SCIENCES SORBONNE UNIVERSITÉ

Exam of M2 course "Introduction à l'analyse géométrique"

20th Dec. 2024. Length: 3 hours. No document allowed.

One recalls the Sobolev embedding theorem:

- $W^{k,p} \subset L^q$ if $\frac{k}{n} - \frac{1}{p} \geqslant -\frac{1}{q}$,
- $W^{k,p} \subset C^0$ if $\frac{k}{n} - \frac{1}{p} > 0$.

Exercice 1. (Course question) Let (M^n, g) be a fixed compact, connected, oriented Riemannian manifold. Let R^g be the scalar curvature. Fix $p = \frac{2n}{n-2}$. One recalls the formula

$$R^{u^{4/(n-2)}g} = 4\frac{n-1}{n-2} u^{-p+1} L_g u, \quad \text{where} \quad L_g u = \Delta u + \frac{n-2}{4(n-1)} R^g u.$$

The Yamabe functional is defined by $I(\tilde{g}) = \int_M R^{\tilde{g}} \operatorname{vol}^{\tilde{g}}$ and the Yamabe invariant is defined by

$$\lambda(M, g) = \inf_{\operatorname{Vol}(\tilde{g})=1, \tilde{g} \text{ conformal to } g} I(\tilde{g}).$$

1° Express the Yamabe functional in terms of u. Prove that if the inf in the definition of $\lambda(M, g)$ is attained at a metric $\tilde{g}$, then $R^{\tilde{g}}$ is constant.

Explain why a direct minimisation method of I is not successful to produce a metric with constant scalar curvature in the conformal class of g.

2° Suppose that on the sphere $\tilde{g} = \varphi^2 \tilde{g}_{S^n}$ has constant scalar curvature. Prove that there exists a conformal transformation f of S^n such that $\tilde{g} = f^* g_{S^n}$.

One recalls the formula:

$$\operatorname{Ric}(g_{S^n}) = \operatorname{Ric}(\tilde{g}) + \varphi^{-1} \left(\nabla d\varphi - (n-1)\frac{|d\varphi|^2}{\varphi} \tilde{g} - (\Delta\varphi)\tilde{g} \right),$$

where all the operators (∇, Δ,...) are calculated with respect to $\tilde{g}$.

3° Come back to the case of a general M but suppose that g is conformally flat with $R^g > 0$. Fix $x_0 \in M$. Explain how one can construct on $M \setminus \{x_0\}$ a metric $\hat{g}$ which is conformal to g, has $R^{\hat{g}} = 0$, and which outside some compact set has a chart diffeomorphic to $\mathbb{R}^n \setminus B_R$ with the following asymptotic behaviour when the coordinate y in $\mathbb{R}^n$ goes to infinity:

$$\hat{g} = \gamma(y) \sum_1^n (dy^i)^2, \quad \gamma(y) = 1 + \frac{A}{|y|^{n-2}} + O(\frac{1}{|y|^{n-1}}).$$

Prove that

$$\lambda(M, g) = \inf_{u \in C_c^\infty(M \setminus \{x_0\})} \frac{\int_{M \setminus \{x_0\}} |du|^2 \operatorname{vol}^{\hat{g}}}{\left(\int_{M \setminus \{x_0\}} u^p \operatorname{vol}^{\hat{g}} \right)^{2/p}}.$$

Explain the role of the constant A in the Yamabe problem (no calculation is requested).

Exercice 2. Let (M^n, g) be a fixed compact, connected, oriented Riemannian manifold. If $\rho > 0$ is small enough (smaller than the injectivity radius of g) then the metric on any ball of radius ρ can be written in normal coordinates as $g = dr^2 + g_r$, where g_r is a metric on the sphere S^{n-1}. One recalls that if the sectional curvature K^g satisfies $-a^2 \leqslant K^g \leqslant b^2$, then

$$\frac{\sin^2(br)}{b^2} g_{S^{n-1}} \leqslant g_r \leqslant \frac{\sinh^2(ar)}{a^2} g_{S^{n-1}}.$$

Let $p = \frac{2n}{n-2}$ the exponent for the Sobolev embedding $\mathrm{H}^1 \subset \mathrm{L}^p$, and μ be the Sobolev constant such that on $\mathbb{R}^n$ we have $\|df\|_{\mathrm{L}^2}^2 \geqslant \mu \|f\|_{\mathrm{L}^p}^2$ for any compactly supported f.

1° Prove that for any $\varepsilon > 0$, there exists $\rho > 0$ such that one can cover M with a finite number of balls B_i of radius ρ with coordinates such that

$$(1+\varepsilon)^{-1} g_{\mathbb{R}^n} \leqslant g \leqslant (1+\varepsilon) g_{\mathbb{R}^n}.$$

2° Given such covering, prove that there exist smooth nonnegative functions (χ_i) such that (χ_i^2) is a partition of unity subordinate to the covering. Prove that

$$\Big(\int_{\mathrm{M}} f^p \operatorname{vol}^g \Big)^{2/p} \leqslant (1 + c\varepsilon) \sum_i \Big(\int_{\mathrm{B}_i} |\chi_i f|^p |dx|^n \Big)^{2/p}$$

where $|dx|^n = dx^1 \cdots dx^n$ is the standard volume form of $\mathbb{R}^n$ in the normal coordinates of B_i, and c is some constant.

3° Prove that for any $\varepsilon > 0$ there exists a constant C_ε such that for any function f on M one has

$$\Big(\int_{\mathrm{M}} f^p \operatorname{vol}^g \Big)^{2/p} \leqslant (\mu^{-1} + \varepsilon) \int_{\mathrm{M}} |df|^2 \operatorname{vol}^g + \mathrm{C}_\varepsilon \int_{\mathrm{M}} f^2 \operatorname{vol}^g .$$

Exercice 3. Let (M^n, g) be a fixed compact, connected, oriented Riemannian manifold.

1° Suppose $p_0, p_1 \in [1, \infty[$ and $\theta \in [0,1]$. Define p_θ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Prove that for any function f one has the interpolation inequality:

$$\|f\|_{\mathrm{L}^{p_\theta}} \leqslant \|f\|_{\mathrm{L}^{p_0}}^{1-\theta} \|f\|_{\mathrm{L}^{p_1}}^{\theta}.$$

2° Prove that for any $\varepsilon > 0$ there exists a constant $c > 0$ such that for any λ and any eigenfunction f of the Laplacian for the eigenvalue λ one has

$$\|f\|_{\mathrm{C}^0} \leqslant c \lambda^{\frac{n}{4}+\varepsilon} \|f\|_{\mathrm{L}^2}.$$

Note. The optimal bound is actually $\lambda^{\frac{n-1}{4}}$ and is obtained on the sphere.

Exercice 4. (*The vortex equation*). Let (S^2, g) be a compact, oriented, Riemannian surface. Let h, φ and ψ be smooth functions on S. We consider the equation for a function f:

$$\Delta f + e^f \varphi - e^{-f} \psi = h.$$

1° Prove the existence and uniqueness of smooth solutions f to this equation under the hypothesis that $\varphi > 0$ and $\psi > 0$.

2° Study the existence and uniqueness of solutions in the case $\psi = 0$ and $\varphi > 0$.

3° We now suppose that $\psi = 0$ and $\varphi \geqslant 0$, but φ can have zeroes. Moreover we suppose that $\Delta \ln \varphi$ is bounded on S.

a. Can you give an example of such function φ on a ball in $\mathbb{R}^2$ with a zero at the origin ?

b. Prove that any smooth solution f of the equation satisfies an a priori bound of the type $\sup e^f \varphi \leqslant \mathrm{C}$ for some fixed constant C.

c. Study the existence of solutions to the equation.