# AMBIGUITY IN OMEGA CONTEXT FREE LANGUAGES

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**Abstract**

We extend the well known notions of ambiguity and of degrees of ambiguity of finitary context free languages to the case of omega context free languages ($\omega$-CFL) accepted by Büchi or Muller pushdown automata. We show that these notions may be defined independently of the Büchi or Muller acceptance condition which is considered. We investigate first properties of the subclasses of omega context free languages we get in that way, giving many examples and studying topological properties of $\omega$-CFL of a given degree of ambiguity.

*Key words:* context free; language; $\omega$-language; ambiguity; Borel hierarchy; Wadge hierarchy.

**Contents**



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## 1 Introduction

The notion of ambiguity is well known in the theory of finitary context free languages. A context free grammar $G$ is said to be non ambiguous if every word $x$ generated by $G$ is generated via a unique leftmost derivation. A context free language (CFL) $L$ is said to be non ambiguous if it is the language $L = L(G)$ generated by a non ambiguous context free grammar $G$; otherwise the CFL $L$ is said to be inherently ambiguous. It was proved by Ginsburg and Ullian that one cannot decide whether an arbitrary CFL is inherently ambiguous [GU66]. Every deterministic CFL is non ambiguous [ABB96] but there exist some non deterministic non ambiguous context free languages [Her97]. Maurer gave a proof of the inherent ambiguity of a simple CFL [Mau69]. Some other examples of CFL are shown to be inherently ambiguous by Flajolet using the theory of analytic functions [Fla85] [Fla86]. The notion of inherent ambiguity has been refined by considering degrees of ambiguity, and this led to CFL which are inherently ambiguous of degree $k$, for $k$ an integer $\geq 2$, or of infinite degree. Examples of CFL which are inherently ambiguous of any degree have been given by Maurer in [Mau68]. Crestin proved that the square of the non ambiguous CFL of palindromes is inherently ambiguous of infinite degree [Cre72]. Further examples of CFL which are inherently ambiguous of infinite degree have been recently given by Naji and Wich [Naj98] [Wic99] [Wic00]. Remark that the notion of ambiguity may be studied in an equivalent way from pushdown automata accepting a given CFL, as stated in [Her97], because one can construct, from a context free grammar $G$ generating $L$, a pushdown automaton $M$ accepting $L$, and conversely, in such a way that there is a one to one correspondence between leftmost derivations of a word $x$ by $G$ and accepting runs of $M$ on $x$.

This paper is a first investigation of the extension of the notion of ambiguity to context free languages of infinite words.

$\omega$-languages accepted by finite automata were first studied by Büchi in the sixties in order to prove the decidability of the monadic second order theory of one successor over the integers [Büc60a]. Since then the so called $\omega$-regular languages have been intensively studied. See [Tho90] and [PP01] for many results and references.

As Pushdown automata are a natural extension of finite automata, Cohen and Gold [CG77] [CG78] and Linna [Lin76] studied the $\omega$-languages accepted by omega pushdown automata, considering various acceptance conditions for omega words. It turned out that the omega languages accepted by omega pushdown automata were also those generated by context free grammars where infinite derivations are considered, also studied by Nivat and Boasson in [Niv77] [Niv78] [BN80]. These languages were then called the omega context free languages ($\omega$-CFL). Topological properties of these $\omega$-languages have been recently studied in [DFR01] [Fin01a] [Fin01b] [Fin00a] [Dup99] [Fin99], in particular in connection with the Borel hierarchy and the Wadge hierarchy which is a great refinement of the Borel hierarchy. See also Staiger's paper [Sta97a] for a survey of general theory of $\omega$-languages, including more powerful accepting devices, like Turing machines.

In this paper we extend the notion of ambiguity and of degrees of ambiguity to omega context free languages, considered as $\omega$-languages accepted by Büchi or Muller pushdown automata, and investigate the subclasses of the class $CFL_\omega$ of omega context free languages we obtain in that way. More precisely:

In section 2 we first review some definitions and results about $\omega$-regular and $\omega$-context free languages.

In section 3, we recall some facts about ambiguity in context free finitary languages and we show that non ambiguous Büchi or Muller pushdown automata define the same class of $\omega$-languages which we call the class of non ambiguous $\omega$-context free languages. We then study first closure properties of the class $NA-CFL_\omega$ of non ambiguous $\omega$-context free languages, showing it is closed under finite disjoint union and intersection with $\omega$-regular languages. We establish some links between the finitary and the infinitary cases which lead to some first examples of inherently ambiguous $\omega$-CFL.

In section 4 we review some definitions and properties of the Borel and projective hierarchies which will be useful in the sequel.

In section 5 we extend to $\omega$-context free languages the usual notion of degree of ambiguity of a finitary context free language. We show that it may be defined independently of the Büchi or Muller acceptance condition which is considered. Then we study first closure properties of the subclasses of $CFL_\omega$ we have obtained. We state some correspondences between the finitary and the infinitary cases which provide some first examples of $\omega$-CFL which are inherently ambiguous of every finite degree or even of infinite degree.

In section 6 we study topological properties of $\omega$-CFL in connection with their degrees of ambiguity. Using Duparc's results on the Wadge hierarchy of Borel sets [Dup01], we prove that, for every integer $n \geq 1$, there exist some $\omega$-CFL which are non ambiguous or inherently ambiguous of each finite degree or even of infinite degree and which are $\mathbf{\Sigma^0_n}$ (respectively $\mathbf{\Pi^0_n}$)-complete Borel sets.

## 2 $\omega$-regular and $\omega$-context free languages

We assume the reader to be familiar with the theory of formal languages and of $\omega$-regular languages, see for example [HU69] [Tho90]. We first recall some of the definitions and results concerning $\omega$-regular and $\omega$-context free languages and omega pushdown automata as presented in [Tho90] [CG77] [CG78].
When $\Sigma$ is a finite alphabet, a finite nonempty string (word) over $\Sigma$ is any sequence $x = x_1 \ldots x_k$ , where $x_i \in \Sigma$ for $i = 1, \ldots, k$ and $k$ is an integer $\geq 1$. The length of $x$ is $k$, denoted by $|x|$. $x^R = x_k \ldots x_1$ is the mirror image of the word $x$.
If $|x| = 0$, $x$ is the empty word (containing zero letters) denoted by $\lambda$.
We write $x(i) = x_i$ and $x[i] = x(1) \ldots x(i)$ for $i \leq k$ and $x[0] = \lambda$.
$\Sigma^\star$ is the set of finite words over $\Sigma$.
The first infinite ordinal is $\omega$.
An $\omega$-word over $\Sigma$ is an $\omega$ -sequence $a_1 \ldots a_n \ldots$, where $a_i \in \Sigma, \forall i \geq 1$.
When $\sigma$ is an $\omega$-word over $\Sigma$, we write $\sigma = \sigma(1)\sigma(2) \ldots \sigma(n) \ldots$
and $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ the finite word of length $n$, prefix of $\sigma$.
The set of $\omega$-words over the alphabet $\Sigma$ is denoted by $\Sigma^\omega$.
An $\omega$-language over an alphabet $\Sigma$ is a subset of $\Sigma^\omega$.

The usual concatenation product of two finite words $u$ and $v$ is denoted $u.v$ (and sometimes just $uv$). This product is extended to the product of a finite word $u$ and an $\omega$-word $v$: the infinite word $u.v$ is then the $\omega$-word such that:
$(u.v)(k) = u(k)$ if $k \leq |u|$ , and
$(u.v)(k) = v(k - |u|)$ if $k > |u|$.

For $V \subseteq \Sigma^\star$, $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega \mid u_i \in V, \forall i \geq 1\}$ is the $\omega$-power of $V$.
For $V \subseteq \Sigma^\star$, the complement of $V$ (in $\Sigma^\star$) is $\Sigma^\star - V$ denoted $V^-$.
For a subset $A \subseteq \Sigma^\omega$, the complement of $A$ is $\Sigma^\omega - A$ denoted $A^-$.

The prefix relation is denoted $\sqsubseteq$: the finite word $u$ is a prefix of the finite word $v$ (denoted $u \sqsubseteq v$) if and only if there exists a (finite) word $w$ such that $v = u.w$.
This definition is extended to finite words which are prefixes of $\omega$-words:
the finite word $u$ is a prefix of the $\omega$-word $v$ (denoted $u \sqsubseteq v$) iff there exists an $\omega$-word $w$ such that $v = u.w$.

For $u \in \Sigma^\star$ or $u \in \Sigma^\omega$, the set of (finite) left factors (or prefixes) of $u$ is

$$LF(u) = \{v \in \Sigma^\star \mid v \sqsubseteq u\}$$

This definition is extended to a finitary language $L \subseteq \Sigma^\star$ or to an $\omega$-language $L \subseteq \Sigma^\omega$:

$$LF(L) = \bigcup_{u \in L} LF(u) = \{v \in \Sigma^\star \mid \exists u \in L \text{ such that } v \sqsubseteq u\}$$

We can now define finite state machines and Büchi and Muller automata:

**Definition 2.1** *: A finite state machine (FSM) is a quadruple $M = (K, \Sigma, \delta, q_0)$, where $K$ is a finite set of states, $\Sigma$ is a finite input alphabet, $q_0 \in K$ is the initial state and $\delta$ is a mapping from $K \times \Sigma$ into $2^K$. A FSM is called deterministic (DFSM) iff : $\delta : K \times \Sigma \rightarrow K$.*
*A Büchi automaton (BA) is a 5-tuple $M = (K, \Sigma, \delta, q_0, F)$ where $M' = (K, \Sigma, \delta, q_0)$ is a finite state machine and $F \subseteq K$ is the set of final states.*
*A Muller automaton (MA) is a 5-tuple $M = (K, \Sigma, \delta, q_0, F)$ where $M' = (K, \Sigma, \delta, q_0)$ is a FSM and $F \subseteq 2^K$ is the collection of designated state sets.*
*A Büchi or Muller automaton is said deterministic if the associated FSM is deterministic.*
*Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an $\omega$-word over $\Sigma$.*
*A sequence of states $r = q_1 q_2 \ldots q_n \ldots$ is called an (infinite) run of $M = (K, \Sigma, \delta, q_0)$ on $\sigma$, starting in state $p$, iff: 1) $q_1 = p$ and 2) for each $i \geq 1$, $q_{i+1} \in \delta(q_i, a_i)$.*
*In case a run $r$ of $M$ on $\sigma$ starts in state $q_0$, we call it simply "a run of $M$ on $\sigma$ ".*
*For every (infinite) run $r = q_1 q_2 \ldots q_n \ldots$ of $M$, $In(r)$ is the set of states in $K$ entered by $M$ infinitely many times during run $r$:*
*$In(r) = \{q \in K \mid \{i \geq 1/q_i = q\}$ is infinite $\}$.*
*For $M = (K, \Sigma, \delta, q_0, F)$ a BA, the $\omega$-language accepted by $M$ is $L(M) = \{\sigma \in \Sigma^\omega \mid$ there exists a run $r$ of $M$ on $\sigma$ such that $In(r) \cap F \neq \emptyset\}$.*
*For $M = (K, \Sigma, \delta, q_0, F)$ a MA, the $\omega$-language accepted by $M$ is $L(M) = \{\sigma \in \Sigma^\omega \mid$ there exists a run $r$ of $M$ on $\sigma$ such that $In(r) \in F\}$.*

The classical result of Mc Naughton [MaN66] established that the expressive power of deterministic MA (DMA) is equal to the expressive power of non deterministic MA (NDMA) which is also equal to the expressive power of non deterministic BA (NDBA).
There is also a characterization of the languages accepted by MA by means of the "$\omega$-Kleene closure":

**Definition 2.2** *For any family $L$ of finitary languages the $\omega$-Kleene closure of $L$, is :*

$$\omega - KC(L) = \{\cup_{i=1}^{n} U_i.V_i^\omega \mid U_i, V_i \in L, \forall i \in [1, n]\}$$

**Theorem 2.3** *For any $\omega$-language $L$, the following conditions are equivalent:*

*(1) $L$ belongs to $\omega - KC(REG)$ , where $REG$ is the class of (finitary) regular languages.*
*(2) There exists a DMA that accepts $L$.*
*(3) There exists a MA that accepts $L$.*
*(4) There exists a BA that accepts $L$.*

*An $\omega$-language $L$ satisfying one of the conditions of the above Theorem is called an $\omega$-regular language. The class of $\omega$-regular languages will be denoted by $REG_\omega$.*

We now define the pushdown machines and the classes of $\omega$-context free languages.

**Definition 2.4** *A pushdown machine (PDM) is a 6-tuple $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$, where $K$ is a finite set of states, $\Sigma$ is a finite input alphabet, $\Gamma$ is a finite pushdown alphabet, $q_0 \in K$ is the initial state, $Z_0 \in \Gamma$ is the start symbol, and $\delta$ is a mapping from $K \times (\Sigma \cup \{\lambda\}) \times \Gamma$ to finite subsets of $K \times \Gamma^\star$ .*
*If $\gamma \in \Gamma^+$ describes the pushdown store content, the leftmost symbol will be assumed to be on " top" of the store. A configuration of a PDM is a pair $(q, \gamma)$ where $q \in K$ and $\gamma \in \Gamma^\star$.*
*For $a \in \Sigma \cup \{\lambda\}$, $\gamma, \beta \in \Gamma^\star$ and $Z \in \Gamma$, if $(p, \beta)$ is in $\delta(q, a, Z)$, then we write $a : (q, Z\gamma) \mapsto_M (p, \beta\gamma)$.*
*$\mapsto_M^\star$ is the transitive and reflexive closure of $\mapsto_M$. (The subscript $M$ will be omitted whenever the meaning remains clear).*
*Let $\sigma = a_1 a_2 \ldots a_n$ be a finite word over $\Sigma$. A finite sequence of configurations $r = (q_i, \gamma_i)_{1 \leq i \leq p}$ is called a run of $M$ on $\sigma$, starting in configuration $(p, \gamma)$, iff:*

*(1) $(q_1, \gamma_1) = (p, \gamma)$*
*(2) for each $i$, $1 \leq i \leq (p-1)$, there exists $b_i \in \Sigma \cup \{\lambda\}$ satisfying $b_i : (q_i, \gamma_i) \mapsto_M (q_{i+1}, \gamma_{i+1})$ such that $a_1 a_2 \ldots a_n = b_1 b_2 \ldots b_{p-1}$*

*Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an $\omega$-word over $\Sigma$. An infinite sequence of configurations $r = (q_i, \gamma_i)_{i \geq 1}$ is called a run of $M$ on $\sigma$, starting in configuration $(p, \gamma)$, iff:*

*(1) $(q_1, \gamma_1) = (p, \gamma)$*
*(2) for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ satisfying $b_i : (q_i, \gamma_i) \mapsto_M (q_{i+1}, \gamma_{i+1})$ such that either $a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$ or $b_1 b_2 \ldots b_n \ldots$ is a finite prefix of $a_1 a_2 \ldots a_n \ldots$*

*The run $r$ is said to be complete when $a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$*

*As for FSM, for every such run, $In(r)$ is the set of all states entered infinitely*

*often during run $r$.*
*A complete run $r$ of $M$ on $\sigma$, starting in configuration $(q_0, Z_0)$, will be simply called " a run of $M$ on $\sigma$ ".*

Recall first the notion of PDA:

**Definition 2.5** *A pushdown automaton (PDA) is a 7-tuple $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ where $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is a PDM and $F \subseteq K$ is the set of final states.*
*The (finitary) language accepted by $M$ is $L(M) = \{\sigma \in \Sigma^\star \mid$ there exists a run $r = (q_i, \gamma_i)_{1 \leq i \leq p}$ of $M$ on $\sigma$ such that $q_p \in F\}$.*

**Definition 2.6** *A finitary language is context free iff it is accepted by a PDA (by final states). The class of context free languages will be denoted $CFL$.*

**Remark 2.7** *Other accepting conditions by PDA have been shown to be equivalent to the acceptation condition by final states. Let us cite, [ABB96]:*

*(a) Acceptation by empty storage.*
*(b) Acceptation by final states and empty storage.*
*(c) Acceptation by topmost stack letter.*
*(d) Acceptation by final states and topmost stack letter.*

Return now to the acceptation of infinite words by pushdown automata:

**Definition 2.8** *A Büchi pushdown automaton (BPDA) is a 7-tuple $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ where $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is a PDM and $F \subseteq K$ is the set of final states. The $\omega$-language accepted by $M$ is $L(M) = \{\sigma \in \Sigma^\omega \mid$ there exists a complete run $r$ of $M$ on $\sigma$ such that $In(r) \cap F \neq \emptyset\}$.*

**Definition 2.9** *A Muller pushdown automaton (MPDA) is a 7-tuple $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ where $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is a PDM and $F \subseteq 2^K$ is the collection of designated state sets. The $\omega$-language accepted by $M$ is $L(M) = \{\sigma \in \Sigma^\omega \mid$ there exists a complete run $r$ of $M$ on $\sigma$ such that $In(r) \in F\}$.*

**Remark 2.10** *We consider here two acceptance conditions for $\omega$-words, the Büchi and the Muller acceptance conditions, respectively denoted 2-acceptance and 3-acceptance in [Lan69] and in [CG78] and $(inf, \sqcap)$ and $(inf, =)$ in [Sta97a].*

Cohen and Gold, and independently Linna, established a characterization Theorem for $\omega$-CFL:

**Theorem 2.11** *Let $CFL$ be the class of context free (finitary) languages. Then for any $\omega$-language $L$ the following three conditions are equivalent:*

*(1) $L \in \omega - KC(CFL)$.*

*(2) There exists a BPDA that accepts $L$.*
*(3) There exists a MPDA that accepts $L$.*

In [CG77] are also studied the $\omega$-languages generated by $\omega$-context free grammars and it is shown that each of the conditions 1), 2), and 3) of the above Theorem is also equivalent to: 4) $L$ is generated by a context free grammar $G$ by leftmost derivations. These grammars are also studied in [Niv77] [Niv78]. So we set the following definition:

**Definition 2.12** *An $\omega$-language is an $\omega$-context free language ($\omega$-CFL) iff it satisfies one of the conditions of the above Theorem.*

Unlike the case of finite automata, deterministic $MPDA$ do not define the same class of $\omega$-languages as non deterministic $MPDA$. Let us now define deterministic pushdown machines.

**Definition 2.13** *A PDM $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is said to be deterministic (DPDM) iff for each $q \in K, Z \in \Gamma$, and $a \in \Sigma$:*

*(1) $\delta(q, a, Z)$ contains at most one element,*
*(2) $\delta(q, \lambda, Z)$ contains at most one element, and*
*(3) if $\delta(q, \lambda, Z)$ is non empty, then $\delta(q, a, Z)$ is empty for all $a \in \Sigma$.*

It turned out that the class of $\omega$-languages accepted by deterministic $BPDA$ is strictly included into the class of $\omega$-languages accepted by deterministic $MPDA$. Let us denote $DCFL_\omega$ this latest class, the class of omega deterministic context free languages ($\omega$-DCFL), and $DCFL$ the class of deterministic context free (finitary) languages. Then recall the following:

**Proposition 2.14** *(1) $DCFL_\omega$ is closed under complementation, but not under union, neither under intersection.*
*(2) $DCFL_\omega \subsetneq \omega - KC(DCFL) \subsetneq CFL_\omega$*
*(these inclusions are strict).*

**Remark 2.15** *If $M$ is a deterministic pushdown machine, then for every $\sigma \in \Sigma^\omega$, there exists at most one run $r$ of $M$ on $\sigma$ determined by the starting configuration.*

## 3 Ambiguity

**Remark 3.1** *From now on we shall have to count the number of accepting runs of a PDA (respectively, of a BPDA, a MPDA) $M$ on a finite word $\sigma = a_1a_2\ldots a_n$ (respectively, on a infinite word $\sigma = a_1a_2\ldots a_n\ldots$) over $\Sigma$.*

*It is then natural to distinguish two computations of $M$ on $\sigma$ for which $M$ enters in the same sequence (respectively, infinite sequence) of configurations but for which $\lambda$-transitions of $M$ do not occur at the same steps of the runs.*

*So we shall slightly modify the definition 2.4 of a run of a PDM $M$ as follows.*

*A run of $M$ on $\sigma$, starting in configuration $(p, \gamma)$ will be a finite sequence $r = (q_i, \gamma_i, \varepsilon_i)_{1 \leq i \leq p}$ where $(q_i, \gamma_i)_{1 \leq i \leq p}$ is a finite sequence of configurations of $M$ and for all $i$, $1 \leq i \leq p$, $\varepsilon_i \in \{0, 1\}$ and:*

*(1)* $(q_1, \gamma_1) = (p, \gamma)$
*(2) for each $i$, $1 \leq i \leq (p-1)$, there exists $b_i \in \Sigma \cup \{\lambda\}$ satisfying*
$b_i : (q_i, \gamma_i) \mapsto_M (q_{i+1}, \gamma_{i+1})$,
*and $[\varepsilon_i = 0$ iff $b_i = \lambda]$*
*and such that $a_1 a_2 \ldots a_n = b_1 b_2 \ldots b_{p-1}$.*
*(3)* $\varepsilon_p = 0$

*Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an $\omega$-word over $\Sigma$. A run of $M$ on $\sigma$, starting in configuration $(p, \gamma)$ will be an infinite sequence $r = (q_i, \gamma_i, \varepsilon_i)_{i \geq 1}$ where $(q_i, \gamma_i)_{i \geq 1}$ is an infinite sequence of configurations of $M$ and, for all $i \geq 1$, $\varepsilon_i \in \{0, 1\}$ and:*

*(1)* $(q_1, \gamma_1) = (p, \gamma)$
*(2) for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ satisfying*
$b_i : (q_i, \gamma_i) \mapsto_M (q_{i+1}, \gamma_{i+1})$
*and $\varepsilon_i = 0$ iff $b_i = \lambda$*
*and such that either* $a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$
*or* $b_1 b_2 \ldots b_n \ldots$ *is a finite prefix of* $a_1 a_2 \ldots a_n \ldots$

*$In(r)$ is still the set of all states entered infinitely often during run $r$. A complete run is defined as above. A complete run $r$ of $M$ on $\sigma$, starting in configuration $(q_0, Z_0)$, will be simply called " a run of $M$ on $\sigma$ ".*

We are going now to recall some known facts about ambiguity.

The notion of ambiguity was first defined for context free grammars generating finite words. A context free grammar $G$ is said to be non ambiguous iff for every word $x$ in the language $L(G)$ generated by $G$, there exists a unique leftmost derivation of $x$ in $G$, [GU66].

One may also consider pushdown automata as accepting devices by final states. But for every context free grammar $G$ one can construct a PDA $M$ such that $L(M) = L(G)$ and vice versa, [ABB96].
Since for every word $x \in L(M) = L(G)$ there is a one to one correspondence between leftmost derivations of $x$ in $G$ and accepting runs of $M$ on $x$, the notion of ambiguity defined from PDA in the following way is equivalent to

the preceding one. Then from now on we shall refer to pushdown automata as accepting devices and forget the generation by grammars.

**Definition 3.2** *Let $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a pushdown automaton where $F \subseteq K$ is the set of final states. The PDA $M$ is said to be non ambiguous iff for every word $\sigma$ in $L(M)$ there exists a unique accepting run of $M$ on $\sigma$.*

**Definition 3.3** *A context free (finitary) language $L$ is said to be non ambiguous iff it is accepted by a non ambiguous $PDA$. In the other case $L$ is said to be inherently ambiguous. In that case each $PDA$ which accepts $L$ is ambiguous. The class of non ambiguous context free languages is denoted $NA-CFL$. The class of inherently ambiguous context free languages is denoted $A-CFL$.*

**Remark 3.4** *One can easily construct from a PDA $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ accepting by final states an equivalent pushdown automaton accepting by final states* **and** *topmost stack letters, i.e. $M' = (K', \Sigma, \Gamma', \delta', q'_0, Z'_0, (F', Z'))$ where $(K', \Sigma, \Gamma', \delta', q'_0, Z'_0)$ is a PDM and $F' \subseteq K$ is the set of final states and $Z' \subseteq \Gamma'$. A word $x \in \Sigma^\star$ is accepted by $M'$ iff there exists a run $r = (q_i, \gamma_i, \varepsilon_i)_{1 \leq i \leq p}$ of $M'$ on $x$ such that $q_p \in F'$* **and** *$\gamma_p = z.\gamma$ for $z \in Z'$ and $\gamma \in \Gamma^\star$.*
*And conversely, from a PDA $M'$ accepting by final states* **and** *topmost stack letters, one can easily construct an equivalent pushdown automaton accepting (only) by final states.*
*Since there is a one to one correspondence between accepting runs of $M$ on $x$ and accepting runs of $M'$ on $x$, the definition of non ambiguous CFL via pushdown automata accepting by final states* **and** *topmost stack letters leads to the same classes $NA - CFL$ and $A - CFL$.*

We now define the notion of non ambiguous $BPDA$ or $MPDA$:

**Definition 3.5** *Let $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a Büchi pushdown automaton where $F \subseteq K$ is the set of final states. The BPDA $M$ is said to be non ambiguous iff for every word $\sigma$ in $L(M)$ there exists a unique accepting run of $M$ on $\sigma$, i.e. a unique complete run $r$ of $M$ on $\sigma$ such that $In(r) \cap F \neq \emptyset$.*

**Definition 3.6** *Let $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a Muller pushdown automaton where $F \subseteq 2^K$ is the collection of designated state sets. The MPDA $M$ is said to be non ambiguous iff for every word $\sigma$ in $L(M)$ there exists a unique accepting run of $M$ on $\sigma$, i.e. a unique complete run $r$ of $M$ on $\sigma$ such that $In(r) \in F$.*

We shall prove that the class of non ambiguous $BPDA$ and the class of non ambiguous $MPDA$ define the same $\omega$-languages:

**Theorem 3.7** *Let $L$ be an omega context free language. Then $L$ is accepted by a non ambiguous $BPDA$ if and only if it is accepted by a non ambiguous $MPDA$.*

**Proof.** Let $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a Büchi pushdown automaton (BPDA) where $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is a PDM and $F \subseteq K$ is the set of final states. Then the $\omega$-language accepted by $M$ is $L(M) = \{\sigma \in \Sigma^\omega$ / there exists a complete run r of $M$ on $\sigma$ such that $In(r) \cap F \neq \emptyset\}$. This $\omega$-language is also accepted by the Muller pushdown automaton $M_1 = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F_1)$ where $F_1 \subseteq 2^K$ is the set of subsets of $K$ which contain at least one state in $F$:

$$F_1 = \{P \subseteq K \mid P \cap F \neq \emptyset\}$$

Then the machines $M$ and $M_1$ differ only by their accepting conditions and it is easy to see that they accept the same language: $L(M) = L(M_1)$. Moreover, for each word $\sigma \in \Sigma^\omega$, there is a one to one correspondence between accepting runs of $M$ on $\sigma$ and accepting runs of $M_1$ on $\sigma$. Thus if $M$ is non ambiguous the $MPDA$ $M_1$ is also non ambiguous.

Conversely assume that $M_2 = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is a Muller pushdown automaton where $F = \{F_1, \ldots, F_n\}$ is the collection of designated state sets. The $\omega$-language accepted by $M_2$ is $L(M_2) = \{\sigma \in \Sigma^\omega$ / there exists a complete run r of $M_2$ on $\sigma$ such that $In(r) \in F\}$. We shall construct a Büchi pushdown automaton which accepts the same language and remain unambiguous if $M_2$ is unambiguous.

Describe first informally the behaviour of the new $BPDA$ $M_3$ we are going to construct.

The $BPDA$ $M_3$ begins to work like $M_2$ but at some appropriate instant, the machine $M_3$ will guess that it is for the last time in a state which is not in $F_i$, for $1 \leq i \leq n$. Then the run will be accepting iff $M_3$ enters infinitely often in each state of $F_i$.

More formally we define

$$M_3 = (K', \Sigma, \Gamma, \delta', q_0', Z_0, F')$$

where

$$K' = K \cup \{q_0'\} \cup \bigcup_{i=1}^{n} \{i\} \times F_i \times (2^{F_i} - \{\emptyset\})$$

$$F' = \{(i, q, F_i) \mid q \in F_i \text{ and } 1 \leq i \leq n\}$$

and the transition relation is defined by the following transition rules:

(a) $(q_0, Z_0) \in \delta'(q_0', \lambda, Z_0)$.
(b) $((i, q_0, \{q_0\}), Z_0) \in \delta'(q_0', \lambda, Z_0)$ iff $q_0 \in F_i$, for $1 \leq i \leq n$.

And for $a \in \Sigma \cup \{\lambda\}$, $\gamma \in \Gamma^\star$, $Z \in \Gamma$ and $(p, \gamma) \in \delta(q, a, Z)$:

(c) If $q \notin F_i$ and $p \in F_i$ then $(p, \gamma) \in \delta'(q, a, Z)$ and $((i, p, \{p\}), \gamma) \in \delta'(q, a, Z)$.
(d) If $P = F_i$ and $p \in F_i$, then $((i, p, \{p\}), \gamma) \in \delta'((i, q, P), a, Z)$.
(e) If $P \neq F_i$ and $p \in F_i$, $((i, p, P \cup \{p\}), \gamma) \in \delta'((i, q, P), a, Z)$.

The equality $L(M_2) = L(M_3)$ holds by construction. Moreover, for each word $\sigma \in \Sigma^\omega$, there is a one to one correspondence between accepting runs of $M_2$ on $\sigma$ and accepting runs of $M_3$ on $\sigma$. Thus if $M_2$ is non ambiguous the $BPDA$ $M_3$ is also non ambiguous. □

Then one can set the following:

**Definition 3.8** *An omega context free language $L$ is said to be non ambiguous iff it is accepted by a non ambiguous $MPDA$ (or equivalently by a non ambiguous $BPDA$). In the other case $L$ is said to be inherently ambiguous. In that case each $MPDA$ or $BPDA$ which accepts $L$ is ambiguous. The class of non ambiguous omega context free languages is denoted $NA{-}CFL_\omega$. The class of inherently ambiguous omega context free languages is denoted $A - CFL_\omega$.*

**Remark 3.9** *The above construction is similar to Arnold's construction of a non ambiguous BA accepting the same $\omega$-regular language as a given deterministic MA (which is non ambiguous). Each $\omega$-regular language is accepted by a deterministic MA thus each $\omega$-regular language is accepted by a non ambiguous BA, [Arn83b]. But in the case of pushdown automata, there exist some omega context free languages which are not accepted by any deterministic pushdown automaton and even by any non ambiguous MPDA. But starting with a non ambiguous MPDA $M_2$ one can construct a non ambiguous BPDA $M_3$ which accepts the same language. We shall see later that the same construction is also useful in the study of degrees of ambiguity for $\omega$-CFL.*

**Example 3.10** *Every $\omega$-CFL which is accepted by a deterministic MPDA is of course a non ambiguous $\omega$-CFL. For example each $\omega$-regular language is a non ambiguous $\omega$-CFL, as well as the following $\omega$-language over the alphabet $\{a, b, c\}$:*

$$L = \{a^n.b^n \mid n \geq 1\}.c.\{a, b, c\}^\omega$$

The class of non ambiguous CFL is closed under intersection with regular languages. This result may be extended to the case of $\omega$-languages:

**Theorem 3.11** *The class of non ambiguous $\omega$-CFL is closed under intersection with $\omega$-regular languages.*

**Proof.** Let $L = L(M)$ be a non ambiguous $\omega$-CFL accepted by a non ambiguous MPDA $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ and $L' = L(M')$ be an $\omega$-regular language accepted by a deterministic MA $M' = (K', \Sigma, \delta', q'_0, F')$.

The $\omega$-language $L \cap L'$ is accepted by a MPDA which is the classical product of the two machines $M$ and $M'$ with appropriate acceptance conditions. More formally let $M''$ be the MPDA $M'' = (K'', \Sigma, \Gamma, \delta'', (q_0, q_0'), Z_0, F'')$, where

$$K'' = K \times K'$$

and the transition relation $\delta''$ is defined by the following transition rules:

(1) $((q, q'), \nu) \in \delta''((p, p'), a, \gamma)$ iff $(q, \nu) \in \delta(p, a, \gamma)$ and $\delta'(p', a) = q'$,
for each $a \in \Sigma$ and $\gamma \in \Gamma$ and $\nu \in \Gamma^\star$ and $p, q \in K$ and $p', q' \in K'$.
(2) $((q, p'), \nu) \in \delta''((p, p'), \lambda, \gamma)$ iff $(q, \nu) \in \delta(p, \lambda, \gamma)$,
for each $\gamma \in \Gamma$ and $\nu \in \Gamma^\star$ and $p, q \in K$ and $p' \in K'$.

And the collection of designated state sets $F''$ is defined by: a subset $S$ of $K''$ is in $F''$ if and only if its projection $proj_K(S)$ onto $K$ is in $F$ **and** its projection $proj_{K'}(S)$ onto $K'$ is in $F'$.

It is clear that the MPDA accepts $L \cap L'$ and that for each $\omega$-word $\sigma \in \Sigma^\omega$ accepted by $M''$, there exists a unique accepting run of $M''$ on $\sigma$ because $M$ is non ambiguous and $M'$ is deterministic. □

In the case of context free languages one can derive first properties of non ambiguous (respectively inherently ambiguous) $\omega$-CFL from the finitary case. Let us state firstly the following result.

**Proposition 3.12** *Let $V \subseteq \Sigma^\star$ be a finitary context free language over the alphabet $\Sigma$ and $d$ be a new letter not in $\Sigma$, then the following equivalences hold:*

*(a) $V.d^\omega$ is in $NA - CFL_\omega$ iff $V$ is in $NA - CFL$.*
*(b) $V.d.(\Sigma \cup \{d\})^\omega$ is in $NA - CFL_\omega$ iff $V$ is in $NA - CFL$.*

**Proof.** (a) Assume first that $V$ is a non ambiguous context free finitary language. Then there exists a non ambiguous $PDA$ $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ where $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is a PDM and $F \subseteq K$ is the set of final states such that $L(M) = V$. We shall define a non ambiguous BPDA which accepts the $\omega$-language $V.d^\omega$. Let $M''$ be the PDM $M'$ to which we add another state $q_f$ and whose transition relation $\delta''$ is just $\delta$ to which we add the following transition rules:
For $q \in F$ and $Z \in \Gamma$, $(q_f, Z)$ is in $\delta''(q, d, Z)$ and $(q_f, Z)$ is in $\delta''(q_f, d, Z)$.

Now consider the BPDA $N = (K \cup \{q_f\}, \Sigma \cup \{d\}, \Gamma, \delta'', q_0, Z_0, \{q_f\})$. The set of final states of $N$ is just $\{q_f\}$. Then it is easy to see that $L(N) = V.d^\omega$ and that $N$ is non ambiguous because $M$ was non ambiguous. Thus the $\omega$-language $V.d^\omega$ is a non ambiguous $\omega$-CFL.

Conversely assume that $V.d^\omega$ is a non ambiguous $\omega$-CFL. By Theorem 2.11 there exist some context free finitary languages $U_i$ and $V_i$, for $1 \leq i \leq n$, such that

$$V.d^\omega = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$$

Recall now the way Linna proved that the class of $\omega$-CFL is equal to the omega Kleene closure of the class of context free finitary languages, [Lin76] [Sta97a].

Let $L = L(M)$ be an $\omega$-CFL accepted by a BPDA $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$. Define for all pairs $(p, z) \in K \times \Gamma$ and all $q \in K$ the following context free languages:

$V_{(p,z)} = \{\sigma \in \Sigma^\star \mid$ there exists a run of $M$ on $\sigma$ starting in configuration $(q_0, Z_0)$ and ending in a configuration $(p, z.\gamma)$ for $\gamma \in \Gamma^\star\}$

This may be written :

$V_{(p,z)} = \{\sigma \in \Sigma^\star \mid \sigma : (q_0, Z_0) \mapsto_M^\star (p, z.\gamma) \text{ for } \gamma \in \Gamma^\star\}$

Define also

$W^q_{(p,z)} = \{\sigma \in \Sigma^+ \mid$ there exists a run of $M$ on $\sigma$ starting in a configuration $(p, z)$ and ending in a configuration $(p, z\gamma)$ for $\gamma \in \Gamma^\star$ and such that M enters in state $q$ during the run$\}$

We can now express the $\omega$-language accepted by the BPDA $M$ by means of the languages $V_{(p,z)}$ and $W^q_{(p,z)}$ which are finitary context free languages:

$$L(M) = \bigcup_{(p,z) \in K \times \Gamma \ and \ q \in F} V_{(p,z)}.(W^q_{(p,z)})^\omega$$

Return now to the case $L(M) = V.d^\omega$, where $V.d^\omega$ is a non ambiguous $\omega$-CFL and $M = (K, \Sigma \cup \{d\}, \Gamma, \delta, q_0, Z_0, F)$ is a non ambiguous BPDA. Another construction of Linna [Lin76] provides a pushdown automaton accepting $V$ from the BPDA $M$. We recall it now because it will be useful for our proof and in the next section.

Denote

$$R = \{(p, z) \mid (\bigcup_{q \in F} W^q_{(p,z)}) \cap \{d\}^+ \neq \emptyset\}$$

Let then $M_1$ be the following $PDA$:

$$M_1 = (K \cup K' \cup \{q_f\}, \Sigma, \Gamma, \delta_1, q_0, Z_0, \{q_f\})$$

where $K' = \{p' \mid p \in K\}$, $q_f$ is a new state not in $K$ and and the transition relation $\delta_1$ is defined by:

(1) $\delta_1(p, a, Z) = \delta(p, a, Z)$ for all $p \in K$, $a \in \Sigma$ and $Z \in \Gamma$,
(2) $\delta_1(p, \lambda, Z) = \delta(p, \lambda, Z) \cup \{(q', \gamma) \mid (q, \gamma) \in \delta(p, d, Z)\}$ if $(p, Z) \notin R$,
(3) $\delta_1(p, \lambda, Z) = \delta(p, \lambda, Z) \cup \{(q_f, \lambda)\}$ if $(p, Z) \in R$,
(4) $\delta_1(p', \lambda, Z) = \{(q', \gamma) \mid (q, \gamma) \in \delta(p, d, Z) \cup \delta(p, \lambda, Z)\}$ if $(p, Z) \notin R$,
(5) $\delta_1(p', \lambda, Z) = \{(q_f, \lambda)\}$ if $(p, Z) \in R$,
(6) otherwise $\delta_1$ is undefined.

The PDA $M_1$ accepts the context free language $V$. In fact $M_1$ while reading a word $u \in \Sigma^\star$, begins the reading as the BPDA $M$. Some $\lambda$-transitions are added to simulate the reading by $M$ of letters $d$ in the $\omega$-word $u.d^\omega$ until the pushdown automaton reaches a configuration $(p, z\gamma)$, for $\gamma \in \Gamma^\star$, such that $(p, z) \in R$, and this ensures that the word $u$ is in $V$, otherwise the word $u$ is not in $V$.

We have assumed that $V.d^\omega$ is non ambiguous and that $M$ is also non ambiguous. From the definition of $M_1$ we can see that for $u \in V$ there exists a unique accepting run of $M$ on $u.d^\omega$ and the simulation of an initial segment of this run by $M_1$ is also unique. Hence there exists a unique accepting run of $M_1$ on $u$. Then $M_1$ is non ambiguous and so is the context free language $V$.

(b) As in the proof of (a), one can easily construct, from a non ambiguous PDA $M$ accepting a language $V \subseteq \Sigma^\star$, a non ambiguous BPDA $N$ accepting the $\omega$-language $V.d.(\Sigma \cup \{d\})^\omega$.

Conversely assume that $V.d.(\Sigma \cup \{d\})^\omega$ is a non ambiguous $\omega$-CFL, where $V \subseteq \Sigma^\star$, and $d$ is a new letter not in $\Sigma$. Then the class of non ambiguous $\omega$-CFL being closed under intersection with $\omega$-regular languages, the $\omega$-language

$$V.d^\omega = V.d.(\Sigma \cup \{d\})^\omega \cap \Sigma^\star.d^\omega$$

is non ambiguous and the proof of (a) implies that $V$ is in $NA - CFL$. □

**Remark 3.13** *In the second part of the above proof of (a), we have constructed a PDA $M_1$ accepting V from a BPDA accepting the $\omega$-language $V.d^\omega$. And if $M$ is non ambiguous, $M_1$ also is non ambiguous. More generally if a word $u.d^\omega$, where $u \in \Sigma^\star$, admits less than $k$ accepting runs of $M$, where $k$ is an integer $\geq 1$, then the word $u$ admits also less than $k$ accepting runs of $M_1$. This will be useful in the next sections for the study of degrees of ambiguity.*

Recall now some examples of inherently ambiguous context free languages, (see [Fla85] [Fla86]), which will provide some inherently ambiguous $\omega$-CFL by proposition 3.12.

**Theorem 3.14 ([Fla85][Fla86])** *For an integer $n \geq 0$ let $\bar{n} = a^n b$ be the unary representation of $n$ over the alphabet $\Sigma = \{a, b\}$. Then the following context free languages $G_{\neq}$, $G_{<}$, $G_{>}$, $G_{=}$, $H_{\neq}$ are inherently ambiguous, where:*

$$G_{\neq} = \{\bar{n_1}\bar{n_2}\ldots\bar{n_p} \mid \textit{ for some } j \ n_j \neq j\}$$

$$G_{<} = \{\bar{n_1}\bar{n_2}\ldots\bar{n_p} \mid \textit{ for some } j \ n_j < j\}$$
$$G_{>} = \{\bar{n_1}\bar{n_2}\ldots\bar{n_p} \mid \textit{ for some } j \ n_j > j\}$$
$$G_{=} = \{\bar{n_1}\bar{n_2}\ldots\bar{n_p} \mid \textit{ for some } j \ n_j = j\}$$
$$H_{\neq} = \{\bar{n_1}\bar{n_2}\ldots\bar{n_p} \mid \textit{ for some } j \ n_j \neq p\}$$

*(where the variable $p$ runs over all integers $\geq 1$ and the $n_j$ over integers $\geq 0$.)*

From the preceding proposition we can infer some other results about $\omega$-CFL:

**Proposition 3.15** *The class of non ambiguous context free $\omega$-languages is not closed under finite union. But it is closed under disjoint finite union.*

**Proof.** It is well known that the class of non ambiguous context free finitary languages is not closed under finite union. For instance let

$$V_1 = \{a^n b^m c^p \mid n, m, p \text{ are integers } \geq 1 \text{ and } n = m\}$$
$$V_2 = \{a^n b^m c^p \mid n, m, p \text{ are integers } \geq 1 \text{ and } m = p\}$$

It is easy to show that the languages $V_1$ and $V_2$ are deterministic context free, i.e. that they are accepted by some deterministic pushdown automata. Hence they are non ambiguous context free languages, [ABB96].

But their union $V_1 \cup V_2$ is a context free language which is known to be inherently ambiguous, [Mau69].

Then if $d$ is a new letter the $\omega$-languages $V_1.d^\omega$ and $V_2.d^\omega$ are non ambiguous $\omega$-CFL but their union

$$V_1.d^\omega \cup V_2.d^\omega = (V_1 \cup V_2).d^\omega$$

is an inherently ambiguous $\omega$-CFL by Proposition 3.12.

In order to prove that the class $NA - CFL_\omega$ is closed under disjoint union, assume now that $L_1$ is accepted by a non ambiguous MPDA $M_1$ and $L_2$ is accepted by a non ambiguous MPDA $M_2$ and that $L_1$ and $L_2$ are disjoint.
It is then easy to show that there exists a non ambiguous MPDA $M$ which accepts the $\omega$-language $L_1 \cup L_2$. We explain informally this result.

One can assume that the state sets $K_1$ and $K_2$ of $M_1$ and $M_2$ are disjoint, and add a new initial state $Q_0$. Then one add some $\lambda$-transitions which are used to choose at the first step of runs of $M$ if the new machine simulates next the MPDA $M_1$ or the MPDA $M_2$.
Let then $\sigma \in L_1 \cup L_2$. One can assume without loss of generality that $\sigma \in L_1$ and then $\sigma \notin L_2$ because $L_1 \cap L_2 = \emptyset$. There is a unique accepting run of $M_1$ on $\sigma$ thus there is a unique accepting run $r$ of $M$ on $\sigma$: the machine $M$ chooses at the first step of the run $r$ (using a $\lambda$-transition) to simulate next the machine $M_1$ and after this $\lambda$-transition the run $r$ of $M$ is identical with the unique accepting run of $M_1$ on $\sigma$.

The case of a finite number $k$ of non ambiguous $\omega$-CFL is proved by induction on the integer $k$. □

**Proposition 3.16** *It is undecidable to determine whether an arbitrary $\omega$-CFL is non ambiguous (respectively inherently ambiguous).*

**Proof.** It is known that it is undecidable to determine whether an arbitrary CFL is non ambiguous (respectively inherently ambiguous), [GU66]. The proposition 3.16 follows then from this result and from proposition 3.12. □

A natural question now arises: does there exist a characterization Theorem analogous to Theorem 2.11 for non ambiguous $\omega$-CFL? The answer is given by the following:

**Theorem 3.17** *The class of non ambiguous $\omega$-CFL is strictly included into the omega Kleene closure of the class of non ambiguous context free finitary languages:*

$$NA - CFL_\omega \subsetneq \omega - KC(NA - CFL)$$

**Proof.** Let $L = L(M)$ be an $\omega$-CFL accepted by a BPDA $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$. Assume that $L(M)$ is non ambiguous and that $M$ is a non ambiguous BPDA.

Recall Linna defined in [Lin76] for all pairs $(p, z) \in K \times \Gamma$ and all $q \in K$ the following context free languages:

$V_{(p,z)} = \{\sigma \in \Sigma^\star \mid$ there exists a run of $M$ on $\sigma$ starting in configuration $(q_0, Z_0)$ and ending in a configuration $(p, z.\gamma)$ for $\gamma \in \Gamma^\star\}$

$V_{(p,z)} = \{\sigma \in \Sigma^\star \mid \sigma : (q_0, Z_0) \mapsto^\star_M (p, z.\gamma) \text{ for } \gamma \in \Gamma^\star\}$

And

$W^q_{(p,z)} = \{\sigma \in \Sigma^+ \mid$ there exists a run of $M$ on $\sigma$ starting in a configuration $(p, z)$ and ending in a configuration $(p, z\gamma)$ for $\gamma \in \Gamma^\star$ and such that M enters in state $q$ during the run$\}$

Then the $\omega$-language accepted by the BPDA $M$ is

$$L(M) = \bigcup_{(p,z)\in K\times\Gamma \ and \ q\in F} V_{(p,z)}.(W^q_{(p,z)})^\omega$$

In fact this union is restricted to the pairs $(p, z)$ such that $V_{(p,z)}$ is non empty and such that there exists a state $q \in F$ such that $W^q_{(p,z)}$ is non empty.

We are going to show that these context free languages $V_{(p,z)}$ and $W^q_{(p,z)}$ are non ambiguous.

Let $\sigma \in \Sigma^\star$ be a finite word in such a language $V_{(p,z)}$. Then there exists an infinite word in $L(M)$ which is in the form $\sigma.u^\omega$ with $u \in W^q_{(p,z)}$ and $q \in F$. But there is a unique accepting run of $M$ on $\sigma.u^\omega$, hence there exists a unique run on $\sigma$ of the pushdown machine associated with $M$ which starts in the initial configuration and ends in a configuration $(p, z\gamma)$ for $\gamma \in \Gamma^\star$. This ensures that the pushdown automaton accepting $V_{(p,z)}$ by final state $p$ and topmost stack letter $z$ is non ambiguous. As remarked above this implies that $V_{(p,z)}$ is also a ccepted by a non ambiguous PDA accepting by final states. Hence the context free language $V_{(p,z)}$ is non ambiguous.

Consider now a language $W^q_{(p,z)}$ such that $V_{(p,z)}$ is non empty and $q \in F$. By definition this language is also accepted by a PDA obtained from $M$ with initial configuration $(p, z)$ and which accepts by accepting states and topmost stack letter $z$. The exact construction is left to the reader.

Let then $\sigma \in \Sigma^\star$ be a finite word in $V_{(p,z)}$ and $u$ be a finite (nonempty) word in $W^q_{(p,z)}$. The word $\sigma.u^\omega$ is in $L(M)$ thus there exists a unique accepting run of $M$ on $\sigma.u^\omega$. As above this implies that the pushdown automaton accepting $W^q_{(p,z)}$ has a unique accepting run on $u$ (it suffices to consider an initial segment of the run on the finite word $\sigma.u$). Thus this pushdown automaton is non ambiguous. Hence the language $W^q_{(p,z)}$ is non ambiguous.

The inclusion

$$NA - CFL_\omega \subseteq \omega - KC(NA - CFL)$$

is then proved. The inclusion is strict because the class $NA - CFL_\omega$ is not closed under finite union but $\omega - KC(NA - CFL)$ is. $\square$

One may ask for a similar result about the class of inherently ambiguous omega context free languages. But we shall prove the following

**Theorem 3.18** *The class of inherently ambiguous $\omega$-CFL is not included into the omega Kleene closure of the class of inherently ambiguous context free* *finitary languages:*

$$A - CFL_\omega \nsubseteq \omega - KC(A - CFL)$$

**Proof.** Let $V \subseteq \Sigma^\star$ be an inherently ambiguous context free finitary language and $d$ be a new letter not in $\Sigma$. By Proposition 3.12 the $\omega$-language $V.d^\omega$ is an inherently ambiguous $\omega$-CFL. On the other hand by Theorem 2.11 there exist some context free languages $U_i$ and $V_i$, for $1 \leq i \leq n$, such that

$$V.d^\omega = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$$

But then for all $i \in [1; n]$, $V_i \subseteq d^\star$. But it is well known that a context free language over an alphabet containing only one letter is a regular language. Thus the languages $V_i$ are deterministic context free languages hence they belong to the class of non ambiguous context free finitary languages. $\square$

We are going now to recall some facts about Borel and projective hierarchies which will be useful in the study of degrees of ambiguity for $\omega$-CFL.

## 4 Borel and projective hierarchies

We assume the reader to be familiar with basic notions of topology which may be found in [Kur66] [LT94] [Sta97a] [PP01].

Topology is an important tool for the study of $\omega$-languages, and leads to characterization of several classes of $\omega$-languages.
For a finite alphabet $X$, we consider $X^\omega$ as a topological space with the Cantor topology. The open sets of $X^\omega$ are the sets in the form $W.X^\omega$, where $W \subseteq X^\star$. A set $L \subseteq X^\omega$ is a closed set iff its complement $X^\omega - L$ is an open set. The class of open sets of $X^\omega$ will be denoted by $\mathbf{G}$ or by $\mathbf{\Sigma^0_1}$. The class of closed sets will be denoted by $\mathbf{F}$ or by $\mathbf{\Pi^0_1}$. Closed sets are characterized by the following:

**Proposition 4.1** *A set $L \subseteq X^\omega$ is a closed subset of $X^\omega$ iff for every $\sigma \in X^\omega$,*

$[\forall n \geq 1, \exists u \in X^\omega$ *such that* $\sigma(1) \ldots \sigma(n).u \in L]$ *implies that* $\sigma \in L$.

Every closed set $L \subseteq X^\omega$ may be obtained as the adherence of a finitary language. We first recall the notion of adherence.

**Definition 4.2** *Let $V \subseteq X^\star$ be a finitary language over the alphabet $X$. The adherence of the language $V$ is*

$$Adh(V) = \{\sigma \in X^\omega \ / \ LF(\sigma) \subseteq LF(V)\}$$

We can now state the following result.

**Proposition 4.3 (see [Sta97a])** *A set $L \subseteq X^\omega$ is a closed set of $X^\omega$ iff there exists a finitary language $V \subseteq X^\star$ such that $L = Adh(V)$.*

Define now the next classes of the Borel Hierarchy:

**Definition 4.4** *The classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ of the Borel Hierarchy on the topological space $X^\omega$ are defined as follows:*
*$\mathbf{\Sigma^0_1}$ is the class of open sets of $X^\omega$.*
*$\mathbf{\Pi^0_1}$ is the class of closed sets of $X^\omega$.*
*$\mathbf{\Pi^0_2}$ or $\mathbf{G}_\delta$ is the class of countable intersections of open sets of $X^\omega$.*
*$\mathbf{\Sigma^0_2}$ or $\mathbf{F}_\sigma$ is the class of countable unions of closed sets of $X^\omega$.*
*And for any integer $n \geq 1$:*
*$\mathbf{\Sigma^0_{n+1}}$ is the class of countable unions of $\mathbf{\Pi^0_n}$-subsets of $X^\omega$.*
*$\mathbf{\Pi^0_{n+1}}$ is the class of countable intersections of $\mathbf{\Sigma^0_n}$-subsets of $X^\omega$.*
*The Borel Hierarchy is also defined for transfinite levels. The classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$, for a countable ordinal $\alpha$, are defined in the following way:*
*$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of $X^\omega$ in $\cup_{\gamma<\alpha}\mathbf{\Pi}^0_\gamma$.*
*$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of $X^\omega$ in $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma$.*

Recall some basic results about these classes, [Mos80]:

**Proposition 4.5**

*(a)* $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subsetneq \mathbf{\Sigma}^0_{\alpha+1} \cap \mathbf{\Pi}^0_{\alpha+1}$, *for each countable ordinal $\alpha \geq 1$.*
*(b)* $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma = \cup_{\gamma<\alpha}\mathbf{\Pi}^0_\gamma \subsetneq \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$, *for each countable limit ordinal $\alpha$.*
*(c) A set $W \subseteq X^\omega$ is in the class $\mathbf{\Sigma}^0_\alpha$ iff its complement is in the class $\mathbf{\Pi}^0_\alpha$.*
*(d)* $\mathbf{\Sigma}^0_\alpha - \mathbf{\Pi}^0_\alpha \neq \emptyset$ *and* $\mathbf{\Pi}^0_\alpha - \mathbf{\Sigma}^0_\alpha \neq \emptyset$ *hold for every countable ordinal $\alpha \geq 1$.*
*(e) For every ordinal $\alpha \geq 1$, the class $\mathbf{\Sigma}^0_\alpha$ is closed under countable unions and the class $\mathbf{\Pi}^0_\alpha$ is closed under countable intersections.*

We shall say that a subset of $X^\omega$ is a Borel set of rank $\alpha$, for a countable ordinal $\alpha$, iff it is in $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

There is a nice characterization of $\mathbf{\Pi^0_2}$-subsets of $X^\omega$. First define the notion of $W^\delta$:

**Definition 4.6** *For $W \subseteq X^\star$, let $W^\delta = \{\sigma \in X^\omega \mid \exists^\omega i \text{ such that } \sigma[i] \in W\}$. ($\sigma \in W^\delta$ iff $\sigma$ has infinitely many prefixes in $W$).*

Then we can state the following Proposition:

**Proposition 4.7 (see [Sta97a])** *A subset $L$ of $X^\omega$ is a $\mathbf{\Pi^0_2}$-subset of $X^\omega$ iff there exists a set $W \subseteq X^\star$ such that $L = W^\delta$.*

For $X$ a finite set, (and this is also true if $X$ is an infinite alphabet) there

are some subsets of $X^\omega$ which are not Borel sets. Indeed there exists another hierarchy beyond the Borel hierarchy, which is called the projective hierarchy and which is obtained from the Borel hierarchy by successive applications of operations of projection and complementation. More precisely, a subset $A$ of $X^\omega$ is in the class $\mathbf{\Sigma^1_1}$ of **analytic** sets iff there exists another finite set $Y$ and a Borel subset $B$ of $(X \times Y)^\omega$ such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$. We denote $(x, y)$ the infinite word over the alphabet $X \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 0$.
Now a subset of $X^\omega$ is in the class $\mathbf{\Pi^1_1}$ of **coanalytic** sets iff its complement in $X^\omega$ is an analytic set.
The next classes are defined in the same manner, $\mathbf{\Sigma^1_{n+1}}$-sets of $X^\omega$ are projections of $\mathbf{\Pi^1_n}$-sets and $\mathbf{\Pi^1_{n+1}}$-sets are the complements of $\mathbf{\Sigma^1_{n+1}}$-sets.

Recall also the notion of completeness with regard to reduction by continuous functions.
Let $\alpha$ be a countable ordinal. A set $F \subseteq X^\omega$ is a $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$)-complete set iff for any set $E \subseteq Y^\omega$ (with $Y$ a finite alphabet):
$E \in \mathbf{\Sigma^0_\alpha}$ (respectively $E \in \mathbf{\Pi^0_\alpha}$) iff there exists a continuous function $f : Y^\omega \to X^\omega$ such that $E = f^{-1}(F)$.
A similar notion exists for the classes of the projective hierarchy: in particular A set $F \subseteq X^\omega$ is a $\mathbf{\Sigma^1_1}$ (respectively $\mathbf{\Pi^1_1}$)-complete set iff for any set $E \subseteq Y^\omega$ ($Y$ a finite alphabet):
$E \in \mathbf{\Sigma^1_1}$ (respectively $E \in \mathbf{\Pi^1_1}$) iff there exists a continuous function $f : Y^\omega \to X^\omega$ such that $E = f^{-1}(F)$.

A $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$)-complete set is a $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$)-set which is in some sense a set of the highest topological complexity among the $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$)-sets.
$\mathbf{\Sigma^0_n}$ (respectively $\mathbf{\Pi^0_n}$)-complete sets, with $n$ an integer $\geq 1$, are thoroughly characterized in [Sta86].

Recall that a set $F \subseteq X^\omega$ is a $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$)-complete set if and only if it is a $\mathbf{\Sigma^0_\alpha}$ but not $\mathbf{\Pi^0_\alpha}$ set (respectively $\mathbf{\Pi^0_\alpha}$ but not $\mathbf{\Sigma^0_\alpha}$ set). This follows from Wadge's study of the now called Wadge hierarchy of Borel sets, see section 6 below and [Wad84] [Dup01].

## 5 Degrees of ambiguity

The notion of inherently ambiguous CFL has been refined by considering the degrees of ambiguity which we now recall. We use partially the notations of the recent paper of Herzog [Her97], but call $\aleph_0$ the cardinal of the countable set of natural numbers, considered also as the supremum of the set of finite cardinals.

**Definition 5.1** *Let $M$ be a PDA accepting finite words over the alphabet $\Sigma$. For $x \in \Sigma^\star$ let $\alpha_M(x)$ be the number of accepting runs of $M$ on $x$, and*

*(a) If* $\sup\{\alpha_M(x) \mid x \in \Sigma^\star\} \in \mathbb{N}$, *then*

$$\alpha_M = \sup\{\alpha_M(x) \mid x \in \Sigma^\star\}$$

*(b) If $\{\alpha_M(x) \mid x \in \Sigma^\star\}$ is unbounded in $\mathbb{N}$, then*

$$\alpha_M = \aleph_0^-$$

We assume now that the set $\mathbb{N} \cup \{\aleph_0^-\}$ is linearly ordered by the relation $<$ and that for each integer $k \in \mathbb{N}$,

$$k < k+1 < \aleph_0^-$$

Then we can set the following definition.

**Definition 5.2** *Let $k \in \mathbb{N} \cup \{\aleph_0^-\}$. Let*

$$PDA(\alpha \leq k) = \{M \mid M \text{ is a } PDA \text{ with } \alpha_M \leq k\}$$

$$PDA(\alpha < k) = \{M \mid M \text{ is a } PDA \text{ with } \alpha_M < k\}$$
$$CFL(\alpha \leq k) = \{L(M) \mid M \text{ is a } PDA \text{ with } \alpha_M \leq k\}$$
$$CFL(\alpha < k) = \{L(M) \mid M \text{ is a } PDA \text{ with } \alpha_M < k\}$$

*A context free language $L$ is said to be inherently ambiguous of degree $k$, for $k$ an integer $\geq 2$ if*

$$L \in A(k) - CFL = CFL(\alpha \leq k) - CFL(\alpha < k)$$

*A context free language $L$ is usually said to be inherently ambiguous of infinite degree if*

$$L \in A(\aleph_0^-) - CFL = CFL(\alpha \leq \aleph_0^-) - CFL(\alpha < \aleph_0^-)$$

*In that case we shall say also that $L$ is of degree of ambiguity $\aleph_0^-$, (and not just $\aleph_0$), for a reason which will be explained later.*

**Remark 5.3** *For a PDA $M$ accepting finite words over the alphabet $\Sigma$, $\alpha_M$ is equal to zero if and only if $L(M)$ is the empty set because in that case $\alpha_M(x) = 0$ for all $x \in \Sigma^\star$. In the other cases $\alpha_M$ may be a finite integer $\geq 1$ or $\aleph_0^-$.*

**Remark 5.4** *The classes $CFL(\alpha \leq k)$ and $CFL(\alpha < k)$ are defined here by means of PDA accepting by final states. If we consider PDA accepting by final states and topmost stack letters, one can see that the classes $CFL(\alpha \leq k)$ and $CFL(\alpha < k)$ remain unchanged, using the fact cited in Remark 3.4.*

**Remark 5.5** *For $k = \aleph_0^-$ the class $CFL(\alpha \leq \aleph_0^-)$ is the whole class $CFL$. For $k = 1$, the class $CFL(\alpha \leq 1)$ is the class of non ambiguous context free languages.*

The inclusions

$$CFL(\alpha \leq k) \subsetneq CFL(\alpha \leq k+1)$$

are strict for every integer $k \geq 0$. And so is the inclusion

$$CFL(\alpha < \aleph_0^-) \subsetneq CFL(\alpha \leq \aleph_0^-)$$

This may be shown using the following examples due to Maurer [Mau68] and also cited in [Her97].

**Example 5.6** *Let $k$ be an integer $\geq 2$ and let $\Sigma_k = \{a_1, a_2, \ldots a_{2k}\}$ be an alphabet having $2k$ letters and let*

$$A_k = \bigcup_{i=1}^{k} \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2i-1}^{n_i} a_{2i}^{n_1} a_{2i+1}^{n_{i+1}} \ldots a_{2k-1}^{n_k} a_{2k}^{n_k} \mid n_1, n_2, \ldots, n_k \geq 1\}$$

*Above $n_1, n_2, \ldots n_k$ are integers $\geq 1$ and in a word*

$$a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2i-1}^{n_i} a_{2i}^{n_1} a_{2i+1}^{n_{i+1}} \ldots a_{2k-1}^{n_k} a_{2k}^{n_k}$$

*for each $j$, $1 \leq j < i$, letters $a_{2j}$ and $a_{2j+1}$ appear in the segment $a_{2j}^{n_{j+1}}.a_{2j+1}^{n_{j+1}}$, for each $j$, $i \leq j < k$, letters $a_{2j+1}$ and $a_{2j+2}$ appear in the segment $a_{2j+1}^{n_{j+1}}.a_{2j+2}^{n_{j+1}}$.*

*Let also*

$$A_\infty = A_2^\star$$

*Then $A_k$ is inherently ambiguous of degree $k$ over the alphabet $\Sigma_k$ and $A_\infty$ is inherently ambiguous of infinite degree over the alphabet $\Sigma_2$.*

We want now to extend these notions in the context of $\omega$-CFL. We shall consider first BPDA and MPDA. We denote $2^{\aleph_0}$ the cardinal of the continuum which is also the cardinal of the set $\Sigma^\omega$ of $\omega$-words over the finite alphabet $\Sigma$ (having at least two letters).

**Lemma 5.7** *Let $M$ be a BPDA or MPDA accepting infinite words over the alphabet $\Sigma$. For $x \in \Sigma^\omega$ let $\alpha_M(x)$ be the cardinal of the set of accepting runs of $M$ on $x$. Then*

$$\alpha_M(x) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$$

**Proof.** As indicated in the proof of Theorem 3.7 one can construct, from a MPDA $M'$, another BPDA $M$ such that $L(M) = L(M')$ and for every word $\sigma$ in $L(M) = L(M')$ there is a one to one correspondence between accepting runs of $M$ on $\sigma$ and accepting runs of $M'$ on $\sigma$. Hence one can prove the lemma for BPDA first.

Let then $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a BPDA, and $\sigma \in \Sigma^\omega$. We shall show that the set of (codes of) accepting runs of $M$ on $\sigma$ is an analytic set.

To an infinite sequence $r = (q_i, \gamma_i, \varepsilon_i)_{i\geq 1}$, (where $(q_i, \gamma_i)_{i\geq 1}$ is an infinite sequence of configurations and, for each $i \geq 1$, $\varepsilon_i \in \{0, 1\}$) we associate first an $\omega$-word $\bar{r}$ over the alphabet $\Gamma \cup K \cup \{0, 1\}$ defined by

$$\bar{r} = q_1.\gamma_1.\varepsilon_1.q_2.\gamma_2.\varepsilon_2 \ldots q_i.\gamma_i.\varepsilon_i \ldots$$

Then to an infinite word $\sigma \in \Sigma^\omega$ and an infinite sequence $r = (q_i, \gamma_i, \varepsilon_i)_{i\geq 1}$, we associate an infinite word $(\sigma \times \bar{r})$ over the alphabet $X = \Sigma \times (\Gamma \cup K \cup \{0, 1\})$ defined by

$$(\sigma \times \bar{r})(j) = (\sigma(j), \bar{r}(j))$$

for each integer $j \geq 1$.

Explain now informally that one can construct a Turing machine $T$ accepting infinite words over the alphabet $X = \Sigma \times (\Gamma \cup K \cup \{0, 1\})$ with a Büchi condition and such that an $\omega$-word $x \in X^\omega$ is accepted by $T$ if and only if it is in the form $(\sigma \times \bar{r})$ where $\sigma \in \Sigma^\omega$ and $r$ is an accepting run of $M$ on $\sigma$.

But it is well known that the $\omega$-language $L(T)$ accepted by a Turing machine $T$ is an analytic set (see for example [Sta97a]).

Let $\sigma \in \Sigma^\omega$ and $L(T)|\sigma$ be the section of $L(T)$ at point $\sigma$ defined by: an $\omega$-word $x \in X^\omega$ is in $L(T)|\sigma$ iff for all $i \geq 1$, $x(i) = (\sigma(i), a(i))$ for some $a(i) \in (\Gamma \cup K \cup \{0, 1\})$.

Thus for a fixed $\omega$-word $\sigma \in \Sigma^\omega$ the set $L(T)|\sigma$ is also an analytic set because it is the intersection of the analytic set $L(T)$ with the closed set $X^\omega|\sigma$.

But by Suslin's Theorem [Mos80], an analytic subset of $X^\omega$ is either countable or has the continuum power, even if the continuum hypothesis fails (the continuum hypothesis says that **every subset** of $X^\omega$ is either countable or has the continuum power, but this has been shown undecidable in the classical axiomatic system $ZF$ of set theory [Sch67]). Then we can infer that the set $L(T)|\sigma$ and hence the set of accepting runs of $M$ on $\sigma$ is either countable (and in that case its cardinal is either an integer or $\aleph_0$) or has the continuum power (and in that case its cardinal is $2^{\aleph_0}$). □

**Remark 5.8** *Let $M$ be a BPDA or MPDA such that $L(M) \subseteq \Sigma^\omega$. The preceding lemma implies that*

$$\sup\{\alpha_M(x) \mid x \in \Sigma^\omega\} \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$$

*In the case* $\sup\{\alpha_M(x) \mid x \in \Sigma^\omega\} = \aleph_0$*, two cases may happen:*

*(a) There exists (at least) one word* $x \in \Sigma^\omega$ *such that* $\alpha_M(x) = \aleph_0$*.*
*(b) There does not exist any word* $x \in \Sigma^\omega$ *such that* $\alpha_M(x) = \aleph_0$*.*

*In order to distinguish these two cases, we shall set the following definition.*

**Definition 5.9** *Let* $M$ *be a BPDA or MPDA accepting infinite words over the alphabet* $\Sigma$*. Then*

*(a) If* $\sup\{\alpha_M(x) \mid x \in \Sigma^\omega\} \in \mathbb{N} \cup \{2^{\aleph_0}\}$*, then*

$$\alpha_M = \sup\{\alpha_M(x) \mid x \in \Sigma^\omega\}$$

*(b) If* $\sup\{\alpha_M(x) \mid x \in \Sigma^\omega\} = \aleph_0$ *and there does not exist any word* $x \in \Sigma^\omega$ *such that* $\alpha_M(x) = \aleph_0$*, then*

$$\alpha_M = \aleph_0^-$$

*(c) If* $\sup\{\alpha_M(x) \mid x \in \Sigma^\omega\} = \aleph_0$ *and there exists (at least) one word* $x \in \Sigma^\omega$ *such that* $\alpha_M(x) = \aleph_0$*, then*

$$\alpha_M = \aleph_0$$

We assume now that the set $\mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$ is linearly ordered by the relation $<$ and that for each integer $k \in \mathbb{N}$,

$$k < k+1 < \aleph_0^- < \aleph_0 < 2^{\aleph_0}$$

We can now define a hierarchy of BPDA and MPDA:

**Definition 5.10** *Let* $k \in \mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$ *and*

$$BPDA(\alpha \leq k) = \{M \mid M \textit{ is a } BPDA \textit{ with } \alpha_M \leq k\}$$

$$MPDA(\alpha \leq k) = \{M \mid M \textit{ is a } MPDA \textit{ with } \alpha_M \leq k\}$$

$$BPDA(\alpha < k) = \{M \mid M \textit{ is a } BPDA \textit{ with } \alpha_M < k\}$$

$$MPDA(\alpha < k) = \{M \mid M \textit{ is a } MPDA \textit{ with } \alpha_M < k\}$$

As in the finitary case, the class $BPDA(\alpha \leq 1)$ (respectively $MPDA(\alpha \leq 1)$) is the class of non ambiguous BPDA (respectively MPDA). We have seen in Theorem 3.7 that Büchi or Muller acceptance conditions lead to the same class of non ambiguous omega context free languages. This result may be extended to the other classes defined above:

**Theorem 5.11** *For all $k \in \mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$,*

$$\{L(M) \mid M \in BPDA(\alpha \leq k)\} = \{L(M) \mid M \in MPDA(\alpha \leq k)\}$$

$$\{L(M) \mid M \in BPDA(\alpha < k)\} = \{L(M) \mid M \in MPDA(\alpha < k)\}$$

**Proof.** Return to the proof of Theorem 3.7. We have shown that one can construct, from a BPDA $M$ accepting the $\omega$-language $L(M) \subseteq \Sigma^\omega$, a MPDA $M_1$ such that $L(M) = L(M_1)$, and vice-versa. Moreover, for every $\omega$-word $\sigma \in \Sigma^\omega$, there was a one to one correspondence between accepting runs of $M$ on $\sigma$ and accepting runs of $M_1$ on $\sigma$. $\square$

Then one can set the following definition:

**Definition 5.12** *For all $k \in \mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$, let*

$$CFL_\omega(\alpha \leq k) = \{L(M) \mid M \in BPDA(\alpha \leq k)\} = \{L(M) \mid M \in MPDA(\alpha \leq k)\}$$

$$CFL_\omega(\alpha < k) = \{L(M) \mid M \in BPDA(\alpha < k)\} = \{L(M) \mid M \in MPDA(\alpha < k)\}$$

*For every $k$, being an integer $\geq 2$ or in $\{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$, a context free $\omega$-language $L$ is said to be inherently ambiguous of degree $k$, if*

$$L \in A(k) - CFL_\omega = CFL_\omega(\alpha \leq k) - CFL_\omega(\alpha < k)$$

**Remark 5.13** *As in the finitary case, for a BPDA or MPDA $M$ accepting infinite words over the alphabet $\Sigma$, $\alpha_M$ is equal to zero if and only if $L(M)$ is the empty set because in that case $\alpha_M(x) = 0$ for all $x \in \Sigma^\omega$.*

**Remark 5.14** *For $k = 2^{\aleph_0}$ the class $CFL(\alpha \leq 2^{\aleph_0})$ is the whole class $CFL_\omega$. For $k = 1$, the class $CFL_\omega(\alpha \leq 1)$ is the class of non ambiguous context free $\omega$-languages.*

We shall now extend Theorem 3.11:

**Theorem 5.15** *For all $k \in \mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$, The classes $CFL_\omega(\alpha \leq k)$ and $CFL_\omega(\alpha < k)$ are closed under intersection with $\omega$-regular languages.*

**Proof.** We use the same construction as in the proof of Theorem 3.11. Let $L = L(M)$ be an $\omega$-CFL in the class $CFL_\omega(\alpha \leq k)$ (respectively $CFL_\omega(\alpha < k)$) accepted by a MPDA $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ in $MPDA(\alpha \leq k)$ (respectively $MPDA(\alpha < k)$) and $L' = L(M')$ be an $\omega$-regular language accepted by a deterministic MA $M' = (K', \Sigma, \delta', q'_0, F')$.

The $\omega$-language $L \cap L'$ is accepted by a MPDA $M''$ which is the classical product of the two machines $M$ and $M'$ with appropriate acceptance conditions, formally written in the above proof.

It is clear that $M''$ is in $MPDA(\alpha \leq k)$ (respectively $MPDA(\alpha < k)$), because each accepting run of $M''$ on a word $\sigma \in L(M'')$ is derived from an accepting run of $M$ on $\sigma$ in a unique way. $\square$

Considering closure under union we shall prove the following result:

**Theorem 5.16** *(a) For all $k \in \mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$, the classes $CFL_\omega(\alpha \leq k)$ and $CFL_\omega(\alpha < k)$ are closed under disjoint finite union.*

*(b) For all $k \in \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$, the classes $CFL_\omega(\alpha \leq k)$ and $CFL_\omega(\alpha < k)$ are closed under finite union.*

*(c) Let $k_1, \ldots, k_n$ be some positive integers and for all $i \in [1, n]$ let $L_i \in CFL_\omega(\alpha \leq k_i)$, then*

$$\bigcup_{1 \leq i \leq n} L_i \in CFL_\omega(\alpha \leq (k_1 + k_2 + \ldots + k_n))$$

**Proof.** In order to prove $(a)$, we can make a similar construction as in the proof of proposition 3.15.

The same construction implies $(b)$ because a finite union of finite sets is a finite set, a finite union of countable sets is a countable set, and a finite union of sets of cardinal $\leq 2^{\aleph_0}$ is a set of cardinal $\leq 2^{\aleph_0}$.

$(c)$ follows easily from the above construction of a BPDA $M$ accepting $\bigcup_{1 \leq i \leq n} L_i$ from BPDA $M_i$ accepting $L_i$. $\square$

We can now study first some links between the case of finite words and the infinitary case.

**Proposition 5.17** *Let $V \subseteq \Sigma^\star$ be a finitary context free language over the alphabet $\Sigma$ and $d$ be a new letter not in $\Sigma$, then the following equivalences hold for all $k \in \mathbb{N} \cup \{\aleph_0^-\}$:*

*(a) $V.d^\omega$ is in $CFL_\omega(\alpha \leq k)$ iff $V$ is in $CFL(\alpha \leq k)$.*

*(b) $V.d.(\Sigma \cup \{d\})^\omega$ is in $CFL_\omega(\alpha \leq k)$ iff $V$ is in $CFL(\alpha \leq k)$.*

**Proof.** $(a)$ Return now to the proof of Proposition 3.12. Let $V \subseteq \Sigma^\star$ be a finitary context free language in $CFL(\alpha \leq k)$, for $k \in \mathbb{N} \cup \{\aleph_0^-\}$. We have constructed from a PDA accepting $V$ a BPDA accepting $V.d^\omega$. This construction preserved non ambiguity and it is easy to see that it preserves also the degrees of ambiguity and that $V.d^\omega \in CFL_\omega(\alpha \leq k)$.

Conversely by the Remark 3.13, the inverse construction implies that $V$ is in $CFL(\alpha \leq k)$ if $V.d^\omega$ is in $CFL_\omega(\alpha \leq k)$.

$(b)$ Assume first that $V$ is in $CFL(\alpha \leq k)$. As for $(a)$, an easy construction shows that $V.d.(\Sigma \cup \{d\})^\omega$ is in $CFL_\omega(\alpha \leq k)$.

Conversely assume that $V.d.(\Sigma \cup \{d\})^\omega$ is in $CFL_\omega(\alpha \leq k)$. Then the class $CFL_\omega(\alpha \leq k)$ being closed under intersection with $\omega$-regular languages, the $\omega$-language

$$V.d^\omega = V.d.(\Sigma \cup \{d\})^\omega \cap \Sigma^\star.d^\omega$$

is in the class $CFL_\omega(\alpha \leq k)$ and then the proof of $(a)$ shows that $V$ is in $CFL(\alpha \leq k)$. □

**Example 5.18** *We have now first examples of $\omega$-CFL which are inherently ambiguous of degree $k$, for $k \geq 2$ and $k \in \mathbb{N} \cup \{\aleph_0^-\}$. We have seen in Example 5.6 that if $k$ is an integer $\geq 2$ and*

$$A_k = \bigcup_{i=1}^{k} \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2i-1}^{n_i} a_{2i}^{n_1} a_{2i+1}^{n_{i+1}} \ldots a_{2k-1}^{n_k} a_{2k}^{n_k} \mid n_1, \ldots, n_k \geq 1\}$$

$$A_\infty = A_2^\star$$

*Then $A_k$ is inherently ambiguous of degree $k$ over the alphabet $\Sigma_k = \{a_1, \ldots a_{2k}\}$ and $A_\infty$ is inherently ambiguous of infinite degree over the alphabet $\Sigma_2$. Then if $d$ is a new letter the $\omega$-languages $A_k.d^\omega$ and $A_k.d.(\Sigma \cup \{d\})^\omega$ are $\omega$-CFL which are inherently ambiguous of degree $k$, and $A_\infty.d^\omega$ and $A_\infty.d.(\Sigma \cup \{d\})^\omega$ are inherently ambiguous of degree $\aleph_0^-$.*

**Remark 5.19** *Using these examples one can prove that, for each integer $k \geq 1$, the class $CFL_\omega(\alpha \leq k)$ is not closed under finite union, as we have already proved that the class $NA - CFL_\omega = CFL_\omega(\alpha \leq 1)$ is not closed under finite union (see proposition 3.15).*

It is natural to ask now whether there exist some $\omega$-CFL which are inherently ambiguous of degree $\aleph_0$ or $2^{\aleph_0}$. We first give examples of $\omega$-CFL in the class $A(\aleph_0) - CFL_\omega$ and next in the class $A(2^{\aleph_0}) - CFL_\omega$.

**Example 5.20** *For an integer $n \geq 0$ let $\bar{n} = ba^n$ be the unary representation of $n$ over the alphabet $\Sigma = \{a, b\}$. Then the following context free $\omega$-languages $G'_1$, $G'_{\neq}$, $G'_{<}$, $G'_{>}$, $G'_{=}$ are inherently ambiguous of degree $\aleph_0$, where:*

$$G'_1 = \{\bar{n}_1 \bar{n}_2 \ldots \bar{n}_p \ldots \mid \ n_j = n_1 \text{ holds for some } j \geq 2\}$$

$$G'_{\neq} = \{\bar{n}_1 \bar{n}_2 \ldots \bar{n}_p \ldots \mid \text{ for some } j \ n_j \neq j\}$$
$$G'_{<} = \{\bar{n}_1 \bar{n}_2 \ldots \bar{n}_p \ldots \mid \text{ for some } j \ n_j < j\}$$
$$G'_{>} = \{\bar{n}_1 \bar{n}_2 \ldots \bar{n}_p \ldots \mid \text{ for some } j \ n_j > j\}$$
$$G'_{=} = \{\bar{n}_1 \bar{n}_2 \ldots \bar{n}_p \ldots \mid \text{ for some } j \ n_j = j\}$$

*(where the variable $p$ runs over all integers $\geq 1$ and the $n_j$ over integers $\geq 0$.)*

It is easy to see that each of the above $\omega$-languages is in the class $CFL_\omega(\alpha \leq \aleph_0)$ by finding a BPDA in $BPDA(\alpha \leq \aleph_0)$ accepting it. Explain for example the behaviour of such a BPDA $M$ accepting $G'_=$. The pushdown store of $M$ is used like a counter. The counter is increased of one when a letter $b$ is read until the BPDA $M$, after reading the $j^{th}$ letter $b$, decreases the counter of one for each letter $a$ read. If then the counter value is zero and the following letter is a $b$ the $BPDA$ works like a deterministic $BA$ and just checks that there is infinitely many letters $b$ in the $\omega$-word. Then the non determinism has been used to guess the integer $j$. And in an $\omega$-word $\bar{n_1}\bar{n_2}\ldots\bar{n_p}\ldots$ such that for some $j$, $n_j = j$, there exist only countably many such choices. Thus an $\omega$-word $\sigma \in G'_=$ has only countably many accepting runs by the BPDA $M$ and $G'_= \in CFL_\omega(\alpha \leq \aleph_0)$.

In a similar manner a BPDA $M_1$ accepting $G'_1$ need only a counter as a pushdown store. The machine $M_1$, when reading an $\omega$-word $\bar{n_1}\bar{n_2}\ldots\bar{n_p}\ldots$ has to keep the integer $n_1$ in the memory of the stack and then it guesses, using the non determinism, an integer $j \geq 2$ and checks that $n_j = n_1$. We can see that for a given $\omega$-word $\sigma \in G'_1$, there are only countably such choices, hence $\sigma \in G'_1$ has only countably many accepting runs by the BPDA $M_1$ and $G'_1 \in CFL_\omega(\alpha \leq \aleph_0)$.

In order to prove that $G'_1$ is not in $CFL_\omega(\alpha < \aleph_0)$, we will use in [Fin00d] an extension of Ogden's Lemma which has been already often used to prove the inherent ambiguity of some context free finitary languages, [Her97].

**Example 5.21** *Let as above*

$$V_1 = \{a^n b^m c^p \mid n, m, p \textit{ are integers } \geq 1 \textit{ and } n = m\}$$

$$V_2 = \{a^n b^m c^p \mid n, m, p \textit{ are integers } \geq 1 \textit{ and } m = p\}$$

*The languages $V_1$ and $V_2$ are deterministic context free, i.e. that there are accepted by some deterministic pushdown automata, hence they are non ambiguous context free languages. But their union $V = V_1 \cup V_2$ is an inherently ambiguous context free language, [Mau69]. And the context free $\omega$-language $V^\omega$ is in $A(2^{\aleph_0}) - CFL_\omega$.*

*As for the preceding example, we will show this result, using Ogden' s Lemma, in another paper [Fin00d].*

Some other examples of context free $\omega$-languages of maximum degree of ambiguity, i.e. of degree $2^{\aleph_0}$, will be given in next section using topological properties of $\omega$-CFL (see example 6.3 below).
We strongly conjecture that some other examples may be provided by the following

**Conjecture 5.22** *Let $V \subseteq \Sigma^\star$ be a finitary context free language over the alphabet $\Sigma$ and $d$ be a new letter not in $\Sigma$, then the following equivalence holds:*

$$(V.d)^\omega \text{ is in } A(2^{\aleph_0}) - CFL_\omega \text{ iff } V \text{ is in } A - CFL$$

Remark that the following proposition is easy to prove:

**Proposition 5.23** *Let $V \subseteq \Sigma^\star$ be a finitary context free language over the alphabet $\Sigma$ and $d$ be a new letter not in $\Sigma$, then the following equivalence holds:*

$$(V.d)^\omega \text{ is in } NA - CFL_\omega \text{ iff } V \text{ is in } NA - CFL$$

**Proof.** If $V$ is in $NA-CFL$ one can easily construct a non ambiguous BPDA $M$ accepting $(V.d)^\omega$.
Assume now that $(V.d)^\omega$ is in $NA-CFL_\omega$ and let $u \in V$. Then the $\omega$-language

$$V.(d.u)^\omega = (V.d)^\omega \cap \Sigma^\star.(d.u)^\omega$$

is also in $NA - CFL_\omega$ because this class is closed under intersection with $\omega$-regular languages. But then with a similar argument as in the proof of proposition 3.12 one can find a non ambiguous PDA accepting the finitary language $V$ thus $V$ is in $NA - CFL$. □

## 6 Topological properties

### *6.1 Recall of previous results*

When considering regular or context free $\omega$-languages, natural questions arise: are they all Borel sets of finite rank, Borel sets, analytic sets ...?

Landweber studied first the topological properties of $\omega$-regular languages. McNaughton's Theorem implies that $\omega$-regular languages are boolean combination of $G_\delta$ sets, [MaN66]. Landweber characterized the $\omega$-regular languages in each of the Borel classes $\mathbf{F}, \mathbf{G}, \mathbf{F}_\sigma, \mathbf{G}_\delta$, and showed that one can decide, for an effectively given $\omega$-regular language $L$, whether $L$ is in $\mathbf{F}, \mathbf{G}, \mathbf{F}_\sigma$, or $\mathbf{G}_\delta$. It turned out that an $\omega$-regular language is in the class $\mathbf{G}_\delta$ iff it is accepted by a $DBA$. These results were extended to deterministic $\omega$-CFL by Linna [Lin77]. In the non deterministic case, Cohen and Gold proved in [CG78] that one cannot decide whether an $\omega$-CFL is in the class $\mathbf{F}, \mathbf{G}$ or $\mathbf{G}_\delta$. We proved in [Fin01a] that there are $\omega$-CFL in each Borel class of finite rank. And that, for any Borel class $\mathbf{\Sigma^0_n}$ or $\mathbf{\Pi^0_n}$, n being an integer, one cannot decide whether an $\omega$-CFL is in $\mathbf{\Sigma^0_n}$ or $\mathbf{\Pi^0_n}$. Every $\omega$-CFL is an analytic set but we showed in [Fin00a] that there exist some $\omega$-CFL which are analytic but non Borel sets.

One may ask for a similar investigation for the subclasses of $CFL_\omega$ defined above by means of the notion of ambiguity.

From a general study of topological properties of transition systems due to Arnold [Arn83a] we know that every non ambiguous $\omega$-CFL is a Borel set (one considers a BPDA as a transition system with infinitely many states). Pierre Simonnet, using a uniformization theorem of descriptive set theory, has extended this result by proving the following Theorem:

**Theorem 6.1 (Simonnet [Sim00])** *Let $L(M)$ be an $\omega$-CFL accepted by a BPDA $M$ such that $L(M)$ is an analytic but non Borel set. Then there exist $2^{\aleph_0}$ $\omega$-words, each of which having $2^{\aleph_0}$ accepting runs by $M$.*

With our notations we can infer from this result the following one:

**Corollary 6.2** *Let $L \subseteq \Sigma^\omega$ be an $\omega$-CFL. If $L$ is non ambiguous or inherently ambiguous of degree $k$, with $k$ an integer $\geq 2$ or $k \in \{\aleph_0^-, \aleph_0\}$, then $L$ is a Borel subset of $\Sigma^\omega$.*

**Example 6.3** *Recall here the construction of a simple context free $\omega$-language which is analytic but not Borel, [Fin00a]. Let $\Sigma = \{0, 1\}$ be an alphabet containing two letters 0 and 1, $A$ be a new letter and*

$$D = \{u.A.v \ / \ u, v \in \Sigma^\star \text{ and } (|v| = 2|u|) \ \text{ or } \ (|v| = 2|u| + 1)\ \}$$

*Then $D$ is a context free language over the alphabet $(\Sigma \cup \{A\})$. Let then $g$ be the substitution $\Sigma \to P((\Sigma \cup \{A\})^\star)$ defined by $a \to a.D$.*

*Let then $W = 0^\star.1$. The image of $W$ by the substitution $g$ is $g(W)$ which is a finitary context free language such that $(g(W))^\omega$ is an analytic but non Borel context free $\omega$-language.*

*Thus Simonnet's Theorem implies that*

$$(g(W))^\omega \in A(2^{\aleph_0}) - CFL_\omega$$

Recall that we have proved in proposition 3.16 that one cannot decide whether an arbitrary $\omega$-CFL $L$ is non ambiguous, i.e. whether $L \in CFL_\omega(\alpha \leq 1)$. We can now extend this result by proving the following Theorem:

**Theorem 6.4** *Let $k$ be an integer $\geq 2$ or $k \in \{\aleph_0^-, \aleph_0\}$. Then it is undecidable to determine whether an arbitrary $\omega$-CFL is in the class $CFL_\omega(\alpha \leq k)$ (respectively $CFL_\omega(\alpha < k)$). So one cannot decide whether an arbitrary $\omega$-CFL is inherently ambiguous of degree $2^{\aleph_0}$.*

**Proof.** Recall that we proved in [Fin00a] that one cannot decide whether an arbitrary $\omega$-CFL is a Borel set. In fact we found a family of $\omega$-CFL $D(X,Y)$ over a finite alphabet $\Sigma_A$ such that only two cases may happen. In the first case, $D(X,Y) = (\Sigma_A)^\omega$, therefore $D(X,Y)$ is an $\omega$-regular language and it is a non ambiguous $\omega$-CFL hence $D(X,Y) \in CFL_\omega(\alpha \leq 1)$ and $D(X,Y) \in CFL_\omega(\alpha \leq k)$ hold for every integer $k \geq 2$ and for $k \in \{\aleph_0^-, \aleph_0\}$.

In the second case $D(X,Y)$ is a $\mathbf{\Sigma}^1_1$-complete subset of $(\Sigma_A)^\omega$, and then by Theorem 6.1, $D(X,Y) \in A(2^{\aleph_0}) - CFL_\omega$ holds and $D(X,Y)$ is not in any class $CFL_\omega(\alpha \leq k)$ or $CFL_\omega(\alpha < k)$ for $k$ an integer $\geq 2$ or $k \in \{\aleph_0^-, \aleph_0\}$.

But one cannot decide which case holds and this ends the proof. $\square$

From now on, we shall pursue the study of links between topological properties and ambiguity of $\omega$-CFL .

We have previously proved that the class of context free $\omega$-languages exhausts the finite ranks of the Borel hierarchy (it meets every finite level of the Borel hierarchy), using previous results of Duparc on the Wadge hierarchy of Borel sets. We shall see that these results will be still useful in the present context. Hence we introduce now the Wadge Hierarchy which is a great refinement of the Borel hierarchy:

### *6.2 Wadge hierarchy*

**Definition 6.5** *For $E \subseteq X^\omega$ and $F \subseteq Y^\omega$, $E$ is said Wadge reducible to $F$ ($E \leq_W F$) iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $E = f^{-1}(F)$.*
*$E$ and $F$ are Wadge equivalent iff $E \leq_W F$ and $F \leq_W E$. this will be denoted by $E \equiv_W F$. And we shall say that $E <_W F$ iff $E \leq_W F$ but not $F \leq_W E$.*

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation.
The equivalence classes of $\equiv_W$ are called Wadge degrees.
A set $E \subseteq X^\omega$ is said to be selfdual iff $E \equiv_W E^-$; otherwise $E$ is said to be non selfdual.
$WH$ is the class of Borel subsets of a set $X^\omega$, where $X$ is a finite set, equipped with $\leq_W$ and with $\equiv_W$.
For $E \subseteq X^\omega$ and $F \subseteq Y^\omega$, if $E \leq_W F$ and $E = f^{-1}(F)$ where $f$ is a continuous function from $X^\omega$ into $Y^\omega$, then $f$ is called a continuous reduction of $E$ to $F$. Intuitively it means that $E$ is less complicated than $F$ because to check whether $x \in E$ it suffices to check whether $f(x) \in F$ where $f$ is a continuous function. Hence the Wadge degree of an $\omega$-language is a measure of its topological complexity.

**Remark 6.6** *In the above definition, we consider that a subset $E \subseteq X^\omega$ is* *given together with the alphabet $X$.*

Then we can define the Wadge class of a set $F$:

**Definition 6.7** *Let $F$ be a Borel subset of $X^\omega$. The wadge class of $F$ is $[F]$ defined by: $[F] = \{E \mid E \subseteq Y^\omega$ for a finite alphabet $Y$ and $E \leq_W F\}$.*

Recall that each Borel class $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ is a Wadge class.

Recall now that a set $X$ is well ordered by a binary relation $<$ iff $<$ is a linear order on $X$ and there is not any strictly decreasing (for $<$) infinite sequence of elements in $X$.

**Theorem 6.8 (Wadge)** *Up to the complement and $\equiv_W$, the class of Borel subsets of $X^\omega$, for $X$ a finite alphabet, is a well ordered hierarchy. There is an ordinal $|WH|$, called the length of the hierarchy, and a map $d^0_W$ from $WH$ onto $|WH|$, such that for all $A, B \in WH$:*
*$d^0_W A < d^0_W B \leftrightarrow A <_W B$ and*
*$d^0_W A = d^0_W B \leftrightarrow [A \equiv_W B$ or $A \equiv_W B^-]$.*

If we restrict the study to Borel sets of finite rank, the Wadge hierarchy has then length ${}^1\epsilon_0$ where ${}^1\epsilon_0$ is the limit of the ordinals $\omega_1(n)$ defined by $\omega_1(1) = \omega_1$ and $\omega_1(n+1) = \omega_1^{\omega_1(n)}$ for $n$ a non negative integer, $\omega_1$ being the first non countable ordinal.
The length of the whole Wadge hierarchy of Borel sets is a much larger ordinal. It is described using Veblen functions in [Wad84] [Dup01].

Wadge gave first a description of the Wadge hierarchy of Borel sets, [Wad84]. Duparc recently got a new proof of Wadge's results and he gave a normal form of Borel sets, i.e. an inductive construction of a Borel set of every given degree [Dup95a] [Dup01]. His proof relies on set theoretic operations which are the counterpart of arithmetical operations over ordinals needed to compute the Wadge degrees.

In fact Duparc studied the Wadge hierarchy via the study of the conciliating hierarchy. He introduced in [Dup95a] [Dup01] conciliating sets which are sets of finite *or* infinite words over an alphabet $X$, i.e. subsets of $X^\star \cup X^\omega = X^{\leq\omega}$. It turned out that the conciliating hierarchy is isomorphic to the Wadge hierarchy of non self dual Borel sets, via the following correspondence:

For $A \subseteq X^{\leq\omega}$ and $d$ a letter not in $X$, define

$$A^d = \{x \in (X \cup \{d\})^\omega \ / \ x(/d) \in A\}$$

where $x(/d)$ is the sequence obtained from $x$ when removing every occurrence of the letter $d$.

Considering conciliating sets, we shall sometimes simply say "$A$ is a Borel set" instead of "$A^d$ is a Borel set". In the same way we shall say "$A$ is in the Borel class $\mathbf{\Sigma}^{\mathbf{0}}_\alpha$ (respectively $\mathbf{\Pi}^{\mathbf{0}}_\alpha$)" instead of "$A^d$ is in the Borel class $\mathbf{\Sigma}^{\mathbf{0}}_\alpha$ (respectively $\mathbf{\Pi}^{\mathbf{0}}_\alpha$)".

The set theoretic operations are then defined over conciliating sets. We shall use in this paper the operation of exponentiation.

### *6.3 Operation of exponentiation of conciliating sets*

We first recall the following:

**Definition 6.9** *Let $X_A$ be a finite alphabet and $\twoheadleftarrow \notin X_A$, let $X = X_A \cup \{\twoheadleftarrow\}$.*
*Let $x$ be a finite or infinite word over the alphabet $X = X_A \cup \{\twoheadleftarrow\}$.*
*Then $x^{\twoheadleftarrow}$ is inductively defined by:*
$\lambda^{\twoheadleftarrow} = \lambda$,
*For a finite word $u \in (X_A \cup \{\twoheadleftarrow\})^\star$:*
$(u.a)^{\twoheadleftarrow} = u^{\twoheadleftarrow}.a$, *if* $a \in X_A$,
$(u.\twoheadleftarrow)^{\twoheadleftarrow} = u^{\twoheadleftarrow}$ *with its last letter removed if* $|u^{\twoheadleftarrow}| > 0$,
$(u.\twoheadleftarrow)^{\twoheadleftarrow} = \lambda$ *if* $|u^{\twoheadleftarrow}| = 0$,
*and for $u$ infinite:*
$(u)^{\twoheadleftarrow} = \lim_{n\in\omega}(u[n])^{\twoheadleftarrow}$, *where, given $\beta_n$ and $v$ in $X_A^\star$,*
$v \sqsubseteq \lim_{n\in\omega}\beta_n \leftrightarrow \exists n \forall p \geq n \quad \beta_p[|v|] = v$.

**Remark 6.10** *For $x \in X^{\leq\omega}$, $x^{\twoheadleftarrow}$ denotes the string $x$, once every $\twoheadleftarrow$ occuring in $x$ has been "evaluated" to the back space operation (the one familiar to your computer!), proceeding from left to right inside $x$. In other words $x^{\twoheadleftarrow} = x$ from which every interval of the form "$a \twoheadleftarrow$ " ($a \in X_A$) is removed. So we may consider the letter $\twoheadleftarrow$ as an eraser.*

For example if $u = (a \twoheadleftarrow)^n$, for $n \geq 1$, $u = (a \twoheadleftarrow)^\omega$ or $u = (a \twoheadleftarrow\twoheadleftarrow)^\omega$ then $(u)^{\twoheadleftarrow} = \lambda$,
if $u = (ab \twoheadleftarrow)^\omega$ then $(u)^{\twoheadleftarrow} = a^\omega$,
if $u = bb(\twoheadleftarrow a)^\omega$ then $(u)^{\twoheadleftarrow} = b$.

We can now define the operation $A \to A^\sim$ of exponentiation of conciliating sets:

**Definition 6.11** *For $A \subseteq X_A^{\leq\omega}$ and $\twoheadleftarrow \notin X_A$, let $X = X_A \cup \{\twoheadleftarrow\}$ and*
$A^\sim = \{x \in (X_A \cup \{\twoheadleftarrow\})^{\leq\omega} \mid x^{\twoheadleftarrow} \in A\}$.

The operation $\sim$ is monotone with regard to the Wadge ordering and produce some sets of higher complexity, in the following sense:

**Theorem 6.12 (Duparc [Dup01] )** *a) For $A \subseteq X_A^{\leq\omega}$ and $B \subseteq X_B^{\leq\omega}$, $A^d$ and $B^d$ borel sets, $A^d \leq_W B^d \leftrightarrow (A^\sim)^d \leq_W (B^\sim)^d$.*

*b) If $A^d \subseteq (X_A \cup \{d\})^\omega$ is a $\mathbf{\Sigma^0_n}$-complete (respectively $\mathbf{\Pi^0_n}$-complete) set ( for an integer $n \geq 1$ ), then $(A^\sim)^d$ is a $\mathbf{\Sigma^0_{n+1}}$-complete (respectively $\mathbf{\Pi^0_{n+1}}$-complete) set.*

We proved in [Fin01a] that the class $CFL_\omega$ is closed under this operation $\sim$.

**Theorem 6.13** *Whenever $A \subseteq X_A^\omega$ is an $\omega$-CFL, then $A^\sim \subseteq (X_A \cup \{\twoheadleftarrow\})^\omega$ is an $\omega$-CFL.*

We shall now extend this result by proving the following:

**Theorem 6.14** *(I) $A \subseteq X_A^\omega$ is in $NA-CFL_\omega$ if and only if $A^\sim \subseteq (X_A \cup \{\twoheadleftarrow\})^\omega$ is in $NA-CFL_\omega$.*

*(II) $A \subseteq X_A^\omega$ is inherently ambiguous of degree $k$, where $k$ is an integer $\geq 2$ or $k \in \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$ if and only if $A^\sim \subseteq (X_A \cup \{\twoheadleftarrow\})^\omega$ is inherently ambiguous of the same degree $k$.*

**Proof.** We reason firstly as in the proof of Theorem 6.13 in [Fin01a].
Let $A$ be an $\omega$-CFL which is accepted by a Büchi pushdown automaton $M = (K, X_A, \Gamma, \delta, q_0, Z_0, F)$.

An $\omega$-word $\sigma \in A^\sim$ may be considered as an $\omega$-word $\sigma^{\twoheadleftarrow} \in A$ to which we possibly add, before the first letter $\sigma^{\twoheadleftarrow}(1)$ of $\sigma^{\twoheadleftarrow}$ (respectively between two consecutive letters $\sigma^{\twoheadleftarrow}(n)$ and $\sigma^{\twoheadleftarrow}(n+1)$ of $\sigma^{\twoheadleftarrow}$), a finite word $v_1$ (respectively $v_{n+1}$ ) where:
$v_{n+1}$ belongs to the context free (finitary) language $L_3$ generated by the context free grammar with the following production rules:
$S \rightarrow aS \twoheadleftarrow S$ with $a \in X_A$,
$S \rightarrow a \twoheadleftarrow S$ with $a \in X_A$,
$S \rightarrow \lambda$ ($\lambda$ being the empty word).
this language $L_3$ corresponds to words where every letter of $X_A$ has been removed after using the back space operation.
And $v_1$ belongs to the finitary language $L_4 = (\twoheadleftarrow)^\star.(L_3.(\twoheadleftarrow)^\star)^\star$. This language corresponds to words where every letter of $X_A$ has been removed after using the back space operation and this operation maybe has been used also when there was not any letter to erase. $L_4$ is a context free language because the class $CFL$ is closed under star operation and concatenation product.

**Remark 6.15** *Recall that a one counter automaton is a pushdown automaton with a pushdown alphabet in the form $\Gamma = \{Z_0, z\}$ where $Z_0$ is the bottom symbol and always remains at the bottom of the pushdown store. And a one counter language is a (finitary) language which is accepted by a one counter automaton by final states. It is easy to see that in fact $L_3$ and $L_4$ are deterministic one-counter languages, i.e. $L_3$ and $L_4$ are accepted by deterministic* *one-counter automata. And for $a \in X_A$, the language $L_3.a$ is also accepted by a deterministic one-counter automaton.*

Then we shall construct from $M$ another BPDA $M^\sim$ which accepts the $\omega$-language $A^\sim$ over the alphabet $X = X_A \cup \{\twoheadleftarrow\}$.
Describe first informally the behaviour of the machine $M^\sim$ when it reads an $\omega$-word $\sigma \in A^\sim$. Recall that this word may be considered as an $\omega$-word $\sigma^{\twoheadleftarrow} \in A$ to which we possibly add, before the first letter $\sigma^{\twoheadleftarrow}(1)$ of $\sigma^{\twoheadleftarrow}$ (respectively between two consecutive letters $\sigma^{\twoheadleftarrow}(n)$ and $\sigma^{\twoheadleftarrow}(n+1)$ of $\sigma^{\twoheadleftarrow}$), a finite word $v_1$ (respectively $v_{n+1}$ ) where $v_1$ belongs to the context free language $L_4$ and $v_{n+1}$ belongs to the context free language $L_3$.

$M^\sim$ starts the reading as a pushdown automaton accepting the language $L_4$. Then $M^\sim$ begins to read as $M$, but at any moment of the computation it may guess (using the non determinism) that it reads a finite segment $v$ of $L_3$ which will be erased (using the eraser $\twoheadleftarrow$). It reads $v$ using an additional stack letter $E$ which permits to simulate a one counter automaton at the top of the stack while keeping the memory of the stack of $M$. Then, after the reading of $v$, $M^\sim$ simulates again the machine $M$ and so on.

More formally $M^\sim = (K^\sim, X_A \cup \{\twoheadleftarrow\}, \Gamma \cup \{E\}, \delta^\sim, q_0', Z_0, F)$, where

$$K^\sim = K \cup \{q_0'\} \cup \{q^1 \mid q \in K\}$$

$E$ is a new letter not in $\Gamma$,

And the transition relation $\delta^\sim$ is defined by the following cases (where the transition rules (a)-(d) are used to simulate a pushdown automaton accepting $L_4$ and the BPDA $M^\sim$ enters in a state $q^1$, for $q \in K$, when it simulates a one counter automaton accepting $L_3$):

(a) $\delta^\sim(q_0', \twoheadleftarrow, Z_0) = (q_0', Z_0)$.
(b) $(q_0', EZ_0) \in \delta^\sim(q_0', a, Z_0)$,
for each $a \in X_A$.
(c) $\delta^\sim(q_0', \twoheadleftarrow, E) = (q_0', \lambda)$.
(d) $\delta^\sim(q_0', a, E) = (q_0', EE)$,
for each $a \in X_A$.
(e) $(q, \nu) \in \delta^\sim(q_0', a, Z_0)$ iff $(q, \nu) \in \delta(q_0, a, Z_0)$,
for each $a \in X_A \cup \{\lambda\}$ and $\nu \in \Gamma^\star$ and $q \in K$.
(f) $(q', \nu) \in \delta^\sim(q, a, \gamma)$ iff $(q', \nu) \in \delta(q, a, \gamma)$,
for each $a \in X_A \cup \{\lambda\}$ and $\gamma \in \Gamma$ and $\nu \in \Gamma^\star$ and $q, q' \in K$.
(g) $(q^1, E\gamma) \in \delta^\sim(q, a, \gamma)$,
for each $a \in X_A$ and $\gamma \in \Gamma$ and $q \in K$.
(h) $\delta^\sim(q^1, a, E) = (q^1, EE)$,
for each $a \in X_A$ and $q \in K$.

(i) $\delta^{\sim}(q^1, \twoheadleftarrow, E) = (q^1, \lambda)$.
(j) $(q', \nu) \in \delta^{\sim}(q^1, a, \gamma)$ iff $(q', \nu) \in \delta(q, a, \gamma)$,
for each $a \in X_A$ and $\gamma \in \Gamma$ and $\nu \in \Gamma^\star$ and $q, q' \in K$.
(k) $(q^1, E\gamma) \in \delta^{\sim}(q^1, a, \gamma)$,
for each $a \in X_A$ and $\gamma \in \Gamma$ and $q \in K$.

(I) We claim now that if $M$ is a non ambiguous BPDA then $M^{\sim}$ is also a non ambiguous BPDA. This comes from the fact that the operation of erasing in an $\omega$-word $\sigma \in A^{\sim}$ is uniquely determined. Moreover the $\lambda$-transitions occurring in the reading of $\sigma$ by $M^{\sim}$ simulate the $\lambda$-transitions occurring in the reading of $\sigma^{\twoheadleftarrow}$ by $M$ and they can not appear from a state of $K^1$ (transition rule $(j)$). Hence they may appear just before but not just after the reading of a segment $v$ which will be erased. This ensures that the accepting run of an $\omega$-word $\sigma \in A^{\sim}$ is unique. And in an $\omega$-word $\sigma$ accepted by $M^{\sim}$ there exist infinitely many letters of $X_A$ which induce an **infinite** word $\sigma^{\twoheadleftarrow}$ of $(X_A)^\omega$ which is in $A$ (because otherwise at some moment $M^{\sim}$ would enter in a state of $K^1$ and then remain in states of $K^1$).

This proves that if $A$ is in $NA - CFL_\omega$ then $A^{\sim}$ is in $NA - CFL_\omega$.

Conversely assume that $A^{\sim}$ is in $NA - CFL_\omega$. Then there exists a non ambiguous BPDA $N$ which accepts the $\omega$-language $A^{\sim}$. But it is easy to see that

$$A^{\sim} \cap (X_A)^\omega = A$$

Hence one can get a BPDA $M$ accepting $A$ from the BPDA $N$ by suppressing the transition rules involving the input letter $\twoheadleftarrow$. Each accepting run of $M$ on an $\omega$-word $\sigma \in A$ comes from an accepting run of $N$ on $\sigma$. Thus $M$ is non ambiguous because $N$ was non ambiguous. And $A = L(M)$ is in $NA - CFL_\omega$.

(II) Assume now that $A \subseteq X_A^\omega$ is inherently ambiguous of degree $k$, where $k$ is an integer $\geq 2$ or $k \in \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$. Let then $M \in BPDA(\alpha \leq k)$ accepting $A$. For the same reasons as for $(I)$ above, $M^{\sim} \in BPDA(\alpha \leq k)$ holds. Thus $A^{\sim} \in CFL_\omega(\alpha \leq k)$. Now if $A^{\sim} \in CFL_\omega(\alpha < k)$ we could construct, from a BPDA $N \in BPDA(\alpha < k)$ accepting $A^{\sim}$, another BPDA $M'$ accepting $A$ by suppressing the transition rules involving the input letter $\twoheadleftarrow$. And then $M'$ would be in $BPDA(\alpha < k)$. This would lead to a contradiction thus $A^{\sim} \notin CFL_\omega(\alpha < k)$ and $A^{\sim} \in A(k) - CFL_\omega$.

Assume conversely that $A^{\sim} \in A(k) - CFL_\omega$, where $k$ is an integer $\geq 2$ or $k \in \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$.

Let $M$ be a BPDA in $BPDA(\alpha \leq k)$ accepting $A^{\sim}$. Like above we can construct from $M$, by suppressing the transition rules involving the input letter $\twoheadleftarrow$, another BPDA $N$ in $BPDA(\alpha \leq k)$ accepting $A$. Thus $A \in CFL_\omega(\alpha \leq k)$.

And if $A \in CFL_\omega(\alpha < k)$ held, the BPDA $M^\sim$ accepting $A^\sim$ we have constructed above would be in $BPDA(\alpha < k)$. This is impossible because $A^\sim \in A(k) - CFL_\omega$. Then $A \notin CFL_\omega(\alpha < k)$ and finally $A \in A(k) - CFL_\omega$.

□

We have just considered above the case of a conciliating set containing only *infinite* words. We are going to consider the case of a conciliating set containing only *finite* words.

**Theorem 6.16** *(I)* $A \subseteq X_A^\star$ *is in* $NA - CFL$ *if and only if* $A^\sim \subseteq (X_A \cup \{\twoheadleftarrow\})^{\leq\omega}$ *is the union of a language in* $NA - CFL$ *and of an* $\omega$*-language in* $NA - CFL_\omega$.

*(II)* $A \subseteq X_A^\star$ *is in* $A(k) - CFL$*, where* $k$ *is an integer* $\geq 2$ *or* $k = \aleph_0^-$*, if and only if* $A^\sim \subseteq (X_A \cup \{\twoheadleftarrow\})^{\leq\omega}$ *is the union of a context free language in* $A(k) - CFL$ *and of an* $\omega$*-language in* $A(k) - CFL_\omega$.

**Proof.** Let $A \subseteq X_A^\star$ be a context free language. The set $A^\sim$ is a subset of $(X_A \cup \{\twoheadleftarrow\})^{\leq\omega}$ which is constituted of finite **and** infinite words. Let h be the substitution: $X_A \rightarrow P((X_A \cup \{\twoheadleftarrow\})^\star)$ defined by $a \rightarrow a.L_3$ where $L_3$ is the context free language defined above. Then it is easy to see that the finite words of $A^\sim$ are obtained by substituting in $A$ the language $a.L_3$ for each letter $a \in X_A$ and concatenating on the left by the language $L_4$.
But $CFL$ is closed under substitution and concatenation [Ber79], then this language is a context free finitary language $D_A$.

The infinite words in $A^\sim$ constitutes the $\omega$-language
$D_A.(L_3 - \{\lambda\})^\omega$ if $\lambda \notin A$, and
$D_{(A-\{\lambda\})}.(L_3 - \{\lambda\})^\omega \cup (L_4 - \{\lambda\})^\omega$ if $\lambda \in A$,

The languages $L_4 - \{\lambda\}$ and $L_3 - \{\lambda\}$ are context free, thus the set of infinite words in $A^\sim$ is an $\omega$-CFL $D'_A$ because $\omega - KC(CFL) \subseteq CFL_\omega$ by Theorem 2.11.

With similar ideas as in the preceding case of a conciliating set $A$ containing only infinite words, we shall construct a PDA accepting $D_A$ and a MPDA accepting $D'_A$.

Let then $A \subseteq X_A^\star$ be a context free language which is accepted by the PDA $M = (K, X_A, \Gamma, \delta, q_0, Z_0, (F, \Gamma))$ by final states **and** topmost stack letters, where $F$ is the set of final states (remark that the accepting condition is just by final states and nonempty storage). And let $M_1^\sim$ be the PDA

$$M_1^\sim = (K_1^\sim, X_A \cup \{\twoheadleftarrow\}, \Gamma \cup \{E\}, \delta_1^\sim, q'_0, Z_0, (F \cup \{q_F\}, \Gamma))$$

where

$$K_1^\sim = K \cup \{q_1 \mid q \in K\} \cup \{q'_0\} \cup \{q_F\}$$

$q_F$ and $q'_0$ are new states not in $K \cup \{q_1 \mid q \in K\}$,
$E$ is a new letter not in $\Gamma$,

and the transition relation $\delta_1^\sim$ is defined by the following cases (where the transition rules (a)-(d) are used to simulate a pushdown automaton accepting $L_4$ and the PDA $M_1^\sim$ enters in a state $q_1$, for $q \in K$, when it simulates a one counter automaton accepting $L_3$):

(a) $\delta_1^\sim(q'_0, \twoheadleftarrow, Z_0) = (q'_0, Z_0)$.
(b) $(q'_0, EZ_0) \in \delta_1^\sim(q'_0, a, Z_0)$,
for each $a \in X_A$.
(c) $\delta_1^\sim(q'_0, \twoheadleftarrow, E) = (q'_0, \lambda)$.
(d) $\delta_1^\sim(q'_0, a, E) = (q'_0, EE)$,
for each $a \in X_A$.
(e) $(q, \nu) \in \delta_1^\sim(q'_0, a, Z_0)$ iff $(q, \nu) \in \delta(q_0, a, Z_0)$,
for each $a \in X_A \cup \{\lambda\}$ and $\nu \in \Gamma^\star$ and $q \in K$.
(f) $(q', \nu) \in \delta_1^\sim(q, a, \gamma)$ iff $(q', \nu) \in \delta(q, a, \gamma)$,
for each $a \in X_A \cup \{\lambda\}$ and $\gamma \in \Gamma$ and $\nu \in \Gamma^\star$ and $q, q' \in K$.
(g) $(q_1, E\gamma) \in \delta_1^\sim(q, a, \gamma)$,
for each $a \in X_A$ and $\gamma \in \Gamma$ and $q \in K$.
(h) $\delta_1^\sim(q_1, a, E) = (q_1, EE)$,
for each $a \in X_A$ and $q \in K$.
(i) $\delta_1^\sim(q_1, \twoheadleftarrow, E) = (q_1, \lambda)$.
(j) $(q', \nu) \in \delta_1^\sim(q_1, a, \gamma)$ iff $(q', \nu) \in \delta(q, a, \gamma)$,
for each $a \in X_A$ and $\gamma \in \Gamma$ and $\nu \in \Gamma^\star$ and $q, q' \in K$.
(k) $(q_1, E\gamma) \in \delta_1^\sim(q_1, a, \gamma)$,
for each $a \in X_A$ and $\gamma \in \Gamma$ and $q \in K$.
(l) $(q_F, \gamma) \in \delta_1^\sim(q_1, \lambda, \gamma)$
for each $\gamma \in \Gamma$ and $q \in F$ .

The transition rules of $M_1^\sim$ are similar to those of the BPDA $M^\sim$ of the preceding proof, except for the last transition rule [(l)] which is used to accept words with a final segment $v \in L_3$ which will be erased using the eraser $\twoheadleftarrow$. We have used a PDA accepting by final states **and** topmost stack letters because we wanted to be able to continue the computation on such a segment and then we had to avoid a final configuration with empty storage after the reading of a word in $A$.

By construction $L(M_1^\sim) = D_A$ holds.

We are going to prove now the parts of (I) and (II) involving the language $D_A$.

Assume now that $A$ is in $NA-CFL$. For the same reasons as in the proof of Theorem 6.14 the language $D_A$ is in $NA-CFL$ (we shall show below that $D'_A$ is in $NA-CFL_\omega$). Conversely assume that $D_A$ is in $NA-CFL$ then one can construct, from a non ambiguous PDA accepting $D_A$, a non ambiguous PDA accepting $A$ because $D_A \cap X_A^\star = A$; hence $A$ is in $NA - CFL$.

Assume that $A$ is in $A(k) - CFL$, where $k$ is an integer $\geq 2$ or $k = \aleph_0^-$. Then if $M$ is a PDA in $PDA(\alpha \leq k)$ accepting $A$ by final states and topmost stack letters, the PDA $M_1^\sim$ we have constructed is in $PDA(\alpha \leq k)$ hence $D_A$ is in $CFL(\alpha \leq k)$. But if $D_A$ was in $CFL(\alpha < k)$, the language $A = D_A \cap X_A^\star$ would be in $CFL(\alpha < k)$ because this class is closed under intersection with regular languages and $X_A^\star$ is regular. Thus $D_A$ is not in $CFL(\alpha < k)$, and $D_A \in A(k) - CFL$. (We shall show below that $D'_A$ is in $A(k) - CFL_\omega$).

Conversely assume that $D_A \in A(k) - CFL$. Then $A = D_A \cap X_A^\star$ is a context free language in $CFL(\alpha \leq k)$ because the class $CFL(\alpha \leq k)$ is closed under intersection with regular languages. If $A$ was in $CFL(\alpha < k)$, the above construction of $M_1^\sim$ would provide, from a PDA $M$ in $PDA(\alpha < k)$ accepting $A$ by final states and topmost stack letters, another PDA in $PDA(\alpha < k)$ accepting $D_A$. This is impossible because $D_A \in A(k) - CFL$, therefore

$$A \in A(k) - CFL = CFL(\alpha \leq k) - CFL(\alpha < k)$$

Return now to the $\omega$-language $D'_A$ which is
$D_A.(L_3 - \{\lambda\})^\omega$ if $\lambda \notin A$, and
$D_{(A-\{\lambda\})}.(L_3 - \{\lambda\})^\omega \cup (L_4 - \{\lambda\})^\omega$ if $\lambda \in A$.

Consider firstly the case where $\lambda \notin A$ and $D'_A = D_A.(L_3 - \{\lambda\})^\omega$ and assume again that $A \subseteq X_A^\star$ is a context free language which is accepted by the PDA $M = (K, X_A, \Gamma, \delta, q_0, Z_0, (F, \Gamma))$ by final states **and** topmost stack letters, where $F$ is the set of final states. We shall construct, from the PDA $M$, a MPDA $M_2^\sim$ accepting $D'_A$, with similar ideas as above for the construction of $M^\sim$ and $M_1^\sim$.

Let $M_2^\sim$ be the MPDA

$$M_2^\sim = (K_2^\sim, X_A \cup \{\twoheadleftarrow\}, \Gamma \cup \{E\}, \delta_2^\sim, q'_0, Z_0, F_2)$$

where

$$K_2^\sim = K \cup \{q_1 \mid q \in K\} \cup \{q_2 \mid q \in K\} \cup \{q'_0\}$$
$$q'_0 \text{ is a new state not in } K \cup \{q_1 \mid q \in K\} \cup \{q_2 \mid q \in K\}$$
$$E \text{ is a new letter not in } \Gamma$$
$$F_2 = \{\{q_1, q_2\} \mid q \in F\}$$

And the transition relation $\delta_2^\sim$ is defined by the following cases (where the transition rules (a)-(d) are used to simulate a pushdown automaton accepting $L_4$ and the PDA $M_2^\sim$ enters in a state $q_1$, for $q \in K$, when it simulates a one counter automaton accepting $L_3$):

(a) $\delta_2^{\sim}(q'_0, \twoheadleftarrow, Z_0) = (q'_0, Z_0)$.
(b) $(q'_0, EZ_0) \in \delta_2^{\sim}(q'_0, a, Z_0)$,
for each $a \in X_A$.
(c) $\delta_2^{\sim}(q'_0, \twoheadleftarrow, E) = (q'_0, \lambda)$.
(d) $\delta_2^{\sim}(q'_0, a, E) = (q'_0, EE)$,
for each $a \in X_A$.
(e) $(q, \nu) \in \delta_2^{\sim}(q'_0, a, Z_0)$ iff $(q, \nu) \in \delta(q_0, a, Z_0)$,
for each $a \in X_A \cup \{\lambda\}$ and $\nu \in \Gamma^\star$ and $q \in K$.
(f) $(q', \nu) \in \delta_2^{\sim}(q, a, \gamma)$ iff $(q', \nu) \in \delta(q, a, \gamma)$,
for each $a \in X_A \cup \{\lambda\}$ and $\gamma \in \Gamma$ and $\nu \in \Gamma^\star$ and $q, q' \in K$.
(g) $(q_1, E\gamma) \in \delta_2^{\sim}(q, a, \gamma)$,
for each $a \in X_A$ and $\gamma \in \Gamma$ and $q \in K$.
(h) $\delta_2^{\sim}(q_1, a, E) = (q_1, EE)$,
for each $a \in X_A$ and $q \in K$.
(i) $\delta_2^{\sim}(q_1, \twoheadleftarrow, E) = (q_1, \lambda)$.
(j) $(q', \nu) \in \delta_2^{\sim}(q_1, a, \gamma)$ iff $(q', \nu) \in \delta(q, a, \gamma)$,
for each $a \in X_A$ and $\gamma \in \Gamma$ and $\nu \in \Gamma^\star$ and $q, q' \in K$.
(k) $(q_2, \gamma) \in \delta_2^{\sim}(q_1, \lambda, \gamma)$
for each $\gamma \in \Gamma$ and $q \in K$.
(l) $(q_1, E\gamma) \in \delta_2^{\sim}(q_2, a, \gamma)$
for each $a \in X_A$ and $\gamma \in \Gamma$ and $q \in K$.

The differences between the transition rules of $M_1^{\sim}$ and those of $M_2^{\sim}$ rely on transition rules [(k)] and [(l)]. We have introduced new states $q_2$, for $q \in K$, which are used in $\lambda$-transitions and appear during the reading of a segment $v = v_i.v_{i+1}$ which is in $L_3$, with $v_i$ and $v_{i+1}$ minimal words in $L_3$ (i.e. containing none strict initial prefix in $L_3$); such a state $q_2$ appears after the reading of the word $v_i$ and before the reading of the word $v_{i+1}$.

By construction $L(M_2^{\sim}) = D'_A$

Assume now that $A$ is in $NA-CFL$ and that $M$ is a non ambiguous PDA. By construction the MPDA $M_2^{\sim}$ is non ambiguous and then $D'_A$ is non ambiguous.

If we assume that $A$ is in $A(k)-CFL$, where $k$ is an integer $\geq 2$ or $k = \aleph_0^-$, then we can construct $M_2^{\sim}$ from a PDA in $PDA(\alpha \leq k)$ accepting $A$ by final states and topmost stack letters, and the resulting MPDA $M_2^{\sim}$ is in $MPDA(\alpha \leq k)$. Thus $D'_A \in CFL_\omega(\alpha \leq k)$. If $D'_A$ was in $CFL_\omega(\alpha < k)$, then for a fixed letter $b \in X_A$, the $\omega$-language

$$A.(b. \twoheadleftarrow)^\omega = D'_A \cap X_A^\star.(b. \twoheadleftarrow)^\omega$$

would be in $CFL_\omega(\alpha < k)$ because the class $CFL_\omega(\alpha < k)$ is closed under intersection with $\omega$-regular languages and the $\omega$-language $X_A^\star.(b. \twoheadleftarrow)^\omega$ is regular.

But then one could construct, from a BPDA in $BPDA(\alpha < k)$ accepting $A.(b. \twoheadleftarrow)^\omega$, a PDA in $PDA(\alpha < k)$ accepting $A$ (with similar ideas as in the proof of Theorem 3.12). But by hypothesis $A$ is in $A(k) - CFL$ hence $A$ is not in $CFL(\alpha < k)$. Thus $D'_A$ is not in $CFL_\omega(\alpha < k)$ and

$$D'_A \in A(k) - CFL_\omega$$

Consider now the case where $\lambda \in A$ and

$$D'_A = D_{(A-\{\lambda\})}.(L_3 - \{\lambda\})^\omega \cup (L_4 - \{\lambda\})^\omega$$

It is easy to show that the $\omega$-language $(L_4 - \{\lambda\})^\omega$ is a non ambiguous context free $\omega$-language. In fact it is accepted by a deterministic BPDA.

Assume now that $A \in NA - CFL$. Then $A - \{\lambda\} = A \cap X_A^+$ is also a non ambiguous context free language because the class $NA - CFL$ is closed under intersection with regular languages. Then the preceding case shows that

$$D_{(A-\{\lambda\})}.(L_3 - \{\lambda\})^\omega \in NA - CFL_\omega$$

We can conclude that $D'_A \in NA - CFL_\omega$ because the class $NA - CFL_\omega$ is closed under disjoint finite union.

Assume now that $A \in A(k) - CFL$, where $k$ is an integer $\geq 2$ or $k = \aleph_0^-$. Then it is easy to show that $A - \{\lambda\}$ is also in $A(k) - CFL$. The preceding case shows that

$$D_{(A-\{\lambda\})}.(L_3 - \{\lambda\})^\omega \in A(k) - CFL_\omega$$

Hence

$$D_{(A-\{\lambda\})}.(L_3 - \{\lambda\})^\omega \in CFL_\omega(\alpha \leq k)$$

and we can infer that $D'_A \in CFL_\omega(\alpha \leq k)$ because the class $CFL_\omega(\alpha \leq k)$ is closed under disjoint finite union.

And if $D'_A$ was in $CFL_\omega(\alpha < k)$, then for a fixed letter $b \in X_A$, the $\omega$-language

$$(A - \{\lambda\}).(b. \twoheadleftarrow)^\omega = D'_A \cap X_A^+.(b. \twoheadleftarrow)^\omega$$

would be in $CFL_\omega(\alpha < k)$ because the class $CFL_\omega(\alpha < k)$ is closed under intersection with $\omega$-regular languages and the $\omega$-language $X_A^+.(b. \twoheadleftarrow)^\omega$ is regular.

But then one could construct, from a BPDA in $BPDA(\alpha < k)$ accepting $(A - \{\lambda\}).(b. \twoheadleftarrow)^\omega$, a PDA in $PDA(\alpha < k)$ accepting $(A - \{\lambda\})$ (with similar ideas as in the proof of Theorem 3.12). But by hypothesis $(A - \{\lambda\})$ is in

$A(k) - CFL$ hence $(A - \{\lambda\})$ is not in $CFL(\alpha < k)$. Thus $D'_A$ is not in $CFL_\omega(\alpha < k)$ and

$$D'_A \in A(k) - CFL_\omega$$

□

Duparc's operation of exponentiation is defined over conciliating sets which are sets of finite **and** infinite words. So we consider now conciliating sets which are unions of a finitary context free language and of a context free $\omega$-language. These conciliating sets are infinitary context free languages studied by D. Beauquier in [Bea84a] [Bea84b] and they also appear in the study of processes which may terminate or not.

Let then

$$L = A \cup B$$

where $A$ is a CFL and $B$ is an $\omega$-CFL over the same alphabet $X_A = X_B$.

With the preceding notations

$$L^\sim = A^\sim \cup B^\sim = D_A \cup D'_A \cup B^\sim$$

It is easy to see that this union is disjoint, because

(1) If $x \in D_A$ then $x$ is a finite word.
(2) If $x \in D'_A$ then $x$ is an infinite word and $x^\twoheadleftarrow$ is a finite word (in $A$).
(3) If $x \in B^\sim$ then $x$ is an infinite word and $x^\twoheadleftarrow$ is an infinite word (in $B$).

The degree of ambiguity of $D_A$ is the same as the degree of ambiguity of $A$ by the preceding Theorem.
We want now to determine the degree of ambiguity of $D'_A \cup B^\sim$ which is an $\omega$-CFL over the alphabet $X_A \cup \{\twoheadleftarrow\}$. Let us state the following proposition.

**Proposition 6.17** *Let $L = A \cup B$ where $A$ is a CFL and $B$ is an $\omega$-CFL over the same alphabet $X_A = X_B$. With the preceding notations the degree of ambiguity of $D'_A \cup B^\sim$ is given by the following cases:*

*(1) If $A \in NA - CFL$ and $B \in NA - CFL_\omega$ then*

$$(D'_A \cup B^\sim) \in NA - CFL_\omega$$

*(2) If $A \in NA - CFL$ and $B \in A(k) - CFL_\omega$ (with $k \geq 2$) then*

$$(D'_A \cup B^\sim) \in A(k) - CFL_\omega$$

*(3) If $A \in A(k) - CFL$ (with $k \geq 2$) and $B \in NA - CFL_\omega$ then*

$$(D'_A \cup B^\sim) \in A(k) - CFL_\omega$$

*(4) If $A \in A(k) - CFL$ (with $k \geq 2$) and $B \in A(k') - CFL_\omega$ (with $k' \geq 2$) then*

$$(D'_A \cup B^\sim) \in A(sup(k,k')) - CFL_\omega$$

**Proof.**
[(1)] Assume that $A \in NA - CFL$ and $B \in NA - CFL_\omega$. By preceding Theorems 6.14 and 6.16, $D'_A \in NA - CFL_\omega$ and $B^\sim \in NA - CFL_\omega$. Thus

$$(D'_A \cup B^\sim) \in NA - CFL_\omega$$

because the class $NA - CFL_\omega$ is closed under finite disjoint union and $(D'_A \cap B^\sim) = \emptyset$.

[(2)] Assume that $A \in NA - CFL$ and $B \in A(k) - CFL_\omega$ where $k$ is an integer $\geq 2$ or $k \in \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$. Then by Theorems 6.14 and 6.16, $D'_A \in NA - CFL_\omega$ and $B^\sim \in A(k) - CFL_\omega$. Hence $D'_A$ and $B^\sim$ are in $CFL_\omega(\alpha \leq k)$ and

$$(D'_A \cup B^\sim) \in CFL_\omega(\alpha \leq k)$$

because the class $CFL_\omega(\alpha \leq k)$ is closed under finite disjoint union.

And if $(D'_A \cup B^\sim)$ was in the class $CFL_\omega(\alpha < k)$ then the $\omega$-language

$$B = (D'_A \cup B^\sim) \cap X_A^\omega$$

would be in $CFL_\omega(\alpha < k)$ because the class $CFL_\omega(\alpha < k)$ is closed under intersection with $\omega$-regular languages. But this is not possible because $B \in A(k) - CFL_\omega$ holds by hypothesis. Thus

$$(D'_A \cup B^\sim) \in A(k) - CFL_\omega = CFL_\omega(\alpha \leq k) - CFL_\omega(\alpha < k)$$

[(3)] Assume that $A \in A(k) - CFL$, where $k$ is an integer $\geq 2$ or $k = \aleph_0^-$, and $B \in NA - CFL_\omega$. Then by Theorems 6.14 and 6.16, $D'_A \in A(k) - CFL_\omega$ and $B^\sim \in NA - CFL_\omega$. Hence

$$(D'_A \cup B^\sim) \in CFL_\omega(\alpha \leq k)$$

because the class $CFL_\omega(\alpha \leq k)$ is closed under finite disjoint union.

And if $(D'_A \cup B^\sim)$ was in the class $CFL_\omega(\alpha < k)$ then for a fixed letter $b \in X_A$, the $\omega$-language

$$A.(b. \twoheadleftarrow)^\omega = X_A^\star.(b. \twoheadleftarrow)^\omega \cap (D'_A \cup B^\sim)$$

would be in $CFL_\omega(\alpha < k)$ because the class $CFL_\omega(\alpha < k)$ is closed under intersection with $\omega$-regular languages. But then, as stated in the proof of

Theorem 6.16, one could construct a PDA in $PDA(\alpha < k)$ accepting $A$. This is not possible because $A \in A(k) - CFL$ holds by hypothesis. Thus

$$(D'_A \cup B^{\sim}) \in A(k) - CFL_\omega = CFL_\omega(\alpha \leq k) - CFL_\omega(\alpha < k)$$

[(4)] Assume that $A \in A(k) - CFL$, where $k$ is an integer $\geq 2$ or $k = \aleph_0^-$, and $B \in A(k') - CFL_\omega$ where $k$ is an integer $\geq 2$ or $k' \in \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$. Then by Theorems 6.14 and 6.16, $D'_A \in A(k) - CFL_\omega$ and $B^{\sim} \in A(k') - CFL_\omega$.

Hence $D'_A$ and $B^{\sim}$ are in $CFL_\omega(\alpha \leq sup(k, k'))$ and

$$(D'_A \cup B^{\sim}) \in CFL_\omega(\alpha \leq sup(k, k'))$$

because this class is closed under finite disjoint union. With similar arguments as in the preceding cases, one can prove that $(D'_A \cup B^{\sim})$ is neither in $CFL_\omega(\alpha < k)$ nor in $CFL_\omega(\alpha < k')$. Hence $(D'_A \cup B^{\sim})$ is not in $CFL_\omega(\alpha < sup(k, k'))$ and finally

$$(D'_A \cup B^{\sim}) \in A(sup(k, k')) - CFL_\omega$$

□

In order to obtain further results about $\omega$-languages from results about conciliating sets, we shall use the above defined correspondence $A \rightarrow A^d$ between the conciliating hierarchy and the Wadge hierarchy. We shall then prove the following proposition.

**Proposition 6.18** *a) if $A \subseteq \Sigma^\star$ is a context free language in $NA - CFL$ (respectively in $A(k) - CFL$), then $A^d$ is an $\omega$-CFL in $NA - CFL_\omega$ (respectively in $A(k) - CFL_\omega$).*

*b) if $A \subseteq \Sigma^\omega$ is an $\omega$-CFL in $NA - CFL_\omega$ (respectively in $A(k) - CFL_\omega$), then $A^d$ is an $\omega$-CFL in $NA - CFL_\omega$ (respectively in $A(k) - CFL_\omega$).*

*c) If $A$ is the union of a finitary context free language $B$ and of an $\omega$-CFL $C$ over the same alphabet $\Sigma$, then $A^d = B^d \cup C^d$ is an $\omega$-CFL such that:*

$$[B \in NA - CFL \text{ and } C \in NA - CFL_\omega] \longrightarrow A^d \in NA - CFL_\omega$$

$$[B \in NA - CFL \text{ and } C \in A(k) - CFL_\omega] \longrightarrow A^d \in A(k) - CFL_\omega$$
$$[B \in A(k) - CFL \text{ and } C \in NA - CFL_\omega] \longrightarrow A^d \in A(k) - CFL_\omega$$
$$[B \in A(k) - CFL \text{ and } C \in A(k') - CFL_\omega] \longrightarrow A^d \in A(sup(k, k')) - CFL_\omega$$

**Proof of a).**
Let $A \subseteq \Sigma^\star$ be a finitary context free language. We have proved in [Fin01a] that $A^d$ is an $\omega$-CFL. An easy construction, left to the reader, provides, from a PDA $M$ such that $L(M) = A$, a BPDA $M'$ such that $L(M') = A^d$. Moreover if $M \in PDA(\alpha \leq k)$ then $M' \in BPDA(\alpha \leq k)$, hence if $A$ is non ambiguous $A^d$ is also a non ambiguous $\omega$-CFL and if $A$ is in $A(k) - CFL$ then one can

construct such a $M'$ in $BPDA(\alpha \leq k)$. It remains to show that if $A$ is in $A(k) - CFL$ then $A^d$ is not in $CFL_\omega(\alpha < k)$. But

$$A.d^\omega = A^d \cap \Sigma^\star.d^\omega$$

Thus if $A^d$ was in $CFL_\omega(\alpha < k)$, the $\omega$-language $A.d^\omega$ would be in $CFL_\omega(\alpha < k)$ because this class is closed under intersection with $\omega$-regular languages. And by proposition 5.17 the language $A$ would then be in $CFL(\alpha < k)$ which would lead to a contradiction because we have assumed that $A$ is in $A(k) - CFL$.

**Proof of b).**
Let $A \subseteq \Sigma^\omega$ be an $\omega$-CFL. We have proved in [Fin01a] that $A^d$ is an $\omega$-CFL. An easy construction, left to the reader, provides, from a BPDA $M$ such that $L(M) = A$, a BPDA $M'$ such that $L(M') = A^d$ (intuitively the BPDA works as $M$ but is blind to letters $d$). Again if $M \in BPDA(\alpha \leq k)$ then $M' \in BPDA(\alpha \leq k)$, hence if $A$ is non ambiguous so is $A^d$. And if $A$ is inherently ambiguous of degree $k$, then one can construct, from a BPDA $M \in BPDA(\alpha \leq k)$ accepting $A$, a BPDA $M' \in BPDA(\alpha \leq k)$ accepting $A^d$. It remains to show that if $A$ is in $A(k) - CFL_\omega$ then $A^d$ is not in $CFL_\omega(\alpha < k)$. But

$$A = A^d \cap \Sigma^\omega$$

Thus if $A^d$ was in $CFL_\omega(\alpha < k)$, the $\omega$-language $A$ would be also in $CFL_\omega(\alpha < k)$ and this is not possible because by hypothesis $A$ was in $A(k) - CFL_\omega$.

**Proof of c).** Let $A$ be the union of a finitary context free language $B$ and of an $\omega$-CFL $C$ over the same alphabet $\Sigma$. Then $a)$ and $b)$ imply that $A^d = B^d \cup C^d$ is an $\omega$-CFL and that

$$\begin{aligned}
[B \in NA - CFL \text{ and } C \in NA - CFL_\omega] &\longrightarrow A^d \in NA - CFL_\omega \\
[B \in NA - CFL \text{ and } C \in A(k) - CFL_\omega] &\longrightarrow A^d \in CFL_\omega(\alpha \leq k) \\
[B \in A(k) - CFL \text{ and } C \in NA - CFL_\omega] &\longrightarrow A^d \in CFL_\omega(\alpha \leq k) \\
[B \in A(k) - CFL \text{ and } C \in A(k') - CFL_\omega] &\longrightarrow A^d \in CFL_\omega(\alpha \leq sup(k, k'))
\end{aligned}$$

because the classes $NA - CFL_\omega$ and $CFL_\omega(\alpha \leq k)$ are closed under finite disjoint union and $B^d \cap C^d = \emptyset$.

It remains to show that in the second and third cases, $A^d$ is not in $CFL_\omega(\alpha < k)$ and in the last case that $A^d$ is not in $CFL_\omega(\alpha < sup(k, k'))$. This may be seen with similar methods as above, because:

$$B.d^\omega = A^d \cap \Sigma^\star.d^\omega$$

$$C = A^d \cap \Sigma^\omega$$

□

### *6.4 Borel hierarchy and non ambiguous ω-CFL*

From preceding theorems we first deduce that non ambiguous $\omega$-CFL exhaust the finite ranks of the Borel hierarchy.

**Theorem 6.19** *For each integer $n \geq 1$, there exist $\mathbf{\Sigma^0_n}$-complete non ambiguous $\omega$-CFL $E_n$ and $\mathbf{\Pi^0_n}$-complete non ambiguous $\omega$-CFL $F_n$.*

**Proof.** For $n = 1$ consider the $\mathbf{\Sigma^0_1}$-complete $\omega$-regular language
$E_1 = \{\alpha \in \{0,1\}^\omega / \exists i \quad \alpha(i) = 1\}$
and the $\mathbf{\Pi^0_1}$-complete $\omega$-regular language
$F_1 = \{\alpha \in \{0,1\}^\omega / \forall i \quad \alpha(i) = 0\}$.
These languages are non ambiguous omega context free languages because $REG_\omega \subseteq NA - CFL_\omega$.
Now consider the $\mathbf{\Sigma^0_2}$-complete $\omega$-regular language
$E_2 = \{\alpha \in \{0,1\}^\omega / \exists^{<\omega} i \quad \alpha(i) = 1\}$
and the $\mathbf{\Pi^0_2}$-complete $\omega$-regular language
$F_2 = \{\alpha \in \{0,1\}^\omega / \exists^{\omega} i \quad \alpha(i) = 0\}$,
where $\exists^{<\omega} i$ means: " there exist only finitely many $i$ such that ..." , and
$\exists^\omega i$ means: " there exist infinitely many $i$ such that ...".
$E_2$ and $F_2$ are non ambiguous omega context free languages because they are $\omega$-regular languages.

To obtain non ambiguous omega context free languages further in the Borel hierarchy, consider now $O_1$ (respectively $C_1$ ) subsets of $\{0,1\}^{\leq\omega}$ such that $(O_1)^d$ (respectively $(C_1)^d$ ) are $\mathbf{\Sigma^0_1}$-complete ( respectively $\mathbf{\Pi^0_1}$-complete ) .
For example $O_1 = \{x \in \{0,1\}^{\leq\omega} \ / \ \exists \ i \ x(i) = 1\}$ and
$C_1 = \{\lambda\}$.

We have to apply $n \geq 1$ times the operation of exponentiation of sets.
More precisely, we define, for a set $A \subseteq X_A^{\leq\omega}$:
$A^{\sim.0} = A$
$A^{\sim.1} = A^\sim$ and
$A^{\sim.(n+1)} = (A^{\sim.n})^\sim$ .

Now apply $n$ times (for an integer $n \geq 1$) the operation $\sim$ (with different new letters $\twoheadleftarrow_1$, $\twoheadleftarrow_2$, $\twoheadleftarrow_3, \ldots$, $\twoheadleftarrow_n$) to $O_1$ and $C_1$.

By Theorem 6.12, it holds that for an integer $n \geq 1$:
$(O_1^{\sim.n})^d$ is a $\mathbf{\Sigma^0_{n+1}}$-complete subset of $\{0, 1, \twoheadleftarrow_1, \ldots, \twoheadleftarrow_n, d\}^\omega$.
$(C_1^{\sim.n})^d$ is a $\mathbf{\Pi^0_{n+1}}$-complete subset of $\{0, 1, \twoheadleftarrow_1, \ldots, \twoheadleftarrow_n, d\}^\omega$.

And it is easy to see that $O_1$ and $C_1$ are in the form $A \cup B$ where $A$ is a finitary regular language and $B$ is a regular $\omega$-language, hence $A$ is a non ambiguous context free finitary language and $B$ is a non ambiguous context free $\omega$-language. Then it follows from Theorem 6.16 and proposition 6.17 that the conciliating sets $O_1^{\sim.n}$ and $C_1^{\sim.n}$ are also in that form. Then the $\omega$-languages $(O_1^{\sim.n})^d$ and $(C_1^{\sim.n})^d$ are in $NA - CFL_\omega$ by proposition 6.18.

Hence the class $NA - CFL_\omega$ exhausts the finite ranks of the hierarchy of Borel sets: we obtain the non ambiguous $\omega$-CFL $E_n = (O_1^{\sim.(n-1)})^d$ and $F_n = (C_1^{\sim.(n-1)})^d$, for $n \geq 3$.

### *6.5 Borel hierarchy and $\omega$-CFL which are inherently ambiguous of degree $\leq \aleph_0^-$*

We shall study $\omega$-CFL which are inherently ambiguous of degree $\leq \aleph_0^-$.

Recall that we have seen in Example 5.6 that if $k$ is an integer $\geq 2$ and

$$A_k = \bigcup_{i=1}^{k} \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2i-1}^{n_i} a_{2i}^{n_1} a_{2i+1}^{n_{i+1}} \ldots a_{2k-1}^{n_k} a_{2k}^{n_k} \mid n_1, \ldots, n_k \geq 1\}$$

$$A_\infty = A_2^\star$$

Then $A_k$ is inherently ambiguous of degree $k$ over the alphabet $\Sigma_k = \{a_1, \ldots, a_{2k}\}$ and $A_\infty$ is inherently ambiguous of degree $\aleph_0^-$ over the alphabet $\Sigma_2$.

Let then $e$ be a new letter and

$$\begin{aligned} B_k &= A_k.e.\{a_1, \ldots, a_{2k}, e\}^\star \\ C_k &= A_k.e.\{a_1, \ldots, a_{2k}, e\}^\omega \\ D_k = B_k \cup C_k &= A_k.e.\{a_1, \ldots, a_{2k}, e\}^{\leq\omega} \\ B_\infty &= A_\infty.e.\{a_1, \ldots, a_4, e\}^\star \\ C_\infty &= A_\infty.e.\{a_1, \ldots, a_4, e\}^\omega \\ D_\infty = B_\infty \cup C_\infty &= A_\infty.e.\{a_1, \ldots, a_4, e\}^{\leq\omega} \end{aligned}$$

We can infer from proposition 5.17 that $C_k$ (respectively $C_\infty$) is an $\omega$-CFL which is inherently ambiguous of degree $k$ (respectively $\aleph_0^-$), and one can easily prove in a similar manner that $B_k$ (respectively $B_\infty$) is a context free language which is inherently ambiguous of degree $k$ (respectively $\aleph_0^-$), i.e.

$$B_k \in A(k) - CFL \text{ and } C_k \in A(k) - CFL_\omega$$

$$B_\infty \in A(\aleph_0^-) - CFL \text{ and } C_\infty \in A(\aleph_0^-) - CFL_\omega$$

Remark now that the $\omega$-language $(D_k)^d = (B_k \cup C_k)^d$ (respectively $(D_\infty)^d = (B_\infty \cup C_\infty)^d$) is an open subset of $\{a_1, \ldots, a_{2k}, e, d\}^\omega$ (respectively $\{a_1, \ldots, a_4, e, d\}^\omega$). But $(D_k)^d$ (respectively $(D_\infty)^d$) is not a closed set because otherwise the $\omega$-word $(a_1)^\omega$ would be in $(D_k)^d$ (respectively $(D_\infty)^d$). This follows from the characterization of closed sets given in proposition 4.1, because for each integer $n \geq 1$ the finite word $(a_1)^n$ is an initial prefix of some $\omega$-word in $(D_k)^d$ (respectively $(D_\infty)^d$). Therefore $(D_k)^d$ (respectively $(D_\infty)^d$) is a $\mathbf{\Sigma_1^0}$-complete subset of $\{a_1, \ldots, a_{2k}, e, d\}$ (respectively $\{a_1, \ldots, a_4, e, d\}$).

Hence we can generate some $\omega$-CFL which are inherently ambiguous of degree $k$ (respectively $\aleph_0^-$) and further in the Borel hierarchy, using the operation of exponentiation of sets.

By Theorem 6.12, for every integer $n \geq 1$, $(D_k^{\sim.n})^d$ (respectively $(D_\infty^{\sim.n})^d$) is a $\mathbf{\Sigma_{n+1}^0}$-complete subset of $(\Sigma_k \cup \{\twoheadleftarrow_1, \ldots, \twoheadleftarrow_n, e, d\})^\omega$ (respectively $(\Sigma_2 \cup \{\twoheadleftarrow_1, \ldots, \twoheadleftarrow_n, e, d\})^\omega$).

$D_k$ (respectively $D_\infty$) is the union of a context free language in $A(k) - CFL$ (respectively $A(\aleph_0^-) - CFL$) and of an $\omega$-CFL in $A(k) - CFL_\omega$ (respectively $A(\aleph_0^-) - CFL_\omega$). But then it follows from Theorem 6.16 and proposition 6.17 that the conciliating sets $D_k^{\sim.n}$ (respectively $D_\infty^{\sim.n}$) are also in that form. Then the $\omega$-languages $(D_k^{\sim.n})^d$ (respectively $(D_\infty^{\sim.n})^d$) are in $A(k) - CFL_\omega$ (respectively $A(\aleph_0^-) - CFL_\omega$) by proposition 6.18. So we have proved the following result:

**Proposition 6.20** *Let $k$ be an integer $\geq 2$ or $k = \aleph_0^-$. For each integer $n \geq 1$, there exist $\mathbf{\Sigma_n^0}$-complete $\omega$-CFL in $A(k) - CFL_\omega$, i.e. which are inherently ambiguous of degree $k$.*

In order to construct $\omega$-CFL which are inherently ambiguous of degree $k \leq \aleph_0^-$ and in the corresponding $\mathbf{\Pi_n^0}$ classes, we look first for some closed $\omega$-CFL.

Return to the above example 5.6, for $k$ an integer $\geq 2$:

$$A_k = \bigcup_{i=1}^{k} \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2i-1}^{n_i} a_{2i}^{n_1} a_{2i+1}^{n_{i+1}} \ldots a_{2k-1}^{n_k} a_{2k}^{n_k} \mid n_1, \ldots, n_k \geq 1\}$$

Then $A_k$ is inherently ambiguous of degree $k$ over the alphabet $\Sigma_k = \{a_1, \ldots, a_{2k}\}$.

Let then $e$ be a new letter and

$$B_k = A_k.e.\{a_1, \ldots, a_{2k}, e\}^\star$$

$$G_k = Adh(B_k)$$

Consider first the case of $G_2 = Adh(B_2)$. It is easy to see that if $\sigma$ is an $\omega$-word in $G_2$ then either the word $\sigma$ contains at least an occurrence of the letter $e$ and $\sigma \in C_2 = A_2.e.\{a_1, a_2, a_3, a_4, e\}^\omega$ or $\sigma$ is in the following form:
$a_1^\omega$, or
$a_1^n a_2^\omega$, for an integer $n \geq 1$, or
$a_1^n a_2^n a_3^\omega$, for an integer $n \geq 1$.

$C_2$ is in $A(2) - CFL_\omega$ and each of the $\omega$-languages: $\{a_1^\omega\}$, $\{a_1^n a_2^\omega \mid n \geq 1\}$, $\{a_1^n a_2^n a_3^\omega \mid n \geq 1\}$, is a deterministic $\omega$-CFL, hence is a non ambiguous $\omega$-CFL. But the union of $C_2$ and of these three $\omega$-languages is disjoint thus $G_2 = Adh(B_2)$ is in $CFL_\omega(\alpha \leq 2)$ because this class is closed under finite disjoint union. But $G_2$ is not in $NA - CFL_\omega$: otherwise $C_2 = G_2 \cap \Sigma_2^\star.e.(\Sigma_2 \cup \{e\})^\omega$ would be non ambiguous because the class $NA - CFL_\omega$ is closed under intersection with $\omega$-regular languages. Thus $G_2 = Adh(B_2)$ is in $A(2) - CFL_\omega$.

$G_2 = Adh(B_2)$ is a closed subset of $(\Sigma_2 \cup \{e\})^\omega$ but $(Adh(B_2))^d$ is not a closed subset of $(\Sigma_2 \cup \{e, d\})^\omega$. If it was closed the word $a_1^n.d^\omega$, for $n \geq 1$, would be in $(Adh(B_2))^d$ by proposition 4.1 because every initial segment $a_1^n.d^p$ of $a_1^n.d^\omega$ may be extended to the $\omega$-word $a_1^n.d^p.a_1^\omega \in (Adh(B_2))^d$ ($a_1^\omega \in Adh(B_2)$). But every word in $(Adh(B_2))^d$ contains infinitely many letters in $\Sigma_2 \cup \{e\}$ ($Adh(B_2)$ contains only infinite words) thus $a_1^n.d^\omega$ is not in $(Adh(B_2))^d$.

We are now looking for a conciliating set in the form $A \cup B \subseteq X^{\leq \omega}$ where $A \in A(2) - CFL$ and $B \in A(2) - CFL_\omega$ and $(A \cup B)^d$ is a closed subset of $(X \cup \{d\})^\omega$.

The set $B_2 \cup Adh(B_2)$ is still not convenient for our purpose: $(B_2 \cup Adh(B_2))^d$ is not closed. If it was closed (for example) the $\omega$-words $a_1^n.d^\omega$, for $n \geq 1$, would be in $(B_2 \cup Adh(B_2))^d$ because every initial segment $a_1^n.d^p$ may be extended to the $\omega$-word $a_1^n.d^p.a_1^\omega$ which is in $(B_2 \cup Adh(B_2))^d$ ($a_1^\omega$ is in $B_2 \cup Adh(B_2)$). But $a_1^n.d^\omega$ is not in $(B_2 \cup Adh(B_2))^d$ because the finite word $a_1^n$ is neither in $B_2$ nor in $Adh(B_2)$.

We see that we must add to $B_2 \cup Adh(B_2)$ the set of finite words $u$ over $\Sigma_2 \cup \{e\}$ such that every prefix of $u$ is a prefix of a word of $B_2$, i.e. the set of prefixes of words of $B_2$. By analogy with the adherence of a language we can set the following definition:

**Definition 6.21** *Let $V \subseteq X^\star$ be a finitary language over the alphabet $X$. The finite adherence of the language $V$ is*

$$Adh^{fin}(V) = \{\sigma \in X^\star \mid LF(\sigma) \subseteq LF(V)\} = LF(V)$$

**Proposition 6.22** *Let $V \subseteq X^\star$ be a finitary language over the alphabet $X$. Then the $\omega$-language $(Adh^{fin}(V) \cup Adh(V))^d$ is a closed subset of $(X \cup \{d\})^\omega$.*

**Proof.** Let $V \subseteq X^\star$ be a finitary language over the alphabet $X$. And let $W = (Adh^{fin}(V) \cup Adh(V))^d$. We shall show, using proposition 4.1 that $W$ is a closed subset of $(X \cup \{d\})^\omega$. Let then $\sigma \in (X \cup \{d\})^\omega$, such that $[\forall n \geq 1, \exists u \in (X \cup \{d\})^\omega$ such that $\sigma(1)\ldots\sigma(n).u \in W]$. We see that for each integer $n \geq 1$ the word $\sigma[n](/d)$, obtained from $\sigma[n]$ by removing every occurrence of the letter $d$, may be extended to a word in $Adh^{fin}(V)$ or to an $\omega$-word in $Adh(V)$. Hence for each integer $n \geq 1$ the word $\sigma[n](/d)$ is a prefix of a word of $V$ and then the (finite or infinite) word $\sigma(/d)$ is in $(Adh^{fin}(V) \cup Adh(V))$. Thus $\sigma$ is in $W = (Adh^{fin}(V) \cup Adh(V))^d$ and this ends the proof that $W$ is a closed subset of $(X \cup \{d\})^\omega$. □

Consider then the set $Adh^{fin}(B_2)$. It is constituted of words of $B_2$ and of words in one of the following sets of words:
$B_2^1 = \{a_1^n \mid n \geq 0\}$,
$B_2^2 = \{a_1^n a_2^p \mid n \geq 1 \text{ and } p \geq 1\}$,
$B_2^3 = \{a_1^n a_2^n a_3^p \mid n \geq 1 \text{ and } p \geq 1\}$,
$B_2^4 = \{a_1^n a_2^n a_3^p a_4^q \mid n \geq 1 \text{ and } p \geq 1 \text{ and } 1 \leq q \leq p\}$,
$B_2^5 = \{a_1^n a_2^p a_3^q \mid n \geq 1 \text{ and } p \geq 1 \text{ and } 1 \leq q \leq p\}$,
$B_2^6 = \{a_1^n a_2^p a_3^p a_4^q \mid n \geq 1 \text{ and } p \geq 1 \text{ and } 1 \leq q \leq n\}$,

It is easy to see that these languages $B_2^i$, for $1 \leq i \leq 6$, are deterministic hence non ambiguous context free languages.

The union $\bigcup_{1 \leq i \leq 4} B_2^i$ is a disjoint union thus the language $\bigcup_{1 \leq i \leq 4} B_2^i$ is in $NA - CFL$. In a similar manner the union $\bigcup_{5 \leq i \leq 6} B_2^i$ is disjoint then the language $\bigcup_{5 \leq i \leq 6} B_2^i$ is in $NA - CFL$.

By a similar result for finitary languages as Theorem 5.16 c) for $\omega$-CFL, we can deduce that $\bigcup_{1 \leq i \leq 6} B_2^i$ is in $CFL(\alpha \leq 2)$ and then that

$$Adh^{fin}(B_2) = B_2 \cup \bigcup_{1 \leq i \leq 6} B_2^i \quad \text{is in} \quad CFL(\alpha \leq 2)$$

because the union of $B_2$ and $\bigcup_{1 \leq i \leq 6} B_2^i$ is a disjoint union. Moreover we can see that

$$Adh^{fin}(B_2) \in A(2) - CFL$$

because otherwise $Adh^{fin}(B_2)$ would be non ambiguous and then

$$B_2 = Adh^{fin}(B_2) \cap \Sigma_2^\star.e.(\Sigma_2 \cup \{e\})^\star$$

would be also non ambiguous because the class $NA - CFL$ is closed under intersection with rational languages.

Now the conciliating set $W_2 = Adh^{fin}(B_2) \cup Adh(B_2)$ is the union of a language in $A(2)-CFL$ and of an $\omega$-language in $A(2)-CFL_\omega$ and

$$(Adh^{fin}(B_2) \cup Adh(B_2))^d$$

is a closed subset of $(\Sigma_2 \cup \{e, d\})^\omega$. The $\omega$-language $(W_2)^d$ is not an open subset of $(\Sigma_2 \cup \{e, d\})^\omega$ because otherwise there would exist a finitary language $V \subseteq (\Sigma_2 \cup \{e, d\})^\star$ such that $(W_2)^d = V.(\Sigma_2 \cup \{e, d\})^\omega$. But the $\omega$-word $d^\omega$ is in $(W_2)^d$ (because the empty word is in $Adh^{fin}(B_2)$) hence there would exist a finite word $v \in V$ such that $v = d^n$ for some integer $n \geq 0$. Then the $\omega$-word $d^n.e^\omega$ would be in $(W_2)^d$ but this is not possible because the $\omega$-word $e^\omega$ is not in $(Adh^{fin}(B_2) \cup Adh(B_2))$.

$(W_2)^d$ is a closed and not open subset of $(\Sigma_2 \cup \{e, d\})^\omega$ hence it is a $\mathbf{\Pi^0_1}$-complete subset of $(\Sigma_2 \cup \{e, d\})^\omega$.

Now as above we can generate some $\omega$-CFL which are inherently ambiguous of degree 2 and of greater topological complexity, using the operation of exponentiation of sets.

By Theorem 6.12, for every integer $n \geq 1$, $(W_2^{\sim .n})^d$ is a $\mathbf{\Pi^0_{n+1}}$-complete subset of $(\Sigma_2 \cup \{\twoheadleftarrow_1, \ldots, \twoheadleftarrow_n, e, d\})^\omega$.

$W_2$ is the union of a context free language in $A(2)-CFL$ and of an $\omega$-CFL in $A(2)-CFL_\omega$. But then it follows from Theorem 6.16 and proposition 6.17 that the conciliating sets $W_2^{\sim .n}$ are also in that form. Then the $\omega$-languages $(W_2^{\sim .n})^d$ are in $A(2)-CFL_\omega$ by proposition 6.18. So we have proved the following result:

**Proposition 6.23** *For each integer $n \geq 1$, there exist $\mathbf{\Pi^0_n}$-complete $\omega$-CFL in $A(2)-CFL_\omega$, i.e. which are inherently ambiguous of degree 2.*

We are going now to extend this result to every finite degree of ambiguity. We then consider

$$B_k = A_k.e.\{a_1, \ldots, a_{2k}, e\}^\star \quad \text{and}$$
$$W_k = Adh^{fin}(B_k) \cup Adh(B_k)$$

Firstly the $\omega$-language $Adh(B_k)$ is the union of the following $\omega$-languages:

$C_k = A_k.e.\{a_1, \ldots, a_{2k}, e\}^\omega$

$A = \{a_1^\omega\}$, and for $1 \leq j < k$:

$A_j.a_{2j+1}^\omega = \bigcup_{i=1}^{j} \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2i-1}^{n_i} a_{2i}^{n_1} a_{2i+1}^{n_{i+1}} a_{2i+2}^{n_{i+1}} \ldots a_{2j+1}^\omega \mid n_1, \ldots, n_j \geq 1\}$,

$C_k^j = \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2j-2}^{n_j} a_{2j-1}^{n_j} a_{2j}^\omega \mid n_1, \ldots, n_j \geq 1\}$.

The $\omega$-language $A$ is regular hence it is a non ambiguous $\omega$-CFL. And for every integer $j$, $1 \leq j < k$, the context free language $A_j$ is inherently ambiguous of degree $j$ hence the $\omega$-CFL $A_j.a_{2j+1}^{\omega}$ is also inherently ambiguous of degree $j$ by proposition 5.17. The union of these $\omega$-languages is disjoint thus

$$A \cup \bigcup_{1 \leq j < k} A_j.a_{2j+1}^{\omega} \quad \in CFL_{\omega}(\alpha < k) = CFL_{\omega}(\alpha \leq k-1)$$

because the class $CFL_{\omega}(\alpha < k)$ is closed under disjoint finite union.

On the other side it is easy to see that, for every integer $j \in [1, k-1]$, the $\omega$-language $C_k^j$ is a deterministic context free $\omega$-CFL, hence $C_k^j \in NA - CFL_{\omega}$. The languages $C_k^j$ are disjoint thus their union is a non ambiguous $\omega$-CFL by Theorem 5.16 (a).

And by Theorem 5.16 (c),

$$A \cup \bigcup_{1 \leq j < k} A_j.a_{2j+1}^{\omega} \cup \bigcup_{1 \leq j < k} C_k^j \quad \in \quad CFL_{\omega}(\alpha \leq (k-1)+1) = CFL_{\omega}(\alpha \leq k)$$

We know also that $C_k \in CFL_{\omega}(\alpha \leq k)$ but $C_k$ and the preceding union are disjoint, hence

$$Adh(B_k) = C_k \cup A \cup \bigcup_{1 \leq j < k} A_j.a_{2j+1}^{\omega} \cup \bigcup_{1 \leq j < k} C_k^j \quad \in \quad CFL_{\omega}(\alpha \leq k)$$

because the class $CFL_{\omega}(\alpha \leq k)$ is closed under finite disjoint union. And $Adh(B_k)$ is not in $CFL_{\omega}(\alpha < k)$ because otherwise the $\omega$-language

$$C_k = Adh(B_k) \cap (\Sigma_k)^{\star}.e.(\Sigma_k \cup \{e\})^{\omega}$$

would be also in $CFL_{\omega}(\alpha < k)$ because this class is closed under intersection with $\omega$-regular languages. But this is not the case as $C_k$ is inherently ambiguous of degree $k$.

Finally we have proved that $Adh(B_k) \in A(k) - CFL_{\omega}$.

Consider now the set of finite words $Adh^{fin}(B_k)$. This set is the union of the following sets of finite words:

$B_k$,

$B = \{a_1^n \mid n \geq 0\}$,

and for $1 \leq j < k$:

$$A_j.(a_{2j+1})^+ = \bigcup_{i=1}^{j} \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2i-1}^{n_i} a_{2i}^{n_1} a_{2i+1}^{n_{i+1}} a_{2i+2}^{n_{i+1}} \ldots a_{2j+1}^{p} \mid n_1, \ldots, n_j, p \geq 1\},$$

$$\bar{A}_j = A_j.\{a_{2j+1}^p.a_{2j+2}^q \mid p \geq 1 \text{ and } 1 \leq q \leq p\},$$

$$B_k^j = \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2j-2}^{n_j} a_{2j-1}^{n_j} a_{2j}^{p} \mid n_1, \ldots, n_j, p \geq 1\},$$

$$\bar{B}_k^j = \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2j-2}^{n_j} a_{2j-1}^{n_j} a_{2j}^{p} a_{2j+1}^{q} \mid n_1, \ldots, n_j, p \geq 1 \text{ and } 1 \leq q \leq p\},$$

$$\bar{B}_k = \{a_1^{n_1} a_2^{n_2} a_3^{n_2} \ldots a_{2k-2}^{n_k} a_{2k-1}^{n_k} a_{2k}^{p} \mid n_1, \ldots, n_k, \geq 1 \text{ and } 1 \leq p \leq n_1\}.$$

The language $B$ is regular thus it is a deterministic, hence non ambiguous, context free language. For each integer $j \in [1, k-1]$, the language $A_j$ is an inherently ambiguous context free language of degree $j$, thus $A_j.(a_{2j+1})^\star$ is also an inherently ambiguous context free language of degree $j < k$, (by a similar result as proposition 5.17 for finitary languages). It is also easy to see that the language $\bar{A}_j$ is in $CFL(\alpha \leq j)$, because one can find a PDA in $PDA(\alpha \leq j)$ accepting $\bar{A}_j$ from a PDA in $PDA(\alpha \leq j)$ accepting $A_j$.

The union $B \cup \bigcup_{1 \leq j < k} A_j.(a_{2j+1})^+ \cup \bigcup_{1 \leq j < k} \bar{A}_j$ is disjoint thus

$$B \cup \bigcup_{1 \leq j < k} A_j.(a_{2j+1})^+ \cup \bigcup_{1 \leq j < k} \bar{A}_j \quad \text{is in} \quad CFL(\alpha < k) = CFL(\alpha \leq k-1)$$

because the class $CFL(\alpha < k)$ is closed under finite disjoint union.

Considering the other languages above, it is easy to see that the languages $B_k^j$, $\bar{B}_k^j$, for $1 \leq j < k$, and $\bar{B}_k$ are deterministic hence non ambiguous context free languages.

These languages are disjoint, then

$$\bar{B}_k \cup \bigcup_{1 \leq j < k} B_k^j \cup \bigcup_{1 \leq j < k} \bar{B}_k^j \text{ is in } NA - CFL$$

because the class $NA - CFL$ is closed under disjoint finite union.

By a similar result for finitary languages as Theorem 5.16 c) for $\omega$-CFL, we can deduce that

$$B \cup \bigcup_{1 \leq j < k} A_j.(a_{2j+1})^+ \cup \bigcup_{1 \leq j < k} \bar{A_j} \cup \bigcup_{1 \leq j < k} B_k^j \cup \bigcup_{1 \leq j < k} \bar{B_k^j} \cup \bar{B}_k$$

is in $CFL(\alpha \leq (k-1)+1) = CFL(\alpha \leq k)$.

And $Adh^{fin}(B_k)$ is the union of $B_k$ and of this above language in $CFL(\alpha \leq k)$. Then $Adh^{fin}(B_k)$ is in $CFL(\alpha \leq k)$ because it is the disjoint union of two languages in $CFL(\alpha \leq k)$.

But $Adh^{fin}(B_k)$ is not in $CFL(\alpha < k)$ because otherwise

$$B_k = Adh^{fin}(B_k) \cap \Sigma_k^\star.e.(\Sigma_k \cup \{e\})^\star$$

would be also in $CFL(\alpha < k)$ because this class is closed under intersection with regular languages. And $B_k \in A(k)-CFL$ hence it is not in $CFL(\alpha < k)$.

Finally we have proved that

$$Adh^{fin}(B_k) \in A(k) - CFL$$

And the conciliating set $W_k = Adh^{fin}(B_k) \cup Adh(B_k)$ is the union of a language in $A(k) - CFL$ and of an $\omega$-language in $A(k) - CFL_\omega$ and

$$(Adh^{fin}(B_k) \cup Adh(B_k))^d$$

is a closed subset of $(\Sigma_k \cup \{e,d\})^\omega$. The $\omega$-language $(W_k)^d$ is not an open subset of $(\Sigma_k \cup \{e,d\})^\omega$ because otherwise there would exist a finitary language $V \subseteq (\Sigma_k \cup \{e,d\})^\star$ such that $(W_k)^d = V.(\Sigma_k \cup \{e,d\})^\omega$. But the $\omega$-word $d^\omega$ is in $(W_k)^d$ (because the empty word is in $Adh^{fin}(B_k)$) hence there would exist a finite word $v \in V$ such that $v = d^n$ for some integer $n \geq 0$. Then the $\omega$-word $d^n.e^\omega$ would be in $(W_k)^d$ but this is not possible because the $\omega$-word $e^\omega$ is not in $(Adh^{fin}(B_k) \cup Adh(B_k))$.

$(W_k)^d$ is a closed and not open subset of $(\Sigma_k \cup \{e,d\})^\omega$ hence it is a $\mathbf{\Pi^0_1}$-complete subset of $(\Sigma_k \cup \{e,d\})^\omega$.

Now we can again generate some $\omega$-CFL which are inherently ambiguous of degree $k$ and of greater topological complexity, using the operation of exponentiation of sets.

By Theorem 6.12, for every integer $n \geq 1$, $(W_k^{\sim.n})^d$ is a $\mathbf{\Pi^0_{n+1}}$-complete subset of $(\Sigma_k \cup \{\twoheadleftarrow_1, \ldots, \twoheadleftarrow_n, e, d\})^\omega$.

$W_k$ is the union of a context free language in $A(k) - CFL$ and of an $\omega$-CFL in $A(k) - CFL_\omega$. But then it follows from Theorem 6.16 and proposition 6.17

that the conciliating sets $W_k^{\sim.n}$ are also in that form. Then the $\omega$-languages $(W_k^{\sim.n})^d$ are in $A(k) - CFL_\omega$ by proposition 6.18. So we have proved the following result:

**Proposition 6.24** *Let $k$ be an integer $\geq 2$. For each integer $n \geq 1$, there exist $\mathbf{\Pi^0_n}$-complete $\omega$-CFL $F_n(k)$ in $A(k) - CFL_\omega$, i.e. which are inherently ambiguous of degree $k$.*

In order to extend this result to $\omega$-CFL which are inherently ambiguous of degree $\aleph_0^-$, we shall use a result of Crestin:

**Theorem 6.25 (Crestin [Cre72])** *Let $\Sigma = \{a, b\}$ and*

$$C = \{u.v \ / \ u, v \in \{a, b\}^+ \text{ and } u^R = u \text{ and } v^R = v\}$$

*Then the language $C$ is a context free language which is inherently ambiguous of infinite degree (i.e. $C \in A(\aleph_0^-) - CFL$).*

In fact $C = L_p^2$ where $L_p = \{v \in \{a, b\}^+ \ / \ v^R = v\}$ is the language of palindromes over the alphabet $\{a, b\}$.

Consider now the language

$$D = C.e.\{a, b, e\}^\star$$

where $e$ is a new letter. Then it is easy to see that

$$Adh(D) = \{a, b\}^\omega \cup C.e.\{a, b, e\}^\omega$$
$$Adh^{fin}(D) = \{a, b\}^\star \cup C.e.\{a, b, e\}^\star$$

because every word $u \in \{a, b\}^\star$ is a prefix of a palindrome (for example of the palindrome $u.u^R$) hence it is also a prefix of a word of $C$ and of a word of $D$.

The $\omega$-language $C$ being in $A(\aleph_0^-) - CFL$, the $\omega$-language $C.e.\{a, b, e\}^\omega$ is an $\omega$-CFL which is in $A(\aleph_0^-) - CFL_\omega$ by proposition 5.17. On the other side the $\omega$-language $\{a, b\}^\omega$ is regular hence it is a non ambiguous $\omega$-CFL. Thus $Adh(D) \in CFL_\omega(\alpha \leq \aleph_0^-)$ because the class $CFL_\omega(\alpha \leq \aleph_0^-)$ is closed under finite disjoint union. And $Adh(D)$ is not in $CFL_\omega(\alpha < \aleph_0^-)$ because otherwise the $\omega$-language

$$C.e.\{a, b, e\}^\omega = Adh(D) \cap \{a, b\}^\star.e.\{a, b, e\}^\omega$$

would be also in $CFL_\omega(\alpha < \aleph_0^-)$ because the class $CFL_\omega(\alpha < \aleph_0^-)$ is closed under intersection with $\omega$-regular languages. But this is not possible because $C.e.\{a, b, e\}^\omega \in A(\aleph_0^-) - CFL_\omega$. Finally we have proved that

$$Adh(D) \in A(\aleph_0^-) - CFL_\omega$$

Consider now the finitary language $Adh^{fin}(D) = \{a, b\}^\star \cup C.e.\{a, b, e\}^\star$. In a similar manner (details are here left to the reader) we can show that

$$Adh^{fin}(D) \in A(\aleph_0^-) - CFL$$

Consider now the conciliating set

$$W = Adh(D) \cup Adh^{fin}(D)$$

which is the union of a language in $A(\aleph_0^-) - CFL$ and of an $\omega$-language in $A(\aleph_0^-) - CFL_\omega$. Then

$$(W)^d = (Adh^{fin}(D) \cup Adh(D))^d$$

is a closed subset of $\{a, b, e, d\}^\omega$. The $\omega$-language $(W)^d$ is not an open subset of $\{a, b, e, d\}^\omega$ because otherwise there would exist a finitary language $V \subseteq \{a, b, e, d\}^\star$ such that $(W)^d = V.\{a, b, e, d\}^\omega$. But the $\omega$-word $d^\omega$ is in $(W)^d$ (because the empty word is in $Adh^{fin}(D)$) hence there would exist a finite word $v \in V$ such that $v = d^n$ for some integer $n \geq 0$. Then the $\omega$-word $d^n.e^\omega$ would be in $(W)^d$ but this is not possible because the $\omega$-word $e^\omega$ is not in $(Adh^{fin}(D) \cup Adh(D))$.

$(W)^d$ is a closed and non open subset of $\{a, b, e, d\}^\omega$ hence it is a $\mathbf{\Pi^0_1}$-complete subset of $\{a, b, e, d\}^\omega$.

Now reasoning as above, by Theorem 6.12, for every integer $n \geq 1$, $(W^{\sim.n})^d$ is a $\mathbf{\Pi^0_{n+1}}$-complete subset of $\{\twoheadleftarrow_1, \ldots, \twoheadleftarrow_n, a, b, e, d\}^\omega$.

$W$ is the union of a context free language in $A(\aleph_0^-) - CFL$ and of an $\omega$-CFL in $A(\aleph_0^-) - CFL_\omega$, thus it follows from Theorem 6.16 and proposition 6.17 that the conciliating sets $W^{\sim.n}$ are also in that form. Then the $\omega$-languages $(W^{\sim.n})^d$ are in $A(\aleph_0^-) - CFL_\omega$ by proposition 6.18. So we have proved the following result:

**Proposition 6.26** *For each integer $n \geq 1$, there exist $\mathbf{\Pi^0_n}$-complete $\omega$-CFL which are inherently ambiguous of degree $\aleph_0^-$.*

We can now summarize the preceding propositions in the following Theorem:

**Theorem 6.27** *Let $k$ be an integer $\geq 2$ or $k = \aleph_0^-$. Then for each integer $n \geq 1$, there exist $\mathbf{\Sigma^0_n}$-complete $\omega$-CFL $E_n(k)$ and $\mathbf{\Pi^0_n}$-complete $\omega$-CFL $F_n(k)$ which are in $A(k) - CFL_\omega$, i.e. which are inherently ambiguous of degree $k$.*

## 7 Concluding remarks and further work

We will pursue the study of the above section by considering in [Fin00d] Borel sets which are $\omega$-CFL in $A(\aleph_0) - CFL_\omega$ or in $A(2^{\aleph_0}) - CFL_\omega$.

The Wadge Hierarchy is a great refinement of the Borel hierarchy and it is natural to ask for the restriction of the Wadge Hierarchy to several classes of $\omega$-languages. In fact there is an effective version of the Wadge Hierarchy restricted to $\omega$-regular languages. This hierarchy is now called the Wagner hierarchy and has length $\omega^\omega$. Wagner [Wag79] gave an automata structure characterization, based on notion of chain and superchain, for an automaton to be in a given class. And one can also compute the Wadge degree of any $\omega$-regular language. Wilke and Yoo proved in [WY95] that one can compute the Wadge degree of any $\omega$-regular language in polynomial time.

The Wadge hierarchy of deterministic $\omega$-CFL has been determined. It has length $\omega^{\omega^2}$. It has been recently studied in [DFR01] [Dup99] [Fin99] [Fin01c].

We proved in [Fin01b] that the length of the Wadge hierarchy of $\omega$-CFL (including non deterministic $\omega$-CFL) is an ordinal greater than or equal to the Cantor ordinal $\varepsilon_0$.

Using the results and methods of this paper and of [Fin01b], we will show in [Fin00b] that the Wadge hierarchy restricted to the class $NA - CFL_\omega$ (respectively $A(k) - CFL_\omega$, for $k \geq 2$) has still a length $\geq \varepsilon_0$.

In the last section we have constructed closed $\omega$-CFL in $A(k) - CFL_\omega$ by taking the adherence of a finitary language in $A(k) - CFL$. One may ask for the preservation of unambiguity (respectively of inherent ambiguity) by the operation: $V \to Adh(V)$ and also by the operations: $V \to V^\omega$ and: $V \to V^\delta$, which are fundamental operations from finitary languages to $\omega$-languages. We prove in another paper that each of these operations preserves neither unambiguity nor inherent ambiguity, [Fin00c].

A further line of research is the study of context free grammars generating context free $\omega$-languages. It seems that one could extend the equivalence between the existence of a non ambiguous grammar generating a CFL L and the existence of a non ambiguous PDA accepting L to the case of $\omega$-languages. We then could study the links between the ambiguity of a finitary language $L(G)$ generated by a context free grammar $G$ and the ambiguity of the context free $\omega$-language $L_\omega(G)$ generated by the same grammar $G$.

In a recent work, Wich distinguishes ambiguous grammars of infinite degree by the growth-rate of their ambiguity with respect to the length of the words. He shows in [Wic99] that each cycle-free context-free grammar $G$ is either exponentially ambiguous or its ambiguity is bounded by a polynomial. It should be possible, as suggested by Simonnet, to show, at least for special classes of grammars, that a grammar which is exponentially ambiguous generates an $\omega$-CFL in $A(2^{\aleph_0}) - CFL_\omega$ and that a grammar (of infinite degree of ambiguity) whose ambiguity is bounded by a polynomial generates an $\omega$-CFL in $A(\aleph_0) - CFL_\omega$.

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