On Decidability Properties of Local Sentences

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Abstract

Local (first order) sentences, introduced by Ressayre, enjoy very nice decidability properties, following from some stretching theorems stating some remarkable links between the finite and the infinite model theory of these sentences [Res88]. Another stretching theorem of Finkel and Ressayre implies that one can decide, for a given local sentence φ and an ordinal $\alpha < \omega^\omega$, whether φ has a model of order type α. This result is very similar to Büchi's one who proved that the monadic second order theory of the structure $(\alpha, <)$, for a countable ordinal α, is decidable. It is in fact an extension of that result, as shown in [Fin01] by considering the expressive power of monadic sentences and of local sentences over languages of words of length α. The aim of this paper is twofold. We wish first to attract the reader's attention on these powerful decidability results proved using methods of model theory and which should find some applications in computer science and we prove also here several additional results on local sentences.
The first one is a new decidability result in the case of local sentences whose function symbols are at most unary: one can decide, for every *regular cardinal* ω_α (the α-th infinite cardinal), whether a local sentence φ has a model of order type ω_α.
Secondly we show that this result can not be extended to the general case. Assuming the consistency of an inaccessible cardinal we prove that the set of local sentences having a model of order type ω_2 is not determined by the axiomatic system $ZFC + GCH$, where GCH is the generalized continuum hypothesis.
Next we prove that for all integers $n, p \geq 1$, if $n < p$ then the local theory of ω_n, i.e. the set of local sentences having a model of order type ω_n, is recursive in the local theory of ω_p and also in the local theory of α where α is any ordinal of cofinality ω_n.

Key words: local sentences, decidability properties, model of ordinal order type α, monadic theory of an ordinal, ω_2-model, Kurepa tree, independence result.

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1 Introduction

A local sentence is a first order sentence which is equivalent to a universal sentence and satisfies some semantic restrictions: closure in its models takes a finite number of steps. Ressayre introduced local sentences in [Res88] and established some remarkable links between the finite and the infinite model theory of these sentences given by some stretching theorems. Assuming that a binary relation symbol belongs to the signature of a local sentence φ and is interpreted by a linear order in every model of φ, the stretching theorems state that the existence of some well ordered models of φ is equivalent to the existence of some finite model of φ, generated by some particular kind of indiscernibles, like special, remarkable or monotonic ones. Another stretching theorem of Finkel and Ressayre establishes the equivalence between the existence of a model of order type α (where α is an infinite ordinal $< \omega^\omega$) and the existence of a finite model (of another local sentence φ_α) generated by N_{φ_α} semi-monotonic indiscernibles (where N_{φ_α} is a positive integer depending on φ_α) [FR96].
This theorem provides some decision algorithms which show the decidability of the following problem: (P) "For a given local sentence φ and an ordinal $\alpha < \omega^\omega$, has φ a model of order type α ?"
This last result is very similar to Büchi's one who proved that the monadic second order theory of the structure $(\alpha, <)$, for a countable ordinal α, is decidable, [Büc62,Tho90,BS73]. Büchi obtained some decision algorithms by proving firstly that, for α-languages (languages of infinite words of length α) over a finite alphabet, definability by monadic second order sentences is equivalent to acceptance by finite automata where a transition relation is added for limit steps.
We can compare the expressive power of monadic sentences and of local sentences, considering languages defined by these sentences. For each ordinal $\alpha < \omega^\omega$, an α-language over a finite alphabet Σ is called local in [Res88,FR96] (or also locally finite in [Fin01,Fin04,Fin02]) iff it is defined by a second order sentence in the form $\exists R_1 \ldots \exists R_k \varphi$, where φ is local in the signature $S(\varphi) = \{<, R_1, \ldots R_k, (P_a)_{a \in \Sigma}\}$, $R_1, \ldots R_k$ are relation or function symbols, and, for each $a \in \Sigma$, P_a is a unary predicate symbol.
The class LOC_α of local α-languages, for $\omega \leq \alpha < \omega^\omega$, is a strict extension of the class REG_α of regular α-languages, defined by monadic second order sentences [Fin01]. Moreover this extension is very large. This can be seen by considering the topological complexity of α-languages and firstly of ω-languages. It is well known that all regular ω-languages are boolean combinations of $\mathbf{\Sigma}^0_2$ Borel sets hence $\mathbf{\Delta}^0_3$ Borel sets, [Tho90,PP04]. On the other hand the class LOC_ω meets all finite levels of the Borel hierarchy, contains some Borel sets of infinite rank and even some analytic but non Borel sets, [Fin02].
Thus the decision algorithm for local sentences provides in fact a very large

extension, for $\alpha < \omega^\omega$, of Büchi's result about the decidability of the monadic second order theory of $(\alpha, <)$. Moreover, at least for $\alpha = \omega$, the algorithm for local sentences is of much lower complexity than the corresponding algorithm for monadic second order sentences [Fin02].

We think that these powerful decidability results proved using methods of model theory should find some applications in computer science and that the study of local sentences could become an interdisciplinary subject for both model theory and computer science communities.

So the aim of this paper is twofold: firstly to attract the reader's attention on these good properties of local sentences and their possible further applications; secondly to prove several new results on local sentences, described below.

Büchi showed that for every ordinal $\alpha < \omega_2$, where ω_2 is the second uncountable cardinal, the monadic theory of $(\alpha, <)$ is decidable. This result cannot be extended to ω_2. Assuming the existence of a weakly compact cardinal (a kind of large cardinal) Gurevich, Magidor and Shelah proved that the monadic theory of $(\omega_2, <)$ is not determined by the set theory axiomatic system ZFC. They proved even much more: for any given $S \subseteq \omega$ there is a model of ZFC where the monadic theory of $(\omega_2, <)$ has the Turing degree of S; in particular it can be non-recursive [GMS83].

Ressayre asked similarly for which ordinals α it is decidable whether a given local sentence φ has a model of order type α. The question is solved in [FR96] for $\alpha < \omega^\omega$ but for larger ordinals the problem was still open.

We firstly consider local sentences whose function symbols are at most unary. We show that these sentences satisfy an extension of the stretching theorem implying new decidability properties. In particular, for each *regular* cardinal ω_α (hence in particular for each ω_n where n is a positive integer), it is decidable whether a local sentence φ has a model of order type ω_α. To know that this restricted class $LOCAL(1)$ of local sentences has more decidability properties is of interest because it has already a great expressive power.

Sentences in $LOCAL(1)$ can define all regular finitary languages [Res88], and all the quasirational languages forming a large class of context free languages containing all linear languages [Fin01].

If we consider their expressive power over infinite words, sentences in $LOCAL(1)$ can define all regular ω-languages [Fin01], but also some $\mathbf{\Sigma^0_n}$-complete and some $\mathbf{\Pi^0_n}$-complete Borel sets for every integer $n \geq 1$, [Fin02].

Next we show that this decidability result can not be extended to local sentences having n-ary function symbols for $n \geq 2$. Assuming the consistency of an inaccessible cardinal, we prove that the local theory of ω_2 (the set of local sentences having a model of order type ω_2) is not determined by the system $ZFC + GCH$, where GCH is the generalized continuum hypothesis. This is also extended to many larger ordinals.

This result is obtained by showing that there is a local sentence which has a model of order type ω_2 if and only if there is a Kurepa tree, i.e. a tree of height ω_1 whose levels are countable and which has more than ω_1 branches of length ω_1. Kurepa trees have been much studied in set theory and their existence

has been shown to be independent of $ZFC + GCH$, via the consistency of $ZFC +$ " there is an inaccessible cardinal".
It is remarkable that our proof needs only the consistency of an inaccessible cardinal which is still a large cardinal but a very smaller cardinal than a weakly compact cardinal. This gives another indication of the great expressive power of local sentences with regard to that of monadic sentences.
We could still expect, as Shelah did in [She75] about the possible monadic theories of ω_2, that there are only finitely many possible local theories of ω_2, and that each of them is decidable. But it seems more plausible that the situation is much more complicated, as it is shown to be the case for monadic theories of ω_2 in [GMS83]: there are in fact continuum many possible monadic theories of ω_2 (in different universes of set theory); moreover, for every set of positive integers $S \subseteq \omega$, there is a monadic theory of ω_2, in some world, which is as complex as S.
We then extend the above results by proving that for all integers $n, p \geq 1$, if $n < p$ then the local theory of ω_n is recursive in the local theory of ω_p and also in the local theory of α where α is any ordinal of cofinality ω_n.
Some of these new results are seemingly far from problems arising in concrete applications studied in computer science. However our main result is obtained by encoding (Kurepa) trees in models of a local sentence and methods used here for such coding might be very useful for problems arising in computer science where (finite or infinite) trees are a widely used tool.

The paper is organized as follows. In section 2 we review some previous definitions and results about local sentences. In section 3 we prove new decidability results. Our main result on the local theory of ω_2 is proved in section 4. Some results on the local theories of ω_n, $n \geq 1$, are stated in section 5.

2 Review of previous results

In this paper the (first order) signatures are finite, always contain one binary predicate symbol $=$ for equality, and can contain both functional and relational symbols.

When M is a structure in a signature Λ, $|M|$ is the domain of M.
If f is a function symbol (respectively, R is a relation symbol, a is a constant symbol) in Λ, then f^M (respectively, R^M, a^M) is the interpretation in the structure M of f (respectively, R, a).
Notice that, when the meaning is clear, the superscript M in f^M, R^M, a^M, will be sometimes omitted in order to simplify the presentation.

For a structure M in a signature Λ and $X \subseteq |M|$, we define:
$cl^1(X, M) = X \cup \bigcup_{\{f \text{ n-ary function of } \Lambda\}} f^M(X^n) \cup \bigcup_{\{a \text{ constant of } \Lambda\}} a^M$

$cl^{n+1}(X, M) = cl^1(cl^n(X, M), M)$ for an integer $n \geq 1$
and $cl(X, M) = \bigcup_{n \geq 1} cl^n(X, M)$ is the closure of X in M.

The signature of a first order sentence φ, i.e. the set of non logical symbols appearing in φ, is denoted $\mathrm{S}(\varphi)$. As usual $M \models \varphi$ means that the sentence φ is satisfied in the structure M, i.e. that M is a model of φ.

Definition 2.1 *A first order sentence φ is local if and only if:*

(a) $M \models \varphi$ and $X \subseteq |M|$ imply $cl(X, M) \models \varphi$
(b) $\exists n \in \mathbb{N}$ such that $\forall M$, if $M \models \varphi$ and $X \subseteq |M|$, then $cl(X, M) = cl^n(X, M)$, (closure in models of φ takes less than n steps).

For a local sentence φ, n_φ is the smallest integer $n \geq 1$ satisfying (b) of the above definition. In this definition, (a) implies that a local sentence φ is always equivalent to a universal sentence, so we may assume that this is always the case.

Example 2.2 *Let φ be the sentence in the signature $S(\varphi) = \{<, P, i, a\}$, where $<$ is a binary relation symbol, P is a unary relation symbol, i is a unary function symbol, and a is a constant symbol, which is the conjunction of:*

(1) $\forall xyz[(x \leq y \vee y \leq x) \wedge ((x \leq y \wedge y \leq x) \leftrightarrow x = y) \wedge ((x \leq y \wedge y \leq z) \rightarrow x \leq z)]$,
(2) $\forall xy[(P(x) \wedge \neg P(y)) \rightarrow x < y]$,
(3) $\forall xy[(P(x) \rightarrow i(x) = x) \wedge (\neg P(y) \rightarrow P(i(y)))]$,
(4) $\forall xy[(\neg P(x) \wedge \neg P(y) \wedge x \neq y) \rightarrow i(x) \neq i(y)]$,
(5) $\neg P(a)$.

We now explain the meaning of the above sentences (1)-(5)*.*
Assume that M is a model of φ. The sentence (1) *expresses that $<$ is interpreted in M by a linear order;* (2) *expresses that P^M is an initial segment of the model M;* (3) *expresses that the function i^M is trivially defined by $i^M(x) = x$ on P^M and is defined from $\neg P^M$ into P^M.* (4) *says that i^M is an injection from $\neg P^M$ into P^M and* (5) *ensures that the element a^M is in $\neg P^M$.*
The sentence φ is a conjunction of universal sentences thus it is equivalent to a universal one, and closure in its models takes at most two steps (one adds the constant a in one step then takes the closure under the function i). Thus φ is a local sentence.
If we consider only the order types of well ordered models of φ, we can easily see that φ has a model of order type α, for every finite ordinal $\alpha \geq 2$ and for every infinite ordinal α which is not a cardinal.

Many more examples of local sentences will be given later in Sections 4 and 5. The reader may also find many other ones in the papers [Res88,FR96,Fin01]

[Fin89,Fin02,Fin04].

The set of local sentences is recursively enumerable but not recursive [Fin01]. However there exists a "recursive presentation" up to logical equivalence of all local sentences.

Theorem 2.3 (Ressayre, see [Fin01]) *There exists a recursive set* $\mathbf{L}$ *of local sentences and a recursive function* $\mathbf{F}$ *such that:*

1) ψ *local* $\longleftrightarrow \exists \psi' \in \mathbf{L}$ *such that* $\psi \equiv \psi'$*.*
2) $\psi' \in \mathbf{L} \longrightarrow n_{\psi'} = \mathbf{F}(\psi')$*.*

The elements of $\mathbf{L}$ are the $\psi \wedge C_n$, where ψ run over the universal formulas and C_n run over the universal formulas in the signature $S(\psi)$ which express that closure in a model takes at most n steps.
$\psi \wedge C_n$ is local and $n_{\psi \wedge C_n} \leq n$. Then we can compute $n_{\psi \wedge C_n}$, considering only finite models of cardinal $\leq m$, where m is an integer depending on n. And each local sentence ψ is equivalent to a universal formula θ, hence $\psi \equiv \theta \wedge C_{n_\psi}$.

From now on we shall assume that the signature of local sentences contain a binary predicate $<$ which is interpreted by a linear ordering in all of their models.
We recall now the stretching theorem for local sentences. Below, semi-monotonic, special, and monotonic indiscernibles are particular kinds of indiscernibles which are precisely defined in [FR96].

Theorem 2.4 ([FR96]) *For each local sentence* φ *there exists a positive integer* N_φ *such that*

(A) φ *has arbitrarily large finite models if and only if* φ *has an infinite model if and only if* φ *has a finite model generated by* N_φ *indiscernibles.*
(B) φ *has an infinite well ordered model if and only if* φ *has a finite model generated by* N_φ *semi-monotonic indiscernibles.*
(C) φ *has a model of order type* ω *if and only if* φ *has a finite model generated by* N_φ *special indiscernibles.*
(D) φ *has well ordered models of unbounded order types in the ordinals if and only if* φ *has a finite model generated by* N_φ *monotonic indiscernibles.*

To every local sentence φ *and every ordinal* α *such that* $\omega \leq \alpha < \omega^\omega$ *one can associate by an effective procedure a local sentence* φ_α*, a unary predicate symbol* P *being in the signature* $S(\varphi_\alpha)$*, such that:*

(C_α*)* φ *has a well ordered model of order type* α *if and only if* φ_α *has a finite model* M *generated by* N_{φ_α} *semi-monotonic indiscernibles into* P^M*.*

The integer N_φ can be effectively computed from n_φ and q where $\varphi = \forall x_1 \ldots \forall x_q$

$\theta(x_1, \ldots, x_q)$ and θ is an open formula, i.e. a formula without quantifiers. If $v(\varphi)$ is the maximum number of variables of terms of complexity $\leq n_\varphi + 1$ (resulting by at most $n_\varphi + 1$ applications of function symbols) and $v'(\varphi)$ is the maximum number of variables of an atomic formula involving terms of complexity $\leq n_\varphi + 1$ then $N_\varphi = max\{3v(\varphi); v'(\varphi) + v(\varphi); q.v'(\varphi)\}$.

From Theorem 2.4 we can prove the decidability of several problems about local sentences. For instance (C) states that a local sentence φ has an infinite well ordered model iff it has a *finite* model generated by N_φ semi-monotonic indiscernibles. Therefore in order to check the existence of an infinite well ordered model of φ one can only consider models whose cardinals are bounded by an integer depending on n_φ and N_φ, because closure in models of φ takes at most n_φ steps. This can be done in a finite amount of time.

Notice that the set of local sentences is not recursive so the algorithms given by the following theorem are applied to local sentences in the recursive set $\mathbf{L}$ given by Proposition 2.3. In particular φ is given with the integer n_φ.

Theorem 2.5 ([FR96]) *It is decidable, for a given local sentence φ, whether*

(1) φ has arbitrarily large finite models.
(2) φ has an infinite model.
(3) φ has an infinite well ordered model.
(4) φ has well ordered models of unbounded order types in the ordinals.
(5) φ has a model of order type α, where $\alpha < \omega^\omega$ is a given ordinal.

These decidable problems $(1) - (4)$ and (5) (at least for $\alpha = \omega$) are in the class $\mathbf{NTIME}(\mathbf{2}^{\mathbf{O(n.\log(n))}})$, (and even probably of lower complexity):
Using non determinism a Turing machine may guess a finite structure M of signature $\mathrm{S}(\varphi)$ generated by N_φ elements $y_1, \ldots y_{N_\varphi}$ in at most n_φ steps. Then, assuming $\varphi = \forall x_1 \ldots \forall x_q \theta(x_1, \ldots, x_q)$ where θ is an open formula, the Turing machine checks that $\theta(x_1, \ldots, x_q)$ holds for all $x_1 \ldots x_q$ in M, and that the elements $y_1, \ldots y_{N_\varphi}$ are indiscernibles (respectively, semi-monotonic, special, monotonic, indiscernibles) in M.
On the other hand Büchi's procedure to decide whether a monadic second order formula of size n of $S1S$ is true in the structure $(\omega, <)$ might run in time $\underbrace{2^{2^{\cdot^{\cdot^{2^n}}}}}_{\mathbf{O(n)}}$, [Büc62,Saf89]. Moreover Meyer proved that one cannot essentially improve this result: the monadic second order theory of $(\omega, <)$ is not elementary recursive, [Mey75].
We know that the expressive power of local sentences is much greater than that of monadic second order sentences hence this is a remarkable fact that decision algorithms for local sentences given by Theorem 2.5 are of much lower complexity than the algorithm for decidability of the monadic second order theory $S1S$ of one successor over the integers.

Notice however that the nonemptiness problem for Büchi automata is known to be logspace-complete for the complexity class **NLOGSPACE** which is included in the class **DTIME**(**Pol**) of problems which can be solved in deterministic polynomial time [VW94,BGG97]. Moreover there is a linear time algorithm for deciding the nonemptiness problem for Büchi automata which is nowadays very useful for many applications in the domain of specification and verification of non terminating systems, see for example [BV98].

3 More decidability results

We assume in this section that the function symbols of a local sentence φ are at most unary. We shall prove in this case some more decidability results which rely on an extension of the stretching Theorem 2.4.

The cardinal of a set X will be denoted by $card(X)$.
We recall that the infinite cardinals are usually denoted by $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$
The cardinal $\aleph_\alpha$ is also denoted by ω_α, as usual when it is considered as an ordinal.

We recall now the notions of cofinality of an ordinal and of regular cardinal which may be found for instance in [Dev84,Jec02].
Let α be a limit ordinal, the cofinality of α, denoted $cof(\alpha)$, is the least ordinal β such that there exists a strictly increasing sequence of ordinals $(\alpha_i)_{i<\beta}$, of length β, such that

$$\forall i < \beta \qquad \alpha_i < \alpha \quad \text{ and }$$

$$\sup_{i<\beta} \alpha_i = \alpha$$

This definition is usually extended to 0 and to the successor ordinals:

$$cof(0) = 0 \text{ and } cof(\alpha + 1) = 1 \text{ for every ordinal } \alpha.$$

The cofinality of a limit ordinal is always a limit ordinal satisfying:

$$\omega \leq cof(\alpha) \leq \alpha$$

$cof(\alpha)$ is in fact a cardinal. A cardinal k is said to be *regular* iff $cof(k) = k$. Otherwise $cof(k) < k$ and the cardinal k is said to be *singular*.

We recall now the notion of special indiscernibles, [FR96], in that particular case where all function symbols hence all terms of $S(\varphi)$ are unary.
A set X included in a structure M, having a linear ordering $<$ in its signature, is a set of indiscernibles iff whenever $\bar{x}$ and $\bar{y}$ are order isomorphic sequences

from X they satisfy in M the same atomic sentences. The indiscernibles of X are special iff they satisfy (i) and (ii):

(i) for all $x < y$ in X and all terms t: $t(x) < y$.

(ii) for all $x < y$ in X and all terms t: $t(y) < x \to t(y) = t(z)$ for all elements $z > x$ of X (i.e. t is constant on $\{z \in X \mid z > x\}$).

Theorem 3.1 *For each local sentence φ whose function symbols are at most unary, there is a positive integer N_φ such that, for each regular cardinal ω_α, the following statements are equivalent:*

(a) φ has an ω-model.
(b) φ has a finite model generated by N_φ special indiscernibles.
(c) φ has a β-model, for all limit ordinals β.
(d) φ has an ω_α-model.

Proof. It is proved in [FR96] that for each local sentence φ there is a positive integer N_φ such that (a) is equivalent to (b).
To prove $(a) \to (c)$ assume that φ has an ω-model M. Then it is proved in [FR96] that there exists an infinite set X of special indiscernibles in M. Recall that every linear order Y can be extended to a model $M(Y)$ of φ, called the stretching of M along Y, so that:

(1) $M(X)$ is the submodel of M generated by the set X.

(2) $Y \subseteq Z$ implies $M(Y) \subseteq M(Z)$.

(3) Every order embedding $f : Y \to Z$ has an extension $M(f)$ which is an embedding of $M(Y)$ into $M(Z)$.

Let then β be a limit ordinal and $M(\beta)$ be the stretching of M along β. We are going to show that $M(\beta)$ is of order type β. The model $M(\beta)$ is generated by the set β in a finite number of steps so there is a finite set T_φ of (unary) terms of the signature $S(\varphi)$ such that the domain of $M(\beta)$ is $\beta \cup \cup_{t \in T_\varphi} \cup_{\gamma < \beta} t(\gamma)$. The indiscernibles are special thus for each term $t \in T_\varphi$, either t is constant on β or for all indiscernibles $x < y < z$ in β we have $x < t(y) < z$. It is then easy to see that $M(\beta)$ is of order type β.
$(c) \to (d)$ is trivial so it remains to prove $(d) \to (a)$.
We assume that α is an ordinal and that M is a model of φ of order type ω_α where ω_α is a *regular* cardinal. We are going to show that there exists in M an infinite set of special indiscernibles. These indiscernibles have to satisfy (i) and (ii) only for terms of complexity $\leq n_\varphi$ because for each term t of complexity greater than n_φ there will be another term t' of complexity $\leq n_\varphi$ such that $t(x) = t'(x)$ for all indiscernibles x. This finite set of terms of complexity $\leq n_\varphi$ will be denoted by $T = \{t_1, t_2, \ldots, t_N\}$.

Using the fact that ω_α is a *regular* cardinal, we can firstly construct by induction a strictly increasing sequence $(x_\delta)_{\delta<\omega_\alpha}$ of elements of M such that for each ordinal $\delta < \omega_\alpha$ and each term $t \in T$ it holds that $t(x_\delta) < x_{\delta+1}$. We denote $X_0 = \{x_\delta \mid \delta < \omega_\alpha\}$; this set has cardinal $\aleph_\alpha$.
We consider now the three following cases:

First case. The set $\{x_\delta \in X_0 - \{x_0\} \mid t_1(x_\delta) = x_0\}$ has cardinal $\aleph_\alpha$. Then we denote this set by X_0^1.
Second case. The set $\{x_\delta \in X_0 - \{x_0\} \mid t_1(x_\delta) < x_0\}$ has cardinal $\aleph_\alpha$ and the first case does not hold. The initial segment $\{x \in M \mid x < x_0\}$ of M has cardinal smaller than $\aleph_\alpha$ thus there is a subset of $\{x_\delta \in X_0 - \{x_0\} \mid t_1(x_\delta) < x_0\}$ which has cardinal $\aleph_\alpha$ and on which t_1 is constant. Then we denote this set by X_0^1.
Third case. The set $\{x_\delta \in X_0 - \{x_0\} \mid t_1(x_\delta) > x_0\}$ has cardinal $\aleph_\alpha$ and the two first cases do not hold. Then we call this set X_0^1.

We can repeat now this process, replacing X_0 by X_0^1 and the term t_1 by the term t_2, so we obtain a new set $X_0^2 \subseteq X_0^1$ having still cardinal $\aleph_\alpha$. Next we repeat the process replacing X_0^1 by X_0^2 and the term t_2 by the term t_3, so we obtain a new set $X_0^3 \subseteq X_0^2$ having still cardinal $\aleph_\alpha$.
After having considered all terms $t_1, t_2, \ldots, t_N$ we have got a set $X_0^N \subseteq X_0^{N-1} \subseteq \ldots \subseteq X_0$. We denote $X_1 = X_0^N$.

Let x_{δ_1} be the first element of X_1. We can repeat all the above process replacing X_0 by X_1 and x_0 by x_{δ_1}. This way, considering successively each of the terms $t_1, t_2, \ldots, t_N$, we construct new sets $X_1^N \subseteq X_1^{N-1} \subseteq \ldots \subseteq X_1^1 \subseteq X_1$, each of them having cardinal $\aleph_\alpha$, and we set $X_2 = X_1^N$.
Assume now that we have applied this process K times for some integer $K \geq 2$. Then we have constructed successively some sets $X_1, X_2, \ldots, X_K$ of cardinal $\aleph_\alpha$. Let now x_{δ_K} be the first element of X_K. We can repeat the above process replacing X_0 by X_K and x_0 by x_{δ_K}. This way we construct a new set $X_{K+1} = X_K^N$ of cardinal $\aleph_\alpha$.
Then we can construct by induction the sets X_K for all integers $K \geq 1$. We set $X = \{x_{\delta_i} \mid 0 \leq i < \omega\}$ where for all i, x_{δ_i} is the first element of X_i.

Let now $X^{[n]}$ be the set of strictly increasing n-sequences of elements of X. Let $\sim$ be the equivalence relation defined on $X^{[v'(\varphi)]}$ by: $x \sim y$ if and only if x and y satisfy in M the same atomic formulas of complexity $\leq n_\varphi + 1$ (i.e. whose terms are of complexity $\leq n_\varphi + 1$). Applying the Infinite Ramsey Theorem, we can now get an infinite set $Y \subseteq X$ such that $Y^{[v'(\varphi)]}$ is contained in a single equivalence class of $\sim$.
Y is a set of indiscernibles in M because if z and z' are two elements of $Y^{[n]}$ for $n \geq v'(\varphi)$, then they satisfy in M the same atomic sentences of complexity $\leq n_\varphi + 1$ hence of any complexity by Fact 1 of [FR96, page 568].
By the above construction of the set X, the indiscernibles of Y are special.

Thus the submodel $M(Y)$ of M generated by Y is a model of φ of order type ω. $\square$

Notice that one cannot omit the hypothesis of the *regularity* of the cardinal ω_α in the above theorem. This is due to the fact that there exists a local sentence whose function symbols are at most unary and which has some well ordered models of order type α, for every ordinal α which is not a *regular* cardinal. Such an example is given in [FR96]. We are going to recall it now because some steps of its construction will be also useful later.

We recall first the operation $\varphi \to \varphi^\star$ over local sentences which was first defined by Ressayre in [Res88] in order to prove that the class of local languages is closed under star operation.
For each local sentence φ, the signature of the first order sentence $\mathrm{S}(\varphi^\star)$ is $\mathrm{S}(\varphi)$ to which is added a unary function symbol I and in which every constant symbol e is replaced by a unary function symbol $e(x)$.
$\varphi^\star$ is the sentence defined by the conjunction of:

(1) ($<$ is a linear order),
(2) $\forall yz[I(y) \leq y$ and $(y \leq z \to I(y) \leq I(z))$ and $(I(y) \leq z \leq y \to I(z) = I(y))]$,
(3) $\forall xy[I(x) = I(y) \to e(x) = e(y)]$, for each constant e of the signature $\mathrm{S}(\varphi)$ of φ,
(4) $\forall x_1 \ldots x_n[(\bigvee_{i,j\leq n} I(x_i) \neq I(x_j)) \to f(x_1 \ldots x_n) = min(x_1 \ldots x_n)]$, for each n-ary function f of $\mathrm{S}(\varphi)$,
(5) $\forall x_1 \ldots x_n[(\bigwedge_{i,j\leq n} I(x_i) = I(x_j)) \to I(f(x_1 \ldots x_n)) = I(x_1)]$, for each n-ary function f of $\mathrm{S}(\varphi)$,
(6) $\forall x\varphi^x$, where φ^x is the local sentence φ in which every constant e is replaced by the term e(x) and each quantifier is relativized to the set $\{y \mid I(y) = I(x)\}$.

We now explain the meaning of sentences (1)-(6). Sentence (2) is used to divide a model M of $\varphi^\star$ into successive segments. The function I^M is constant on each of these segments and the image $I^M(x)$ of an element x is the first element of the segment containing x. Sentence (3) expresses that each unary function e^M obtained from a constant symbol $e \in \mathrm{S}(\varphi)$ is constant on every segment of the model. (4) and (5) express that, for each function symbol $f \in \mathrm{S}(\varphi)$, each segment of the model is closed under the function f^M and that f^M is trivially defined by $f^M(x_1 \ldots x_n) = min(x_1 \ldots x_n)$ when at least two of the elements x_i belong to different segments. Finally sentence (6) expresses that each structure which is obtained by restricting some segment of the model to the signature of φ is a model of φ. This implies that models of $\varphi^\star$ are essentially direct sums of models of φ.
It is easy to see that $n_{\varphi^\star} = n_\varphi + 1$. Closure in models of $\varphi^\star$ takes at most $n_\varphi + 1$ steps: one takes the closure under the function I then the closure under

functions of $S(\varphi)$ in n_φ steps.

We recall now the operation $(\varphi, \psi) \to \varphi^{\star\psi}$ over local sentences which is an extension of the operation $\varphi \to \varphi^\star$ and is defined in [FR96].

We assume that $S(\varphi^\star) \cap S(\psi) = \{<\}$. Then $S(\varphi^{\star\psi}) = S(\varphi^\star) \cup S(\psi) \cup \{P\}$, where P is a new unary predicate symbol not in $S(\varphi) \cup S(\psi)$.
$\varphi^{(\star\psi)}$ is the conjunction of :

(1) $\varphi^\star$,
(2) $\forall x[P(x) \leftrightarrow I(x) = x]$,
(3) $\forall x_1 \ldots x_k[(\bigwedge_{i=1}^k P(x_i)) \to P(t(x_1 \ldots x_k))]$, for each k-ary function t of $S(\psi)$,
(4) $P(a)$, for each constant a of $S(\psi)$,
(5) $\forall x_1 \ldots x_n[(\bigvee_{i=1}^n \neg P(x_i)) \to t(x_1 \ldots x_n) = min(x_1 \ldots x_n)]$, for each n-ary function t in $S(\psi)$.
(6) $\forall x_1 \ldots x_k[Q(x_1 \ldots x_k) \to P(x_1) \wedge \ldots \wedge P(x_k)]$, for each k-ary predicate symbol Q of $S(\psi)$
(7) $\forall x_1 \ldots x_n[(\bigwedge_{i=1}^n P(x_i)) \to \psi_1(x_1 \ldots x_n)]$, where $\psi = \forall x_1 \ldots x_n \psi_1(x_1 \ldots x_n)$ and ψ_1 is an open formula,

We now explain the meaning of (1)-(7). Sentence (1) is $\varphi^\star$ so it expresses that a model M is essentially a direct sum of models of φ . (2) says that in such a model M, P^M is the set of first elements of the segments of M defined with the function I^M. (3)-(5) are used to ensure that P^M is closed under functions of $S(\psi)$ and that these functions are trivially defined elsewhere. (6) says that for every k-ary predicate Q in $S(\psi)$ the set Q^M is included into $(P^M)^k$. Sentence (7) expresses that the restriction of M to the set P^M and to the signature of ψ is a model of ψ.
It is easy to see that $n_{\varphi^{(\star\psi)}} = n_\varphi + n_\psi + 1$; to take closure of a set X in a model of $\varphi^{(\star\psi)}$ one takes the closure under the function I, then under the functions of $S(\psi)$ in n_ψ steps, then under the functions of $S(\varphi)$ in n_φ steps.
The models of $\varphi^{(\star\psi)}$ essentially are direct sums of models of φ , these models being ordered by the order type of a model of ψ.

We are mainly interested in this paper by *well ordered* models of local sentences, so we now recall the notion of spectrum of a local sentence φ. As usual the class of all ordinals is denoted by **On**.

Definition 3.2 *Let φ be a local sentence; the spectrum of φ is*

$$Sp(\varphi) = \{\alpha \in \mathbf{On} \mid \varphi \textit{ has a model of order type } \alpha\}$$

and the infinite spectrum of φ is

$$Sp_\infty(\varphi) = \{\alpha \in \mathbf{On} \mid \alpha \geq \omega \textit{ and } \varphi \textit{ has a model of order type } \alpha\}$$

The spectrum of $\varphi^{(\star\psi)}$ depends on the spectra of the local sentences φ and ψ ans is given by the following proposition.

Proposition 3.3 *Let φ and ψ be some local sentences, then $\varphi^{(\star\psi)}$ is a local sentence and its spectrum is*

$$Sp(\varphi^{(\star\psi)}) = \{\sum_{\alpha<\nu} a_\alpha \mid \nu \in Sp(\psi) \text{ and } \forall \alpha < \nu \; a_\alpha \in Sp(\varphi)\}$$

We can now construct a local sentence which has models of order type α for every infinite ordinal α which is not a regular cardinal [FR96].
Let θ be a local sentence in the signature $\mathrm{S}(\theta)=\{<, a\}$ which just expresses that the constant symbol a is interpreted by the last element of a model. Then the spectrum of θ is the class of successor ordinals.
And let β be a local sentence in the signature $\mathrm{S}(\beta)=\{<, P, s\}$ where P is a unary predicate symbol and s is a unary function symbol, which expresses that in a model M, the set P^M is an initial segment of the model and that s^M is a strictly non decreasing involution from P^M onto $\neg P^M$. Then the spectrum of β is the class of ordinals of the form $\alpha.2$ for some ordinal α.
It holds by construction that there are not in the signature of $\theta^{\star\beta}$ any function symbols of arity greater than 1 and we can verify that $Sp_\infty(\theta^{\star\beta}) = \{\alpha \geq \omega \mid \alpha \text{ is not a regular cardinal }\}$. In particular the sentence $\theta^{\star\beta}$ has a model of order type ω_α for every singular cardinal ω_α but it has no model of order type ω. So the hypothesis of the regularity of the cardinal ω_α was necessary in Theorem 3.1.

Return now to decision algorithms given by stretching theorems. By Theorem 2.5 it is decidable whether a local sentence φ has an ω-model so Theorem 3.1 implies also the following decidability result.

Theorem 3.4 *It is decidable, for a given local sentence φ whose function symbols are at most unary, and a given regular cardinal ω_α, whether:*

(1) φ has an ω_α-model
(2) φ has a β-model for all limit ordinals β.

So in particular one can decide, for a given local sentence φ whose function symbols are at most unary, whether φ has a model of order type ω_1, (respectively, ω_2, ω_n where n is a positive integer).

As mentioned in the introduction it is interesting to know that the class $LOCAL(1)$ of local sentences with at most unary function symbols has more decidability properties because it has already a great expressive power.
In particular $LOCAL(1)$ can define all regular ω-languages [Fin01], but also some $\mathbf{\Sigma^0_n}$-complete and some $\mathbf{\Pi^0_n}$-complete Borel sets for every integer $n \geq 1$, [Fin02].

Moreover it is easy to see that local ω-languages satisfy an extension of Büchi's lemma. Recall that this lemma states that a regular ω-language is non-empty if and only if it contains an ultimately periodic ω-word, i.e. an ω-word in the form $u.v^\omega$ for some *finite* words u and v.
On the other hand by the proof of the Stretching Theorem 2.4 (C) we know that a local ω-language $L(\varphi) \subseteq \Sigma^\omega$ is non-empty if and only if it contains an ω-word which is the reduction to the signature $\Lambda_\Sigma = \{<, (P_a)_{a\in\Sigma}\}$ (of words over Σ) of an ω-model of φ generated by special indiscernibles.
If the function symbols of the local sentence φ are at most unary then it is easy to see that such a reduction of an ω-model of φ generated by special indiscernibles is always an ultimately periodic ω-word.

4 The local theory of ω_2

It was proved in [FR96] that there exists a local sentence ψ (whose signature contains binary function symbols) having well ordered models of order type α for every ordinal α in the segment $[\omega; 2^{\aleph_0}]$ but not any well ordered model of order type α for $card(\alpha) > 2^{\aleph_0}$. On the other hand it is well known that the continuum hypothesis CH is independent of the axiomatic system ZFC. This means that there are some models of ZFC in which $2^{\aleph_0} = \aleph_1$ and some others in which $2^{\aleph_0} \geq \aleph_2$. Therefore the statement "ψ has a model of order type ω_2" is independent of ZFC.
However if we assume the continuum hypothesis and even the generalized continuum hypothesis GCH saying that, for every cardinal $\aleph_\alpha$, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, then the above result of [FR96] does not imply a similar independence result.

Nevertheless we are going to prove the existence of a local sentence Φ such that "Φ has a model of order type ω_2" is independent of $ZFC + GCH$.
For that purpose we shall use results about Kurepa trees which we now recall.

A partially ordered set $(T, \prec_T)$ is called a tree if for every $t \in T$ the set $\{s \in T \mid s \prec_T t\}$ is well ordered under $\prec_T$. Then the order type of the set $\{s \in T \mid s \prec_T t\}$ is called the height of t in T and is denoted by $ht(t)$. We shall not distinguish a tree from its base set.
For every ordinal α the α-th level of T is $T_\alpha = \{t \in T \mid ht(t) = \alpha\}$.
The height of T, denoted by $ht(T)$, is the smallest ordinal α such that $T_\alpha = \emptyset$.
A branch of T will be a linearly ordered subset of T intersecting every non-empty level of T. The set of all branches of T will be denoted $\mathcal{B}(T)$.
A tree T is called an ω_1-tree if $card(T) = \aleph_1$ and $ht(T) = \omega_1$. An ω_1-tree T is called a Kurepa tree if $card(\mathcal{B}(T)) > \aleph_1$ and for every ordinal $\alpha < \omega_1$, $card(T_\alpha) < \aleph_1$.

Recall now the well known results about Kurepa trees, [Dev84]:

Theorem 4.1

(1) *If ZF is consistent so too is the theory: ZFC+GCH+" there is a Kurepa tree ".*
(2) *If the theory ZFC + " there is an inaccessible cardinal " is consistent so too is the theory $ZFC + GCH$ + " there are no Kurepa trees ".*
(3) *If the theory ZFC + " there are no Kurepa trees " is consistent so too is the theory ZFC + " there is an inaccessible cardinal ".*

In order to use the above result in the context of local sentences we state now the main technical result of this section.

Theorem 4.2 *There exists a local sentence Φ such that:*

$$[\Phi \text{ has an } \omega_2\text{-model }] \quad \longleftrightarrow \quad [\text{ there is a Kurepa tree }].$$

To prove this theorem we shall firstly state the two following lemmas.

Lemma 4.3 *There exists a local sentence φ_0 such that φ_0 has a well ordered model of order type ω but has no well ordered model of order type $> \omega$.*

Proof. Such a sentence is given in [FR96] in the signature $S(\varphi_0) = \{<, P, f, p_1, p_2\}$, where P is a unary predicate, f is a binary function, and p_1, p_2 are unary functions. □

Lemma 4.4 *There exists a local sentence φ_1 such that φ_1 has well ordered models of order type α, for every ordinal $\alpha \in [\omega, \omega_1]$, but has no well ordered model of order type $> \omega_1$.*

Proof. We give below the sentence φ_1 in the signature $S(\varphi_1) = S(\varphi_0) \cup \{Q, g\} = \{<, P, f, p_1, p_2, Q, g\}$, where Q is a unary predicate and g is a binary function. φ_1 is the conjunction of the following sentences (1)-(10) whose meaning is explained below:

(1) $\forall xyz[(x \leq y \vee y \leq x) \wedge ((x \leq y \wedge y \leq x) \leftrightarrow x = y) \wedge ((x \leq y \wedge y \leq z) \rightarrow x \leq z)]$,
(2) $\forall xy[(Q(x) \wedge \neg Q(y)) \rightarrow x < y]$,
(3) $\forall xy[(Q(x) \wedge Q(y)) \rightarrow f(x, y) \in Q]$,
(4) $\forall x[Q(x) \rightarrow Q(p_i(x))]$, for each $i \in [1, 2]$,
(5) $\forall xy[(\neg Q(x) \vee \neg Q(y)) \rightarrow f(x, y) = x]$,
(6) $\forall x[\neg Q(x) \rightarrow p_i(x) = x]$, for each $i \in [1, 2]$,
(7) $\forall x_1 \ldots x_j \in Q[\varphi_0'(x_1, \ldots, x_j)]$, where $\varphi_0 = \forall x_1 \ldots x_j \varphi_0'(x_1, \ldots, x_j)$ with φ_0' an open formula,
(8) $\forall xy[(\neg Q(x) \wedge \neg Q(y) \wedge y < x) \rightarrow Q(g(x, y))]$,
(9) $\forall xyz[(\neg Q(x) \wedge \neg Q(y) \wedge \neg Q(z) \wedge y < z < x) \rightarrow g(x, y) \neq g(x, z)]$,
(10) $\forall xy[(Q(x) \vee Q(y) \vee \neg(y < x)) \rightarrow g(x, y) = x]$.

We now explain the meaning of the above sentences (1)-(10).

Assume that M is a model of φ_1. The sentence (1) expresses that $<$ is interpreted in M by a linear order; (2) expresses that Q^M is an initial segment of the model M; (3) and (4) state that Q^M is closed under the functions of $S(\varphi_0)$ while (5) and (6) state that these functions are trivially defined elsewhere; (7) means that the restriction of the model M to the domain Q^M and to the signature of $S(\varphi_0)$ is a model of φ_0; Finally (8) and (9) ensure that, for each $x \in \neg Q$, the binary function g realizes an injection from the segment $\{y \in \neg Q \mid y < x\}$ into Q and (10) states that the function g is trivially defined where it is not useful for that purpose.

The sentence φ_1 is a conjunction of universal sentences thus it is equivalent to a universal one, and closure in its models takes at most $n_\varphi + 1$ steps: one applies first the function g and then the functions of $S(\varphi_0)$. Thus the sentence φ_1 is local.

Consider now a well ordered model M of φ_1. The restriction of M to the domain Q^M and to the signature of $S(\varphi_0)$ is a well ordered model of φ_0 hence it is of order type $\leq \omega$. But the function g defines an injection from each initial segment of $\neg Q$ into Q thus each initial segment of $\neg Q$ is countable and this implies that the order type of $\neg Q^M$ is smaller than or equal to ω_1. Finally we have proved that the order type of M is $\leq \omega_1$.

Conversely it is easy to see that every ordinal $\alpha \in [\omega, \omega_1]$ is the order type of some model of φ_1. □

Return now to the construction of the sentence Φ given by Theorem 4.2. We are going to explain this construction by several successive steps.

A model M of Φ will be totally ordered by $<$ and will be the disjoint union of four successive segments. This will be expressed by the following sentence Φ_1 in the signature $S(\Phi_1) = \{P_0, P_1, P_2, P_3\}$, where P_0, P_1, P_2, P_3, are unary predicate symbols. Φ_1 is the conjunction of:

(1) $\forall xyz[(x \leq y \vee y \leq x) \wedge ((x \leq y \wedge y \leq x) \leftrightarrow x = y) \wedge ((x \leq y \wedge y \leq z) \rightarrow x \leq z)]$,

(2) $\forall xy \bigwedge_{0 \leq i < j \leq 3} [(P_i(x) \wedge P_j(y)) \rightarrow x < y]$.

We want now to ensure that, if M is a well ordered model of Φ, then P_0^M is of order type $\leq \omega$ and P_1^M is of order type $\leq \omega_1$. For that purpose, the signature of Φ will contain the signature $S(\varphi_1) = S(\varphi_0) \cup \{Q, g\} = \{<, P, f, p_1, p_2, Q, g\}$ and Φ will express that if M is a model of Φ, then $P_0^M = Q^M$ and the restriction of the model M to $(P_0^M \cup P_1^M)$ and to the signature of φ_1 is a model of φ_1. This is expressed by the following sentence Φ_2 which is the conjunction of:

(1) $\forall x[Q(x) \leftrightarrow P_0(x)]$,

(2) $\forall xy[(x \in P_0 \cup P_1 \wedge y \in P_0 \cup P_1) \rightarrow f(x, y) \in P_0 \cup P_1]$,

(3) $\forall xy[(x \in P_0 \cup P_1 \wedge y \in P_0 \cup P_1) \rightarrow g(x, y) \in P_0 \cup P_1]$,

(4) $\forall x[(x \in P_0 \cup P_1) \rightarrow p_i(x) \in P_0 \cup P_1]$, for each $i \in [1, 2]$,
(5) $\forall xy[(x \notin P_0 \cup P_1 \vee y \notin P_0 \cup P_1) \rightarrow f(x, y) = x]$,
(6) $\forall xy[(x \notin P_0 \cup P_1 \vee y \notin P_0 \cup P_1) \rightarrow g(x, y) = x]$,
(7) $\forall x[x \notin P_0 \cup P_1 \rightarrow p_i(x) = x]$, for each $i \in [1, 2]$,
(8) $\forall x_1 \ldots x_k \in (P_0 \cup P_1)[\varphi_1'(x_1, \ldots, x_k)]$, where $\varphi_1 = \forall x_1 \ldots x_k \varphi_1'(x_1, \ldots, x_k)$ with φ_1' an open formula.

Above sentences (2)-(4) state that in a model M the set $(P_0 \cup P_1)^M$ is closed under the functions of $S(\varphi_1)$ while (5)-(7) state that these functions are trivially defined elsewhere; (8) means that the restriction of the model M to the domain $(P_0 \cup P_1)^M$ and to the signature of $S(\varphi_1)$ is a model of φ_1.

We want now that, in a model M of Φ, the set P_2^M represents the base set of a tree $(T, \prec)$. We shall use a binary relation symbol $\prec$. The following sentence Φ_3 is the conjunction of:

(1) $\forall xy[x \prec y \rightarrow P_2(x) \wedge P_2(y)]$,
(2) $\forall xyz[((x \preccurlyeq y \wedge y \preccurlyeq x) \leftrightarrow x = y) \wedge ((x \prec y \wedge y \prec z) \rightarrow x \prec z)]$.
(3) $\forall xy[x \prec y \rightarrow x < y]$.

Above sentences (1)-(2) express that $\prec$ is a partial order on P_2 and the sentence (3) ensures that, in a well ordered (for $<$) model M of Φ_3, for every $t \in P_2$, the set $\{s \in P_2 \mid s \prec t\}$ is well ordered under $\prec$ because M itself is well ordered under $<$.

Moreover we want now that in an ω_2-model M of Φ, the set P_2^M represents the base set of an ω_1-tree T whose levels are countable.
We have firstly to distinguish the different levels of the tree T. We shall use for that purpose unary functions I and p and the following sentence Φ_4 conjunction of:

(1) $\forall xy \in P_2[(I(y) \leq y) \wedge (y \leq x \rightarrow I(y) \leq I(x)) \wedge (I(y) \leq x \leq y \rightarrow I(x) = I(y))]$.
(2) $\forall xy \in P_2[x \prec y \rightarrow I(x) < I(y)]$,
(3) $\forall xyz \in P_2[(x \prec y \wedge z \prec y \wedge I(x) = I(z)) \rightarrow x = z]$,
(4) $\forall xy \in P_2[I(x) < I(y) \rightarrow (I(p(I(x), y) = I(x) \wedge p(I(x), y) \prec y)]$,
(5) $\forall xy[(\neg P_2(x) \vee \neg P_2(y) \vee I(x) \neq x \vee I(x) \geq I(y)) \rightarrow p(x, y) = x]$,
(6) $\forall x[\neg P_2(x) \rightarrow I(x) = x]$.

Above the sentence (1) is used to divide the segment P_2 of a model of Φ_4 into successive segments. The function I is constant on each of these segments and the image $I(x)$ of an element $x \in P_2$ is the first element of the segment containing x.
Sentences (2)-(3) ensure that if $y \in P_2$ then every element $x \in P_2$ such that $x \prec y$ belongs to some segment $I_z = \{w \in P_2 \mid I(w) = I(z)\}$ for some $z < I(y)$. Moreover for each $z < I(y)$, the segment I_z contains at most one

element of $\{x \in P_2 \mid x \prec y\}$.
The function p is used to ensure that, for each $z < I(y)$, the segment I_z contains in fact exactly one element $x \in P_2$ such that $x \prec y$: the element $p(I(z), y)$. This is implied by the sentence (4).
Thus Φ_4 will imply that each segment I_z is really a level of the tree T.
If $y \in P_2$ is at level α of the tree T and if $x \in P_2$ and I_x represents the β-th level T_β of the tree T for some $\beta < \alpha$ (so $I(x) < I(y)$), then the element $p(I(x), y)$ is the unique element $t \in T_\beta$ such that $t \prec y$.
Finally sentences (5)-(6) are used to trivially define the functions p and I where they are not useful as explained above.

The following sentence Φ_5 will imply that all levels of the tree T are countable and that $ht(T) \leq \omega_1$ hence also $card(T) \leq \aleph_1$. The signature of Φ_5 is $\{<, P_0, P_1, P_2, I, i, j\}$, where i and j are two new unary function symbols, and Φ_5 is the conjunction of:

(1) $\forall x[P_2(x) \rightarrow P_0(i(x))]$,
(2) $\forall xy[(P_2(x) \wedge P_2(y) \wedge I(x) = I(y) \wedge x \neq y) \rightarrow i(x) \neq i(y)]$,
(3) $\forall x[P_2(x) \rightarrow P_1(j(x))]$,
(4) $\forall xy[(P_2(x) \wedge P_2(y) \wedge x < y) \rightarrow j(x) < j(y)]$,
(5) $\forall x[\neg P_2(x) \rightarrow i(x) = x]$,
(6) $\forall x[\neg P_2(x) \rightarrow j(x) = x]$.

Above sentences (1)-(2) say that the function i is defined from P_2 into P_0 and that it is an injection from any level of the tree T into P_0. We have seen that in a well ordered model M of Φ the set P_0^M will be of order type $\leq \omega$ thus each level of the tree will be countable.
Sentences (3)-(4) say that the function j is strictly increasing from P_2 into P_1 thus in a well ordered model M of Φ the set P_1^M hence also P_2^M will be of order type $\leq \omega_1$. So we shall have $ht(T) \leq \omega_1$ and $card(T) \leq \aleph_1$.
Finally sentences (5)-(6) are used to trivially define the functions i and j on $\neg P_2 = P_0 \cup P_1 \cup P_3$.

In a well ordered model M of Φ of order type ω_2, the set P_2^M will be the base set of an ω_1-tree T and the set P_3^M will be identified to a set of branches of T. For that purpose we use two new binary function symbols h and k and the following sentence Φ_6, conjunction of:

(1) $\forall xy[(P_2(x) \wedge P_3(y)) \rightarrow (P_2(h(I(x), y)) \wedge I(h(I(x), y)) = I(x))]$,
(2) $\forall xyz[(P_2(x) \wedge P_2(y) \wedge P_3(z) \wedge I(x) < I(y)) \rightarrow h(I(x), z) \prec h(I(y), z)]$,
(3) $\forall xy[(\neg P_2(x) \vee \neg P_3(y) \vee x \neq I(x)) \rightarrow h(x, y) = x]$,
(4) $\forall xy[(P_3(x) \wedge P_3(y) \wedge x \neq y) \rightarrow (I(k(x, y)) = k(x, y) \wedge P_2(k(x, y)))]$,
(5) $\forall xy[(P_3(x) \wedge P_3(y) \wedge x \neq y) \rightarrow h(k(x, y), x) \neq h(k(x, y), y))]$,
(6) $\forall xy[(\neg P_3(x) \vee \neg P_3(y) \vee x = y) \rightarrow k(x, y) = x]$.

Above sentences (1)-(2) are used to associate a branch $b(z)$ of T to an element

$z \in P_3$. For each level T_α of the tree which is represented by the segment of P_2 whose first element is $I(x)$, the sentence (1) says that $h(I(x), z)$ is an element at the same level T_α and (2) says that the elements $h(I(x), z)$, for $x \in P_2$, are linearly ordered for $\prec$ hence they form a branch $b(z)$ of the tree T.
The function k is used to associate to two different elements x and y of P_3 a level of the tree T, which is represented by the element $k(x, y)$: the first element of the segment of P_2 representing this level. This is expressed by the sentence (4).
The sentence (5) says that, for two distinct elements x and y of P_3, the branches $b(x)$ and $b(y)$ differ at the level represented by $k(x, y)$.
Finally sentences (3) and (6) are used to trivially define the functions h and k in other cases.

We have seen that in a well ordered model M of Φ, P_1^M and P_2^M will be of order type $\leq \omega_1$. The following sentence Φ_7 will then imply that P_3^M is of order type $\leq \omega_2$. Its signature is $\{<, P_1, P_3, l\}$, where l is a binary function symbol, and Φ_7 is the conjunction of:

(1) $\forall xy[(P_3(x) \wedge P_3(y) \wedge y < x) \rightarrow P_2(l(x, y))]$,
(2) $\forall xyz[(P_3(x) \wedge P_3(y) \wedge P_3(z) \wedge y < z < x) \rightarrow l(x, y) \neq l(x, z)]$,
(3) $\forall xy[(\neg P_3(x) \vee \neg P_3(y) \vee \neg(y < x)) \rightarrow l(x, y) = x]$.

Above sentences (1)-(3) are in fact very similar to sentences (8)-(10) used in the construction of the sentence φ_1.
(1) and (2) ensure that, for each $x \in P_3$, the binary function l realizes an injection from the segment $\{y \in P_3 \mid y < x\}$ into P_2 and (3) states that the function l is trivially defined where it is not useful for that purpose.

We can now define the sentence

$$\Phi = \bigwedge_{1 \leq i \leq 7} \Phi_i$$

in the signature

$$S(\Phi) = \bigwedge_{1 \leq i \leq 7} S(\Phi_i) = \{<, P_0, P_1, P_2, P_3, Q, p_1, p_2, f, g, \prec, p, I, i, j, h, k, l\}.$$

Φ is a conjunction of universal sentences thus it is equivalent to a universal sentence and closure in its models takes at most 7 steps: one takes firstly closure under the function l then under the functions I and k, then under the functions h and p, then under i and j, then under the function g, then under the functions p_1 and p_2, and finally under the function f. Notice that the two last steps are due to the construction of φ_0 and the fact that $n_{\varphi_0} = 2$ (see

[FR96]).
Assume now that M is a well ordered model of Φ. By construction P_0^M is of order type $\leq \omega$, P_1^M and P_2^M are of order types $\leq \omega_1$, and P_2^M is the base set of a tree T whose levels are countable. Moreover every strict initial segment of P_3^M is of cardinal $\leq \aleph_1$, so P_3^M is of order type $\leq \omega_2$. Finally we have got that M itself is of order type $\leq \omega_2$.
Suppose now that M is of order type ω_2. Then P_3^M also is of order type ω_2 and for every strict initial segment J of P_3^M there is an injection from J into P_2^M thus P_2^M is of cardinal $\aleph_1$. But its order type is $\leq \omega_1$, hence it is in fact equal to ω_1.
The tree T is then really an ω_1-tree and all its levels are countable. Moreover the set P_3^M can be identified to a set of branches of T thus $card(\mathcal{B}(T)) > \aleph_1$ and T is a Kurepa tree.
Conversely if there exists a Kurepa tree, we can easily see that Φ has an ω_2-model. $\square$

We can now infer from Theorems 4.1 and 4.2 the following result which shows that the local theory of ω_2 is not determined by the axiomatic system $ZFC + GCH$.

Theorem 4.5 *If the theory ZFC + " there is an inaccessible cardinal " is consistent then " Φ has an ω_2-model " is independent of $ZFC + GCH$.*

Notice that this result can be extended easily to ordinals larger than ω_2. For instance reasoning as in the construction of the local sentence φ_1 from the local sentence φ_0 (see Lemma 4.4 above), we can construct by induction, for each integer $n \geq 2$, a local sentence Ψ_n such that: for all ordinals $\alpha \in]\omega_n, \omega_{n+1}]$,
(Ψ_n has an α-model) iff (Φ has an ω_2-model) iff (there is a Kurepa tree).
This implies the following extension of Theorem 4.5.

Theorem 4.6 *If the theory ZFC + " there is an inaccessible cardinal " is consistent then for each integer $n \geq 2$ and each ordinal $\alpha \in]\omega_n, \omega_{n+1}]$, " Ψ_n has an α-model " is independent of $ZFC + GCH$.*

A similar result can be obtained for larger ordinals of cofinality ω_n, for an integer $n \geq 2$.
We can first construct the local sentence $\Theta_2 = \theta^{\star\Phi}$, from the local sentence θ given in section 3 whose spectrum is the class of successor ordinals, and the local sentence Φ we have constructed above.
It is then easy to see that Θ_2 has not any well ordered model whose order type is an ordinal α having a cofinality greater than ω_2. Moerover if α is an ordinal of cofinality ω_2 then the local sentence Θ_2 has a model of order type α if and only if Φ has an ω_2-model.
In the same way, for each integer $n \geq 2$, we can construct the local sentence $\Theta_{n+1} = \theta^{\star\Psi_n}$, from θ and the local sentence Ψ_n cited in the above theorem.

It is then easy to see that Θ_{n+1} has not any well ordered model whose order type is an ordinal having a cofinality greater than ω_{n+1} because by construction the local sentence Ψ_n has no well ordered model of order type greater than ω_{n+1}. Moreover if α is an ordinal of cofinality ω_{n+1} then the local sentence Θ_{n+1} has a model of order type α iff Ψ_n has an ω_{n+1}-model iff Φ has an ω_2-model iff there is a Kurepa tree.
So we have got the following extension of Theorem 4.6.

Theorem 4.7 *If the theory ZFC + " there is an inaccessible cardinal " is consistent then for each integer $n \geq 2$ and each ordinal α of cofinality ω_n, " Θ_n has an α-model " is independent of $ZFC + GCH$.*

5 The local theories of ω_n, $n \geq 1$

We have already mentioned in the introduction that it would be still possible that there are only finitely many possible local theories of ω_2 and that each of them is decidable, but that it is more plausible that the situation is much more complicated.

On the other hand the above method cannot be applied to study the local theory of ω_1. We are going to prove in this section that the local theory of ω_1 is recursive in the local theory of ω_2, and more generally that, for all integers n, p, $1 \leq n < p$, the local theory of ω_n is recursive in the local theory of ω_p.

Lemma 5.1 *For each integer $n \geq 0$, there exists a local sentence φ_n such that $Sp_\infty(\varphi_n) = [\omega, \omega_n]$.*

Proof. We have already proved this result in the cases $n = 0$ and $n = 1$ by proving Lemmas 4.3 and 4.4. We can now construct by induction on the integer n a local sentence φ_n such that $Sp_\infty(\varphi_n) = [\omega, \omega_n]$. The local sentence φ_n is constructed from the local sentence φ_{n-1} in a similar manner as in the construction of the local sentence φ_1 from the local sentence φ_0 (see the proof of Lemma 4.4). Details are here left to the reader. □

Lemma 5.2 *For every integer $n \geq 1$, there exists a recursive function $\mathcal{S}_n$ defined on the set of first order sentences (whose signatures contain the binary symbol $<$) such that, for a first order sentence φ, [φ is local] if and only if [$\mathcal{S}_n(\varphi)$ is local] and [φ has an ω_n-model] if and only if [$\mathcal{S}_n(\varphi)$ has an ω_{n+1}-model].*

Proof. Let n be an integer ≥ 1 and φ be a first order sentence in a signature $\mathrm{S}(\varphi)$. We are going to explain informally the construction of the sentence $\mathcal{S}_n(\varphi)$ from the sentence φ using similar methods as in the preceding section.

We can assume that S(φ)$\cap$ S(φ_n)=$\{<\}$. The signature of $\mathcal{S}_n(\varphi)$ is equal to S(φ_n) $\cup$ S(φ) $\cup$ $\{s, t, R_1, R_2, R_3\}$, where s is a new unary function symbol, t is a new binary function symbol, and R_1, R_2, R_3 are three unary predicate symbols not in S(φ_n) $\cup$ S(φ).

The sentence $\mathcal{S}_n(\varphi)$ expresses that a model M is linearly ordered by the binary relation $<^M$, and that R_1^M, R_2^M, and R_3^M are three successive segments of M. Then $\mathcal{S}_n(\varphi)$ expresses that the restriction of R_1^M to the signature of φ_n is a model of φ_n and the restriction of R_2^M to the signature of φ is a model of φ (a k-ary function of S(φ_n) is trivially defined out of R_1 by $f(x_1, \ldots, x_k) = x_1$ and similarly functions of S(φ) are trivially defined out of R_2).
The function s is a strictly non decreasing function from R_2^M into R_1^M and is trivially defined by $s(x) = x$ elsewhere.
The function t is used to realize, for every element $a \in R_3^M$, an injection from $\{x \in R_3 \mid x < a\}$ into R_2 (as in the proof of Lemma 4.4) and it is trivially defined where it is not useful for that purpose.

The function $\mathcal{S}_n$ is clearly recursive and it is easy to see that φ is local iff $\mathcal{S}_n(\varphi)$ is local.
In that case it holds that $n_{\mathcal{S}_n(\varphi)} = n_\varphi + n_{\varphi_n} + 2$. Indeed to take the closure of a set X in a model M of $\mathcal{S}_n(\varphi)$ one takes the closure under the function t, then under the functions of S(φ) in n_φ steps, then under the function s, then under the functions of S(φ_n) in n_{φ_n} steps.

Assume now that $\mathcal{S}_n(\varphi)$ has an ω_{n+1}-model M. In this model $(R_1^M, <^M)$ and $(R_2^M, <^M)$ have order types smaller than or equal to ω_n because $Sp_\infty(\varphi_n) = [\omega, \omega_n]$ and there is a strictly non decreasing function s^M from R_2^M into R_1^M. Thus $(R_3^M, <^M)$ must have order type ω_{n+1}. Every strict initial segment of R_3^M is injected in R_2^M so R_2^M has cardinality $\aleph_n$ and its order type is exactly ω_n. This implies that the restriction of the model M to R_2^M and to the signature of φ is an ω_n-model of φ.
Conversely it is easy to see that by construction if there is an ω_n-model of φ then there is an ω_{n+1}-model of $\mathcal{S}_n(\varphi)$. □

We can now state the following result. Recall that the local theory of an ordinal α is the set of local sentences having a model of order type α; it will be denoted by $LT(\alpha)$.

Theorem 5.3 *For all integers $n, p \geq 1$, if $n < p$ then the local theory of ω_n is recursive in the local theory of ω_p.*

Proof. It follows directly from Lemma 5.2 that for each integer $n \geq 1$ the local theory of ω_n is recursive in the local theory of ω_{n+1} because

$$LT(\omega_n) = \mathcal{S}_n^{-1}(LT(\omega_{n+1}))$$

where $\mathcal{S}_n$ is a recursive function. We can now infer, by induction on the integer $p > n$, that if $n < p$ then the local theory of ω_n is recursive in the local theory of ω_p. $\square$

Remark 5.4 *We have called here local theory of α the set of* **all** *local sentences having a model of order type α. We could have restricted this set to local sentences in the recursive set* **L** *given by Proposition 2.3.*
We can get a similar result in that case, defining firstly the recursive function $\mathcal{S}_n$ only on this set **L** *with values in* **L**. *This is possible because we have seen that for a local sentence φ it holds that $n_{\mathcal{S}_n(\varphi)} = n_\varphi + n_{\varphi_n} + 2$. Thus we can compute $n_{\mathcal{S}_n(\varphi)}$ from n_φ.*

Theorem 5.3 states that if $n < p$ then the local theory of ω_n is less "complicated" than the local theory of ω_p because there is a recursive reduction of the first one to the second one.
We are going to prove the following similar result.

Theorem 5.5 *For all integers $n \geq 1$, if α is an ordinal of cofinality ω_n then the local theory of ω_n is recursive in the local theory of α.*

We shall proceed by successive lemmas.

Lemma 5.6 *Let ψ be a local sentence and n be an integer ≥ 1, then there exists another local sentence ψ'_n such that $Sp_\infty(\psi'_n) \subseteq [\omega, \omega_n]$ and the following equivalence holds: [ψ has a model of order type ω_n] iff [ψ'_n has a model of order type ω_n].*

Proof. Let ψ be a local sentence (with $<\in$ S(ψ)) and n be an integer $n \geq 1$. We now explain informally the construction of a local sentence ψ'_n such that $Sp_\infty(\psi'_n) \subseteq [\omega, \omega_n]$ and [ψ has a model of order type ω_n] iff [ψ'_n has a model of order type ω_n].

The signature of ψ'_n is S(ψ'_n)=S(φ_{n-1}) $\cup$ S(ψ) $\cup\{R, t\}$ where R is a new unary predicate symbol and t is a new binary function symbol not in S(φ_{n-1}) $\cup$ S(ψ). The sentence ψ'_n expresses that in a model M, R^M is an initial segment of the model which is closed under functions of S(φ_{n-1}); and the restriction of M to R^M and to the signature S(φ_{n-1}) is a model of φ_{n-1}. In the same way the restriction of M to $\neg R^M$ and to the signature S(ψ) is a model of ψ. The function t is used to realize, for every element $a \in \neg R$, an injection from $\{x \in \neg R \mid x < a\}$ into R (as in the proof of Lemma 4.4) and it is trivially defined where it is not useful for that purpose.
We know that the sentence φ_{n-1} given by Lemma 5.1 has infinite spectrum $Sp_\infty(\varphi_{n-1}) = [\omega, \omega_{n-1}]$ so in a well ordered model M of ψ'_n the initial segment R^M will have order type $\leq \omega_{n-1}$. Moreover every strict initial segment of $\neg R^M$ will be of cardinal $\leq \aleph_{n-1}$ because it is injected into R^M, so $\neg R^M$ will be of order type $\leq \omega_n$ thus M will be also of order type $\leq \omega_n$. We have then proved

that $Sp_\infty(\psi'_n) \subseteq [\omega, \omega_n]$.
It is now easy to see that if ψ'_n has a model M of order type ω_n, then the restriction of M to $\neg R^M$ and to the signature $\mathrm{S}(\psi)$ is a model of ψ whose order type is ω_n; conversely if ψ has an ω_n-model then there is an ω_n-model of ψ'_n.

The sentence ψ'_n is equivalent to a universal sentence and closure in its models takes at most $n_{\psi'_n} = n_\psi + 1 + n_{\varphi_{n-1}}$. One takes closure under functions of $\mathrm{S}(\psi)$ in n_ψ steps, then closure under the function t in one step, then closure under functions of $\mathrm{S}(\varphi_{n-1})$ in $n_{\varphi_{n-1}}$ steps. Thus ψ'_n is a local sentence. $\square$

Lemma 5.7 *For every integer $n \geq 1$, there exists a recursive function $\mathcal{T}_n$, defined on the set of first order sentences ψ with $<\in S(\psi)$, such that, for every first order sentence ψ, [ψ is local] if and only if [$\mathcal{T}_n(\psi)$ is local] and, for every local sentence ψ, [ψ has an ω_n-model] if and only if [$\mathcal{T}_n(\psi)$ has an α-model] where α is any ordinal of cofinality ω_n.*

Proof. Let n be an integer ≥ 1 and ψ be a first order sentence in a signature $\mathrm{S}(\psi)$. We define $\mathcal{T}_n(\psi) = \theta^{\star\psi'_n}$ where θ is the local sentence whose spectrum is the class of successor ordinals, and ψ'_n is the first order sentence constructed as above from the sentence ψ.
Notice that in preceding lemma the sentence ψ'_n is constructed from a *local sentence* ψ but we can easily extend the construction to all *first order sentences* ψ. Then it holds that ψ is local iff ψ'_n is local.
The sentence $\theta^{\star\psi'_n}$ can also be defined even if ψ'_n is not local, with slight modifications, in such a way that models of $\theta^{\star\psi'_n}$ are still essentially direct sums of models of θ, these models being ordered by the order type of a model of ψ'_n. Moreover it holds also that [$\theta^{\star\psi'_n}$ is local] iff [ψ'_n is local]. Thus [ψ is local] iff [$\mathcal{T}_n(\psi)$=$\theta^{\star\psi'_n}$ is local].
Consider now a local sentence ψ and an ordinal α having cofinality ω_n. Then by Lemma 5.6 the sentence ψ has an ω_n-model iff ψ'_n has an ω_n-model and $Sp_\infty(\psi'_n) \subseteq [\omega, \omega_n]$. This implies that [$\psi$ has an ω_n-model] iff [$\theta^{\star\psi'_n}$ has an α-model] because α has cofinality ω_n. $\square$

We can now end the proof of Theorem 5.5. It follows from Lemma 5.7 that for each integer $n \geq 1$ the local theory of ω_n is recursive in the local theory of α, where α is an ordinal having cofinality ω_n. Indeed

$$LT(\omega_n) = \mathcal{T}_n^{-1}(LT(\alpha))$$

where $\mathcal{T}_n$ is a recursive function. $\square$

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