Onset of instability for a class of non-linear PDE systems

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1.Introduction

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Existence and uniqueness are important for an evolution equation, but of little interest without some inequalities controlling the size of the solution $u(t)$ at a positive time t by the size of the initial datum $u(0)$ in some appropriate functional space.

A typical example of an ill-posed problem (i.e. not well-posed) is the Cauchy problem for the $\overline{\partial}$ equation :

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For an ill-posed problem, large oscillations in the initial datum trigger exponential increasing in time of the solution.

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" It was pointed out very emphatically by Hadamard that it is not natural to consider only analytic solutions and source functions even for an operator with analytic coefficients. This reduces the interest of the Cauchy-Kovalevskaya theorem which . . .does not distinguish between classes of differential operators which have, in fact, very different properties such as the Laplace operator and the Wave operator."

Let us start over with the toy model

$$\partial_t u + i\partial_x u = 0 \text{ on } t > 0, \quad u(0,x) = u_0(x),$$

and assume that $\operatorname{supp} \widehat{u_0} \subset \mathbb{R}_+$. With $v(t,\xi) = \hat{u}(t,\xi)$, we get

$$\dot{v} = \xi v, \quad \hat{u}(t,\xi) = v(t,\xi) = e^{t\xi} v(0,\xi) = e^{t|\xi|} \widehat{u_0}(\xi).$$

Assuming now that $u(T)$ belongs to $L^2(\mathbb{R})$ for some $T > 0$ (not that stringent an assumption), we obtain

$$\widehat{u_0}(\xi) = e^{-T|\xi|} \hat{u}(T,\xi)$$

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Without the assumption on the spectrum, it is possible for that simple model to use the projection $\mathbb{P}_+$ on the subspace of functions with non-negative spectrum and to obtain analyticity for $\mathbb{P}_+ u_0$.

More generally, it is easy to reproduce that backward regularization property for some quasi-linear equations whose characteristics do not stay in the real line.

This is also an instability result, since the very existence of a solution implies some strong regularity property for the initial datum. For instance, obtaining analyticity for the initial datum will ruin existence of a solution if we perturb an analytic initial datum by a smooth flat function at a point.

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Quasi-linear first-order systems. We consider the quasi-linear system,

$$(\sharp) \qquad \partial_t u + A(t,x,u)\cdot\partial_x u = b(t,x,u), \quad u_{|t=0} = u_0(x),$$

$$A(t,x,u)\cdot\partial_x = \sum_{1\le j\le d} A_j(t,x,u)\partial_{x_j},$$

$t \in \mathbb{R}$ is the time-variable, $x \in \mathbb{R}^d$ stands for the space variables, $u(t,x),\ b(t,x,u) \in \mathbb{R}^N$, A_j are real $N \times N$ matrices. We define for $\xi \in \mathbb{R}^d$,

$$\mathcal{A}_u(t,x,\xi) = \sum_{1\le j\le d} A_j(t,x,u(t,x))\xi_j, \qquad (N \times N \text{ real matrix}),$$

$$p_u(\mu;t,x,\xi) = \det\big(\mathcal{A}_u(t,x,\xi) - \mu\,\mathrm{Id}_N\big), \quad (\text{characteristic polynomial of } \mathcal{A}_u).$$

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Considering the Cauchy problem for a quasi-linear real $N \times N$ system

$$(\sharp) \qquad \partial_t u + \sum_{1\le j\le d} A_j(t,x,u)\partial_{x_j} u = b(t,x,u), \quad u(0,x) = u_0(x),$$

we define $\mathcal{A}_u(t,x,\xi) = \sum_{1\le j\le d} A_j(t,x,u)\xi_j$.

We shall say that the system is hyperbolic when the eigenvalues of $\mathcal{A}_u$ are real. Note that if the eigenvalues of

$$\mathcal{A}_u(0,x_0,\xi) = \sum_{1\le j\le d} A_j(0,x_0,u_0(x_0))\xi_j$$

are real and simple for all $\xi \in \mathbb{S}^{d-1}$, then they stay real and simple for the matrix $\mathcal{A}_u(t,x,\xi)$ nearby (strict hyperbolicity) : the characteristic roots are continuous functions $\lambda(t,x,\xi)$, homogeneous of degree one with respect to ξ, and if they were non-real, since the matrix $\mathcal{A}_u$ is real, the roots $\lambda, \bar{\lambda}$ would merge to a double real root.

Strict hyperbolicity implies local well-posedness (see A. Majda, G. Métivier).

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Conversely, even a very weak assumption of well-posedness implies (weak) hyperbolicity : this type of result has now the generic name of Lax-Mizohata theorems and many authors were involved in proving and stating them : P. Lax, S. Mizohata for linear equations, V. Ivrii & V. Petkov for existence of solutions for general C^∞ data for linear equations, S. Wakabayashi, K. Yagdjian for non-linear equations with different notions of stability.

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$$\text{Summing-up : } \begin{cases} \text{Strict hyperbolicity} \Longrightarrow \text{Well-posedness} \\ \text{Well-posedness} \Longrightarrow \text{Weak hyperbolicity} \end{cases}$$

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What happens if $\mathcal{A}_u(0,x,\xi) = \sum_{1 \leq j \leq d} A_j(0,x,u_0(x))\xi_j$

is only weakly hyperbolic ?

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Strict hyperbolicity $\Longrightarrow$ Well-posedness $\Longrightarrow$ Weak hyperbolicity

What if $\mathcal{A}_u(0,x,\xi) = \sum_{1\leq j\leq d} A_j(0,x,u_0(x))\xi_j$ is only weakly hyperbolic ?

We need to look at the behaviour of the characteristic roots for $t>0$, and see if the roots intend to visit the complex flesh around the real line : if that is so, instability will be present.

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We do not expect a system of PDE to behave as a collection of (coupled) scalar equations, but we want to single out typical models of unstable systems.

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As a result, the k-jet of $\mathcal{A}_u$ at $t=0$ depends only on the data. We want conditions depending only on the data (!). The jet of the characteristic polynomial $\det\big(\mathcal{A}_u(t,x,\xi) - \mu\,\mathrm{Id}_N\big)$ at $t=0$ should be easy to calculate.

2. Our results

Definition of Hadamard instability. We assume that we have a reference local solution $\phi(t,x)$ with regularity $H^m, m > 1 + \frac{d}{2}$, near a distinguished point x_0,

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Ill-posedness means : let $0 < T \le T_0$, $U \subset U_0$ a neighborhood of x_0, $\theta \in (1/2, 1]$ be given. There is no neighborhood $\mathscr{U}$ of ϕ_0 in $H^m(U)$ such that for all $u_0 \in \mathscr{U}$, the above PDE system has a solution in $L^\infty([0,T], W^{1,\infty}(U))$ with initial value u_0 satisfying

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• m could be very large (e.g. when a CK solution is available), this is not enough to control the first derivative of the deviation in L^∞.

Assumptions. We describe now some sufficient conditions triggering instability.

Our reference solution

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If for every neighborhood U of x_0, there exists $x \in U, \xi \in \mathbb{S}^{d-1}, \mu \in \mathbb{C} \backslash \mathbb{R}$, such that $p_\phi(\mu; 0, x, \xi) = 0$, this is essentially the "elliptic case", for which Lax-Mizohata theorems prove instability.

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We may thus assume that there exists a neighborhood U_0 of x_0 such that for all $x \in U_0$, all $\xi \in \mathbb{S}^{d-1}$, $p_\phi(\mu;0,x,\xi) = 0 \Longrightarrow \mu \in \mathbb{R}$, i.e. we have weak hyperbolicity near x_0 at time 0. If all the roots at x_0 are simple, this is the strictly hyperbolic case (which is well-posed), so we may assume as well that there is a multiple root at x_0.

- Initial hyperbolicity near x_0 : $\exists\ U_0 \in \mathscr{V}_{x_0}$ such that $\forall (x,\xi) \in U_0 \times \mathbb{S}^{d-1},\ p_\phi(\mu;0,x,\xi) = 0 \Longrightarrow \mu \in \mathbb{R}$.
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This condition is a non-linear one which depends on the first jet of $\mathcal{A}_\phi$ at time 0, since the term $\partial p_\phi/\partial t$ can be calculated using the fact that $(\partial\phi/\partial t)(t=0,x)$ can be expressed (thanks to the equation) as a function of tangential derivatives $\partial\phi_0/\partial x$ and $\phi_0(x)$.

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Since $\frac{\partial p}{\partial t} \neq 0$, this gives with $e_0 e_1 > 0$

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so that the roots are such that

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Theorem and examples

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When Condition (H) holds, the $N \times N$ quasi-linear PDE system above is unstable in the Hadamard sense, i.e. there is no neighborhood $\mathscr{U}$ of ϕ_0 in $H^m(U)$ such that for all $u_0 \in \mathscr{U}$, the above PDE system has a solution in $L^\infty([0,T], W^{1,\infty}(U))$ with initial value u_0 satisfying

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Let's try our hand on a significant example, mentioned by MÉTIVIER.

Example 1 : Van der Waals. Consider the compressible Euler equations in one space dimension, in Lagrangian coordinates :

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x q(u) = 0, \end{cases}$$

with q analytic. The polynomial $p(\mu, t, x, \xi)$ is

$$p(\mu, t, x, \xi) = \begin{vmatrix} -\mu & \xi \\ q'(u)\xi & -\mu \end{vmatrix} = \mu^2 - \xi^2 q'(u(t,x))$$

Assuming $q'(u_0(x_0)) = 0$, we have a double root $\mu = 0$, $\frac{\partial^2 p}{\partial \mu^2} = 2$,

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Take for instance $q(u) = u(u^2 - 1)$, $u_0(x_0) = 3^{-1/2}$, $v_0'(x_0) > 0$, $(q'(u) = 3u^2 - 1)$

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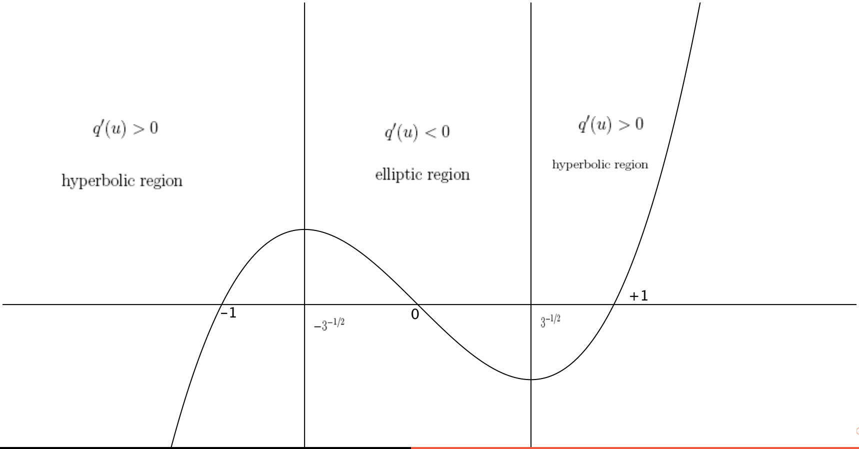

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The roots ($\sim \pm i t^{1/2}\xi$) are not smooth, which is not surprising because of multiple characteristics. The system resembles to

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However, it is interesting in that case to check direcly that a classical second order ODE, the Airy equation, describes the instability of this system pretty well.

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Fourier transform $v(t,\xi) = \hat{u}(t,\xi)$:

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$$c_0 \lambda^{-N_0 - 1} e^{c_1 \lambda t^{3/2}} \leq \|u(t)\|_{H^{-N_0}(|x| \leq 1/\lambda)}, \quad \|u(0)\|_{H^{N_0}(|x| \leq R_0)} \leq C_1 \lambda^{N_0}$$

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Example 2 : Klein-Gordon-Zakharov.

Another example is the family of systems in one space dimension

$$\begin{cases} \partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} v \\ u \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} n \\ m \end{pmatrix} = & (n+1) \begin{pmatrix} v \\ -u \end{pmatrix}, \\ \partial_t \begin{pmatrix} n \\ m \end{pmatrix} + c\partial_x \begin{pmatrix} m \\ n \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} = & \partial_x \begin{pmatrix} 0 \\ u^2 + v^2 \end{pmatrix}, \end{cases}$$

indexed by $\alpha, c \in \mathbb{R}$. The symbol of the first-order operator is

$$A_{\mathrm{KGZ}}(t,x,\xi) = \begin{pmatrix} 0 & 1 & \alpha & 0 \\ 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & c \\ -2u & -2v & c & 0 \end{pmatrix} \xi.$$

In the case $c \notin \{-1,1\}$ and $\alpha = 0$, it has four distinct eigenvalues

$$\{\pm c, \pm 1\}$$

for any values of u, v. This implies that this system is strictly hyperbolic, hence locally well-posed in H^s, for $s > 3/2$.

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[Colin-Ebrard-Gallice-Texier] proved that if $c \notin \{-1, 1\}$ and $\alpha = 0$, the system is locally well-posed in $H^s(\mathbb{R})$, for $s > 1/2$.

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Burgers-like complex systems

We consider a complex scalar quasi-linear equation

$$(\ddagger) \qquad \partial_t u + \sum_{j=1}^{d} a_j(t,x,u)\partial_{x_j} u = b(t,x,u), \qquad u(0,x) = \omega(x).$$

$\mathcal{L} = \partial_t + \sum_{j=1}^{d} a_j(t,x,v)\partial_{x_j} + b(t,x,v)\partial_v$, holomorphic vector field,

$$\nu_0 = (a_1, \dots, a_d),$$
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- And so on : with $\nu_2 = \mathcal{L}\nu_1, \quad \nu_1 = \mathcal{L}\nu_0, \quad \nu_0 = a,$ $\mathcal{L} = \partial_t + a \cdot \partial_x + b\partial_v,$

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Theorem (N.L, Y. Morimoto, C.-J.Xu, Amer. J. Math. 132, (2010))

Let $k \in \mathbb{N}$. If the Cauchy problem

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So the existence of a merely continuous solution forces the initial datum to have some analyticity properties. This triggers instability since "most" initial data won't give rise to a solution. If u_0 analytic, use Cauchy Kovalevskaya to get a local solution, then perturb in C^∞ that u_0 : **no solution.** Métivier proved that result in the elliptic case ($k=0$).

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The C^∞ wave-front-set : $(x_0, \xi_0) \notin WF_\infty u$: $\exists \chi \in C_c^\infty, \chi(x_0) \neq 0$, $\exists W_0 \in \mathcal{V}_{\xi_0/|\xi_0|}$ with

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The analytic wave-front-set $WF_A(u) \supset WF_\infty(u)$ is such that

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It is convenient to use the Fourier-Bros-Iagolnitzer transform of $v \in \mathcal{E}'(\mathbb{R}^d)$,

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$$\exists W_0 \in \mathcal{V}_{x_0 - i\xi_0}, \exists \chi_0 \in C_c^\infty(\Omega), \chi_0(x) = 1 \text{ near } x_0, \exists \epsilon_0 > 0 \quad \text{with}$$

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$$\sup_{\lambda \geq 1, z \in W_0} e^{\epsilon_0 \lambda} |(T\chi_0 u)(z, \lambda)| e^{-\pi\lambda(\operatorname{Im} z)^2} < +\infty.$$

The C^∞ wave-front-set : $(x_0,\xi_0) \notin WF_\infty u$: $\exists \chi \in C_c^\infty, \chi(x_0) \neq 0$, $\exists W_0 \in \mathcal{V}_{\xi_0/|\xi_0|}$ with

$$\forall N, \exists C_N, \forall \xi \in W_0, \forall \lambda \geq 1, \quad |\widehat{\chi u}(\lambda\xi)| \leq C_N \lambda^{-N}.$$

We have of course $p_1(WF_\infty u) = \operatorname{singsupp} u \subset \operatorname{singsupp}_A u$.

The analytic wave-front-set $WF_A(u) \supset WF_\infty(u)$ is such that

$$p_1(WF_A u) = \operatorname{singsupp}_A u.$$

It is convenient to use the Fourier-Bros-Iagolnitzer transform of $v \in \mathcal{E}'(\mathbb{R}^d)$,

$$(Tv)(z,\lambda) = \int_{\mathbb{R}^d} e^{-\pi\lambda(z-x)^2} v(x)dx, \qquad z \in \mathbb{C}, \lambda > 0.$$

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$$\exists W_0 \in \mathcal{V}_{x_0 - i\xi_0}, \exists \chi_0 \in C_c^\infty(\Omega), \chi_0(x) = 1 \text{ near } x_0, \exists \epsilon_0 > 0 \quad \text{with}$$

$$\sup_{\lambda \geq 1, z \in W_0} e^{\epsilon_0 \lambda} |(T\chi_0 u)(z, \lambda)| e^{-\pi\lambda(\operatorname{Im} z)^2} < +\infty.$$

3. Proofs

First reductions. Our reference solution ϕ on $[0, T_0] \times U_0(x_0)$:

$$\partial_t \phi + \sum_{1 \leq j \leq d} A_j(t, x, \phi) \partial_{x_j} \phi = b(t, x, \phi), \quad \phi(0, x) = \phi_0(x).$$

A perturbed datum :

$$u_\epsilon(0, x) = \phi_0(x) + \epsilon^N \varphi_0\Big(\frac{x - x_0}{\epsilon^\kappa}\Big), \quad N \text{ large}, \ \kappa > 0,$$

which is assumed to giving rise to some solution

$$\partial_t u_\epsilon + \sum_{1 \leq j \leq d} A_j(t, x, u_\epsilon) \partial_{x_j} u_\epsilon = b(t, x, u_\epsilon).$$

We write the equation satisfied by

$$u_\epsilon - \phi = v_\epsilon, \quad v_\epsilon(t = 0) = \epsilon^N \varphi_0\Big(\frac{x - x_0}{\epsilon^\kappa}\Big).$$

$$\partial_t(\phi + v_\epsilon) + \sum_{1\leq j\leq d} A_j(t,x,\phi+v_\epsilon)\partial_{x_j}(\phi+v_\epsilon) = b(t,x,\phi+v_\epsilon).$$

$$\begin{aligned}\partial_t\phi + \partial_t v_\epsilon + \Big\{A(t,x,\phi+v_\epsilon) - A(t,x,\phi)\Big\}\cdot\nabla_x(\phi+v_\epsilon)\\ +A(t,x,\phi)\cdot\nabla_x\phi + A(t,x,\phi)\cdot\nabla_x v_\epsilon\\ = b(t,x,\phi+v_\epsilon) - b(t,x,\phi) + b(t,x,\phi).\end{aligned}$$

$$\begin{aligned}\partial_t v_\epsilon + A(t,x,\phi+v_\epsilon)\cdot\nabla_x v_\epsilon + \Big\{A(t,x,\phi+v_\epsilon) - A(t,x,\phi)\Big\}\cdot\nabla_x(\phi)\\ = b(t,x,\phi+v_\epsilon) - b(t,x,\phi).\end{aligned}$$

$$\partial_t v_\epsilon + A(t,x,\phi+v_\epsilon)\cdot\nabla_x v_\epsilon = -\widetilde{A}(t,x,\phi,v_\epsilon)v_\epsilon\nabla_x(\phi) + B(t,x,\phi,v_\epsilon)\cdot v_\epsilon$$

$$\partial_t v_\epsilon + A(t,x,\phi)\cdot\nabla_x v_\epsilon = -\widetilde{A}(t,x,\phi,v_\epsilon)v_\epsilon\nabla_x v_\epsilon - \widetilde{A}(t,x,\phi,v_\epsilon)v_\epsilon\nabla_x(\phi) + B(t,x,\phi,v_\epsilon)\cdot v_\epsilon$$

$$\begin{cases} \partial_t v_\epsilon + A(t,x,\phi)\cdot\nabla_x v_\epsilon = C_1(t,x,\phi,v_\epsilon)v_\epsilon\nabla_x v_\epsilon + C_0(t,x,\phi,v_\epsilon)\cdot v_\epsilon \\ v_\epsilon(0,x) = \epsilon^N \varphi_0\left(\frac{x}{\epsilon^\kappa}\right) \qquad \text{(we took } x_0 = 0\text{).} \end{cases}$$

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We define

$$v_\epsilon(t,x) = \epsilon^N w_\epsilon\Big(t, \overbrace{\frac{x}{\epsilon^\kappa}}^{y}\Big)$$

and we find

$$\begin{cases} \epsilon^N\partial_t w_\epsilon + \epsilon^{N-\kappa}A(t,\epsilon^\kappa y,\phi(t,\epsilon^\kappa y))\cdot\nabla_y w_\epsilon \\ \qquad = C_1(t,\epsilon^\kappa y,\phi(t,\epsilon^\kappa y),\epsilon^N w_\epsilon)\epsilon^N w_\epsilon\epsilon^{N-\kappa}\nabla_y w_\epsilon \\ \qquad\qquad + C_0(t,\epsilon^\kappa y,\phi(t,\epsilon^\kappa y),\epsilon^N w_\epsilon)\cdot\epsilon^N w_\epsilon \\ w_\epsilon(0,y) = \varphi_0(y) \end{cases}$$

$$\begin{cases} \partial_t w_\epsilon + \epsilon^{-1} A(t, \epsilon^\kappa y, \phi(t, \epsilon^\kappa y)) \cdot \epsilon^{1-\kappa} (\nabla_y w_\epsilon)(t, y) \\ \quad = \epsilon^{-1} C_1\big(t, \epsilon^\kappa y, \phi(t, \epsilon^\kappa y), \epsilon^N w_\epsilon(t, y)\big) w_\epsilon(t, y) \epsilon^N \epsilon^{1-\kappa} (\nabla_y w_\epsilon)(t, y) \\ \qquad\qquad + C_0\big(t, \epsilon^\kappa y, \phi(t, \epsilon^\kappa y), \epsilon^N w_\epsilon(t, y)\big) \cdot w_\epsilon(t, y) \\ w_\epsilon(0, y) = \varphi_0(y) \end{cases}$$

$$\begin{cases} \partial_t w_\epsilon + \epsilon^{-1} A(t, \epsilon^\kappa y, \phi(t, \epsilon^\kappa y)) \cdot \epsilon^{1-\kappa} (\nabla_y w_\epsilon)(t, y) \\ \quad = \epsilon^{N-1} \Omega_1\big(t, \epsilon^\kappa y, \epsilon^N w_\epsilon(t, y)\big) w_\epsilon(t, y) \epsilon^{1-\kappa} (\nabla_y w_\epsilon)(t, y) \\ \qquad\qquad + \Omega_0\big(t, \epsilon^\kappa y, \epsilon^N w_\epsilon(t, y)\big) \cdot w_\epsilon(t, y) \\ w_\epsilon(0, y) = \varphi_0(y) \end{cases}$$

• $\kappa = 0$ corresponds to the already known elliptic case where a non-real root exists :

$$\begin{cases} \partial_t w_\epsilon + A(t,y,\phi(t,y)) \cdot (\nabla_y w_\epsilon)(t,y) \\ \quad = \epsilon^N \Omega_1\big(t, \epsilon y, \epsilon^N w_\epsilon(t,y)\big) w_\epsilon(t,y)(\nabla_y w_\epsilon)(t,y) \\ \qquad\qquad\qquad + \Omega_0\big(t,y,\epsilon^N w_\epsilon(t,y)\big) \cdot w_\epsilon(t,y) \\ w_\epsilon(0,y) = \varphi_0(y), \end{cases}$$

leading to a Lax-Mizohata type instability result.

• $\kappa = 1/3$ corresponds to our Airy-like case, Van der Waals & Klein-Gordon-Zakharov examples :

$$\begin{cases} \partial_t w_\epsilon + \epsilon^{-1} A(t, \epsilon^{1/3} y, \phi(t, \epsilon^{1/3} y)) \epsilon^{2/3} (\nabla_y w_\epsilon)(t, y) \\ \quad = \epsilon^N \epsilon^{-1} \Omega_1 \big(t, \epsilon^{1/3} y, \epsilon^N w_\epsilon(t, y)\big) w_\epsilon(t, y) \epsilon^{2/3} (\nabla_y w_\epsilon)(t, y) \\ \qquad\qquad + \Omega_0 \big(t, \epsilon^{1/3} y, 0\big) w_\epsilon(t, y) \\ \qquad\qquad\qquad + \epsilon^N \Omega_2 \big(t, \epsilon^{1/3} y, \epsilon^N w_\epsilon(t, y)\big) w_\epsilon(t, y)^2 \\ w_\epsilon(0, y) = \varphi_0(y). \end{cases}$$

∘ The term

$$\epsilon^N \epsilon^{-1} \Omega_1 \big(t, \epsilon^{1/3} y, \epsilon^N w_\epsilon(t, y)\big) w_\epsilon(t, y) \epsilon^{2/3} (\nabla_y w_\epsilon)(t, y)$$

is a non-linear perturbation of the lhs.

∘ The term $\Omega_0 \big(t, \epsilon^{1/3} y, 0\big) w_\epsilon(t, y)$ is a linear term, eligible for the lhs.

∘ The term $\epsilon^N \Omega_2 \big(t, \epsilon^{1/3} y, \epsilon^N w_\epsilon(t, y)\big) w_\epsilon(t, y)^2$ will be considered as a source term.

• $\kappa = 1/3$ corresponds to our Airy-like case, Van der Waals & Klein-Gordon-Zakharov examples :

$$\begin{cases} \partial_t w_\epsilon + \epsilon^{-1} P\big(t, \epsilon^{1/3} y, w_\epsilon(t,y)\big) \epsilon^{2/3} \nabla_y w_\epsilon + Q\big(t, \epsilon^{1/3} y\big) w_\epsilon(t,y) \\ \qquad\qquad = \epsilon^N \Omega_2\big(t, \epsilon^{1/3} y, \epsilon^N w_\epsilon(t,y)\big) w_\epsilon(t,y)^2 \\ w_\epsilon(0,y) = \varphi_0(y), \end{cases}$$

where P is close to $A(t, \epsilon^{1/3} y, \phi(t, \epsilon^{1/3} y))$ and the source term

$$\epsilon^N \Omega_2\big(t, \epsilon^{1/3} y, \epsilon^N w_\epsilon(t,y)\big) w_\epsilon(t,y)^2$$

is small.

Duhamel's principle and pseudodifferential flows

We shall use pseudodifferential operators with matrix-valued symbols Q satisfying

$$|(\partial_y^k \partial_\eta^l Q)(t,y,\eta)| \leq C_{kl}\epsilon^{-1}\epsilon^{k/3}\epsilon^{2l/3}, \quad k+l \leq N,$$

for instance defined as

$$Q(t,y,\eta) = \epsilon^{-1} Q_1(t,\epsilon^{1/3}y,\epsilon^{2/3}\eta),$$

where the matrix Q_1 has N derivatives bounded. This version of a semi-classical calculus can be provided with a graded algebra of pseudodifferential operators.

We solve the system of ODE for $S(t;\tau)$,

$$\partial_t S + \epsilon^{-1} Q_1(t, \epsilon^{1/3} y, \epsilon^{2/3} \eta) S = 0, \quad S(\tau, \tau) = \mathrm{Id}\,.$$

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$$\partial_t S + \epsilon^{-1} Q_1(t, \epsilon^{1/3} y, \epsilon^{2/3} \eta) S = 0, \quad S(\tau, \tau) = \mathrm{Id}\,.$$

We use $\mathcal{S}(t,\tau) = \mathrm{Op}(S(t,\tau,y,\eta))$ as an approximate parametrix for our Cauchy problem and we find

$$w_\epsilon = \mathcal{S}(t,0)\varphi_0 + \epsilon^N \int_0^t \mathcal{S}(t,\tau)\Omega_2 d\tau + \rho_\epsilon.$$

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• The term ρ_ϵ is a small remainder, thanks to a semi-classical pseudodifferential argument.

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• Condition (H) implies some exponential increase for $\mathcal{S}(t,0)\varphi_0$, provided we choose the vector-valued φ_0 properly, namely a cutoff function $\times$ an eigenvector.

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• Next, we have also some upper bounds for $\mathcal{S}(t,\tau)$ and the integral term must be shown as not perturbing the exponential increase.

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$$w_\epsilon = \mathcal{S}(t,0)\varphi_0 + \epsilon^N \int_0^t \mathcal{S}(t,\tau)\Omega_2 d\tau + \rho_\epsilon.$$

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• Condition (H) implies some exponential increase for $\mathcal{S}(t,0)\varphi_0$, provided we choose the vector-valued φ_0 properly, namely a cutoff function $\times$ an eigenvector.

• Next, we have also some upper bounds for $\mathcal{S}(t,\tau)$ and the integral term must be shown as not perturbing the exponential increase.

• Two assets for this : the ϵ^N in front and, using reductio ad absurdum, we may assume that we have a priori bounds on w_ϵ (the term Ω_2 depends non-linearly on w_ϵ).

Stratification of the boundary of the instability region

• We have seen that a toy model for Hadamard instability in the presence of a non-real root is the scalar equation

$$\partial_t + i\partial_x.$$

• Assuming that the roots are real and at most double, our toy model is no longer a scalar equation, but is a system

$$\mu = 2, \quad \nu = 1, \qquad \begin{pmatrix} 0 & 1 \\ -t & 0 \end{pmatrix}, \; \lambda^2 + t = 0 \text{ has singular roots,}$$

$$\mu = 2, \quad \nu = 2, \quad \begin{pmatrix} 0 & 1 \\ -t^2 & 0 \end{pmatrix}, \; \lambda^2 + t^2 = 0 \text{ has smooth roots,}$$

are two examples in the non-semi-simple case .

The semi-simple case is easier

$$\mu = 2, \qquad \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, \; \lambda^2 + t^2 = 0 \text{ has smooth roots.}$$

- Now assume that for our PDE system, hyperbolic at initial time,

$$\partial_t u + A(t,x,u)\cdot\partial_x u = b(t,x,u) \text{ on } t>0, \qquad u_{|t=0} = u_0(x).$$

$$A_u(t,x,\xi) = \sum_{1\le j\le d} A_j(t,x,u(t,x))\xi_j, \qquad p_u(\lambda;t,x,\xi) = \det\big(A_u(t,x,\xi) - \lambda\,\mathrm{Id}_N\big),$$

we have a triple root

$$p = \frac{\partial p}{\partial\lambda} = \frac{\partial^2 p}{\partial\lambda^2} = 0, \quad \frac{\partial^3 p}{\partial\lambda^3} \neq 0, \text{ at } t=0, x=x_0, \xi=\xi_0\in\mathbb{S}^{d-1}.$$

We check the (nilpotent) matrix (the semi-simple-case should be easier to handle and the case where the minimal polynomial has degree two is dealt with before)

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and its perturbations} \quad \begin{pmatrix} 0 & 1 & b_1 t \\ a_1 t & 0 & 1 \\ a_2 t & a_3 t & 0 \end{pmatrix}$$

We have

$$\begin{vmatrix} -\lambda & 1 & b_1 t \\ a_1 t & -\lambda & 1 \\ a_2 t & a_3 t & -\lambda \end{vmatrix}$$
$$= (-\lambda)(\lambda^2 - a_3 t) - a_1 t(-\lambda - a_3 b_1 t^2) + a_2 t(1 + b_1 t \lambda)$$
$$= -\lambda^3 + \lambda t\big(a_3 + a_1 + a_2 b_1 t\big) + a_1 a_3 b_1 t^3 + a_2 t,$$

and the discriminant is

$$-\Delta(t) = -4t^3\big(a_3 + a_1 + a_2 b_1 t\big)^3 + 27\big(a_1 a_3 b_1 t^3 + a_2 t\big)^2.$$

Assuming $a_2 \neq 0$, we find that $\Delta(t) < 0$ near $t = 0$ (and 0 at $t = 0$), so that the polynomial has two complex conjugate non-real roots and one real root.

It seems interesting to check the one-dimensional 3×3 system

$$\partial_t u + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & 0 & 0 \end{pmatrix} \partial_x u$$

and to calculate the solution of

$$\dot{M} + i\xi \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & 0 & 0 \end{pmatrix} M = 0$$

$$\begin{vmatrix} -X & 1 & 0 \\ 0 & -X & 1 \\ t & 0 & -X \end{vmatrix} = -X^3 + t, \quad \text{roots } \{t^{1/3}, t^{1/3}j, t^{1/3}j^2\}$$

$$\{i\xi t^{1/3},\ i\xi t^{1/3}j,\ i\xi t^{1/3}j^2\}$$

and if $\xi > 0, t > 0,$

$$i\xi t^{1/3}j = i\xi t^{1/3}(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = \xi t^{1/3}(-\frac{i}{2} - \frac{\sqrt{3}}{2}), \quad \mathrm{Re}\big(i\xi t^{1/3}j\big) < 0$$

It turns out that this is related to special functions solutions of the fourth-order scalar equation

$$f^{(4)}(t) + atf'(t) + bf(t) = 0, \quad a, b \text{ non-zero complex parameters,}$$

an ODE that can be solved explicitly, thanks to the fact that the Fourier transform of $tv(t)$ is $i\frac{d}{d\tau}\hat{v}$ so that the above equation on the Fourier side is first-order with 0 as a regular singular point,

$$(i\tau)^4 \hat{f}(\tau) + ai\frac{d}{d\tau}(i\tau\hat{f}(\tau)) + b\hat{f}(\tau) = 0,$$

$$g = \hat{f}, \quad a\tau g' = (b-a)g + \tau^4 g,$$

$$\tau g' = (c + a^{-1}\tau^4)g, \quad c = (b-a)a^{-1}.$$

• We could go on : assume that for our PDE system, hyperbolic at initial time, with size $N \times N$,

$$\partial_t u + A(t,x,u)\cdot \partial_x u = b(t,x,u) \text{ on } t>0, \qquad u_{|t=0} = u_0(x).$$

$$A_u(t,x,\xi) = \sum_{1\le j\le d} A_j(t,x,u(t,x))\xi_j, \qquad p_u(\lambda;t,x,\xi) = \det\big(A_u(t,x,\xi) - \lambda\,\mathrm{Id}_N\big),$$

has a root with multiplicity $\nu \geq 2$,

$$p = \frac{\partial p}{\partial \lambda} = \cdots = \frac{\partial^{\nu-1} p}{\partial \lambda^{\nu-1}} = 0, \quad \frac{\partial^{\nu} p}{\partial \lambda^{\nu}} \neq 0, \text{ at } t=0, x=x_0, \xi=\xi_0 \in \mathbb{S}^{d-1}.$$

We check the (nilpotent) matrix with size ν

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and its perturbation

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with characteristic polynomial $(-1)^\nu(\lambda^\nu + t)$ and eigenvalues $t^{1/\nu} e^{i\pi(\frac{2k-1}{\nu})}, 0 \le k < \nu$ with imaginary part $t^{1/\nu} \sin(\frac{2\pi k - \pi}{\nu})$, so that

$$\operatorname{Re}\big(i\xi t^{1/\nu} e^{i\pi(\frac{2k-1}{\nu})}\big) = -\xi t^{1/\nu} \sin(\frac{2\pi k - \pi}{\nu}) < 0,$$

if for instance

$$t > 0,\ \xi > 0, \quad 1 < \frac{2k-1}{\nu} < 2, \quad \text{i.e. } \frac{\nu+1}{2} < k < \frac{2\nu+1}{2},$$

$$\text{for } \nu = 2,\ k = 2, \qquad \text{for } \nu \ge 3,\ \frac{2\nu+1}{2} - \frac{\nu+1}{2} = \frac{\nu}{2} > 1.$$

Of course many other perturbations are relevant, each of it giving rise to another model such as

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -t & t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This would produce a lot of special functions which could be of interest in the study of instability for systems of PDE.

$$\partial_t u + A(t,x,u) \cdot \partial_x u = b(t,x,u) \text{ on } t > 0, \quad u_{|t=0} = u_0(x),$$

$$A_u(t,x,\xi) = \sum_{1 \leq j \leq d} A_j(t,x,u(t,x))\xi_j, \qquad p_u(\mu;t,x,\xi) = \det\big(A_u(t,x,\xi) - \mu \operatorname{Id}_N\big).$$

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◦ When the hyperbolicity is strict, local well-posedness occurs.

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• Hadamard's well-posedness requires hyperbolicity : when a non-real root shows up at time 0, instability occurs : this is the "elliptic" case and the related model is a scalar equation, the $\bar\partial$ equation.

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• When weak hyperbolicity occurs at $t=0$ with roots intending to exit the real line, instability occurs. When the roots are at most double, our Condition (H) above, a non-linear condition depending only on the data, ensures instability. The related model is no longer scalar, but is a 2×2 system closely related to Airy's equation.

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• When weak hyperbolicity occurs at $t=0$, with a root of multiplicity $\nu\ge 2$, it is quite likely that some sufficient non-linear conditions (depending only on the data) for instability can be described "macroscopically" (without actually computing the roots). The typical models will be some $\nu\times\nu$ system which are related in some cases to higher-order scalar ODE involving some special functions.

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Thank you for your attention