exists a constant $C$ such that, for all $u \in \mathscr{S}(\mathbb{R}^n)$,*

$$\operatorname{Re}\langle a(x,D)u, u\rangle + C\|u\|^2_{H^{\frac{m-1}{2}}(\mathbb{R}^n)} \ge 0. \tag{4.4.1}$$

*Proof. First reductions.* We may assume that $m = 1$: in fact, the statement for $m = 1$ implies the result by considering, for a nonnegative $a \in S^m_{1,0}$, the operator $\langle D\rangle^{\frac{1-m}{2}} a(x,D)\langle D\rangle^{\frac{1-m}{2}}$ which, according to Theorem 4.3.8 has a symbol in $S^1_{1,0}$, which belongs to $\langle\xi\rangle^{1-m}a(x,\xi) + S^0_{1,0}$. Applying the result for $m = 1$, and the $L^2$-boundedeness of operators with symbols in $S^0_{1,0}$, we get for all $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\operatorname{Re}\langle \langle D\rangle^{\frac{1-m}{2}} a(x,D)\langle D\rangle^{\frac{1-m}{2}} u, u\rangle + C\|u\|^2_{L^2(\mathbb{R}^n)} \ge 0,$$

which gives the sought result when applied to $u = \langle D\rangle^{\frac{m-1}{2}} v$. We may also replace $a(x,D)$ by $a^w$, where $a^w$ is the operator with Weyl symbol $a$ . In fact, according to Lemma 4.1.8, $J^{1/2}a - a \in S^0_{1,0}$ and $\operatorname{Op}(S^0_{1,0})$ is $L^2$-bounded.
*Main step: a result with a small parameter.* We consider a nonnegative $a \in S^1_{1,0}$ and

$$\varphi \in C^\infty_c((0,+\infty);\mathbb{R}_+) \text{ such that } \int_0^{+\infty} \varphi(h)\frac{dh}{h} = 1. \tag{4.4.2}$$

This implies

$$a(x,\xi) = \int_0^{+\infty} \underbrace{\varphi(\langle\xi\rangle h)a(x,\xi)}_{=a_h(x,\xi)} \frac{dh}{h}. \tag{4.4.3}$$

We have, with $\Gamma_h(x,\xi) = 2^n \exp -2\pi(h^{-1}|x|^2 + h|\xi|^2)$ and $X = (x,\xi)$,

$$(a_h * \Gamma_h)(X) = a_h(X) + \int_0^1 (1-\theta)a_h''(X+\theta Y)Y^2\Gamma_h(Y)dYd\theta$$
$$= a_h(X) + r_h(X). \tag{4.4.4}$$

The main step of the proof is that $(a_h * \Gamma_h)^w \ge 0$, a result following from the next calculation (for $u \in \mathscr{S}(\mathbb{R}^n)$), due to Definition 4.2.1. We have, with $\Omega_{u,u}$ defined in (4.2.4),

$$\begin{aligned}\langle (a_h * \Gamma_h)^w u, u\rangle &= \iint (a_h * \Gamma_h)(x,\xi)\left(\int e^{-2i\pi z\cdot\xi} u(x+\frac{z}{2})\bar u(x-\frac{z}{2})dz\right) dxd\xi \\ &= \iint a(y,\eta)(\Omega_{u,u}(1/2) * \Gamma_h)(y,\eta)dyd\eta,\end{aligned}$$

and since $(\Omega_{u,u}(1/2) * \Gamma_h)(x,\xi) =$

$$\begin{aligned}&\iiint e^{-2i\pi z\cdot(\xi-\eta)} u(x-y+\frac{z}{2})\bar u(x-y-\frac{z}{2}) 2^n \exp -2\pi(h^{-1}|y|^2+h|\eta|^2)dzdyd\eta \\ &= \iint e^{-2i\pi z\cdot\xi} u(x-y+\frac{z}{2})\bar u(x-y-\frac{z}{2}) 2^{n/2} e^{-2\pi h^{-1}|y|^2} h^{-n/2} e^{-\frac{\pi}{2h}|z|^2} dzdy \\ &= \iint u(x-y_1)\bar u(x-y_2) e^{-2i\pi(y_2-y_1)\cdot\xi} 2^{n/2} h^{-n/2} e^{-\frac{\pi}{2h}|y_1+y_2|^2} e^{-\frac{\pi}{2h}|y_1-y_2|^2} dy_1 dy_2 \\ &= 2^{n/2} h^{-n/2}\left|\int u(x-y_1) e^{2i\pi y_1\cdot\xi} e^{-\pi h^{-1}|y_1|^2} dy_1\right|^2 \ge 0,\end{aligned}$$

we get indeed $(a_h * \Gamma_h)^w \ge 0$. From (4.4.3) and (4.4.4), we get

$$a^w = \int_0^{+\infty} a_h^w h^{-1} dh = \int_0^{+\infty} (a_h * \Gamma_h)^w h^{-1} dh - \int_0^{+\infty} r_h^w h^{-1} dh \ge \\ - \int_0^{+\infty} r_h^w h^{-1} dh.$$

*Last step:* $\int_0^{+\infty} r_h^w h^{-1} dh$ *is* $L^2$*-bounded.* This is a technical point, where the main difficulty is coming from the integration in $h$. We have from (4.4.4) and the fact that $\Gamma_h$ is an even function,

$$r_h(X) = \frac{1}{8\pi}\operatorname{trace}_h a_h''(X) + \frac{1}{3!}\iint_0^1 (1-\theta)^3 a_h^{(4)}(X+\theta Y)Y^4\Gamma_h(Y)dYd\theta,$$

with $\operatorname{trace}_h a_h''(X) = h \operatorname{trace}\partial_x^2 a_h + h^{-1}\operatorname{trace}\partial_\xi^2 a_h$. Since $\varphi \in C_c^\infty((0,+\infty))$, we have

$$\int_0^{+\infty} h \operatorname{trace}\partial_x^2 a_h h^{-1} dh = \operatorname{trace}\partial_x^2 a(x,\xi)\int_0^{+\infty}\varphi(\langle\xi\rangle h)dh = c\operatorname{trace}\partial_x^2 a\langle\xi\rangle^{-1},$$

with $c = \int_0^{+\infty}\varphi(t)dt$. The symbol $c\operatorname{trace}\partial_x^2 a\langle\xi\rangle^{-1}$ belongs to $S^0_{1,0}$ as well as the other term $\int_0^{+\infty} h^{-1}\operatorname{trace}\partial_\xi^2 a_h(x,\xi)h^{-1}dh$: we have

$$\begin{aligned}(\partial_\xi a_h)(x,\xi) &= (\partial_\xi a)(x,\xi)\varphi(h\langle\xi\rangle) + a(x,\xi)\varphi'(h\langle\xi\rangle)h\langle\xi\rangle^{-1}\xi \\ (\partial_\xi^2 a_h)(x,\xi) &= (\partial_\xi^2 a)(x,\xi)\varphi(h\langle\xi\rangle) + 2\partial_\xi a(x,\xi)\varphi'(h\langle\xi\rangle)h \\ &\quad + a(x,\xi)\varphi''(h\langle\xi\rangle)h^2\langle\xi\rangle^{-2}\xi^2 + a(x,\xi)\varphi'(h\langle\xi\rangle)h\partial_\xi\bigl(\xi\langle\xi\rangle^{-1}\bigr),\end{aligned}$$

and checking for instance the term $\int_0^{+\infty} h^{-1}(\partial_\xi^2 a)(x,\xi)\varphi(h\langle\xi\rangle)\frac{dh}{h}$, we see that it is equal to

$$(\partial_\xi^2 a)(x,\xi)\int_0^{+\infty} h^{-1}\varphi(h\langle\xi\rangle)\frac{dh}{h} = (\partial_\xi^2 a)(x,\xi)\langle\xi\rangle\int_0^{+\infty} h^{-1}\varphi(h)\frac{dh}{h}$$
$$= c_1(\partial_\xi^2 a)(x,\xi)\langle\xi\rangle \in S^0_{1,0},$$

whereas the other terms are analogous. We are finally left with the term

$$\rho(X) = \frac{1}{3!}\iiint_0^1 (1-\theta)^3 a_h^{(4)}(X+\theta Y)Y^4\Gamma_h(Y)dY h^{-1}dh d\theta,$$

and we note that on the integrand of (4.4.3), the product $h\langle\xi\rangle$ is bounded above and below by fixed constants and that integral can in fact be written as

$$a(x,\xi) = \int_{\kappa_0\langle\xi\rangle^{-1}}^{\kappa_1\langle\xi\rangle^{-1}} \varphi(\langle\xi\rangle h)a(x,\xi)dh/h$$

with $0 < \kappa_0 = \min\operatorname{supp}\varphi < \kappa_1 = \max\operatorname{supp}\varphi$. Consequently the symbol $a_h$ satisfies the following estimates:

$$|\partial_\xi^\alpha\partial_x^\beta a_h| \le C_{\alpha\beta}h^{-1+|\alpha|}$$

where the $C_{\alpha\beta}$ are some semi-norms of $a$ (and thus independent of $h$). As a result, the above estimates can be written in a more concise and convenient way, using the multilinear forms defined by the derivatives. We have, with $T=(t,\tau)\in\mathbb{R}^n\times\mathbb{R}^n$,

$$|a_h^{(l)}(X)T^l| \le C_l h^{-1}g_h(T)^{l/2}, \quad \text{with } g_h(t,\tau) = |t|^2 + h^2|\tau|^2.$$

We calculate

$$\rho^{(k)}(X)T^k = \frac{1}{3!}\iiint_0^1 (1-\theta)^3 a_h^{(4+k)}(X+\theta Y)Y^4T^k\Gamma_h(Y)dY h^{-1}dh d\theta,$$

which satisfies with $\omega_h(t,\tau) = h^{-1}g_h(t,\tau)$,

$$\begin{aligned}
&|\rho^{(k)}(X)T^k| \\
&\quad\le \frac{C_{4+k}}{4!}\iint \mathbf{1}\{h\le\kappa_1\}h^{-1}g_h(T)^{k/2}\underbrace{g_h(Y)^2}_{=h^2\omega_h(Y)^2}2^n e^{-2\pi\omega_h(Y)}dY h^{-1}dh \\
&\quad\le \frac{C_{4+k}}{4!}g_h(T)^{k/2}\iint \omega_h(Y)^2\mathbf{1}\{h\le\kappa_1\}2^n e^{-2\pi\omega_h(Y)}dY dh \le \tilde{C}_k(|t|+|\tau|)^k
\end{aligned}$$

and this proves that the function $\rho$ belongs to $C_b^\infty(\mathbb{R}^{2n})$, as well as $J^{1/2}\rho$ (Lemma 4.1.8) and thus $\rho^w = (J^{1/2}\rho)(x,D)$ is bounded on $L^2$ (Theorem 4.1.4). The proof is complete. □

*Remark* 4.4.2. Theorem 4.4.1 remains valid for systems, even in infinite dimension. For definiteness, let us assume simply that $a(x,\xi)$ is a $N\times N$ Hermitian non-negative matrix of symbols in $S^1_{1,0}$. Then for all $u\in\mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$, the inequality (4.4.1) holds. The vector space $\mathbb{C}^N$ can be replaced in the above statement by an infinite-dimensional complex Hilbert space $H$ with $a$ valued in $\mathscr{L}(H)$ and the proof above requires essentially no change.

## 4.5 The semi-classical calculus

A semiclassical symbol of order $m$ is defined as a family of smooth functions $a(\cdot,\cdot,h)$ defined on the phase space $\mathbb{R}^{2n}$, depending on a parameter $h \in (0,1]$, such that, for all multi-indices $\alpha,\beta$

$$\sup_{(x,\xi,h)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,1]} |(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi,h)| h^{m-|\alpha|} < +\infty. \tag{4.5.1}$$

The set of semi-classical symbols of order $m$ will be denoted by $S^m_{scl}$. A typical example of such a symbol of order 0 is a function $a_1(x, h\xi)$ where $a_1$ belongs to $C_b^\infty(\mathbb{R}^{2n})$: we have indeed $\partial_\xi^\alpha\partial_x^\beta\big(a_1(x,h\xi)\big) = (\partial_\xi^\alpha\partial_x^\beta a_1)(x,h\xi)h^{|\alpha|}$. It turns out that this version of the semi-classical calculus is certainly the easiest to understand and that Theorem 4.1.4 is implying the main continuity result for these symbols. The reader has also to keep in mind that we are not dealing here with a single function defined on the phase space, but with a family of symbols depending on a (small) parameter $h$, a way to express that the constants occurring in (4.5.1) are "independent of $h$". We shall review the results of the section on the $S^m_{1,0}$ class of symbols and show how they can be transferred to the semi-classical framework, *mutatis mutandis* and almost without any new argument. To understand the correspondence between symbols in $S^m_{1,0}$ and semi-classical symbols, it is essentially enough to think of the $S_{1,0}$ calculus as a semi-classical calculus with small parameter $\langle\xi\rangle^{-1}$.

We can define, for $a \in S^m_{scl}, k \in \mathbb{N}$,

$$\gamma_{k,m}(a) = \sup_{(x,\xi,h)\in\mathbb{R}^{2n}\times(0,1],|\alpha|+|\beta|\le k} |(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi,h)| h^{m-|\alpha|}. \tag{4.5.2}$$

**Theorem 4.5.1.** *Let $a \in S^m_{scl}$. Then the operator $a(x,D,h)h^m$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself with constants independent of $h \in (0,1]$.*

*Proof.* We have $a(x,D,h) = \operatorname{Op}(a(x,\xi,h))$. The set $\big\{a(x,\xi,h)h^m\big\}_{h\in(0,1]}$ is bounded in $C_b^\infty(\mathbb{R}^{2n})$, so that we can use Theorem 4.1.2 to get the result. □

**Theorem 4.5.2.** *Let $a \in S^m_{scl}$. Then the operator $a(x,D,h)h^m$ is bounded on $L^2(\mathbb{R}^n)$ with a norm bounded above independently of $h \in (0,1]$.*

*Proof.* The set $\big\{a(x,\xi,h)h^m\big\}_{h\in(0,1]}$ being bounded in $C_b^\infty(\mathbb{R}^{2n})$, it follows from Theorem 4.1.4. □

**Theorem 4.5.3.** *Let $m_1, m_2$ be real numbers and $a_1 \in S^{m_1}_{scl}, a_2 \in S^{m_2}_{scl}$. Then the composition $a_1(x,D,h)a_2(x,D,h)$ makes sense as a continuous operator from $\mathscr{S}(\mathbb{R}^n)$ into itself, as well as a bounded operator on $L^2(\mathbb{R}^n)$ and*

$$a_1(x,D,h)a_2(x,D,h) = (a_1 \diamond a_2)(x,D,h)$$

*where $a_1 \diamond a_2$ belongs to $S^{m_1+m_2}_{scl}$ and is given by the formula*

$$(a_1 \diamond a_2)(x,\xi,h) = (\exp 2i\pi D_y\cdot D_\eta)\Big(a_1(x,\xi+\eta,h)a_2(y+x,\xi,h)\Big)_{|y=0,\eta=0}. \tag{4.5.3}$$

*Proof.* This is a direct consequence of Theorem 4.1.5 since

$$\cup_{j=1,2}\{h^{m_j}a_j(x,\xi,h)\}_{h\in(0,1]} \quad \text{is bounded in } C_b^\infty(\mathbb{R}^{2n}).$$

□

**Theorem 4.5.4.** *Let $m_1, m_2$ be real numbers and $a_1 \in S_{scl}^{m_1}, a_2 \in S_{scl}^{m_2}$. Then $a_1(x,D,h)a_2(x,D,h) = (a_1 \diamond a_2)(x,D,h)$, the symbol $a_1 \diamond a_2$ belongs to $S_{scl}^{m_1+m_2}$ and we have the asymptotic expansion, for all $N \in \mathbb{N}$,*

$$a_1 \diamond a_2 = \sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha a_1 \partial_x^\alpha a_2 + r_N(a_1,a_2), \tag{4.5.4}$$

*with $r_N(a_1,a_2) \in S_{scl}^{m_1+m_2-N}$. Note that $D_\xi^\alpha a_1 \partial_x^\alpha a_2$ belongs to $S_{scl}^{m_1+m_2-|\alpha|}$.*

*Proof.* Since $h^{m_j}a_j(x,\xi,h), j=1,2$, belongs to $S_{scl}^0$, we may assume that $m_1 = m_2 = 0$. We can use the formula (4.3.3) and apply the formula (4.3.8) to get the desired formula with

$$r_N(a_1,a_2)(x,\xi,h) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} e^{2i\pi\theta D_z\cdot D_\zeta} (2i\pi D_z \cdot D_\zeta)^N \big(a_1(x,\zeta,h)a_2(z,\xi,h)\big) d\theta_{|z=x,\zeta=\xi}. \tag{4.5.5}$$

The function $(z,\zeta) \mapsto b_{x,\xi,h}(z,\zeta) = (2i\pi D_z \cdot D_\zeta)^N a_1(x,\zeta,h)a_2(z,\xi,h)$ belongs to $S_{scl}^{-N}(\mathbb{R}^{2n}_{z,\zeta})$ uniformly with respect to the parameters $(x,\xi) \in \mathbb{R}^{2n}$: it satisfies, using the notation (4.5.2), for $\max(|\alpha|,|\beta|) \le k$,

$$|\partial_\zeta^\alpha \partial_z^\beta b_{x,\xi,h}(z,\zeta)| \le \gamma_{k,m_1}(a_1)\gamma_{k,m_2}(a_2)h^{N+|\alpha|}.$$

Applying Lemma 4.1.8, we obtain that the function

$$\rho_{x,\xi,h}(z,\zeta) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} (J^\theta b_{x,\xi,h})(z,\zeta) d\theta$$

belongs to $S_{scl}^{-N}(\mathbb{R}^{2n}_{z,\zeta})$ uniformly with respect to $x,\xi,h$, so that in particular

$$\sup_{(x,\xi,z,\zeta)\in\mathbb{R}^{4n}, h\in(0,1]} |\rho_{x,\xi,h}(z,\zeta)h^{-N}| = C_0 < +\infty.$$

Since $r_N(a_1,a_2)(x,\xi) = \rho_{x,\xi,h}(x,\xi)$, we obtain

$$|r_N(a_1,a_2)(x,\xi)| \le C_0 h^N. \tag{4.5.6}$$

Using the formula (4.5.5) above gives as well the smoothness of $r_N(a_1,a_2)$ and with the identities (consequences of $\partial_{x_j}(a_1 \diamond a_2) = (\partial_{x_j}a_1) \diamond a_2 + a_1 \diamond (\partial_{x_j}a_2)$)

$$\partial_{x_j}\big(r_N(a_1,a_2)\big) = r_N(\partial_{x_j}a_1,a_2) + r_N(a_1,\partial_{x_j}a_2)$$
$$\partial_{\xi_j}\big(r_N(a_1,a_2)\big) = r_N(\partial_{\xi_j}a_1,a_2) + r_N(a_1,\partial_{\xi_j}a_2),$$

it is enough to reapply (4.5.6) to get the result $r_N \in S_{scl}^{-N}$. □

Lemma 4.1.8 and Taylor's expansion (4.5.5) give the following result.

**Theorem 4.5.5.** *Let $a \in S^m_{scl}$. Then $a^* = J\bar{a}$ and the mapping $a \mapsto a^*$ is continuous from $S^m_{scl}$ into itself. Moreover, for all integers $N$, we have*

$$a^* = \sum_{|\alpha|<N} \frac{1}{\alpha!} D_\xi^\alpha \partial_x^\alpha \bar{a} + r_N(a), \quad r_N(a) \in S^{m-N}_{scl}.$$

**Corollary 4.5.6.** *Let $a_j \in S^{m_j}_{scl}, j = 1, 2$. Then we have*

$$a_1 \diamond a_2 \equiv a_1 a_2 \mod S^{m_1+m_2-1}_{scl}, \tag{4.5.7}$$

$$a_1 \diamond a_2 - a_2 \diamond a_1 \equiv \frac{1}{2i\pi}\{a_1, a_2\} \mod S^{m_1+m_2-2}_{scl}, \tag{4.5.8}$$

$$\text{For } a \in S^m_{scl}, \quad a^* \equiv \bar{a} \mod S^{m-1}_{scl}. \tag{4.5.9}$$

**Lemma 4.5.7.** *Let $\mu \in \mathbb{R}$ and $(c_j)_{j\in\mathbb{N}}$ be a sequence of symbols such that $c_j \in S^{\mu-j}_{scl}$. Then there exists $c \in S^\mu_{scl}$ such that*

$$c \sim \sum_j c_j, \qquad i.e. \quad \forall N \in \mathbb{N}, \quad c - \sum_{0\le j<N} c_j \in S^{\mu-N}_{scl}.$$

*Proof.* The proof is almost identical to the proof of Lemma 4.3.13.

Let $\omega \in C^\infty_b(\mathbb{R}; \mathbb{R}_+)$ such that $\omega(t) = 0$ for $t \le 1$ and $\omega(t) = 1$ for $t \ge 2$. Let $(\lambda_j)_{j\in\mathbb{N}}$ be a sequence of numbers $\ge 1$. We want to define

$$c(x, \xi, h) = \sum_{j\ge 0} c_j(x, \xi, h)\omega(h^{-1}\lambda_j^{-1}), \tag{4.5.10}$$

and we shall show that a suitable choice of $\lambda_j$ will provide the answer. Multiplying the $c_j$ by $h^\mu$, we may assume that $\mu = 0$. We have then

$$|c_j(x, \xi, h)|\omega(h^{-1}\lambda_j^{-1}) \le \gamma_0(c_j)h^j \mathbf{1}_{1\ge h\lambda_j} \le \gamma_0(c_j)\lambda_j^{-j},$$

so that,

$$\forall j \ge 1,\ \lambda_j \ge 2\gamma_0(c_j)^{\frac{1}{j}} = \mu_j^{(0)} \Longrightarrow \forall j \ge 1,\ |c_j(x, \xi, h)|\omega(h^{-1}\lambda_j^{-1}) \le 2^{-j},$$

showing that the function $c$ can be defined as above in (4.5.10) and is a continuous bounded function. Let $1 \le k \in \mathbb{N}$ be given. Calculating (with $\omega_j = \omega(h^{-1}\lambda_j^{-1})$) the derivatives $\omega_j \partial_\xi^\alpha \partial_x^\beta(c_j)$ for $|\alpha| + |\beta| = k$, we get

$$\omega_j|\partial_\xi^\alpha \partial_x^\beta(c_j)| \le \gamma_k(c_j)h^{j+|\alpha|}\mathbf{1}_{1\ge h\lambda_j} \le \gamma_k(c_j)\lambda_j^{-j/2}h^{|\alpha|+\frac{j}{2}},$$

so that

$$\forall j \ge k,\ \lambda_j \ge 2^2\bigl(\gamma_k(c_j)\bigr)^{\frac{2}{j}} = \mu_j^{(k)} \Longrightarrow \forall j \ge k,\ |\partial_\xi^\alpha \partial_x^\beta(c_j\omega_j)| \le 2^{-j}h^{|\alpha|+\frac{j}{2}}, \tag{4.5.11}$$

showing that the function $c$ can be defined as above in (4.5.10) and is a $C^k$ function such that

$$|(\partial_\xi^\alpha \partial_x^\beta c)(x, \xi, h)| \le \sum_{0\le j<k} \gamma_k(c_j)h^{j+|\alpha|} + \sum_{j\ge k} 2^{-j}h^{|\alpha|} \le C_k h^{|\alpha|}.$$

It is possible to fulfill the conditions on the $\lambda_j$ above for all $k \in \mathbb{N}$: just take $\lambda_j \geq \sup_{0\leq k\leq j} \mu_j^{(k)}$. The function $c$ belongs to $S^0_{scl}$ and, with $S^{-\infty}_{scl} = \cap_{m\in\mathbb{R}} S^m_{scl}$,

$$r_N = c - \sum_{0\leq j<N} c_j = \sum_{0\leq j<N} \underbrace{(\omega_j - 1)c_j}_{\in S^{-\infty}_{scl}} + \sum_{j\geq N} c_j\omega_j,$$

and for $|\alpha| + |\beta| = k$, using the estimates (4.5.11), we obtain

$$\sum_{j\geq N} |\partial_\xi^\alpha \partial_x^\beta (c_j\omega_j)(x,\xi,h)| \leq \sum_{N\leq j<\max(2N,k)} \overbrace{|\partial_\xi^\alpha \partial_x^\beta (c_j\omega_j)(x,\xi,h)|}^{\lesssim h^{|\alpha|+j}\lesssim h^{|\alpha|+N}} + \sum_{j\geq \max(2N,k)} \underbrace{|\partial_\xi^\alpha \partial_x^\beta (c_j\omega_j)(x,\xi,h)|}_{\lesssim 2^{-j}h^{|\alpha|+\frac{j}{2}}\lesssim 2^{-j}h^{|\alpha|+N}},$$

proving that $r_N \in S^{-N}_{scl}$. The proof of the lemma is complete. □

*Remark* 4.5.8. These asymptotic results (as well as the example $a_1(x, h\xi)$ with $a_1 \in C_b^\infty(\mathbb{R}^{2n})$ see page 111) led many authors to set a slightly different framework for the semiclassical calculus; instead of dealing with a family of symbols $a(x, \xi, h)$ satisfying the estimates (4.5.1), one deals with a function $a \in C_b^\infty(\mathbb{R}^{2n})$ and consider the operator $a(x, hD_x)$ or the operator $a(x, h\xi)^w$; another way to express this is to modify the quantization formula and to define for instance

$$(a^{w_h}u)(x) = \iint e^{\frac{2i\pi}{h}\langle x-y,\xi\rangle} a(\frac{x+y}{2}, \xi) u(y) dy d\xi h^{-n}, \text{ i.e. } a^{w_h} = a(x, h\xi)^w. \tag{4.5.12}$$

Then, using Lemma 4.5.7, given a sequence $(a_j)_{j\geq 0}$ in $C_b^\infty(\mathbb{R}^{2n})$, it is possible to consider $a(x, \xi, h) \in S^0_{scl}$ with

$$a(x,\xi,h) \sim \sum_{j\geq 0} h^j a_j(x, h\xi), \quad \text{i.e.} \quad \forall N,\ a(x,\xi,h) - \sum_{0\leq j<N} h^j a_j(x,h\xi) \in S^{-N}_{scl}.$$

The symbol $a_0$ is the *principal symbol* and

$$a(x,\xi,h)^w \sim \sum_{j\geq 0} h^j a_j^{w_h}, \quad \text{i.e.} \quad \forall N,\ a(x,\xi,h)^w - \sum_{0\leq j<N} h^j a_j^{w_h} = h^N r_{N,h}^{w_h},$$

where $\{r_{N,h}\}_{0<h\leq 1}$ is bounded in $C_b^\infty(\mathbb{R}^{2n})$: in fact we have from Theorem 4.5.4,

$$h^N r_{N,h}(x, h\xi) = s_N(x,\xi,h), \quad s_N \in S^{-N}_{scl}, \quad \text{i.e. } r_{N,h}(x,\xi) = h^{-N} s_N(x, h^{-1}\xi, h),$$

and thus

$$|(\partial_\xi^\alpha \partial_x^\beta r_{N,h})(x,\xi) = h^{-N-|\alpha|}(\partial_\xi^\alpha \partial_x^\beta s_N)(x, h^{-1}\xi, h)| \leq h^{-N-|\alpha|}\gamma_{\alpha,\beta,N} h^{N+|\alpha|}.$$

If $a, b \in S^0_{scl}$ and $a \sim \sum_{j\geq 0} h^j a_j(x, h\xi), b \sim \sum_{j\geq 0} h^j b_j(x, h\xi)$ as above, then one can prove, using Corollary 4.5.6 and Lemma 4.1.8

$$a^w b^w \equiv (a_0 b_0)^{w_h} \mod h(S^0_{scl})^w, \tag{4.5.13}$$

$$[a^w, b^w] \equiv \frac{h}{2i\pi}\{a_0, b_0\}^{w_h} \mod h^2(S^0_{scl})^w. \tag{4.5.14}$$

There are many variations on this theme, and in particular, one can replace the space $C_b^\infty(\mathbb{R}^{2n})$ by a more general one, involving some weight functions, for instance with polynomial growth at infinity. At this point, we are leaving an introduction to the pseudodifferential calculus and can use our more general approach of Chapter 2, involving metrics on the phase space, which incorporate all these variations. Expecting these generalizations, we shall not use the $w_h$ quantization in this book, except for the present remark.

**Theorem 4.5.9.** *Let $a$ be a symbol in $S^0_{scl}$ such that*

$$\inf_{(x,\xi)\in\mathbb{R}^{2n}, h\in(0,1]} |a(x,\xi,h)| > 0.$$

*Then there exists $b \in S^0_{scl}$ such that*

$$\begin{aligned} b(x,D,h)a(x,D,h) &= \operatorname{Id} + l(x,D,h), \\ a(x,D,h)b(x,D,h) &= \operatorname{Id} + r(x,D,h), \end{aligned} \qquad r, l \in S^{-\infty}_{scl} = \cap_\nu S^\nu_{scl}.$$

*Proof.* The only change to perform in the proof of Theorem 4.3.12 to get this result is to replace everywhere $S_{1,0}$ by $S_{scl}$. □

**Theorem 4.5.10.** *Let $\chi$ be a symbol in $S^0_{scl}$ and let $a$ be a symbol in $S^0_{scl}$ such that $\inf_{h\in(0,1],(x,\xi)\in\operatorname{supp}\chi(\cdot,\cdot,h)} |a(x,\xi,h)| > 0$. Let $\psi$ be a symbol in $S^0_{scl}$ such that $\operatorname{supp}\psi(\cdot,\cdot,h) \subset \{(x,\xi), \chi(x,\xi,h) = 1\}$. Then there exists $b \in S^0_{scl}$ such that*

$$b(x,D,h)a(x,D,h) = \psi(x,D,h) + l(x,D,h), \qquad l \in S^{-\infty}_{scl}.$$

*Proof.* Here also we have only to follow the proof of Theorem 4.3.15 and use Lemma 4.5.7 instead of Lemma 4.3.13 in the course of the proof. □

**Theorem 4.5.11.** *Let $a$ be a nonnegative symbol in $S^0_{scl}$. Then there exists a constant $C$ such that, for all $u \in \mathscr{S}(\mathbb{R}^n)$,*

$$\operatorname{Re}\langle a(x,D,h)u, u\rangle + hC\|u\|^2_{L^2(\mathbb{R}^n)} \geq 0. \tag{4.5.15}$$

*Equivalently, there exists $C \geq 0$ such that $a^w + Ch \geq 0$.*

*Proof.* The proof of Theorem 4.4.1 is containing a proof of this result: noticing that it is harmless to replace the standard quantization by the Weyl quantization for this result, since $J^{1/2}a - a$ belongs to $S^{-1}_{scl}$ (see the formula (4.3.8) and Lemma 4.3.9), we use the formula (4.4.4) to obtain than $(a * \Gamma_h)^w \geq 0$. The difference $a * \Gamma_h - a$ is $\int_0^1 (1-\theta)\int_{\mathbb{R}^{2n}} a''(X+\theta Y, h)Y^2\Gamma_h(Y)dYd\theta$, which belongs to $S^0_{scl}$. □

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