A FACTORISATION THEOREM FOR CURVES WITH VANISHING SELF-INTERSECTION

P. DINGOYAN

1. Introduction

1.1. We give, in some cases, a positive answer to a question of F. Campana:

Let $i : C_0 \to Y$ be a smooth curve in a compact Kähler surface. Assume:

i) the self-intersection of C_0 is vanishing $C_0.C_0 = 0$,

ii) the image of the fundamental group of C_0 in Y is of infinite index $[\pi_1(Y) : i_*\pi_1(C_0)] = +\infty$.

Question: Is C_0 a fiber of a holomorphic map $f : Y \to B$ from Y to a curve ?

Let $p : X \to Y$ be the infinite covering of Y with fundamental group $i_*\pi_1(C_0)$. Then $i : C_0 \to Y$ lifts to an embedding $s : C_0 \to X$ with $s(C_0).s(C_0) = 0$. We give the following positive answer:

Theorem 1.1.1. *Assume that a finite covering of $p : X \to Y$ is Galois or admits a proper Green function. Then there exists proper holomorphic maps $f' : X \to B'$ and $f : Y \to B$ to curves B and B' such that $s(C_0)$ is a fiber of f' and C_0 is a fiber of f. In particular X is holomorphically convex.*

The theorem applies in particular when $i_*\pi_1(C_0)$ is finite and $\pi_1(Y)$ is infinite. We give below a more general statement for an effective divisor in a Kähler compact manifold.

1.2. The strategy is to solve the Poincaré-Lelong equation $i\partial\overline{\partial}u = s(C_0)$ (in the sense of currents) and then use the logarithmic form ∂u to produce the required fibration. In general, the obstruction to solve this equation lies in the first Chern class of the line bundle $[s(C_0)]$. Our basic observation is that $s(C_0).s(C_0) = 0$ implies that the first Chern class of $[s(C_0)]$ vanishes in cohomology. This fact is a consequence of the Hodge index theorem in complete Kähler manifolds of infinite volume (see sect. 3). In particular:

Proposition 1.2.1. *Let C_0 be a divisor in a smooth Kähler surface which satisfies $i)$ and $ii)$ above. Then its first Chern class $c_1([C_0]) \in H^2(Y, \mathbb{Z})$ lies in $H^2(\pi_1(Y), \mathbb{Z})$.*

We then find conditions which imply that the $\partial\overline{\partial}-$lemma of Kähler geometry is valid in the non compact setting: in the case of a Galois cover, we use the Galois $\partial\overline{\partial}-$lemma that was proved by the author in [11]. In the case where a proper Green function exists, we follow the approach of Gromov [13] and Napier-Ramachandran [19], [20]: to solve the $\partial\overline{\partial}-$equation, it is enough to solve the Laplace equation on functions and integrate by parts if the geometry allows.

1.3. Comparison with some previous works. In [13], [19], [20], the authors deduce factorisation theorems from the study of an analytic $1-$form associated to a square integrable or compactly supported $1-$form. Here one uses a $2-$form with compact support and a positivity assumption and produces a logarithmic $1-$form. Then the study of the foliation follows almost the same pattern than the aformentioned articles. However, in the case of a Galois covering, the use of the automorphism group allows a more elementary construction.

M. Nori [22] studied the image of the fundamental group of the normalization $\pi_1(\bar{C}) \to \pi_1(Y)$ of a non smooth curve C whose irreducible components have positive self-intersection. This lead to a weak Lefschetz theorem: the image is of finite index. From this point of view, the weak lefschetz theorem was reconsidered in [21].

F. Campana [6] studied consequences of the negativity of the self-intersection of a compact curve in an infinite covering and gave some factorisation theorems (see loc. cit. sect. 4): if a smooth curve in a Kähler compact surface satisfies *ii*) and has a torsion normal bundle then C is a fiber of a holomorphic map. His proof used Ueda's theory [28] describing a neighborhood of a smooth compact curve with vanishing self-intersection in a complex surface.

B.Tortaro [27] and J.Pereira [23] proves some factorisation theorem for divisors with some vanishing hypothesis on Chern class. For instance, assume three smooth disjoint divisors in a manifold X lie in the same rational cohomology class. Then they prove there exists a holomorphic map from X to a curve such that each divisor is a fiber. B.Tortaro [27] gives examples of two smooth disjoint divisors in a surface X which lie in the same rational cohomology class but which are not fiber of a map to a curve. One can check that the image of the fundamental group of each divisor is of finite indice in the fundamental group of X.

The topic of Shafarevitch conjecture and Shafarevich dimension is discussed in [7], [17]. A different point of view is given by Gurjar and his co-authors (see [14], [15]): they studied some fibrations $p : Y \to B$ of compact surfaces and proved that the image of the fundamental group of any fiber in the fundamental group of Y is finite.

1.4. Acknowledgement. I warmly thank F.Campana for the discussions around this subject. He is at the root of the question and discussions with him lead to some of the conclusions. Our contribution is mainly technical. I thank also the complex analysis team of Nancy for its pleasant welcome to its seminar.

2. PRELIMINARIES

In this section we recall that a local proper fibration always extends to a global proper fibration in a manifold of bounded geometry. We refer to [13] or [25] for a definition of a manifold of bounded geometry. We note that a covering of a compact Riemannian manifold with the pull-back structure is of bounded geometry.

2.1. Notations. If D is a divisor, let $|D|$ be its support and let $[D]$ be the line bundle it defines. We first recall:

Lemma 2.1.1. *Let $f : X \to Y$ be a continuous map between locally compact spaces. Let $p \in X$ be such that $f^{-1}f(p)$ is compact. Then there exists neighborhoods U of $f^{-1}f(p)$ and U_1 of $f(p)$ such that $f : U \to U_1$ is proper.*

Proof. see [26] p.77 or Sätz 9 or [4] p.27. □

Definition 2.1.2. *Let D_1 and D_2 be two divisors in a complex manifold X. Assume D_1 is compact. If $\dim_{\mathbb{C}} X \geq 3$, let ω be a fixed Kähler form. Let h be a smooth metric on $[D_1]$ such that the Chern form $c_1(h)$ has compact support. Then we defined the intersection number by*

$$D_1.D_2 := \int_X c_1(D_1) \wedge c_1(D_2) \wedge \omega^{n-2} \,. \quad (1)$$

From [9] sect.2 (in particular 2.11 and 2.12), one can check that the intersection number of divisors without common irreducible components is positive. In particular if $\{D_i\}$ is a (finite) family of compact effective divisor such that $\cup_i |D_i|$ is connected then the generalized Zariski's lemma holds for the intersection form on $\{D_i\}$ (see [6] appendix A.A or [4] Chap.1 sect.2).

Lemma 2.1.3 (Zariski's lemma)**.** *Let $(X, Y) \mapsto X.Y$ be a bilinear form defined on $\oplus_{i=1}^{n} \mathbb{R} e_i$ such that $e_i.e_j \geq 0$ if $i \neq j$ and the matrix $(e_i.e_j)_{i,j}$ is irreducible (the graph, with an edge between each pair $\{e_i, e_j\}$ such that $e_i.e_j \neq 0$, is connected). Then only one of the following possibility occurs for the quadratic form it defines:*

i) *Q is negative definite.*
ii) *Q is semi-definite of rank $n-1$, with a one dimensional annihilator spanned by a strictly positive vector $X_0 > 0$ (i.e. its coordinate are positive).*
iii) *$Q(X) > 0$ for some X, then there exists $X_0 > 0$ such that $\forall i$, $X_0.e_i > 0$.*

One deduces:

Lemma 2.1.4. *Let D be an effective divisor with connected compact support $|D|$ in a complex manifold X. If $\dim_{\mathbb{C}} X \geq 3$, assume there exists a Kähler form ω and define self-intersection of a compact divisor as in (1). Assume $D.D = 0$ and there exists an holomorphic map $f : X \to \Delta$ to the unit disc. Then $|D|$ is a connected component of a fiber of f. Hence f is proper in a neighborhood of $|D|$. Moreover if D is not a multiple of an effective divisor and $f(|D|) = 0$, there exists $l \in \mathbb{N}^*$ such that the divisor defined by the fiber over 0 is equal to $l.D$*

The following theorem is noteworthy: a local proper holomorphic map to a curve extends to a global proper holomorphic map. The second point says that a divisor moves if it moves in a covering. We refer to [13], [2], [19], [21] and [27] for variations on this theorem. For the sake of completness, we sketch a proof using Barlet's space.

Theorem 2.1.5.
1) *Let $H \to X$ be a compact hypersurface in a Kähler complex manifold of bounded geometry. Assume that there exists a neighborhood W of H and an holomorphic map $f : W \to \Delta$ to the unit disc such that $f^{-1}(0) = H$. Then there exists a curve B and a proper map $a : X \to B$ extending f. In particular X is holomorphically convex.*
2) *Let D_0 be a divisor in a compact Kähler manifold Y. Assume $D_0.D_0 = 0$ and D_0 is not a sum of non trivial effective divisors with this property. Assume there exists a covering $p : X \to Y$ such that a compact connected component T of $p^{-1}(|D_0|)$ admits a non constant holomorphic function on one of its neighborhood. Then there exists a holomorphic map $b : Y \to B$ to a curve such that a multiple of D_0 is a fiber.*

Proof.
1.0) Using lemma 2.1.1, we may assume that f is proper. Let $f^*(0)$ be the divisor defined by the fiber over 0. Let D be the smallest effective divisor supported on $f^{-1}(0)$ such that $D.D = 0$. Zariski's Lemma 2.1.3 implies $f^*(0) = l.D$ for some $l \in \mathbb{N}^*$. Let $\mathcal{C}_{l.D}(X)$ be the connected component of Barlet's space [3] of X which contained $l.D$. Let $n : n\mathcal{C}_{l.D}(X) \to \mathcal{C}_{l.D}(X)$ be a normalization of this space. If s is a point of Barlet's space, let D_s be the associated cycle. From the definition of Barlet's space, the map f defines a morphism $i : \Delta \to \mathcal{C}_{l.D}(X)$ that maps 0 to $l.D$. Shrinking Δ if necessary, we may assume that a fiber over Δ^* is smooth.
1.1) From [3] Lemma 1 p.37 and the fact that a cycle D_s is homologuous to $l.D$, we deduce that $i : \Delta \to \mathcal{C}_{l.D}(X)$ is a local parametrization and $\mathcal{C}_{l.D}(X)$ is a smooth curve at $l.D$.
1.2) Let $\mathcal{D} = \{(x, s) \in X \times n\mathcal{C}_D(X)$ such that $x \in D_s\}$ be the universal divisor with associated projections f_1, f_2. From [3] th1 p.38 and 1.1), we deduce that D_s is reduced and irreducible if s is a generic point in $n\mathcal{C}_D(X)$. But if $s, s' \in n\mathcal{C}_D(X)$, then $D_s.D_{s'} = 0$, hence for generic s, s', these divisors are either disjoint or equal. This implies:

$$ii)\, \mathcal{D} \to X \text{ is finite} \quad \text{and} \quad ii)\, \mathcal{D} \to X \text{ is generically one to one.}$$

1.3) The main point is that $\mathcal{D} \to X$ is proper, hence onto, if (X, ω) is of bounded geometry: in such a manifold, homologuous irreducible cycles have bounded volume and bounded diameter. Hence Bishop's compactness theorem may be applied. For a proof, see e.g. [7] prop.3.12.
1.4) The map $\mathcal{D} \to X$ is one to one and a biholomorphism for X is normal. Let $g = f_1^{-1} : X \to \mathcal{D}$. Then $a = f_2 \circ g : X \to n\mathcal{C}_D(X)$ is a proper holomorphic map such that $f_2 \circ g_{|W} = i \circ f$ and $f^{-1}(0) = l.D$.
2) Let $p : X \to Y$ be an etale covering. Assume a connected component T of $p^{-1}(|D_0|)$ is compact. Let D be the divisor with support $|D| = T$ and structure scheme defined by $p^*(D_0)$.

Let W' be a neighborhood of $|D|$ such that $p' = p_{|W'} : W' \to p(W')$ is a finite covering of order m and W' is the connected component of $p^{-1}p(W')$ which contains $|D|$ (one uses a retract of a neighborhood of $|D_0|$ onto $|D_0|$).

Let $\sigma_i(f)$ be the $i-$th symetric function of f with respect to p'. Then $\sigma_i(f)$ vanishes on $|D_0| = p(|D|)$. But $D_0.D_0 = 0$ implies that $(\sigma_i(f) = 0) = \alpha_i D_0 + L_i$, where L_i is a divisor with support disjoint from $|D_0|$. Using that D is a connected component of a pullback, one easily sees that $\alpha_m = m.l$.

We deduce from the first case of the theorem that $\mathcal{C}_{m.l.D_0}(Y)$ is a smooth curve at $m.l.D_0$ and there exists a map $b : Y \to n\mathcal{C}_{m.l.D_0}(Y)$ induced by $\sigma_m(f)$. □

3. The Hodge index theorem for square integrable $(1,1)-$forms.

3.1. The Hodge index theorem. This section contains our main lemma.

Lemma 3.1.1. *Let (X,ω) be a complete Kähler manifold of infinite volume.*

1) *Any l^2-harmonic $(1,1)-$form is primitive. In particular the intersection form is negative definite on $\mathcal{H}^{1,1}_{(2)}(X)$ the space of square integrable harmonic $(1,1)-$forms.*

2) *Let α be a closed $(1,1)-$form with compact support such that $\int_X \alpha\wedge\bar\alpha\wedge\omega^{n-2}\geq 0$, then α is orthogonal to the harmonic forms and α is $d-$exact.*

Proof. see also [10] for (1) and (2). For notational convenience, if a,b are $(1,1)-$forms on X, one sets $\int a\wedge b := \int_X a\wedge b\wedge\omega^{n-2}$.

1) Let $h\in\mathcal{H}^{1,1}_{(2)}(X)$ then $\Lambda h\in\mathcal{H}^{0}_{(2)}(X)$. However square integrable harmonic functions are identically nul on a complete manifold of infinite volume. This is a consequence of Gaffney theorem ([12], [1] p.93, [8] th.26). Hence Λh is vanishing. This prove that the form $h\wedge\bar h\wedge\omega^{n-2}$ is negative and $||h||^2=-h\wedge\bar h\wedge\omega^{n-2}$ (see [16] sect. 1.2).

2) Let $\alpha=h+e$ be the decomposition of α in its harmonic part h plus a form $e\in\overline{\mathrm{R}an(d)}=\mathrm{K}erd^{*\perp}$. Then $\int\alpha\wedge\bar\alpha=\int h\wedge\bar h+\int e\wedge\bar e+2\mathrm{Re}\int e\wedge\bar h$. But $\int e\wedge\bar h=\pm(e,\star h)=0$ for $\star h$ is harmonic if h is. Also $\int e\wedge\bar e=\int\alpha\wedge\bar e=\lim_i\int\alpha\wedge\overline{d\theta_i}$ for the form is compactly supported. Integrating by parts, one sees that $\int e\wedge\bar e=0$. Hence $-||h||^2=\int h\wedge\bar h\geq 0$.

To conclude, we use that if a closed form with compact support is orthogonal to the l^2-harmonic forms then it is exact (see e.g. [18] lemma 1.92). □

3.2. Negativity of self-intersection of compact divisors. As a corollary, we obtain a general result on self-intersection of a compact divisor in a complete Kähler manifold of infinite volume (see [21], [6] sect. 2).

Corollary 3.2.1.

1) *Let X be a complete Kähler manifold of infinite volume. Let C be a compact divisor. Then $C.C\leq 0$ and if $C.C=0$ then the cohomology class $c_1([C])$ is vanishing in $H^2(X,\mathbb{R})$.*

2) *Let Y be a compact Kähler manifold. Let $i: C\to Y$ be a divisor in Y such that $i_*\pi_1(C)$ is of infinite index in $\pi_1(Y)$ and $C.C=0$. Then $c_1([C])\in H^2(Y,\mathbb{Z})$ belongs to $H^2(\pi_1(Y),\mathbb{Z})$. Hence the pullback of this class to the universal covering is trivial.*

Proof.

1) is a direct consequence of the lemma.

2) Let $\tilde C_i$ be a connected component of $\pi^{-1}(C)$. Let $f_i: X_i = X^u/G_i\to Y$ be the quotient of the universal covering X^u of Y by the stabiliser of $\tilde C_i$. Then $f_i:\tilde C/G_i=C_i\to C$ is a compact divisor in X_i which is biholomorphic to C, so that $C_i.C_i=0$. This implies that $c_1([C_i])$ is trivial in $H^2(X_i,\mathbb{R})$ hence also $c_1([\tilde C_i])=f_i^{-1}c_1([C_i])$. Therefore the cohomology class of $\pi^*(C)$ is trivial. The class $c_1([C])$ defines a class in $H^2(\Pi_1(Y),\mathbb{Z})$ (cohomology classes on Y which are exact on X^u). □

We may compare this with the case of a fibration: let $f: Y\to W$ be a holomorphic fibration onto a curve of genus greater than one. Then W is a $K(G,1)$, hence $\mathbb{Z}=H^2(W,\mathbb{Z})\simeq H^2(\pi_1(W),\mathbb{Z})$. We obtain a natural morphism $H^2(W,\mathbb{Z})\to H^2(\Pi_1(Y),\mathbb{Z})$. It maps the cohomology of a point $p\in W$, to the cohomology of the fiber $f^{-1}(p)$ in Y.

Corollary 3.2.2 (Weak Lefschetz)**.** *Let Y be a compact complex Kähler manifold. Let $i: C\to Y$ be a divisor in Y such that $C.C>0$. Then the index of $i_*\pi_1(|C|)$ in $\pi_1(Y)$ is finite.*

4. The case of a compact divisor in an infinite Galois covering.

4.1. Notations and definitions. Let $p : X \to Y$ be a covering. In general, the space of square integrable sections of a pull-back bundle will be defined through the pull-back metrics.

a) Let Γ be the deck transformation group of the covering $p : X \to Y$. Then $\mathbb{C}[\Gamma]$ acts on $l^2(X)$ by a representation λ such that $\lambda(\sum_{g\in\Gamma} a_g\delta_g).(F)(x) = \sum_{g\in\Gamma} a_g F(g^{-1}x)$. The weak closure (or the bicommutant) of $\lambda(\mathbb{C}[\Gamma])$ is given by the action of the Von Neumann algebra $\mathrm{M}(\Gamma)$ of the discret group Γ.

b) An element $\lambda_s(f) \in \mathrm{M}(\Gamma)$ is caracterised by its value $f = \lambda_s(f)(\delta_e) \in l^2(\Gamma)$ on the Dirac mass at the neutral element δ_e. Then $\lambda_s(f) : l^2(\Gamma) \to l^2(\Gamma)$ is given by the left convolution of f on $l^2(\Gamma)$ and $\lambda(f) : l^2(X) \to l^2(X)$ is defined through the weak limit $\sum_{g\in\Gamma} f(g)\lambda(\delta_g).F$ in $l^2(X)$.

c) By definition, an element of $\mathrm{M}(\Gamma)$ is a weak isomorphism (or is almost invertible) if it is injective with dense range. The fact that $\mathrm{M}(\Gamma)$ admits a finite trace implies that it is enough to be injective or with dense range.

We will use the Von Neumann algebra $\mathrm{M}(\Gamma)$ through the following lemma ([11] cor.3.4.6):

Lemma 4.1.1 (A Galois $\partial\overline{\partial}-$lemma)**.** *Let $p : X \to Y$ be a Galois covering of a compact Kähler manifold. Let α be a $d-$closed square integrable $(p, q)-$form on X which is orthogonal to harmonic forms. Then there exists a weak isomorphism $r \in M(\Gamma)$, there exists a square integrable form β on X such that $\lambda(r).\alpha = \partial\overline{\partial}\beta$.*

4.2. Factorisation in the Galois case.

Theorem 4.2.1. *Let $p : X \to Y$ be an infinite Galois covering of a compact Kähler manifold. Let C be a compact divisor in X such that $C.C = \int_X c_1([C])^2 \wedge \omega^{n-2} = 0$. Then*

1) *there exists a proper map $f' : X \to B'$ to a curve such that C is a fiber (up to multiplicity). In particular X is holomorphically convex.*
2) *Moreover there exists a holomorphic map $f : Y \to B$ to a curve such that $p_*(C)$ is a fiber of f (up to multiplicity).*

Proof. From 3.2.1, we may assume C connected and effective.

a) We first prove that 1) implies 2). Using theorem 2.1.5 (2), it is enough to prove that $|C|$ is a connected componant of $p^{-1}p(|C|)$: if not, another irreducible component T in the connected component of $p^{-1}p(C)$ which contains C is compact for p is Galois. If T intersects C then $T.C > 0$. However the Hodge index theorem and $C.C = 0$ implies that $c_1(C) = du$ is exact. But $c_1(T)$ is closed with compact support, hence
$$T.C = \int_X c_1(T) \wedge du \wedge \omega^{n-2} = 0.$$

b) We may assume that C is not a multiple of an effective divisor, and that C is not the sum of non trivial compact effective divisors with vanishing self-intersection. Let W be a small connected neighborhood of $p(|C|)$ such that W', the connected component of $p^{-1}(W)$ which contains C, satisfies $p^{-1}p(C) \cap W' = C$. Then $p' = p_{|W'} : W' \to W$ is a finite covering. Hence the direct image $p_*(C)$ is a divisor with vanishing self-intersection. Zariski's lemma 2.1.3 implies there exists a minimal effective divisor C_0 with support $p(|C|)$ such that $C_0.C_0 = 0$. Then $p'^*(C_0) = C$.

Let s be a section of $[C]$ such that $(s) = C$. We fix a metric on the line bundle $[C]$ such that $|s| = 1$ outside a given compact neighborhood of C. Using a pull-back metric by p', we may even assume that $|s|$ is invariant by the stabiliser subgroup of C.

Let α be the metric first Chern form of $[C]$. The Poincaré-Lelong equation reads
$$dd^c \ln |s|^2 = C - \alpha\,.$$

Lemma 3.1.1 proves that α is orthogonal to the l^2-harmonic forms for $0 = C.C = \int_X \alpha \wedge \alpha$.
Let Γ be the deck transformations group of $p : X \to Y$. The Galois $\partial\overline{\partial}-$lemma 4.1.1 implies that

there exists a weak isomorphism $\lambda(f) \in \mathrm{M}(\Gamma)$ and a function u square integrable in the domain of d and dd^c, such that:

$$\lambda(f).\alpha = dd^c u\,.$$

The function $\ln|s|^2$ has compact support, hence $\lambda(f).\ln|s|^2$ is well defined. We obtain the following equation (the action of the Von Neumann algebra on a current with compact support is defined by duality):

$$\lambda(f).C = \lambda(f).\left(dd^c \ln|s|^2 + \alpha\right) \tag{2}$$

$$= dd^c\left(\lambda(f).\ln|s|^2 + u\right) \tag{3}$$

This proves that the function $P = \left(\lambda(f).\ln|s|^2 + u\right)$ is pluriharmonic on $X \setminus \Gamma.|C|$ with a logarithmic pole along $\Gamma.|C|$ (P extends has a difference of two plurisubharmonic functions for f is not positive in general).

Note that d^cP is square integrable in the fiber of $j^*p = p_{|X\setminus\Gamma|C|}$ and is closed over $Y \setminus |C_0|$. It defines a non trivial class of weight two in $\mathbb{H}^1(Y \setminus |C_0|, p_{*(2)}\mathbb{C})$ for its residue along $p^{-1}(|C_0|)$ is $\lambda(f).[C]$, which is a non vanishing current (for $\lambda(f)$ is injective). The degeneracy of $\mathcal{U}(\Gamma)-$spectral sequence implies that the $(1,0)-$part of this class is represented by a square integrable logarithmic one form L. From [11],[10], we know that L is closed.

In our situation, the form $\partial P = \partial\left(\lambda(f).\ln|s|^2 + u\right)$ is a representative of the $(1,0)-$part of L: it is a logarithmic one form which is closed on $X \setminus p^{-1}(|C_0|)$ and square integrable as a section of $p^*\Omega^1_X(log|C_0|)$.

Using translation, one sees that the space A of closed square integrable logarithmic one forms is infinite dimensional for it is non trivial. Hence, for any finite number of compact connected component $C_1, \ldots, C_k$ of $p^{-1}(|C_0|)$, the linear map $\oplus_{i=1}^k Res_{C_i} : A \to \oplus_{i=1}^k \mathbb{C}$ has a non trivial kernel. A logarithmic meromorphic form with vanishing residue on $C_1, \ldots, C_k$ is holomorphic. This form is therefore a closed holomorphic one form on $(X \setminus p^{-1}(|C_0|)) \cup C_1 \ldots \cup C_k$.

Lemma 4.2.2. *Let X' be a complex manifold. Assume that a space Λ of closed holomorphic one forms is infinite dimensional. Then for any compact subset K, there exists a non constant holomorphic function in a neighborhood of K whose derivative lies in Λ.*

Proof. X' is exhausted by relatively compact domains U with smooth boundaries. Hence $H_1(\bar{U})$ is finite dimensional and the natural map given by integration $\Lambda \to H_1(\bar{U})^*$ has a non trivial kernel. A closed holomorphic one form with vanishing period is exact, on $\bar{U}$. Its primitive is $\bar{\partial}-$closed. □

We conclude the proof of the theorem using lemma 2.1.4 and theorem 2.1.5 (1) and (2).

□

Corollary 4.2.3. *Let $i : C \to Y$ be a divisor in a compact Kähler manifold. Assume that a subgroup of finite index of $i_*\pi_1(|C|)$ has a normal closure of infinite index in $\pi_1(Y)$ and that $C.C = 0$. Then each connected component of the support of C is a fiber of a holomorphic map to a curve.*

Proof. Hypothesis implies that there exists an infinite Galois covering $p : X \to Y$ such that each connected componant of $p^{-1}(|C|)$ is compact. We conclude using the previous theorem and the Hodge index theorem 3.2.1. □

Corollary 4.2.4 (Campana)**.** *Let $i : C \to Y$ be a curve with vanishing self-intersection in a surface and assume that $i_*\pi_1(C)$ is finite. Then the universal covering of X is holomorphically convex.*

5. Manifolds of bounded geometry with a proper Green function.

In this section we give a proof using potential theory, following the theory developped in [24], [19], [20]. we will however restrict ourselves to a non compact covering (X, ω) of a compact Kähler manifold.

5.1. Solving the $\partial\overline{\partial}$−equation from the Δ−equation.

Lemma 5.1.1. *Let (X,ω) be a complete Kähler manifold which admits a Green function. Let α be a closed $(1,1)$−form with compact support such that $\int_X \alpha\wedge\bar{\alpha}\wedge\omega^{n-2} = 0$ then α is $\partial\overline{\partial}$−exact. Moreover if the Green function vanishes at infinity, a solution of $\partial\overline{\partial}u = \alpha$ exists such that $\lim_{x\to\infty} u(x) = 0$.*

Remark 1.
In case λ_1, the first eigenvalue of Δ on square integrable functions, is strictly positive then the function u will belong to the domain of d acting on square integrable functions.

Proof. We may assume that α is real. Let G be the Green function of the hyperbolic complete Kähler manifold (X,ω). Then $G(\Lambda\alpha) = u$ is a function such that $\Lambda\alpha = -\Delta u = \Lambda i\partial\overline{\partial}u$. Moreover du is square integrable ([19] lemma 1.6 p.815). Hence the closed form $\alpha - dd^c u$ is pointwise primitive. Let θ be a positive smooth function with compact support such that $\theta = 1$ on a neighborhood of $supp(\alpha)$. Then:

$$0 \geq \int_X \theta\,(\alpha - dd^c u)\wedge\overline{(\alpha - dd^c u)} \quad = \quad -\int_X dd^c\theta(du\wedge d^c u). \tag{4}$$

There exists a positive exhaustion function f on X with bounded gradient and bounded hessian for (X,ω) has bounded geometry. Let $\theta = \chi(\epsilon f)$ where χ is a smooth positive function on $\mathbb{R}$, $\chi_{|]-\infty,1]} = 1$, $\chi_{|[2,+\infty[} = 0$. Letting ϵ goes to zero, one sees that $\int_X (\alpha - dd^c u)\wedge\overline{(\alpha - dd^c u)} = 0$. Therefore

$$\alpha = dd^c u\,.$$

□

5.2. Setting. We consider a compact divisor C in a complex Kähler manifold of bounded geometry admitting a Green function. We assume that $C.C = 0$.

With the notations of the lemma, we know that $C = dd^c\ln|s|^2 + \alpha = dd^c(\ln|s|^2 + u)$ is cohomologically trivial. Assume moreover the Green function is proper then

i) the plurisubharmonic function $(\ln|s|^2 + u)$ is strictly negative on X,
ii) the set $\{\ln|s|^2 + u \leq -a\}$ $(a > 0)$ are compact subsets in X.

Indeed, u is pluriharmonic on $X \setminus supp(\alpha)$ and it vanishes at infinity (if $\lambda_1 > 0$ then $u_{|X\setminus supp(\alpha)}$ is a square integrable pluriharmonic function in a manifold of bounded geometry, hence it vanishes at infinity). Therefore $(\ln|s|^2 + u)$ is strictly negative for it is non constant. The second assertion follows for $supp(\ln|s|^2)$ is compact.

In the case of a proper fibration to a curve $f : X \to B$, the function $(\ln|s|^2 + u)$ is the pull-back of the Green function with a pole on $f(C)$.

Remark 2. Any non compact irreducible analytic set of strictly positive dimension in X intersects $supp(\alpha)$: assume the opposite, u restricted to such a set would be pluriharmonic and would vanish at infinity. Hence u would be vanishing and $\ln|s|^2 + u$ would be constant. In particular C intersects any non compact positive dimensional irreducible analytic set in X for we can shrink $supp(\alpha)$ in an arbitrary neighborhood of C.

We will use the following lemma (see e.g. [20] lemma 2.1 p.389 or [23] for a statement using foliations):

Lemma 5.2.1. *Let u, v be pluriharmonic functions defined on a complex manifold M. Assume the level set of v are compact or empty. Then $\partial u\wedge\partial v = 0$ and*

1) *either there is a non closed leaf and $\partial u = c\partial v$ for some constant c,*
2) *or every leaf is closed, there exists a holomorphic map $f : M \to \mathbb{P}^1$ such that f restricted to the leaf is constant and $\partial u = f\partial v$.*

We deduce a second version of a factorisation theorem for divisors with vanishing self-intersection.

Proposition 5.2.2. *Let $C_0 \to Y$ be an effective divisor with vanishing self-intersection in a compact Kähler manifold (Y, ω_0). Assume that $G = \mathrm{Im}(\pi_1(C_0) \to \pi_1(Y))$ has infinite index . Let $p : X \to Y$ be the covering of Y with fundamental group G.*

Assume that $(X, p^*\omega_0)$ admits a proper Green function (e.g. λ_1, the first eigenvalue of the laplacian acting on functions, is strictly positive). Then there exists a holomorphic map $f : Y \to B$ such that a multiple of C_0 is a multiple of a fiber of f.

Proof. We may assume that C_0 is connected and minimal with the above properties. By hypothesis there exists a divisor C in X that is mapped biholomorphically to C_0 by p.

Then there exists a neighborhood W' of C which is biholomorphic to a neighborhood W of C. The existence of a section $\sigma : W \to W'$, defines a plurisubharmonic function $v = (\ln |s|^2 + u) \circ \sigma$ on W which is pluriharmonic away from C_0. Hence C_0 admits a basis of Levi flat neighborhoods. Let $(C_j)_{j \in \mathbb{N}^*}$ be the connected components of $p^{-1}(C_0)$. Assume that $C = C_1$. Remark 5.2 implies that C_k is compact. Note that p is an infinite covering hence there are an infinite number of connected component.

For $a > A$ large enough, a connected component $B_k(-a)$ of $p^{-1}(\{v < -a\})$ contains only one connected component C_k of $p^{-1}(C_0)$. Let $k \neq 1$. Let $v_k = v \circ p_{|B_k(-A)}$ be the restriction to a neighborhood of C_k of the pullback by p of v.

If the leaves of $\{v_k = -a\}$ were non integrable, there would exist a complex number c such that on $B_k(-A) \setminus C_k$, $\partial u = c \partial v_k$. However ∂v_k has a logarithmic pole on C_k but u is smooth on a neighborhood of C_k. Hence the leaves are closed and there exists a holomorphic map $f_k : B_k(-A) \setminus C_k \to \mathbb{P}^1$ such that $\partial u = f_k \partial v_k$. But u is smooth and ∂v_k has a logarithmic pole on C_k, hence f_k extends as a holomorphic map to $B_k(-A)$ vanishing on C_k. We apply theorem 2.1.5 to conclude that a multiple of C_k and of C_0 are fibers of non constant holomorphic maps to a curve. □

The corollary applies in particular if $i_*(\pi_1(C))$ is amenable and $\pi_1(Y)$ is not amenable using [5]. Also it applies in the case of a Galois covering with Deck transformation group G which is not a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$ (see the arguments in [24], [20] p.399).

Remark 3. F. Campana indicated to us that the original problem has a positive answer in the case of a smooth rational curve or a smooth elliptic curve in a complex Kähler surface. The proof is algebraic and uses the Kodaira's classification.

References

[1] Aldo Andreotti and Edoardo Vesentini. Carleman estimates for the Laplace-Beltrami equation on complex manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (25):81–130, 1965.

[2] D. Arapura, P. Bressler, and M. Ramachandran. On the fundamental group of a compact Kähler manifold. *Duke Math. J.*, 68(3):477–488, 1992.

[3] Daniel Barlet. Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie. In *Fonctions de plusieurs variables complexes, II (Sém. François Norguet, 1974–1975)*, pages 1–158. Lecture Notes in Math., Vol. 482. Springer, Berlin, 1975.

[4] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.

[5] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[6] F. Campana. Negativity of compact curves in infinite covers of projective surfaces. *J. Algebraic Geom.*, 7(4):673–693, 1998.

[7] Frédéric Campana. Remarques sur le revêtement universel des variétés kählériennes compactes. *Bull. Soc. Math. France*, 122(2):255–284, 1994.

[8] Georges de Rham. *Differentiable manifolds*, volume 266 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Forms, currents, harmonic forms, Translated from the French by F. R. Smith, With an introduction by S. S. Chern.

[9] Jean-Pierre Demailly. Monge-Ampère operators, Lelong numbers and intersection theory. In *Complex analysis and geometry*, Univ. Ser. Math., pages 115–193. Plenum, New York, 1993.

[10] Pascal Dingoyan. Applications to mhs on l^2-cohomology groups. in preparation.

[11] Pascal Dingoyan. Some mixed Hodge structures on l^2-cohomology groups of coverings of Kähler manifolds. *Math. Ann.*, 357(3):1119–1174, 2013.

[12] Matthew P. Gaffney. A special Stokes's theorem for complete Riemannian manifolds. *Ann. of Math. (2)*, 60:140–145, 1954.

[13] M. Gromov. Kähler hyperbolicity and L_2-Hodge theory. *J. Differential Geom.*, 33(1):263–292, 1991.

[14] R. V. Gurjar and Sagar Kolte. Fundamental group of some genus-2 fibrations and applications. *Internat. J. Math.*, 23(8):1250080, 20, 2012.

[15] R. V. Gurjar and B. P. Purnaprajna. On the Shafarevich conjecture for genus-2 fibrations. *Math. Ann.*, 343(4):791–800, 2009.

[16] Daniel Huybrechts. *Complex geometry.* Universitext. Springer-Verlag, Berlin, 2005. An introduction.

[17] János Kollár. *Shafarevich maps and automorphic forms.* M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1995.

[18] Wolfgang Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2002.

[19] T. Napier and M. Ramachandran. Structure theorems for complete Kähler manifolds and applications to Lefschetz type theorems. *Geom. Funct. Anal.*, 5(5):809–851, 1995.

[20] T. Napier and M. Ramachandran. Hyperbolic Kähler manifolds and proper holomorphic mappings to Riemann surfaces. *Geom. Funct. Anal.*, 11(2):382–406, 2001.

[21] Terrence Napier and Mohan Ramachandran. The L^2 $\overline{\partial}$-method, weak Lefschetz theorems, and the topology of Kähler manifolds. *J. Amer. Math. Soc.*, 11(2):375–396, 1998.

[22] Madhav V. Nori. Zariski's conjecture and related problems. *Ann. Sci. École Norm. Sup. (4)*, 16(2):305–344, 1983.

[23] Jorge Vitório Pereira. Fibrations, divisors and transcendental leaves. *J. Algebraic Geom.*, 15(1):87–110, 2006. With an appendix by Laurent Meersseman.

[24] Mohan Ramachandran. A Bochner-Hartogs type theorem for coverings of compact Kähler manifolds. *Comm. Anal. Geom.*, 4(3):333–337, 1996.

[25] M. A. Shubin. Spectral theory of elliptic operators on noncompact manifolds. *Astérisque*, (207):5, 35–108, 1992. Méthodes semi-classiques, Vol. 1 (Nantes, 1991).

[26] Karl Stein. Analytische Zerlegungen komplexer Räume. *Math. Ann.*, 132:63–93, 1956.

[27] Burt Totaro. The topology of smooth divisors and the arithmetic of abelian varieties. *Michigan Math. J.*, 48:611–624, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[28] Tetsuo Ueda. On the neighborhood of a compact complex curve with topologically trivial normal bundle. *J. Math. Kyoto Univ.*, 22(4):583–607, 1982/83.

P. Dingoyan, Université Paris 6, case 247, 4 place Jussieu, 75252 Paris Cedex 05, France.
E-mail address: pascal.dingoyan@imj-prg.fr