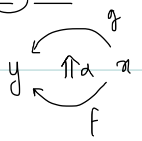
(Marco)

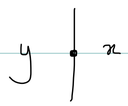
STRING DIAGRAMS AND CATEGORIFIED QUANTUM $\mathfrak{sl}_2$

I - String diagrams

Idea



Poincaré $\rightsquigarrow$ duality



More rigorously : $\mathbb{R}\times I \supseteq \mathbb{Z}\times\{0,1\}$ specified points

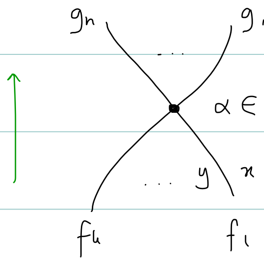
$\mathcal{C}$ a 2-category

$\mathcal{C}$-diagram : labeled finite graph in $\mathbb{R}\times I$ s.t

(i) endpoints are in $\mathbb{Z}\times\{0,1\}$

(ii) proj. on I factor has no critical pt on interior of edges

(iii) labeling is compatible around each vertex

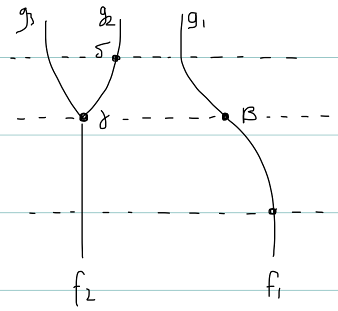


$\alpha \in 2\text{-Hom}_{\mathcal{C}}(f_k \cdots f_1, g_n \cdots g_1)$

$f_1 \in 1\text{-Hom}_{\mathcal{C}}(x,y)$

Convention : don't draw id-edges and id-vertices

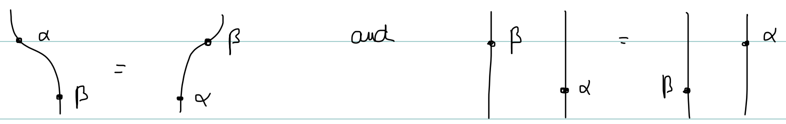
Recipe



$\eta : f_2 f_1 \Rightarrow g_3 g_2 g_1$

$\eta = (\mathrm{id}_{g_3} * \delta * \mathrm{id}_{g_1}) \circ (\gamma * \beta) \circ (\mathrm{id}_{f_2} * \alpha)$

Remark :



We can identify diagrams which yield the same 2-morphism

$\Longrightarrow$ well-defined horizontal and vertical composition for diagrams

Adjoint 1-morphism : $f : x \to y$, $g : y \to x$ with unit $\eta : \mathrm{id}_x \Longrightarrow gf$

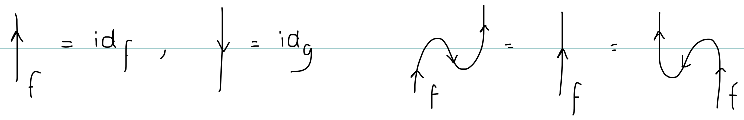
counit $\varepsilon : fg \Longrightarrow \mathrm{id}_y$

with the properties (as diagrams)

[expresses $f \Rightarrow fgf \Rightarrow f$ is id_f
$g \Rightarrow gfg \Rightarrow g$ is id_g]

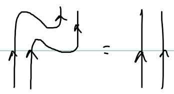
$f \dashv g \dashv f \Rightarrow (f,g)$ biadjoint

Convention : don't draw ε, η vertices, g-labels, we orient f-edges



Remark : $f : x \to y$, $f' : y \to z$ and $(f,g), (f',g')$ biadjoint

then $(f'f, gg')$ biadjoint since



II - Lusztig's version $\dot{\mathbb{U}}$ of $U_q(\mathfrak{sl}_2)$

$U_q(\mathfrak{sl}_2)$ generated by E, F, K, K^{-1} + relations [cf. Ruslan's talk]

$\dot{\mathbb{U}}$: $\mathbb{Q}(q)$-alg. generated by $\bigoplus_{n,m\in\mathbb{Z}} 1_n U_q(\mathfrak{sl}_2) 1_m$ where

- $1_m 1_n = \delta_{m,n} \cdot 1_n$ [idempotents]
- $K^{\pm 1} 1_n = 1_n K^{\pm 1} = q^{\pm n} 1_n$
- $E 1_n = 1_{n+2} E$
- $F 1_n = 1_{n-2} F$

Divided powers : $E^{(a)} = \dfrac{E^a}{[a]!}$, $F^{(a)} = \dfrac{F^a}{[a]!}$ with $[a] = \dfrac{q^a - q^{-a}}{q - q^{-1}}$, $[a]! = [a][a-1]\cdots[1]$

${}_{\mathcal{A}}\dot{\mathbb{U}}$ is $\mathbb{Z}[q,q^{-1}]$-alg generated by $1_{n+2a} E^{(a)} 1_n$ and $1_{n-2b} F^{(a)} 1_n$ $a \geq 0$, $n \in \mathbb{Z}$

Canonical basis : $\dot{\mathbb{B}}$ $E^{(a)} F^{(b)} 1_n$, $n \leq b-a$ with $E^{(a)} F^{(b)} 1_{b-a} = F^{(b)} E^{(a)} 1_{b-a}$

$F^{(b)} E^{(a)} 1_n$ $n \geq b-a$

structure constants are in $\mathbb{N}[q,q^{-1}]$

$_{\mathcal{A}}\dot{\mathbb{U}}$ as a category :
- objects $n \in \mathbb{Z}$
- morphisms : $\mathrm{Hom}_{_{\mathcal{A}}\dot{\mathbb{U}}}(n,m) = 1_n \, {_{\mathcal{A}}\dot{\mathbb{U}}} \, 1_m$

$\rightsquigarrow$ natural to look for a 2-category $\dot{\mathcal{U}}$ with

- objects : $n \in \mathbb{Z}$
- morphisms : $\mathrm{Hom}_{\dot{\mathcal{U}}}(n,m)$ additive category with translations $\{t\} : \mathrm{Hom}_{\dot{\mathcal{U}}}(n,m) \circlearrowleft \quad \forall t \in \mathbb{Z}$

  with $K_0(\mathrm{Hom}_{\dot{\mathcal{U}}}(n,m)) \simeq 1_m \, {_{\mathcal{A}}\dot{\mathbb{U}}} \, 1_n$

Idea : indecomposable 1-morphisms $\xleftrightarrow{1-1}$ elts of $\dot{\mathbb{B}}$

How ? (i) Construct intermediate 2-cat. $\mathcal{U}$

(ii) Define $\dot{\mathcal{U}} = \mathrm{Kar}(\mathcal{U})$ Karoubian envelope [idempotent completion]

III Diagrammatics of $\mathfrak{sl}_2$

def : $\mathcal{U}$ is the additive 2-category with translation s.t

- objects : $n \in \mathbb{Z}$
- 1-morphisms : generated by $\mathbb{1}_n : n \longrightarrow n$, $\mathcal{E}\mathbb{1}_n : n \longrightarrow n+2$, $\mathcal{F}\mathbb{1}_n : n \longrightarrow n-2$

  and their translates

  $\Rightarrow$ $1\text{-}\mathrm{Hom}_{\mathcal{U}}(n,m) = \left\{ \mathcal{E}^{a_1}\mathcal{F}^{b_1} \cdots \mathcal{E}^{a_k}\mathcal{F}^{b_k}\mathbb{1}_n\{t\} \;\middle|\; t \in \mathbb{Z},\ a_i, b_i \in \mathbb{N},\ m = n + 2\sum_{i=1}^{k}(a_i - b_i) \right\}$

- 2-morphisms :
  $\emptyset^n : \mathbb{1}_n\{t\} \Longrightarrow \mathbb{1}_n\{t\}$

  $\uparrow^n : \mathcal{E}\mathbb{1}_n\{t\} \Longrightarrow \mathcal{E}\mathbb{1}_n\{t\}$

  $\downarrow^n : \mathcal{F}\mathbb{1}_n\{t\} \Longrightarrow \mathcal{F}\mathbb{1}_n\{t\}$

  $\uparrow : \mathcal{E}\mathbb{1}_n\{t+2\} \Longrightarrow \mathcal{E}\mathbb{1}_n\{t\}$ and $\downarrow : \mathcal{F}\mathbb{1}_n\{t+2\} \longrightarrow \mathcal{F}\mathbb{1}_n\{t\}$

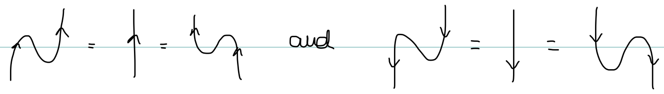
  $\times^n : \mathcal{E}^2\mathbb{1}_n\{t-2\} \Longrightarrow \mathcal{E}^2\mathbb{1}_n\{t\}$

  $\smile^n : \mathbb{1}_n\{t+1-n\} \Longrightarrow \mathcal{E}\mathcal{F}\mathbb{1}_n\{t\}$ and $\smile^n : \mathbb{1}_n\{t+1+n\} \Longrightarrow \mathcal{F}\mathcal{E}\mathbb{1}_n\{t\}$

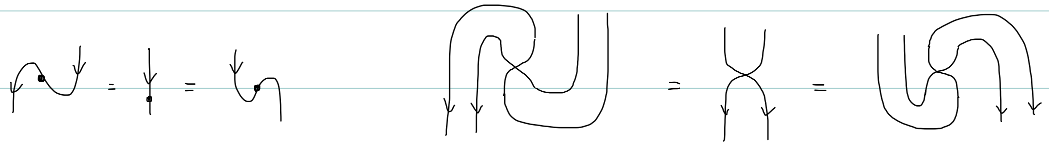
  $\frown^n : \mathcal{E}\mathcal{F}\mathbb{1}_n\{t+1-n\} \Longrightarrow \mathbb{1}_n\{t\}$ and $\frown^n : \mathcal{E}\mathcal{F}\mathbb{1}_n\{t+1+n\} \Longrightarrow \mathbb{1}_n\{t\}$

with relations:

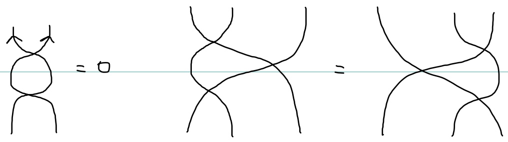
(1) Biadjunction:

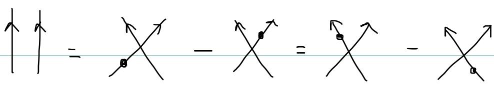


(2)

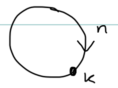


(3) Nil-Hecke:

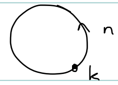




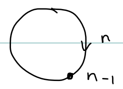
(4) Positivity of bubbles:



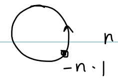
$= 0 \qquad n \leq 1 \, , \; k < n-1$



$= 0 \qquad n \leq -1 \, , \; k < -n-1$

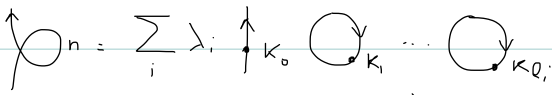


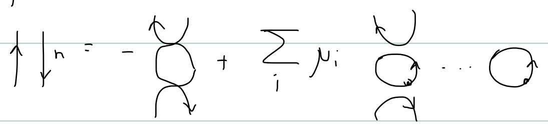
$= id_n \qquad n \geq 1 \, ,$



$= id_{1_n} \qquad n \leq -1$

(5) $\mathfrak{sl}_2$-relations:





Thm: if $\dot{\mathcal{U}} = \mathrm{Kar}(\mathcal{U})$ then $K_0(\mathrm{Hom}_{\dot{\mathcal{U}}}(n,m)) \cong 1_n \, {}_{\mathbb{Z}}\dot{U} 1_m$