WARTHOG 2018
DELIGNE–LUSZTIG THEORY

The classification of finite simple groups achieved in 1983 states that every finite simple group belongs to one of the following families:

- The cyclic groups $\mathbb{Z}/p\mathbb{Z}$ of prime order;
- The alternating groups $\mathfrak{A}_n$ for $n \geq 5$;
- The simple groups of Lie type (such as $\mathrm{PSL}_n(q)$, $\mathrm{PSp}_{2n}(q)$,... with some restrictions on n and q);
- The 26 sporadic groups.

The third family is doubly infinite, and contains most of the finite simple groups. The representation theory of these groups will be the motivation for these lectures.

We will introduce the geometric methods due to Deligne–Lusztig [5] to construct and study the representation theory of the finite groups of Lie type. They defined a family of algebraic varieties, the *Deligne–Lusztig varieties*, on which the finite groups act. Representations are obtained by taking a suitable linear invariant of these varieties, the ℓ-adic cohomology. It turns out that every complex irreducible representation occurs in the cohomology of some Deligne–Lusztig varieties. These representations were later classified by Lusztig in a series of paper culminating in [12].

The story does not stop here. If one wants to develop a modular analogue of the theory of Deligne and Lusztig (for representations over fields of positive characteristic), one needs to go further in the determination of the cohomology of Deligne–Lusztig varieties. Based on the special case of varieties attached to Coxeter elements and strong numerical evidence, Broué, Malle and Michel [4, 2] stated a series of conjectures on the cohomology of Deligne–Lusztig varieties which produced a geometric version of Broué's abelian defect group conjecture [1]. These will be presented first in the case of complex representations, and then for modular representations using the language of derived categories and derived equivalences.

Here is a brief summary of the lectures:

(Day 1) We will recall the structure of connected reductive groups and their remarkable subgroups (maximal tori, Borel subgroups) [13, Part I]. We will explain how to go from the algebraic group $\mathbf{G}$ (over $\overline{\mathbb{F}}_q$) to the finite group $\mathbf{G}(\mathbb{F}_q)$ [13, Part III]. We will then use the remarkable subgroups (in the finite case) to construct representations by induction [12, §8]. The algebraic object which will show up here is the Hecke algebra [8, §7]. This algebra will control part of the representation theory of the finite group, and this is as far as we will go using only algebraic methods.

(Day 2) We will start by presenting the flag variety and its decomposition into Schubert cells [13, §11]. This decomposition is indexed by elements in the Weyl group. A similar construction will produce the Deligne–Lusztig

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varieties [5, §1]. We will explain some of their basic geometric properties, and study the various natural actions they are endowed with. We will be particularly interested by actions of Braid monoids [4]. We will illustrate Deligne and Lusztig's construction in the case of the variety for $\mathrm{GL}_n(q)$ attached to an n-cycle of $\mathfrak{S}_n$ (the Weyl group of GL_n) [5, §2].

(Day 3) We will state the properties of the ℓ-adic cohomology we will need in the following lectures [6, §10]. These include the usual properties to be expected from a cohomology theory: Künneth formula, Mayer-Vietoris sequence... and some deep properties which are only available for varieties over $\overline{\mathbb{F}}_p$ (Lefschetz trace formula). We will also explain what happens when we consider varieties acted on by a finite group.

(Day 4) We will start by constructing the virtual representations coming from the ℓ-adic cohomology of Deligne–Lusztig varieties, the *Deligne–Lusztig characters* [5, §1]. We will then focus on a subset of representations which will control them all, the *unipotent representations*. We will briefly discuss their classification due to Lusztig [11, §4]. An important step towards this classification is the computation of the cohomology of the varieties attached to Coxeter elements (such as the variety for $\mathrm{GL}_n(q)$ discussed during the first day) [11]. We will present Lusztig's result on these varieties and how they can be conjecturally extended to other varieties, following the idea of Broué, Malle, Michel and Rouquier [2, 4]. This will involve a generalisation of the Hecke algebras we saw on the first day [3].

(Day 5) The motivation for extending Lusztig's result to other Deligne–Lusztig varieties is to develop a modular analogue of his work, for representations over fields of positive characteristic. We will first recall the basic notions of modular representation theory of finite groups (blocks, defect groups) [10] as well as the language of homological algebra we will be using (derived categories) [9]. In this framework we will formulate the geometric version of Broué's abelian defect group conjecture [1] and prove it for Deligne–Lusztig varieties attached to Coxeter elements [7].

References

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