Topology and ambiguity in ω-context free languages

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Abstract

We study the links between the topological complexity of an ω-context free language and its degree of ambiguity. In particular, using known facts from classical descriptive set theory, we prove that non Borel ω-context free languages which are recognized by Büchi pushdown automata have a maximum degree of ambiguity. This result implies that degrees of ambiguity are really not preserved by the operation $W \to W^\omega$, defined over finitary context free languages. We prove also that taking the adherence or the δ-limit of a finitary language preserves neither ambiguity nor inherent ambiguity. On the other side we show that methods used in the study of ω-context free languages can also be applied to study the notion of ambiguity in infinitary rational relations accepted by Büchi 2-tape automata and we get first results in that direction.

Keywords: context free languages; infinite words; infinitary rational relations; ambiguity; degrees of ambiguity; topological properties; borel hierarchy; analytic sets.

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1 Introduction

ω-context free languages (ω-CFL) form the class CFL_ω of ω-languages accepted by pushdown automata with a Büchi or Muller acceptance condition. They were firstly studied by Cohen and Gold, Linna, Boasson, Nivat, [CG77] [Lin76] [BN80] [Niv77], see Staiger's paper for a survey of these works [Sta97a]. A way to study the richness of the class CFL_ω is to consider the topological complexity of ω-context free languages when the set Σ^ω of infinite words over the alphabet Σ is equipped with the usual Cantor topology. It is well known that all ω-CFL as well as all ω-languages accepted by Turing machines with a Büchi or a Muller acceptance condition are analytic sets. ω-CFL accepted by deterministic Büchi pushdown automata are $\mathbf{\Pi}^0_2$-sets, while ω-CFL accepted by deterministic Muller pushdown automata are boolean combinations of $\mathbf{\Pi}^0_2$-sets. It was recently proved that the class CFL_ω exhausts the finite ranks of the Borel hierarchy, [Fin01], that there exists some ω-CFL which are Borel sets of infinite rank, [Fin03b], or even analytic but non Borel sets, [Fin03a].

Using known facts from Descriptive Set Theory, we prove here that non Borel ω-CFL have a maximum degree of ambiguity: if $L(\mathcal{A})$ is a non Borel ω-CFL which is accepted by a Büchi pushdown automaton (BPDA) $\mathcal{A}$ then there exist $2^{\aleph_0}$ ω-words α such that $\mathcal{A}$ has $2^{\aleph_0}$ accepting runs reading α, where $2^{\aleph_0}$ is the cardinal of the continuum.

The above result of the second author led the first author to the investigation of the notion of ambiguity and of degrees of ambiguity in ω-context free languages, [Fin03c]. There exist some non ambiguous ω-CFL of every finite Borel rank, but all known examples of ω-CFL which are Borel sets of infinite rank are accepted by ambiguous BPDA. Thus one can make the hypothesis that there are some links between the topological complexity and the degree of ambiguity for ω-CFL and such connections were firstly studied in [Fin03c].
The operations $W \to Adh(W)$ and $W \to W^\delta$, where $Adh(W)$ is the adherence of the finitary language $W \subseteq \Sigma^\star$ and W^δ is the δ-limit of W, appear in the characterization of $\mathbf{\Pi}^0_1$ (i.e. closed)-subsets and $\mathbf{\Pi}^0_2$-subsets of Σ^ω, for an alphabet Σ, [Sta97a]. Moreover it turned out that the first one is useful in the study of topological properties of ω-context free languages of a given degree of ambiguity [Fin03c]. We show that each of these operations preserves neither unambiguity nor inherent ambiguity from finitary to ω-context free languages. We deduce also from the above results that neither unambiguity nor inherent ambiguity is preserved by the operation $W \to W^\omega$. This important operation is defined over finitary languages and is involved in the characterization of the class of ω-regular languages (respectively, of ω-context free languages) as the ω-Kleene closure of the class of regular (respectively, context free) languages [Tho90] [PP02] [Sta97a] [Sta97b].

On the other side we prove that the same theorems of classical descriptive set theory can also be applied in the case of infinitary rational relations accepted by 2-tape Büchi automata. The topological complexity of infinitary rational relations has been studied by the first author who showed in [Fin03d] that there exist some infinitary rational relations which are not Borel. Moreover some undecidability properties have been established in [Fin03e]. We then prove some first results about ambiguity in infinitary rational relations.

The paper is organized as follows. In section 2, we recall definitions and results about ω-CFL and ambiguity. In section 3, Borel and analytic sets are defined. In section 4, we study links between topology and ambiguity in ω-CFL. In section 5, we show some results about infinitary rational relations.

2 ω-context free languages

We assume the reader to be familiar with the theory of formal languages and of ω-regular languages, [Ber79] [Tho90] [Sta97a] [PP02]. We shall use usual notations of formal language theory. When Σ is a finite alphabet, a non-empty finite word over Σ is any sequence $x = a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 1. The length of x is k, denoted by $|x|$. We write $x(i) = a_i$ and $x[i] = x(1) \ldots x(i)$ for $i \leq k$. We write also $x[0] = \lambda$, where λ is the empty word, which has no letter; its length is $|\lambda| = 0$. $\Sigma^\star$ is the set of finite words over Σ, and Σ^+ is the set of finite non-empty words over Σ. The mirror image of a finite word u will be denoted by u^R.

The first infinite ordinal is ω. An ω-word over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where $\forall i \geq 1 \quad a_i \in \Sigma$. The set of ω-words over the alphabet Σ is denoted by Σ^ω. An ω-language over an alphabet Σ is a subset of Σ^ω. For $V \subseteq \Sigma^\star$, the ω-power of V is the ω-language $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega \mid \forall i \geq 1 \ u_i \in V - \{\lambda\}\}$. $LF(v)$ is the set of finite prefixes (or left factors) of the word v, and $LF(V) = \cup_{v \in V} LF(v)$ for every language V of finite or infinite words.

We introduce now ω-context free languages via Büchi pushdown automata.

Definition 2.1 *A Büchi pushdown automaton is a 7-tuple* $\mathcal{A} = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, *where K is a finite set of states, Σ is a finite input alphabet, Γ is a finite pushdown alphabet, $q_0 \in K$ is the initial state, $Z_0 \in \Gamma$ is the start symbol, $F \subseteq K$ is the set of final states, and δ is a mapping from $K \times (\Sigma \cup \{\lambda\}) \times \Gamma$ to finite subsets of $K \times \Gamma^\star$.*
If $\gamma \in \Gamma^+$ describes the pushdown store content, the leftmost symbol will be assumed to be on "top" of the store. A configuration of the BPDA $\mathcal{A}$ is a pair (q, γ) where $q \in K$ and $\gamma \in \Gamma^\star$.
For $a \in \Sigma \cup \{\lambda\}$, $\gamma, \beta \in \Gamma^\star$ and $Z \in \Gamma$, if (p, β) is in $\delta(q, a, Z)$, then we write $a : (q, Z\gamma) \mapsto_{\mathcal{A}} (p, \beta\gamma)$.
Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an ω-word over Σ. A run of $\mathcal{A}$ on σ is an infinite sequence $r = (q_i, \gamma_i, \varepsilon_i)_{i \geq 1}$ where $(q_i, \gamma_i)_{i \geq 1}$ is an infinite sequence of configurations of $\mathcal{A}$ and, for all $i \geq 1$, $\varepsilon_i \in \{0, 1\}$ and:

1. $(q_1, \gamma_1) = (q_0, Z_0)$

2. *for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ satisfying*
   $b_i : (q_i, \gamma_i) \mapsto_{\mathcal{A}} (q_{i+1}, \gamma_{i+1})$
   and ($\varepsilon_i = 0$ iff $b_i = \lambda$)
   and such that $a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$

$In(r)$ is the set of all states entered infinitely often during run r.
The ω-language accepted by $\mathcal{A}$ is

$$L(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \text{ there exists a run } r \text{ of } \mathcal{A} \text{ on } \sigma \text{ such that } In(r) \cap F \neq \emptyset\}$$

The class CFL_ω of ω-context free languages is the class of ω-languages accepted by Büchi pushdown automata. It is also the ω-Kleene closure of the class CFL of context free finitary languages, where for any family $\mathcal{L}$ of finitary languages, the ω-Kleene closure of $\mathcal{L}$, is: $\omega - KC(\mathcal{L}) = \{\cup_{i=1}^{n} U_i.V_i^\omega \mid \forall i \in [1, n] \quad U_i, V_i \in \mathcal{L}\}$.

If we omit the pushdown stack and the λ-transitions, we get the classical notion of Büchi automaton. Recall that the class REG_ω of ω-regular languages is the class of ω-languages accepted by finite automata with a Büchi acceptance condition. It is also the ω-Kleene closure of the class REG of regular finitary languages, [Tho90] [Sta97a] [PP02].

Notice that we introduced in the above definition the numbers $\varepsilon_i \in \{0, 1\}$ in order to distinguish runs of a BPDA which go through the same infinite sequence of configurations but for which λ-transitions do not occur at the same steps of the computations. We can now briefly recall some definitions of [Fin03c] about ambiguity.

We shall denote $\aleph_0$ the cardinal of ω, and $2^{\aleph_0}$ the cardinal of the continuum. It is also the cardinal of the set of real numbers and of the set Σ^ω for every finite alphabet Σ having at least two letters.

Definition 2.2 *Let $\mathcal{A}$ be a BPDA accepting infinite words over the alphabet Σ. For $x \in \Sigma^\omega$ let $\alpha_{\mathcal{A}}(x)$ be the cardinal of the set of accepting runs of $\mathcal{A}$ on x.*

Lemma 2.3 ([Fin03c]) *Let $\mathcal{A}$ be a BPDA accepting infinite words over the alphabet Σ. Then for all $x \in \Sigma^\omega$ it holds that* $\alpha_{\mathcal{A}}(x) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

Definition 2.4 *Let $\mathcal{A}$ be a BPDA accepting infinite words over the alphabet Σ.*

(a) *If $\sup\{\alpha_{\mathcal{A}}(x) \mid x \in \Sigma^\omega\} \in \mathbb{N} \cup \{2^{\aleph_0}\}$, then $\alpha_{\mathcal{A}} = \sup\{\alpha_{\mathcal{A}}(x) \mid x \in \Sigma^\omega\}$.*

(b) *If $\sup\{\alpha_{\mathcal{A}}(x) \mid x \in \Sigma^\omega\} = \aleph_0$ and there is no word $x \in \Sigma^\omega$ such that $\alpha_{\mathcal{A}}(x) = \aleph_0$, then $\alpha_{\mathcal{A}} = \aleph_0^-$.*
   ($\aleph_0^-$ does not represent a cardinal but is a new symbol that we introduce to conveniently speak of this situation).

(c) *If $\sup\{\alpha_{\mathcal{A}}(x) \mid x \in \Sigma^\omega\} = \aleph_0$ and there exists (at least) one word $x \in \Sigma^\omega$ such that $\alpha_{\mathcal{A}}(x) = \aleph_0$, then $\alpha_{\mathcal{A}} = \aleph_0$*

Notice that for a BPDA $\mathcal{A}$, $\alpha_{\mathcal{A}} = 0$ iff $\mathcal{A}$ does not accept any ω-word.
$\mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$ is linearly ordered by the relation $<$ defined by $\forall k \in \mathbb{N}$, $k < k+1 < \aleph_0^- < \aleph_0 < 2^{\aleph_0}$. Now we can define a hierarchy of ω-CFL:

Definition 2.5 *For $k \in \mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$ let*
$CFL_\omega(\alpha \leq k) = \{L(\mathcal{A}) \mid \mathcal{A}$ *is a BPDA with* $\alpha_{\mathcal{A}} \leq k\}$
$CFL_\omega(\alpha < k) = \{L(\mathcal{A}) \mid \mathcal{A}$ *is a BPDA with* $\alpha_{\mathcal{A}} < k\}$
$NA - CFL_\omega = CFL_\omega(\alpha \leq 1)$ *is the class of non ambiguous ω-context free languages.*
For every integer k such that $k \geq 2$, or $k \in \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$,
$A(k) - CFL_\omega = CFL_\omega(\alpha \leq k) - CFL_\omega(\alpha < k)$
If $L \in A(k) - CFL_\omega$ with $k \in \mathbb{N}, k \geq 2$, or $k \in \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$, then L is said to be inherently ambiguous of degree k.

Recall that one can define in a similar way the degree of ambiguity of a finitary context free language. If M is a pushdown automaton accepting finite words by final states (or by final states and topmost stack letter) then $\alpha_M \in \mathbb{N}$ or $\alpha_M = \aleph_0^-$ or $\alpha_M = \aleph_0$. However every context free language is accepted by a pushdown automaton M with $\alpha_M \leq \aleph_0^-$, [ABB96]. We shall denote, with similar notations as above, the class of non ambiguous context free languages by $NA - CFL$ and the class of inherently ambiguous context free languages of degree $k \geq 2$ by $A(k) - CFL$. Then $A(\aleph_0^-) - CFL$ is usually called the class of context free languages which are inherently ambiguous of infinite degree, [Her97].

Now we can state some links between cases of finite and infinite words.

Proposition 2.6 ([Fin03c]) *Let $V \subseteq \Sigma^\star$ be a finitary context free language and d be a new letter not in Σ, then the following equivalence holds for all $k \in \mathbb{N} \cup \{\aleph_0^-\}$:*

$$V.d^\omega \text{ is in } CFL_\omega(\alpha \leq k) \text{ iff } V \text{ is in } CFL(\alpha \leq k)$$

3 Borel and analytic sets

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80] [LT94] [Kec95] [Sta97a] [PP02].
For a finite alphabet X we shall consider X^ω as a topological space with the Cantor topology. The open sets of X^ω are the sets in the form $W.X^\omega$, where $W \subseteq X^\star$. A set $L \subseteq X^\omega$ is a closed set iff its complement $X^\omega - L$ is an open set.
Define now the hierarchy of Borel sets of finite ranks:

Definition 3.1 *The classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ of the Borel hierarchy on the topological space X^ω are defined as follows:*
$\mathbf{\Sigma^0_1}$ *is the class of open sets of X^ω.*
$\mathbf{\Pi^0_1}$ *is the class of closed sets of X^ω.*
And for any integer $n \geq 1$:
$\mathbf{\Sigma^0_{n+1}}$ *is the class of countable unions of $\mathbf{\Pi^0_n}$-subsets of X^ω.*
$\mathbf{\Pi^0_{n+1}}$ *is the class of countable intersections of $\mathbf{\Sigma^0_n}$-subsets of X^ω.*

The Borel hierarchy is also defined for transfinite levels, but we shall not need them in the present study. The class of Borel subsets of X^ω is the closure of the class of open subsets of X^ω under complementation and countable unions (hence also under countable intersections) There are also some subsets of X^ω which are not Borel. In particular the class of Borel subsets of X^ω is strictly included into the class $\mathbf{\Sigma^1_1}$ of analytic sets which are obtained by projection of Borel sets.
Notice that if Σ and Γ are two finite alphabets then the product $\Sigma^\omega \times \Gamma^\omega$ can be identified with the space $(\Sigma \times \Gamma)^\omega$ and we always consider in the sequel that such a space $\Sigma^\omega \times \Gamma^\omega$ is equipped with the Cantor topology.

Definition 3.2 *A set $A \subseteq \Sigma^\omega$ is an analytic set if there is a finite alphabet Γ and a Borel set $B \subseteq \Sigma^\omega \times \Gamma^\omega$ such that $A = \{\alpha \in \Sigma^\omega \mid \exists \beta \in \Gamma^\omega \quad (\alpha, \beta) \in B\}$.*
A set $C \subseteq \Sigma^\omega$ is coanalytic if its complement $\Sigma^\omega - C$ is analytic. The class of analytic sets is denoted $\mathbf{\Sigma^1_1}$ and the class of coanalytic sets is denoted $\mathbf{\Pi^1_1}$.

Recall also the notion of completeness with regard to reduction by continuous functions. For an integer $n \geq 1$, a set $F \subseteq X^\omega$ is said to be a $\mathbf{\Sigma^0_n}$ (respectively, $\mathbf{\Pi^0_n}$, $\mathbf{\Sigma^1_1}$, $\mathbf{\Pi^1_1}$)-complete set iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma^0_n}$ (respectively, $E \in \mathbf{\Pi^0_n}$, $E \in \mathbf{\Sigma^1_1}$, $E \in \mathbf{\Pi^1_1}$) iff there exists a continuous function $f : Y^\omega \to X^\omega$ such that $E = f^{-1}(F)$.
$\mathbf{\Sigma^0_n}$ (respectively, $\mathbf{\Pi^0_n}$)-complete sets, with n an integer ≥ 1, are thoroughly characterized in [Sta86].

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Let Σ and X be two finite alphabets. If $B \subseteq \Sigma^\omega \times X^\omega$ and $\alpha \in \Sigma^\omega$, the section in α of B is $B_\alpha = \{\beta \in X^\omega \mid (\alpha, \beta) \in B\}$ and the projection of B on Σ^ω is the set $PROJ_{\Sigma^\omega}(B) = \{\alpha \in \Sigma^\omega \mid B_\alpha \neq \emptyset\} = \{\alpha \in \Sigma^\omega \mid \exists \beta \ (\alpha, \beta) \in B\}$.

We are going to prove the following lemma which will be useful in the sequel:

Lemma 4.1 *Let Σ and X be two finite alphabets having at least two letters and B be a Borel subset of $\Sigma^\omega \times X^\omega$ such that $PROJ_{\Sigma^\omega}(B)$ is not a Borel subset of Σ^ω. Then there are $2^{\aleph_0}$ ω-words $\alpha \in \Sigma^\omega$ such that the section B_α has cardinality $2^{\aleph_0}$.*

Proof. Let Σ and X be two finite alphabets having at least two letters and B be a Borel subset of $\Sigma^\omega \times X^\omega$ such that $PROJ_{\Sigma^\omega}(B)$ is not Borel.

In a first step we shall prove that there are uncountably many $\alpha \in \Sigma^\omega$ such that the section B_α is uncountable.

Recall that by a Theorem of Lusin and Novikov, see [Kec95, page 123], if for all $\alpha \in \Sigma^\omega$, the section B_α of the Borel set B was countable, then $PROJ_{\Sigma^\omega}(B)$ would be a Borel subset of Σ^ω.

Thus there exists at least one $\alpha \in \Sigma^\omega$ such that B_α is uncountable. In fact we have not only one α such that B_α is uncountable.

For $\alpha \in \Sigma^\omega$ we have $\{\alpha\} \times B_\alpha = B \cap [\{\alpha\} \times X^\omega]$. But $\{\alpha\} \times X^\omega$ is a closed hence Borel subset of $\Sigma^\omega \times X^\omega$ thus $\{\alpha\} \times B_\alpha$ is Borel as intersection of two Borel sets.

If there was only one $\alpha \in \Sigma^\omega$ such that B_α is uncountable, then $C = \{\alpha\} \times B_\alpha$ would be Borel so $D = B - C$ would be borel because the class of Borel sets is closed under boolean operations.
But all sections of D would be countable thus $PROJ_{\Sigma^\omega}(D)$ would be Borel by Lusin and Novikov's Theorem. Then $PROJ_{\Sigma^\omega}(B) = \{\alpha\} \cup PROJ_{\Sigma^\omega}(D)$ would be also Borel as union of two Borel sets, and this would lead to a contradiction.

In a similar manner we can prove that the set $U = \{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\}$ is uncountable, otherwise $U = \{\alpha_0, \alpha_1, \ldots \alpha_n, \ldots\}$ would be Borel as the countable union of the closed sets $\{\alpha_i\}$, $i \geq 0$.
For each $n \geq 0$ the set $\{\alpha_n\} \times B_{\alpha_n}$ would be Borel, and $C = \cup_{n \in \omega} \{\alpha_n\} \times B_{\alpha_n}$ would be Borel as a countable union of Borel sets. So $D = B - C$ would be borel too.
But all sections of D would be countable thus $PROJ_{\Sigma^\omega}(D)$ would be Borel by Lusin and Novikov's Theorem. Then $PROJ_{\Sigma^\omega}(B) = U \cup PROJ_{\Sigma^\omega}(D)$ would be also Borel as union of two Borel sets, and this would lead to a contradiction.

So we have proved that the set $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\}$ is uncountable.

On the other hand we know from another Theorem of Descriptive Set Theory that the set $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is countable }\}$ is a $\mathbf{\Pi}^1_1$-subset of Σ^ω, see [Kec95, page 123].
Thus its complement $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\}$ is analytic. But by Suslin's Theorem an analytic subset of Σ^ω is either countable or has cardinality $2^{\aleph_0}$, [Kec95, p. 88]. Therefore the set $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\}$ has cardinality $2^{\aleph_0}$.

Recall now that we have already seen that, for each $\alpha \in \Sigma^\omega$, the set $\{\alpha\} \times B_\alpha$ is Borel. We can then infer that B_α itself is Borel by considering the function $h : X^\omega \to \Sigma^\omega \times X^\omega$ defined by $h(\sigma) = (\alpha, \sigma)$ for all $\sigma \in X^\omega$. The function h is continuous and $B_\alpha = h^{-1}(\{\alpha\} \times B_\alpha)$. So B_α is Borel because the inverse image of a Borel set by a continuous function is a Borel set. Again by Suslin's Theorem B_α is either countable or has cardinality $2^{\aleph_0}$. ¿From this we deduce that $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable }\} = \{\alpha \in \Sigma^\omega \mid B_\alpha \text{ has cardinality } 2^{\aleph_0}\}$ has cardinality $2^{\aleph_0}$. $\square$

We can now infer some results for ω-context free languages.

Theorem 4.2 *Let $L(\mathcal{A})$ be an ω-CFL accepted by a BPDA $\mathcal{A}$ such that $L(\mathcal{A})$ is an analytic but non Borel set. The set of ω-words, which have $2^{\aleph_0}$ accepting runs by $\mathcal{A}$, has cardinality $2^{\aleph_0}$.*

Proof. Let $\mathcal{A} = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a BPDA such that $L(\mathcal{A})$ is an analytic but non Borel set.

To an infinite sequence $r = (q_i, \gamma_i, \varepsilon_i)_{i \geq 1}$, where for all $i \geq 1$, $q_i \in K$, $\gamma_i \in \Gamma^+$ and $\varepsilon_i \in \{0, 1\}$, we associate an ω-word $\bar{r}$ over the alphabet $X = \Gamma \cup K \cup \{0, 1\}$ defined by

$$\bar{r} = q_1.\gamma_1.\varepsilon_1.q_2.\gamma_2.\varepsilon_2 \ldots q_i.\gamma_i.\varepsilon_i \ldots$$

Then to an infinite word $\sigma \in \Sigma^\omega$ and an infinite sequence $r = (q_i, \gamma_i, \varepsilon_i)_{i \geq 1}$, we associate the couple $(\sigma, \bar{r}) \in \Sigma^\omega \times (\Gamma \cup K \cup \{0, 1\})^\omega$.

Recall now that $\mathbf{\Pi^0_2}$-subsets of a Cantor set Σ^ω are characterized in the following way. For $W \subseteq \Sigma^\star$ the δ-limit W^δ of W is the set of ω-words over Σ having infinitely many prefixes in W: $W^\delta = \{\sigma \in \Sigma^\omega \mid \exists^\omega i \text{ such that } \sigma(1) \ldots \sigma(i) \in W\}$. Then a subset L of Σ^ω is a $\mathbf{\Pi^0_2}$-subset of Σ^ω iff there exists a set $W \subseteq \Sigma^\star$ such that $L = W^\delta$, [Sta97a] [PP02].

It is then easy to see that the set

$$R = \{(\sigma, \bar{r}) \mid \bar{r} \text{ is the code of an accepting run of } \mathcal{A} \text{ over } \sigma\}$$

is a $\mathbf{\Pi^0_2}$-subset of $\Sigma^\omega \times X^\omega = (\Sigma \times X)^\omega$ as intersection of two $\mathbf{\Pi^0_2}$-sets. In fact we have $R = (R')^\delta \cap (R'')^\delta$ where $R' \subseteq (\Sigma \times X)^+$ is the set of couples of words (u, v) in the form:

$$u = a_1.a_2.\ldots.a_p$$

$$v = q_1.\gamma_1.\varepsilon_1.q_2.\gamma_2.\varepsilon_2 \ldots q_n.\gamma_n.\varepsilon_n$$

where for each $i \in [1, p]$ $a_i \in \Sigma$, for each $i \in [1, n]$ $q_i \in K$, $\gamma_i \in \Gamma^+$ and $\varepsilon_i \in \{0, 1\}$. Moreover $|u| = |v|$, $\varepsilon_n = 1$, and

1. $(q_1, \gamma_1) = (q_0, Z_0)$
2. for each $i \in [1, n-1]$, there exists $b_i \in \Sigma \cup \{\lambda\}$ satisfying
   $b_i : (q_i, \gamma_i) \mapsto_{\mathcal{A}} (q_{i+1}, \gamma_{i+1})$
   and ($\varepsilon_i = 0$ iff $b_i = \lambda$)
   and such that $b_1 b_2 \ldots b_{n-1}$ is a prefix of $u = a_1.a_2.\ldots.a_p$.

And $R'' \subseteq (\Sigma \times X)^+$ is the set of couples of words $(u, v) \in \Sigma^+ \times X^+$ such that $|u| = |v|$ and the last letter of v is an element $q \in F$.

In particular R is a Borel subset of $\Sigma^\omega \times X^\omega$. But by definition of R it turns out that $PROJ_{\Sigma^\omega}(R) = L(\mathcal{A})$ so $PROJ_{\Sigma^\omega}(R)$ is not Borel. Thus Lemma 4.1 implies that there are $2^{\aleph_0}$ ω-words $\alpha \in \Sigma^\omega$ such that R_α has cardinality $2^{\aleph_0}$. This means that these words have $2^{\aleph_0}$ accepting runs by the Büchi pushdown automaton $\mathcal{A}$. $\square$

Example 4.3 *Let $\Sigma = \{0, 1\}$ and d be a new letter not in Σ and*

$$D = \{u.d.v \mid u, v \in \Sigma^\star \text{ and } (|v| = 2|u|) \quad \text{or} \quad (|v| = 2|u| + 1) \ \}$$

$D \subseteq (\Sigma \cup \{d\})^\star$ is a context free language. Let $g : \Sigma \to \mathcal{P}((\Sigma \cup \{d\})^\star)$ be the substitution defined by $g(a) = a.D$. As $W = 0^\star 1$ is regular, $g(W)$ is a context free language, thus $(g(W))^\omega$ is an ω-CFL. It is proved in [Fin03a] that $(g(W))^\omega$ is $\mathbf{\Sigma^1_1}$-complete. In particular $(g(W))^\omega$ is an analytic non Borel set. Thus every BPDA accepting $(g(W))^\omega$ has the maximum ambiguity and $(g(W))^\omega \in A(2^{\aleph_0}) - CFL_\omega$.

On the other hand we can prove that $g(W)$ is a non ambiguous context free language.

For that purpose consider a (finite) word $x \in g(W)$; then $x \in g(0^n.1)$ for some integer $n \geq 0$. Therefore x may be written in the form

$$x = 0.u_1.d.v_1.0.u_2.d.v_2 \ldots 0.u_n.d.v_n.1.u_{n+1}.d.v_{n+1}$$

where $u_i.d.v_i \in D$ holds for all $i \in [1, n+1]$. It is easy to see that the length $|v_{n+1}|$ and the word v_{n+1} are determined by the word x: v_{n+1} is the suffix of x following the last letter d of x, and $|v_{n+1}| = 2|u_{n+1}|$ (if $|v_{n+1}|$ is even) or $|v_{n+1}| = 2|u_{n+1}| + 1$ (if $|v_{n+1}|$ is odd) thus $|u_{n+1}|$ is determined by $|v_{n+1}|$ hence u_{n+1} is also determined. Next one can see that v_n also is fixed by x (the word $v_n.1.u_{n+1}$ is the segment of x which is located between the n^{th} and the $(n+1)^{th}$ occurrences of the letter d in x and knowing u_{n+1} gives us v_n).
We can similarly prove by induction on the integer k that the words v_{n+1-k} and u_{n+1-k}, for $k \in [0, n]$, are uniquely determined by x.
Therefore the word x admits a unique decomposition in the above form. We can then easily construct a pushdown automaton (and even a one counter automaton) which accepts the language $g(W)$ and which is non ambiguous. So the language $g(W)$ is a non ambiguous context free language.

The above example shows that the ω-power of a non ambiguous context free language may have maximum ambiguity. Conversely consider the context free language $V = V_1 \cup V_2 \subseteq \{a, b, c\}^\star$ where $V_1 = \{a^n b^n c^p \mid n \geq 1, \ p \geq 1\}$ and $V_2 = \{a^n b^p c^p \mid n \geq 1, \ p \geq 1\}$. V_1 and V_2 are deterministic context free, hence they are non ambiguous context free languages. But their union V is an inherently ambiguous context free language [Mau69]. $V^\star$ is a context free language which is inherently ambiguous of infinite degree (and it is proved in [Naj98] that it is even exponentially ambiguous in the sense of Naji and Wich, see also [Wic99] about this notion). Let then $L = V^\star \cup \{a, b, c\}$. The language L is still a context free language which is inherently ambiguous of infinite degree and $L^\omega = \{a, b, c\}^\omega$ is an ω-regular language hence it is a non ambiguous ω-context free language.

We have then proved that neither unambiguity nor inherent ambiguity is preserved by the operation $L \to L^\omega$:

Proposition 4.4

1. *There exists a non ambiguous context free finitary language L such that L^ω is in $A(2^{\aleph_0}) - CFL_\omega$.*

2. *There exists a context free finitary language L, which is inherently ambiguous of infinite degree, such that L^ω is a non ambiguous ω-context free language.*

We can also consider the above mentioned language $g(W)$ in the context of code theory. We have proved that $g(W)$ is a non ambiguous context free language. By a similar reasoning we can prove that $g(W)$ is a code, i.e. that every (finite) word $y \in g(W)^+$ has a unique decomposition $y = x_1.x_2 \ldots x_n$ in words $x_i \in g(W)$.
On the other side $g(W)$ is not an ω-code, i.e. some words $z \in g(W)^\omega$ have several decompositions in the form $z = x_1.x_2 \ldots x_n \ldots$ where for all $i \geq 1$ $x_i \in g(W)$. In fact we can get a much stronger result, using Lemma 4.1:

Fact 4.5 *There are $2^{\aleph_0}$ ω-words in $g(W)^\omega$ which have $2^{\aleph_0}$ decompositions in words in $g(W)$.*

Proof. We can fix a recursive enumeration θ of the set $g(W)$. So the function $\theta : \mathbb{N} \rightarrow g(W)$ is a bijection and we denote $u_i = \theta(i)$.
Let now $\mathcal{D}$ be the set of couples $(\sigma, x) \in \{0,1\}^\omega \times (\Sigma \cup \{d\})^\omega$ such that:

1. $\sigma \in (0^\star.1)^\omega$, so σ may be written in the form

$$\sigma = 0^{n_1}.1.0^{n_2}.1.0^{n_3}.1 \ldots 0^{n_p}.1.0^{n_{p+1}}.1 \ldots$$

where $\forall i \geq 1 \quad n_i \geq 0$, and

2. $$x = u_{n_1}.u_{n_2}.u_{n_3} \ldots u_{n_p}.u_{n_{p+1}} \ldots$$

$\mathcal{D}$ is a Borel subset of $\{0,1\}^\omega \times (\Sigma \cup \{d\})^\omega$ because it is accepted by a deterministic Turing machine with a Büchi acceptance condition [Sta97a]. On the other hand $PROJ_{(\Sigma \cup \{d\})^\omega}(\mathcal{D}) = g(W)^\omega$ is not Borel and Lemma 4.1 implies that there are $2^{\aleph_0}$ ω-words x in $g(W)^\omega$ such that $\mathcal{D}_x$ has cardinality $2^{\aleph_0}$. This means that there are $2^{\aleph_0}$ ω-words $x \in g(W)^\omega$ which have $2^{\aleph_0}$ decompositions in words in $g(W)$.
We can say that the code $g(W)$ is really not an ω-code ! □

The result given by Theorem 4.2 may be compared with a general study of topological properties of transition systems due to Arnold [Arn83a]. If we consider a BPDA as a transition system with infinitely many states, Arnold's results imply that every non ambiguous ω-CFL is a Borel set. On the other side deterministic ω-CFL have not a great topological complexity, because they are boolean combinations of $\mathbf{\Pi}^0_2$-sets. We know some examples of non ambiguous ω-CFL of every finite Borel rank, but none of infinite Borel rank. These results led the first author to the following question: are there some more links between

the topological complexity of an ω-CFL and the ambiguity of BPDA which accept it? In [Fin03c] the well known notions of degrees of ambiguity for CFL are extended to ω-CFL and such supposed connections are investigated. In particular, using results of Duparc on the Wadge hierarchy, which is a great refinement of the Borel hierarchy [Dup01], it is proved that for each k such that k is an integer ≥ 2 or $k = \aleph_0^-$ and for each integer $n \geq 1$, there exist in $A(k) - CFL_\omega$ some $\mathbf{\Sigma^0_n}$-complete ω-CFL and some $\mathbf{\Pi^0_n}$-complete ω-CFL.
In the proofs of these results is used the operation $W \rightarrow Adh(W)$ where for a finitary language $W \subseteq \Sigma^\star$, $Adh(W) = \{\sigma \in \Sigma^\omega \mid LF(\sigma) \subseteq LF(W)\}$ is the adherence of W. We recall that a set $L \subseteq \Sigma^\omega$ is a closed set of Σ^ω iff there exists a finitary language $W \subseteq \Sigma^\star$ such that $L = Adh(W)$.
It is well known that if W is a context free language, then $Adh(W)$ is in CFL_ω. Moreover every closed (deterministic) ω-CFL is the adherence of a (deterministic) context free language, [Sta97a].
So the question of the preservation of ambiguity by the operation $W \rightarrow Adh(W)$ naturally arises.

Proposition 4.6 *Neither unambiguity nor inherent ambiguity is preserved by taking the adherence of a finitary context free language.*

Proof. (I) We are firstly looking for a non ambiguous finitary context free language which have an inherently ambiguous adherence. Let then the following finitary language over the alphabet $\{a, b, c, d\}$:

$$L_1 = \{a^n b^n c^p.d^{2i} \mid n, p, i \text{ are integers } \geq 1\} \cup \{a^n b^p c^p.d^{2i+1} \mid n, p, i \text{ are integers } \geq 1\}$$

L_1 is the disjoint union of two deterministic (hence non ambiguous) finitary context free languages thus it is a non ambiguous CFL because the class $NA - CFL$ is closed under finite disjoint union. It is easy to see that the adherence of L_1 is

$$Adh(L_1) = \{a^\omega\} \bigcup a^+.b^\omega \bigcup \{a^n b^n \mid n \geq 1\}.c^\omega \bigcup (V_1 \cup V_2).d^\omega$$

where $V_1 = \{a^n b^n c^p \mid n \geq 1,\ p \geq 1\}$ and $V_2 = \{a^n b^p c^p \mid n \geq 1,\ p \geq 1\}$. Then it holds that $Adh(L_1) \cap a^+.b^+.c^+.d^\omega = (V_1 \cup V_2).d^\omega = V.d^\omega$, where $V = V_1 \cup V_2$.
By proposition 2.6, the ω-context free language $V.d^\omega$ is inherently ambiguous because V is inherently ambiguous [Mau69]. Thus $Adh(L_1)$ is inherently ambiguous because otherwise $V.d^\omega$ would be non ambiguous because the class $NA - CFL_\omega$ is closed under intersection with ω-regular languages [Fin03c], and $a^+.b^+.c^+.d^\omega$ is an ω-regular language.

(II) We are now looking for an inherently ambiguous context free language which have a non ambiguous adherence. We shall use a result of Crestin, [Cre72]: the language $C = \{u.v \mid u, v \in \{a, b\}^+ \text{ and } u^R = u \text{ and } v^R = v\}$ is a context free language which is inherently ambiguous (of infinite degree). In fact $C = L_p^2$ where $L_p = \{v \in \{a, b\}^+ \mid v^R = v\}$ is the language of palindromes over the alphabet $\{a, b\}$. Consider now the adherence of the language C. $Adh(C) = \{a, b\}^\omega$ holds because every word $u \in \{a, b\}^\star$ is a prefix of a palindrome (for example of the palindrome $u.u^R$) hence it is also a prefix of a word of C.

Thus C is inherently ambiguous and $Adh(C)$ is a non ambiguous ω-context free language because it is an ω-regular language. $\square$

We have seen that closed sets are characterized as adherences of finitary languages. Similarly we have already seen, in the proof of Theorem 4.2, that $\mathbf{\Pi^0_2}$-subsets of Σ^ω are characterized as δ-limits W^δ of finitary languages $W \subseteq \Sigma^\star$.
Recall that $W \in REG$ implies that $W^\delta \in REG_\omega$. But there exist some context free languages L such that L^δ is not in CFL_ω; see [Sta97a] for an example of such a language L. In the case $W \in CFL$ **and** $W^\delta \in CFL_\omega$, the question naturally arises of the preservation of ambiguity by the operation $W \to W^\delta$. The answer is given by the following:

Proposition 4.7 *Neither unambiguity nor inherent ambiguity is preserved by taking the δ-limit of a finitary context free language.*

Proof. (I) Let again L_1 be the following finitary language over the alphabet $\{a, b, c, d\}$:

$$L_1 = \{a^n b^n c^p.d^{2i} \mid n, p, i \text{ are integers } \geq 1\} \cup \{a^n b^p c^p.d^{2i+1} \mid n, p, i \text{ are integers } \geq 1\}$$

L_1 is a non ambiguous CFL. And the δ-limit of the language L_1 is $(L_1)^\delta = (V_1 \cup V_2).d^\omega = V.d^\omega$. We have already seen that this ω-language is an inherently ambiguous ω-CFL.

(II) Consider now the inherently ambiguous context free language $V = \{a^n b^n c^p \mid n, p \geq 1\} \cup \{a^n b^p c^p \mid n, p \geq 1\}$. Its δ-limit is $V^\delta = \{a^n.b^n \mid n \geq 1\}.c^\omega$. It is easy to see that V^δ is a deterministic ω-CFL hence it is a non ambiguous ω-CFL. $\square$

5 Topology and ambiguity in infinitary rational relations

Infinitary rational relations are subsets of $\Sigma^\omega \times \Gamma^\omega$, where Σ and Γ are finite alphabets, which are accepted by 2-tape Büchi automata.
We are going to see in this section that some above methods can also be used in the case of infinitary rational relations.

Definition 5.1 *A 2-tape Büchi automaton (2-BA) is a sextuple $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$, where K is a finite set of states, Σ and Γ are finite alphabets, Δ is a finite subset of $K \times \Sigma^\star \times \Gamma^\star \times K$ called the set of transitions, q_0 is the initial state, and $F \subseteq K$ is the set of accepting states.*
A computation $\mathcal{C}$ of the 2-tape Büchi automaton $\mathcal{T}$ is an infinite sequence of transitions

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \ldots (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \ldots$$

The computation is said to be successful iff there exists a final state $q_f \in F$ and infinitely many integers $i \geq 0$ such that $q_i = q_f$.

The input word of the computation is $u = u_1.u_2.u_3\dots$
The output word of the computation is $v = v_1.v_2.v_3\dots$
Then the input and the output words may be finite or infinite.
The infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ *accepted by the 2-tape Büchi automaton* $\mathcal{T}$ *is the set of couples* $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ *such that* u *and* v *are the input and the output words of some successful computation* $\mathcal{C}$ *of* $\mathcal{T}$.
The set of infinitary rational relations will be denoted RAT_ω.

One can define degrees of ambiguity for 2-tape Büchi automata and for infinitary rational relations as in the case of BPDA and ω-CFL.

Definition 5.2 *Let* $\mathcal{T}$ *be a 2-BA accepting couples of infinite words of* $\Sigma^\omega \times \Gamma^\omega$. *For* $(u, v) \in \Sigma^\omega \times \Gamma^\omega$, *let* $\alpha_\mathcal{T}(u, v)$ *be the cardinal of the set of accepting computations of* $\mathcal{T}$ *on* (u, v).

Lemma 5.3 *Let* $\mathcal{T}$ *be a 2-BA accepting couples of infinite words* $(u, v) \in \Sigma^\omega \times \Gamma^\omega$. *Then for all* $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ *it holds that* $\alpha_\mathcal{T}(u, v) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

The proof that a value between $\aleph_0$ and $2^{\aleph_0}$ is impossible follows from Suslin's Theorem because one can obtain the set of codes of accepting computations of $\mathcal{T}$ on (u, v) as a section of a Borel set (see proof of next theorem) hence as a Borel set. A similar reasoning was used in the proof of Lemma 2.3, [Fin03c].

Definition 5.4 *Let* $\mathcal{T}$ *be a 2-BA accepting couples of infinite words* $(u, v) \in \Sigma^\omega \times \Gamma^\omega$.

(a) If $\sup\{\alpha_\mathcal{T}(u, v) \mid (u, v) \in \Sigma^\omega \times \Gamma^\omega\} \in \mathbb{N} \cup \{2^{\aleph_0}\}$, *then* $\alpha_\mathcal{T} = \sup\{\alpha_\mathcal{T}(u, v) \mid (u, v) \in \Sigma^\omega \times \Gamma^\omega\}$.

(b) If $\sup\{\alpha_\mathcal{T}(u, v) \mid (u, v) \in \Sigma^\omega \times \Gamma^\omega\} = \aleph_0$ *and there is no* $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ *such that* $\alpha_\mathcal{T}(u, v) = \aleph_0$, *then* $\alpha_\mathcal{T} = \aleph_0^-$.

(c) If $\sup\{\alpha_\mathcal{T}(u, v) \mid (u, v) \in \Sigma^\omega \times \Gamma^\omega\} = \aleph_0$ *and there exists (at least) one couple* $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ *such that* $\alpha_\mathcal{T}(u, v) = \aleph_0$, *then* $\alpha_\mathcal{T} = \aleph_0$

The set $\mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$ is linearly ordered as above by the relation $<$.

Definition 5.5 *For* $k \in \mathbb{N} \cup \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$, *let*
$RAT_\omega(\alpha \le k) = \{R(\mathcal{T}) \mid \mathcal{T}$ *is a* $2 - BA$ *with* $\alpha_\mathcal{T} \le k\}$
$RAT_\omega(\alpha < k) = \{R(\mathcal{T}) \mid \mathcal{T}$ *is a* $2 - BA$ *with* $\alpha_\mathcal{T} < k\}$
$NA - RAT_\omega = RAT_\omega(\alpha \le 1)$ *is the class of non ambiguous infinitary rational relations.*
For every integer $k \ge 2$, *or* $k \in \{\aleph_0^-, \aleph_0, 2^{\aleph_0}\}$,
$A(k) - RAT_\omega = RAT_\omega(\alpha \le k) - RAT_\omega(\alpha < k)$ *is the class of infinitary rational relations which are inherently ambiguous of degree* k.

As for ω-context free languages, one can use Lemma 4.1 to prove the following result.

Theorem 5.6 *Let $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ be an infinitary rational relation accepted by a 2-tape Büchi automaton $\mathcal{T}$ such that $R(\mathcal{T})$ is an analytic but non Borel set. The set of couples of ω-words, which have $2^{\aleph_0}$ accepting computations by $\mathcal{T}$, has cardinality $2^{\aleph_0}$.*

Proof. It is very similar to proof of Theorem 4.2. Let $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ be an infinitary rational relation accepted by a 2-tape Büchi automaton $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$. We assume also that $R(\mathcal{T})$ is an analytic but non Borel set. To an infinite sequence

$$\mathcal{C} = (q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \ldots (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \ldots$$

where for all $i \geq 0$, $q_i \in K$, for all $i \geq 1$, $u_i \in \Sigma^\star$ and $v_i \in \Gamma^\star$, we associate an ω-word $\bar{\mathcal{C}}$ over the alphabet $X = K \cup \Sigma \cup \Gamma \cup \{e\}$, where e is an additional letter. $\bar{\mathcal{C}}$ is defined by:

$$\bar{\mathcal{C}} = q_0.u_1.e.v_1.q_1.u_2.e.v_2.q_2 \ldots q_i.u_{i+1}.e.v_{i+1}.q_{i+1} \ldots$$

Then the set

$$\{(u, v, \bar{\mathcal{C}}) \in \Sigma^\omega \times \Gamma^\omega \times X^\omega \mid \bar{\mathcal{C}} \text{ is the code of an accepting computation of } \mathcal{T} \text{ over } (u, v)\}$$

is accepted by a deterministic Turing machine with a Büchi acceptance condition thus it is a $\mathbf{\Pi^0_2}$-set. We can conclude as in proof of Theorem 4.2. □

The first author showed that there exist some $\mathbf{\Sigma^1_1}$-complete, hence non Borel, infinitary rational relations [Fin03d]. So we can deduce the following result.

Corollary 5.7 *There exist some infinitary rational relations which are inherently ambiguous of degree $2^{\aleph_0}$.*

Remark 5.8 *Looking carefully at the example of non Borel infinitary rational relation given in [Fin03d], we can find a rational relation S over finite words such that S is non ambiguous and S^ω is non Borel. So S is a finitary rational relation which is non ambiguous but S^ω has maximum ambiguity because $S^\omega \in A(2^{\aleph_0}) - RAT_\omega$ holds by Theorem 5.6.*

Moreover the question of the decidability of ambiguity for infinitary rational relations naturally arises. It can be solved, using another recent result of the first author.

Proposition 5.9 ([Fin03e]) *Let X and Y be finite alphabets containing at least two letters, then there exists a family $\mathcal{F}$ of infinitary rational relations which are subsets of $X^\omega \times Y^\omega$, such that, for $R \in \mathcal{F}$, either $R = X^\omega \times Y^\omega$ or R is a $\mathbf{\Sigma^1_1}$-complete subset of $X^\omega \times Y^\omega$, but one cannot decide which case holds.*

Corollary 5.10 *Let k be an integer ≥ 2 or $k \in \{\aleph_0^-, \aleph_0\}$. Then it is undecidable to determine whether a given infinitary rational relation is in the class $RAT_\omega(\alpha \leq k)$ (respectively $RAT_\omega(\alpha < k)$).*
In particular one cannot decide whether a given infinitary rational relation is non ambiguous or is inherently ambiguous of degree $2^{\aleph_0}$.

Proof. Consider the family $\mathcal{F}$ given by Proposition 5.9 and let $R \in \mathcal{F}$.
If $R = X^\omega \times Y^\omega$ then R is obviously non ambiguous but if R is a $\mathbf{\Sigma}^1_1$-complete subset of $X^\omega \times Y^\omega$ then by Theorem 5.6 the infinitary rational relation R is inherently ambiguous of degree $2^{\aleph_0}$. But one cannot decide which case holds and this ends the proof. □

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