Topological and Arithmetical Properties of Infinitary Rational Relations

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Abstract

We prove that there exist some infinitary rational relations which are analytic but non Borel sets, giving an answer to a question of Simonnet [Sim92]. Then we show that for every countable ordinal α one cannot decide whether a given infinitary rational relation is in the Borel class $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$). Furthermore one cannot decide whether a given infinitary rational relation is a Borel set or a $\mathbf{\Sigma}^1_1$-complete set. We prove some recursive analogues to these properties. In particular one cannot decide whether an infinitary rational relation is an arithmetical set. We then deduce from the proof of these results some other ones, like: one cannot decide whether the complement of an infinitary rational relation is also an infinitary rational relation

Keywords: infinitary rational relations; topological properties; Borel and analytic sets; arithmetical properties; decision problems.

1 Introduction

Rational relations on finite words were studied in the sixties and played a fundamental role in the study of families of context free languages [Ber79]. Their extension to rational relations on infinite words was firstly investigated by Gire and Nivat [Gir81] [GN84]. Infinitary rational relations are subsets of $\Sigma_1^\omega \times \Sigma_2^\omega$, where Σ_1 and Σ_2 are finite alphabets, which are recognized by Büchi transducers or by 2-tape finite Büchi automata with asynchronous reading heads (there exists an extension to subsets of $\Sigma_1^\omega \times \Sigma_2^\omega \times \ldots \times \Sigma_n^\omega$ recognized by n-tape Büchi automata, with $\Sigma_1, \ldots, \Sigma_n$ some finite alphabets, but we shall not need to consider it). So the class RAT_ω of infinitary rational relations extends the class RAT of finitary rational relations **and** the class of ω-regular languages (firstly considered by Büchi in order to study the decidability of the monadic second order theory of one successor over the integers [Büc62], see [Tho90] [Sta97] [PP01] for many results and references).
Infinitary rational relations and rational functions over infinite words they can define have been much studied, see for example [CG99] [BC00] [Sim92] [Sta97] [Pri00] [Pri01] for many results and references.

The question of the complexity of such relations on infinite words naturally arises. A way to investigate the complexity of infinitary rational relations is to consider their topological complexity and particularly to locate them with regard to the Borel and the projective hierarchies. It is well known that every ω-language accepted by a Turing machine with a Büchi or Muller acceptance condition is an analytic set, [Sta97], thus every infinitary rational relation is an analytic set.
We show that there exist some infinitary rational relations which are $\mathbf{\Sigma_1^1}$-complete hence non Borel sets, giving an answer to a question of Simonnet [Sim92].

The question of the decidability of the topological complexity of infinitary rational relations also naturally arises.
Mac Naughton's Theorem implies that every ω-regular language is a boolean combination of $\mathbf{\Pi_2^0}$-sets, [Tho90] [Sta97] [PP01] and Landweber proved that one can decide, for a given ω-regular language R, whether R is in the Borel class $\mathbf{\Sigma_1^0}$ (respectively, $\mathbf{\Pi_1^0}$, $\mathbf{\Sigma_2^0}$, $\mathbf{\Pi_2^0}$), [Lan69].
Using an example of $\mathbf{\Sigma_1^1}$-complete infinitary rational relation, we show that the above decidability results can not be extended to rational relations over infinite words: for every countable ordinal α one cannot decide whether a given infinitary rational relation R is in the Borel class $\mathbf{\Sigma}_\alpha^\mathbf{0}$ (respectively $\mathbf{\Pi}_\alpha^\mathbf{0}$). Furthermore one cannot even decide whether a given infinitary rational relation R is a Borel set or a $\mathbf{\Sigma_1^1}$-complete set. Then we prove some recursive analogues to these properties. In particular one cannot decide whether an infinitary rational relation is an arithmetical set. The proof of the above results implies some other properties like the undecidability of the rationality of the complement of an infinitary rational relation.

We give in this paper a short presentation of the above results; the complete proofs are included in two papers which are submitted for publication, [Fin01a] [Fin01c].

The paper is organized as follows. In section 2 we introduce the notion of rational relations over finite or infinite words. In section 3 we recall definitions of Borel and analytic sets. We sketch the proof of the existence of $\mathbf{\Sigma_1^1}$-complete infinitary rational relation in section 4. The undecidability of topological properties is proved in section 5. Recursive analogues are proved in section 6 and other undecidability results in section 7.

2 Rational relations

Let us now introduce notations for words.
Let Σ be a finite alphabet whose elements are called letters. A finite word over Σ is a finite sequence of letters: $x = a_1 a_2 \ldots a_n$ where $\forall i \in [1; n]$ $a_i \in \Sigma$. We shall denote $x(i) = a_i$ the i^{th} letter of x and $x[i] = x(1) \ldots x(i)$ for $i \leq n$. The length of x is $|x| = n$. The empty word will be denoted by λ and has 0 letter. Its length is 0. The set of finite words over Σ is denoted $\Sigma^\star$. $\Sigma^+ = \Sigma^\star - \{\lambda\}$ is the set of non empty words over Σ. A (finitary) language L over Σ is a subset of $\Sigma^\star$. The usual concatenation product of u and v will be denoted by $u.v$ or just uv. For $V \subseteq \Sigma^\star$, we denote $V^\star = \{v_1 \ldots v_n / n \in \mathbb{N} \quad and \quad v_i \in V \quad \forall i \in [1; n]\}$.

The complement $\Sigma^\star - L$ of a finitary language $L \subseteq \Sigma^\star$ will be denoted L^-.

The first infinite ordinal is ω. An ω-word over Σ is an ω -sequence $a_1a_2 \ldots a_n \ldots$, where $a_i \in \Sigma, \forall i \geq 1$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2) \ldots \sigma(n) \ldots$ and $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ the finite word of length n, prefix of σ. The set of ω-words over the alphabet Σ is denoted by Σ^ω. An ω-language over an alphabet Σ is a subset of Σ^ω. For $V \subseteq \Sigma^\star$, $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega / u_i \in V, \forall i \geq 1\}$ is the ω-power of V. The concatenation product is extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that: $(u.v)(k) = u(k)$ if $k \leq |u|$, and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.
The prefix relation is denoted $\sqsubseteq$: the finite word u is a prefix of the finite word v (respectively, the infinite word v), denoted $u \sqsubseteq v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u.w$.
The complement $\Sigma^\omega - L$ of an ω-language $L \subseteq \Sigma^\omega$ will be denoted L^-.

We now assume the reader to have some familiarity with the theory of formal languages and of rational relations over finite or infinite words, see [Büc62] [Ber79] [GN84] [Tho90] [Sta97] [PP01] [Pri00] for many results and references.

A relation over finite words is a subset of $\Sigma^\star \times \Gamma^\star$ where Σ and Γ are two finite alphabets, so it is a set of couples of words.
The complement $(\Sigma^\star \times \Gamma^\star) - R$ of a relation $R \subseteq \Sigma^\star \times \Gamma^\star$ will be denoted R^-.
The usual concatenation product can be extended to couples of words: if $(u, v) \in \Sigma^\star \times \Gamma^\star$ and $(w, t) \in \Sigma^\star \times \Gamma^\star$ then $(u, v).(w, t) = (u.w, v.t)$. Then the star operation is defined for $U \subseteq \Sigma^\star \times \Gamma^\star$ by $U^\star = \cup_{n \geq 1} U^n \cup \{(\lambda, \lambda)\}$ where $U^n = \{(u_1.u_2 \ldots u_n, v_1.v_2 \ldots v_n) \mid \forall i \geq 1 \ (u_i, v_i) \in U\}$.
The set $RAT(\Sigma^\star \times \Gamma^\star)$ of rational relations is the smallest family of subsets of $\Sigma^\star \times \Gamma^\star$ which contains the emptyset, the singletons $\{(a, \lambda)\}$ and $\{(\lambda, b)\}$ for $a \in \Sigma$ and $b \in \Gamma$, and closed under finite union, concatenation product and star operation.
We call RAT the union of the sets $RAT(\Sigma^\star \times \Gamma^\star)$ where Σ and Γ are two finite alphabets.
Rational relations may also be seen as relations recognized by finite transducers or accepted by 2-tape finite automata accepting couple of words by final states [Ber79].
We shall detail these notions below in the case of **infinitary** rational relations.

Recall that ω-regular languages form the class of ω-languages accepted by finite automata with a Büchi acceptance condition and this class is the omega Kleene closure of the class of regular finitary languages, [Tho90] [Sta97] [PP01].
A relation over infinite words (or infinitary relation) is a subset of $\Sigma^\omega \times \Gamma^\omega$ where Σ and Γ are two finite alphabets, so it is a set of couples of infinite words.
The complement $(\Sigma^\omega \times \Gamma^\omega) - R$ of an infinitary relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$ will be denoted R^-.

We are going now to introduce the notion of **infinitary rational relation** which extends the notion of ω-regular language, via definition by Büchi transducers:

Definition 2.1 *A Büchi transducer is a sextuple $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$, where K is a finite set of states, Σ and Γ are finite sets called the input and the output*

alphabets, Δ is a finite subset of $K \times \Sigma^\star \times \Gamma^\star \times K$ called the set of transitions, q_0 is the initial state, and $F \subseteq K$ is the set of accepting states.
A computation $\mathcal{C}$ of the transducer $\mathcal{T}$ is an infinite sequence of transitions

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \ldots (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \ldots$$

The computation is said to be successful iff there exists a final state $q_f \in F$ and infinitely many integers $i \geq 0$ such that $q_i = q_f$.
The input word of the computation is $u = u_1.u_2.u_3 \ldots$
The output word of the computation is $v = v_1.v_2.v_3 \ldots$
Then the input and the output words may be finite or infinite.
The infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ recognized by the Büchi transducer $\mathcal{T}$ is the set of couples $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ such that u and v are the input and the output words of some successful computation $\mathcal{C}$ of $\mathcal{T}$.
The set of infinitary rational relations will be denoted RAT_ω.

Remark 2.2 *Gire and Nivat have shown in [GN84] that if $\mathcal{T}$ is a Büchi transducer recognizing the infinitary relation $R(\mathcal{T})$ then there exists another Büchi transducer $\mathcal{T}'$ such that $R(\mathcal{T}) = R(\mathcal{T}')$ and for every successful computation $\mathcal{C}'$ of $\mathcal{T}'$ the input and the output words are both infinite. The idea of this construction may be found in [Pri00].*

Remark 2.3 *Let Σ and Γ be finite alphabets and $R \subseteq \Sigma^\omega \times \Gamma^\omega$; then R is an infinitary rational relation if and only if it is accepted by a 2-tape finite automaton with asynchronous reading heads accepting words with a Büchi condition.*
One can also consider n-tape finite automata with asynchronous reading heads accepting words with a Büchi condition and this leads to a generalization: the notion of infinitary rational relation $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega \times \ldots \times \Sigma_n^\omega$ where $\Sigma_1, \ldots \Sigma_n$ are finite alphabets. But we shall restrict here our attention to rational relations $R \subseteq \Sigma^\omega \times \Gamma^\omega$ where Σ and Γ are finite alphabets.

As in the case of ω-regular languages it turned out that an infinitary relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$ is rational if and only if it is in the form $R = \cup_{1 \leq i \leq n} S_i.R_i^\omega$ where for all integers $i \in [1, n]$ S_i and R_i are rational relations over finite words and the ω-power U^ω of a finitary rational relation U is naturally defined by $U^\omega = \{u_1.u_2 \ldots u_n \ldots \mid \forall i \ u_i \in U\}$.

Remark 2.4 *An infinitary rational relation is a subset of $\Sigma^\omega \times \Gamma^\omega$ for two finite alphabets Σ and Γ. One can also consider that it is an ω-language over the finite alphabet $\Sigma \times \Gamma$. If $(u, v) \in \Sigma^\omega \times \Gamma^\omega$, one can consider this couple of infinite words as a single infinite word $(u(1), v(1)).(u(2), v(2)).(u(3), v(3)) \ldots$ over the alphabet $\Sigma \times \Gamma$.*
We shall use this fact to investigate the topological complexity of infinitary rational relations.

3 Borel and projective hierarchies

We assume the reader to be familiar with basic notions of topology which may be found in [Kur66] [Mos80] [Kec95] [LT94] [Sta97] [PP01].

Topology is an important tool for the study of subsets of a set X^ω, where X is a finite or infinite set. We study here ω-languages which are defined over a finite alphabet. Thus we shall restrict our study to subsets of spaces in the form X^ω, where X is a finite set (called here an alphabet). We shall consider X^ω as a topological space with the Cantor topology. The open sets of X^ω are the sets in the form $W.X^\omega$, where $W \subseteq X^\star$. A set $L \subseteq X^\omega$ is a closed set iff its complement $X^\omega - L$ is an open set. The class of open sets of X^ω will be denoted by $\mathbf{\Sigma^0_1}$. The class of closed sets will be denoted by $\mathbf{\Pi^0_1}$. Closed sets are characterized by the following:

Proposition 3.1 *A set $L \subseteq X^\omega$ is a closed subset of X^ω iff for every $\sigma \in X^\omega$, $[\forall n \geq 1, \exists u \in X^\omega$ such that $\sigma(1)\ldots\sigma(n).u \in L]$ implies that $\sigma \in L$.*

Define now the next classes of the Borel Hierarchy:

Definition 3.2 *The classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ of the Borel Hierarchy on the topological space X^ω are defined as follows:*
$\mathbf{\Sigma^0_1}$ is the class of open subsets of X^ω.
$\mathbf{\Pi^0_1}$ is the class of closed subsets of X^ω.
And for any integer $n \geq 1$:
$\mathbf{\Sigma^0_{n+1}}$ is the class of countable unions of $\mathbf{\Pi^0_n}$-subsets of X^ω.
$\mathbf{\Pi^0_{n+1}}$ is the class of countable intersections of $\mathbf{\Sigma^0_n}$-subsets of X^ω.
The Borel Hierarchy is also defined for transfinite levels. The classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$, for a countable ordinal $\alpha \geq 1$, are defined in the following way:
$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of X^ω in $\cup_{\gamma<\alpha}\mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of X^ω in $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma$.

Recall some basic results about these classes, [Mos80]:

Theorem 3.3

(a) $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subsetneq \mathbf{\Sigma}^0_{\alpha+1} \cap \mathbf{\Pi}^0_{\alpha+1}$, for each countable ordinal $\alpha \geq 1$.

(b) $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma = \cup_{\gamma<\alpha}\mathbf{\Pi}^0_\gamma \subsetneq \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$, for each countable limit ordinal α.

(c) A set $W \subseteq X^\omega$ is in the class $\mathbf{\Sigma}^0_\alpha$ iff its complement is in the class $\mathbf{\Pi}^0_\alpha$.

(d) $\mathbf{\Sigma}^0_\alpha - \mathbf{\Pi}^0_\alpha \neq \emptyset$ and $\mathbf{\Pi}^0_\alpha - \mathbf{\Sigma}^0_\alpha \neq \emptyset$ for every countable ordinal $\alpha \geq 1$.

We shall say that a subset of X^ω is a Borel set of rank α, for a countable ordinal α, iff it is in $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

There is a characterization of $\mathbf{\Pi^0_2}$-subsets of X^ω, involving the δ-limit W^δ of a finitary language W.

Definition 3.4 (see [Sta97]) *For $W \subseteq X^\star$, let $W^\delta = \{\sigma \in X^\omega / \exists^\omega i$ such that $\sigma[i] \in W\}$. ($\sigma \in W^\delta$ iff σ has infinitely many prefixes in W).*

Proposition 3.5 *A subset L of X^ω is a $\mathbf{\Pi^0_2}$-subset of X^ω iff there exists a set $W \subseteq X^\star$ such that $L = W^\delta$.*

Example 3.6 *Let $\Sigma = \{0, 1\}$ and $\mathcal{A} = (0^\star.1)^\omega \subseteq \Sigma^\omega$. $\mathcal{A}$ is the set of ω-words over the alphabet Σ with infinitely many occurrences of the letter 1. It is well known that $\mathcal{A}$ is a $\mathbf{\Pi^0_2}$-subset of Σ^ω because $\mathcal{A} = ((0^\star.1)^+)^\delta$*

There exists another hierarchy beyond the Borel hierarchy, which is called the projective hierarchy and which is obtained from the Borel hierarchy by successive operations of projection and complementation. More precisely, a subset A of X^ω is in the class $\mathbf{\Sigma^1_1}$ of **analytic** sets iff there exists another finite set Y and a Borel subset B of $(X \times Y)^\omega$ such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$, where (x, y) is the infinite word over the alphabet $X \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 1$. The class of Borel subsets of X^ω is strictly included in the class of analytic subsets of X^ω. Now a subset of X^ω is in the class $\mathbf{\Pi^1_1}$ of **coanalytic** sets iff its complement in X^ω is an analytic set. The next classes are defined in the same manner, (but they will not be useful in the sequel of this paper): $\mathbf{\Sigma^1_{n+1}}$-sets of X^ω are projections of $\mathbf{\Pi^1_n}$-sets and $\mathbf{\Pi^1_{n+1}}$-sets are the complements of $\mathbf{\Sigma^1_{n+1}}$-sets.

Recall the notion of completeness with regard to reduction by continuous functions. If α is a non null countable ordinal, a set $F \subseteq X^\omega$ is said to be a $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$)-complete set iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma^0_\alpha}$ (respectively $E \in \mathbf{\Pi^0_\alpha}$) iff there exists a continuous function $f : Y^\omega \to X^\omega$ such that $E = f^{-1}(F)$. In the same way a set $F \subseteq X^\omega$ is a $\mathbf{\Sigma^1_1}$ (respectively $\mathbf{\Pi^1_1}$)-complete set iff for any set $E \subseteq Y^\omega$ (Y a finite alphabet): $E \in \mathbf{\Sigma^1_1}$ (respectively $E \in \mathbf{\Pi^1_1}$) iff there exists a continuous function f such that $E = f^{-1}(F)$.
A $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$, $\mathbf{\Sigma^1_1}$)-complete set is a $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$, $\mathbf{\Sigma^1_1}$)-set which is in some sense a set of the highest topological complexity among the $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$, $\mathbf{\Sigma^1_1}$)-sets. $\mathbf{\Sigma^0_n}$ (respectively $\mathbf{\Pi^0_n}$)-complete sets, with n an integer ≥ 1, are thoroughly characterized in [Sta86].

Mac Naughton's Theorem implies that every ω-regular language is a boolean combination of $\mathbf{\Pi^0_2}$-sets, hence a $\mathbf{\Delta^0_3} = (\mathbf{\Pi^0_3} \cap \mathbf{\Sigma^0_3})$-set, [Tho90] [Sta97] [PP01]. Landweber studied first the topological properties of ω-regular languages and characterized the ω-regular languages in each of the Borel classes $\mathbf{\Sigma^0_1}$, $\mathbf{\Pi^0_1}$, $\mathbf{\Sigma^0_2}$, $\mathbf{\Pi^0_2}$, [Lan69]. In particular $\mathcal{A} = (0^\star.1)^\omega$ given in Example 3.6 is a well known example of $\mathbf{\Pi^0_2}$-complete subset of $\{0, 1\}^\omega$ and its complement $\{0, 1\}^\omega - (0^\star.1)^\omega$ is a $\mathbf{\Sigma^0_2}$-complete subset of $\{0, 1\}^\omega$.

It is well known that every ω-language accepted by a Turing machine with a Büchi or Muller acceptance condition is an analytic set, [Sta97], thus every infinitary rational relation is an analytic set.
We have shown in [Fin01b] that there exist some infinitary rational relations which are $\mathbf{\Sigma^0_3}$-complete and some others which are $\mathbf{\Pi^0_3}$-complete. We pursue below the study of the topological complexity of infinitary rational relations.

4 $\mathbf{\Sigma^1_1}$-complete infinitary rational relation

Theorem 4.1 ([Fin01a]) *There exist some $\mathbf{\Sigma^1_1}$-complete, hence non Borel, infinitary rational relations.*

In order to prove this result, we use here results about languages of infinite binary trees whose nodes are labelled in a finite alphabet Σ. A node of an infinite binary tree is represented by a finite word over the alphabet $\{l, r\}$ where

r means "right" and l means "left". Then an infinite binary tree whose nodes are labelled in Σ is identified with a function $t : \{l, r\}^\star \to \Sigma$. The set of infinite binary trees labelled in Σ will be denoted T_Σ^ω.
There is a natural topology on this set T_Σ^ω [Mos80], [LT94], [Sim92]. It is defined by the following distance. Let t and s be two distinct infinite trees in T_Σ^ω. Then the distance between t and s is $\frac{1}{2^n}$ where n is the smallest integer such that $t(x) \neq s(x)$ for some word $x \in \{l, r\}^\star$ of length n.
If $card(\Sigma) \geq 2$ then the topological space T_Σ^ω is homeomorphic to the Cantor set hence also to Γ^ω for every finite alphabet Γ having at least two letters.
The open subsets of T_Σ^ω are then in the form $T_0.T_\Sigma^\omega$ where T_0 is a set of finite labelled trees. $T_0.T_\Sigma^\omega$ is the set of infinite binary trees which extend some finite labelled binary tree $t_0 \in T_0$, t_0 is here a sort of prefix, an "initial subtree" of a tree in $t_0.T_\Sigma^\omega$. The Borel hierarchy and the projective hierarchy on T_Σ^ω are defined from open sets in the same manner as in the case of the topological space Σ^ω.

Let t be a tree. A branch B of t is a subset of the set of nodes of t which is linearly ordered by the tree partial order $\sqsubseteq$ and which is closed under prefix relation, i.e. if x and y are nodes of t such that $y \in B$ and $x \sqsubseteq y$ then $x \in B$. A branch B of a tree is said to be maximal iff there is not any other branch of t which strictly contains B. Let t be an infinite binary tree in T_Σ^ω. If B is a maximal branch of t, then this branch is infinite. Let $(u_i)_{i \geq 0}$ be the enumeration of the nodes in B which is strictly increasing for the prefix order. The infinite sequence of labels of the nodes of such a maximal branch B, i.e. $t(u_0)t(u_1)....t(u_n).....$ is called a path. It is an ω-word over the alphabet Σ.
Let then $L \subseteq \Sigma^\omega$ be an ω-language over Σ. We denote $Path(L)$ the set of infinite trees t in T_Σ^ω such that t has at least one path in L. It is well known that if $L \subseteq \Sigma^\omega$ is an ω-language over Σ which is a $\mathbf{\Pi_2^0}$-complete subset of Σ^ω (or a Borel set of higher complexity in the Borel hierarchy) then the set $Path(L)$ is a $\mathbf{\Sigma_1^1}$-complete subset of T_Σ^ω. Hence in particular $Path(L)$ is not a Borel set, [Niw85] [Sim93] [Sim92] [PP01, exercise].

In order to use this result we firstly code an infinite binary tree t labelled in Σ by an ω-word $h(t)$ over the finite alphabet $(\Sigma \cup \{A\}) \times (\Sigma \cup \{A\})$ where A is supposed to be a new letter not in Σ. This coding is chosen such that the function

$$h : T_\Sigma^\omega \to ((\Sigma \cup \{A\}) \times (\Sigma \cup \{A\}))^\omega$$

is continuous.

In a second step we construct, for every regular ω-language $B \subseteq \Sigma^\omega$, a rational relation $\mathcal{R} \subseteq (\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega$ such that $Path(B) = h^{-1}(\mathcal{R})$.

Every word of $\mathcal{R}$ may be seen as a couple $y = (y_1, y_2)$ of ω-words over the alphabet $\Sigma \cup \{A\}$ and then $y = (y_1, y_2)$ is in $\mathcal{R}$ if and only if it is in the form

$$y_1 = x(1).u_1.A.v_2.x(3).u_3.A.v_4.x(5).u_5.A. \ldots A.v_{2n}.x(2n+1).u_{2n+1}.A \ldots$$
$$y_2 = \quad v_1.x(2).u_2.A.v_3.x(4).u_4.A. \ldots A.v_{2n+1}.x(2n+2).u_{2n+2}.A \ldots$$

where for all integers $i \geq 1$, $x(i) \in \Sigma$ and $u_i, v_i \in \Sigma^\star$ and

$$|v_i| = 2|u_i| \quad \text{ or } \quad |v_i| = 2|u_i| + 1$$

and the ω-word $x = x(1)x(2) \ldots x(n) \ldots$ is in B.

Now if B is $\mathbf{\Pi^0_2}$-complete then $Path(B)$ is $\mathbf{\Sigma^1_1}$-complete. Thus $Path(B) = h^{-1}(\mathcal{R})$ implies that the infinitary rational relation $\mathcal{R}$ is $\mathbf{\Sigma^1_1}$-complete.

5 Undecidability of topological properties

We shall say that an infinitary rational relation is effectively given if a Büchi transducer recognizing it or a rational expression defining it is given. we firstly prove the following result.

Proposition 5.1 *Let X and Y be finite alphabets containing at least two letters, then there exists a family $\mathcal{F}$ of infinitary rational relations which are subsets of $X^\omega \times Y^\omega$, such that, for $R \in \mathcal{F}$, either $R = X^\omega \times Y^\omega$ or R is a $\mathbf{\Sigma^1_1}$-complete subset of $X^\omega \times Y^\omega$, but one cannot decide which case holds.*

Proof. Recall that if Σ is an alphabet having at least two letters then it is undecidable to determine, for a given rational relation (over finite words) $\mathcal{S} \subseteq \Sigma^\star \times \Sigma^\star$, whether $\mathcal{S} = \Sigma^\star \times \Sigma^\star$, see [Ber79].

We define, from a $\mathbf{\Sigma^1_1}$-complete infinitary rational relation $\mathcal{R} \subseteq ((\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega)$ and a given rational relation $\mathcal{S} \subseteq \Sigma^\star \times \Sigma^\star$, the following relation:

$$\mathcal{R}^{\mathcal{S}} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$$

where

$$\mathcal{S}_1 = \mathcal{S}.(A, A).((\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega)$$
$$\mathcal{S}_2 = (\Sigma^\star \times \Sigma^\star).(A, A).\mathcal{R}$$
$$\mathcal{S}_3 = [(\Sigma \cup \{A\})^\omega \times \Sigma^\omega] \cup [\Sigma^\omega \times (\Sigma \cup \{A\})^\omega]$$

$\mathcal{R}^{\mathcal{S}}$ is the union of three infinitary rational relations thus $\mathcal{R}^{\mathcal{S}} \in RAT_\omega$ because the class RAT_ω is closed under finite union.

Now two cases may happen.

(1) **First case.** $\mathcal{S} = \Sigma^\star \times \Sigma^\star$ therefore $\mathcal{R}^{\mathcal{S}} = ((\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega)$.

(2) **Second case.** $\mathcal{S} \neq \Sigma^\star \times \Sigma^\star$ therefore there is some $(u, v) \in \Sigma^\star \times \Sigma^\star$ such that $(u, v) \notin \mathcal{S}$. But then, for $(w, t) \in (\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega$, $(u, v).(A, A).(w, t) \in \mathcal{R}^{\mathcal{S}}$ if and only if $(w, t) \in \mathcal{R}$.
Consider now the function

$$\varphi_{(u,v)} : ((\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega) \to ((\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega)$$

defined by

$$\varphi_{(u,v)}((w, t)) = (u, v).(A, A).(w, t)$$

It is easy to see that $\varphi_{(u,v)}$ is a continuous function and that, for all $(w,t) \in ((\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega)$,

$$\varphi_{(u,v)}((w,t)) \in \mathcal{R}^{\mathcal{S}} \text{ if and only if } (w,t) \in \mathcal{R}$$

This means that $\mathcal{R} = \varphi^{-1}_{(u,v)}(\mathcal{R}^{\mathcal{S}})$. But we know that $\mathcal{R}$ is $\mathbf{\Sigma^1_1}$-complete and this implies that $\mathcal{R}^{\mathcal{S}}$ is also $\mathbf{\Sigma^1_1}$-complete.
Remark that we already knew that $\mathcal{R}^{\mathcal{S}}$ was a $\mathbf{\Sigma^1_1}$-set because it is an infinitary rational relation as the union of three infinitary rational relations.

But one cannot decide which case holds. So we have got the family $\mathcal{F}$ in the case of two alphabets X and Y having both three elements. It is now easy to prove the result for two alphabets X and Y having at least two elements. □

We can now state the following results.

Theorem 5.2 *Let Σ and Γ be finite alphabets having at least two letters and α be a countable ordinal ≥ 1. Then for an effectively given infinitary rational relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$ it is undecidable to determine whether:*

(a) R is in the Borel class $\mathbf{\Sigma^0_\alpha}$.

(b) R is in the Borel class $\mathbf{\Pi^0_\alpha}$.

(c) R is a Borel set.

(d) R is a $\mathbf{\Sigma^1_1}$-complete set.

Proof. Let Σ and Γ be finite alphabets having at least two letters and $\mathcal{F}$ be the family of infinitary rational relations included in $\Sigma^\omega \times \Gamma^\omega$ obtained in the proof of proposition 5.1. Then two cases may happen for $F \in \mathcal{F}$: either $F = \Sigma^\omega \times \Gamma^\omega$ or F is a $\mathbf{\Sigma^1_1}$-complete subset of $\Sigma^\omega \times \Gamma^\omega$.

In the first case F is an open and closed subset of $\Sigma^\omega \times \Gamma^\omega$ thus, for every countable ordinal $\alpha \geq 1$, it is in the class $\mathbf{\Sigma^0_\alpha}$ and also in the class $\mathbf{\Pi^0_\alpha}$.
In the second case F is not a Borel set because a $\mathbf{\Sigma^1_1}$-complete set is not Borel.
But one cannot decide which case holds and this ends the proof of Theorem 5.2.
□

6 Undecidability of arithmetical properties

We recall first the definition of the Arithmetical hierarchy of ω-languages, [Sta97]. Let X be a finite alphabet. An ω-language $L \subseteq X^\omega$ is in the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times X^\star$ such that

$$L = \{\sigma \in X^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots, a_{n-1}, \sigma[a_n + 1]) \in R_L\}$$

where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq X^\omega$ belongs to the class Π_n if and only if its complement $X^\omega - L$ belongs to the class Σ_n.
The inclusion relations that hold between the classes Σ_n and Π_n are the same as for the corresponding classes of the Borel hierarchy.

Proposition 6.1 (see [Sta97]) *a)* $\Sigma_n \cup \Pi_n \subsetneq \Sigma_{n+1} \cap \Pi_{n+1}$*, for each integer* $n \geq 1$.

b) $\Sigma_n - \Pi_n \neq \emptyset$ *and* $\Pi_n - \Sigma_n \neq \emptyset$ *hold for each integer* $n \geq 1$.

The classes Σ_n and Π_n are strictly included in the respective classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Sigma^0_n}$ of the Borel hierarchy: for each integer $n \geq 1$, $\Sigma_n \subsetneq \mathbf{\Sigma^0_n}$ and $\Pi_n \subsetneq \mathbf{\Pi^0_n}$, [Sta97].

As in the case of the Borel hierarchy, projections of arithmetical sets (of the second Π-class) lead beyond the Arithmetical hierarchy, to the Analytical hierarchy of ω-languages. The first class of this hierarchy is the class Σ^1_1 (light face). An ω-language $L \subseteq X^\omega$ belongs to the class Σ^1_1 if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N}) \times \{0,1\}^\star \times X^\star$ such that:

$$L = \{\sigma \in X^\omega \mid \exists \tau(\tau \in \{0,1\}^\omega \wedge \forall n \exists m((n, \tau[m], \sigma[m]) \in R_L))\}$$

Then an ω-language $L \subseteq X^\omega$ is in the class Σ^1_1 iff it is the projection of an ω-language over the alphabet $X \times \{0,1\}$ which is in the class Π_2 of the arithmetical hierarchy.

It turned out that an ω-language $L \subseteq X^\omega$ is in the class Σ^1_1 iff it is accepted by a non deterministic Turing machine (reading ω-words) with a Muller or Büchi acceptance condition [Sta97]. This class is denoted $NT(inf, =)$ (where $(inf, =)$ indicates the Muller condition) in [Sta97] and also called the class of recursive ω-languages REK_ω [1]. In particular the class RAT_ω is strictly included into the class REK_ω of recursive ω-languages. There exist some infinitary rational relations which are in $\Sigma^1_1 - \cup_{n\geq 1}\Sigma_n$: for example each $\mathbf{\Sigma^1_1}$-complete $R \in RAT_\omega$ is not in $\cup_{n\geq 1}\Sigma_n$ because it is not a Borel set.
The following undecidability results directly follow from Proposition 5.1.

Theorem 6.2 *Let* Σ *and* Γ *be finite alphabets having at least two letters and* j *be an integer* ≥ 1*. Then for an effectively given infinitary rational relation* $R \subseteq \Sigma^\omega \times \Gamma^\omega$ *it is undecidable to determine whether:*

(a) R *is in the class* Σ_j.

(b) R *is in the class* Π_j.

(c) R *is an arithmetical set in* $\cup_{n\geq 1}\Sigma_n$.

Proof. Let Σ and Γ be finite alphabets having at least two letters and $\mathcal{F}$ be the family of infinitary rational relations included in $\Sigma^\omega \times \Gamma^\omega$ obtained in the proof of proposition 5.1. Then two cases may happen for $F \in \mathcal{F}$: either $F = \Sigma^\omega \times \Gamma^\omega$ or F is a $\mathbf{\Sigma^1_1}$-complete subset of $\Sigma^\omega \times \Gamma^\omega$.

In the first case F is in both classes Σ_1 and Π_1 thus it is in all classes Σ_n and Π_n. In the second case F is not a Borel set hence it is not in $\cup_{n\geq 1}\Sigma_n$ because each arithmetical class Σ_n (respectively Π_n) is included in the Borel class $\mathbf{\Sigma^0_n}$ (respectively $\mathbf{\Pi^0_n}$). But one cannot decide which case holds. □

[1] In another presentation, as in [Rog67], the recursive ω-languages are those which are in the intersection $\Sigma_1 \cap \Pi_1$, see also [LT94].

7 Other undecidability results

Proposition 5.1 establishes a strong undecidability result which implies other ones. We can firstly infer from proposition 5.1 an already known result:

Theorem 7.1 *Let Σ and Γ be finite alphabets having at least two letters. Then it is undecidable to determine, for an effectively given infinitary rational relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$, whether R is accepted by a deterministic Büchi (respectively, Muller) 2-tape finite automaton.*

Proof. Let Σ and Γ be finite alphabets having at least two letters and $\mathcal{F}$ be the family of infinitary rational relations included in $\Sigma^\omega \times \Gamma^\omega$ obtained in the proof of proposition 5.1. Then two cases may happen for $F \in \mathcal{F}$: either $F = \Sigma^\omega \times \Gamma^\omega$ or F is a $\mathbf{\Sigma}^1_1$-complete subset of $\Sigma^\omega \times \Gamma^\omega$.

In the first case F is obviously accepted by a deterministic Büchi (respectively, Muller) 2-tape finite automaton.
In the second case F is $\mathbf{\Sigma}^1_1$-complete thus it cannot be accepted by any deterministic finite machine with a Büchi (respectively Muller) acceptance condition because otherwise it would be a boolean combination of Π_2-sets hence a $\mathbf{\Delta}^0_3$-set. In fact ω-languages accepted by deterministic Büchi Turing machines form the class Π_2 and ω-languages accepted by deterministic Muller Turing machines form the class of boolean combinations of Π_2-sets, see [Sta97]. □

Thus we have also proved the following result showing that the "intrinsic determinism" of infinitary rational relations is undecidable:

Theorem 7.2 *Let Σ and Γ be finite alphabets having at least two letters. Then it is undecidable to determine, for an effectively given infinitary rational relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$, whether R is accepted by a deterministic Büchi (respectively, Muller) Turing machine.*

We consider now the problem of the rationality of the complement of an infinitary rational relation.

Theorem 7.3 *Let Σ and Γ be finite alphabets having at least two letters. Then it is undecidable to determine, for an effectively given infinitary rational relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$, whether its complement $(\Sigma^\omega \times \Gamma^\omega) - R$ is an infinitary rational relation.*

Proof. Let Σ and Γ be finite alphabets having at least two letters and $\mathcal{F}$ be the family of infinitary rational relations included in $\Sigma^\omega \times \Gamma^\omega$ obtained in the proof of proposition 5.1. Then two cases may happen for $F \in \mathcal{F}$: either $F = \Sigma^\omega \times \Gamma^\omega$ or F is a $\mathbf{\Sigma}^1_1$-complete subset of $\Sigma^\omega \times \Gamma^\omega$.

In the first case $F^- = \emptyset$ is in RAT_ω. In the second case F is $\mathbf{\Sigma}^1_1$-complete thus its complement is a $\mathbf{\Pi}^1_1$-complete subset of $\Sigma^\omega \times \Gamma^\omega$. It is well known that a $\mathbf{\Pi}^1_1$-complete set is not a $\mathbf{\Sigma}^1_1$-set thus it cannot be in RAT_ω. But one cannot decide which case holds. □

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