Math. Ann. 320, 211–238 (2001) **Mathematische Annalen**

Digital Object Identifier (DOI) 10.1007/s002080000193

Monge-Ampère currents over pseudoconcave spaces

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Received February 22, 1999 / Accepted August 24, 2000 /
Published online March 12, 2001 – © Springer-Verlag 2001

Mathematics Subject Classification (2000): 32Fxx, 32d20

Introduction

This paper is an attempt to understand growth of Monge-Ampère masses along pseudoconcave ends in a complex manifold.

This problem arises in differential geometry when studying compactification of complete Kähler manifolds under certain curvature conditions (see *e.g.* articles of Mok-Zhong [23], Nadel-Tsuji [24], Siu-Yau [32]). In complex analysis, bounds on Monge-Ampère masses of a closed positive current near a pluripolar set implies an extension of this current through the set (see *e.g.* works of El Mir [22], Sibony [30], Skoda [34]). In this direction, the L^2-Riemann-Roch inequality of Nadel-Tsuji (see [24]) implies that a complete Kähler Hodge metric on a pseudoconcave manifold is of finite volume.

Our first result is obtained in the framework of pluripotential theory. Let M be a complex manifold, $\dim M = n \geq 2$, and let ω be a closed positive $(1, 1)-$current. Assume that ω admits local locally bounded potentials. To each open subset U of M is associated an extremal admissible function φ^*, which is defined on a suitable pseudoconvex hull U_1 of U. It satisfies the Monge-Ampère equation $(\omega+dd^c\varphi^*)^n = 0$ on $U_1 \setminus \bar{U}$, as (n, n)-current of order zero. Identifying a (n, n)-current of order zero with the Borel measure it defines, we deduce the following estimate (we work in the relative topology of U_1).

Theorem. *In the above situation, let X be a connected component of $U_1 \setminus \bar{U}$ which has a compact boundary. Assume that $\{\varphi^* \leq \varphi^*(p)\} \cap X$ is relatively compact in U_1 for any $p \in X$. Then*

$$\int_{\bar{X}} \omega^n \leq \int_{\partial X} \left(\omega + dd^c\varphi^*\right)^n < +\infty\ .$$

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Here, to check the hypothesis we restrict ourself to domains on projective manifolds. It allows us to obtain a complex analytic treatment of the problem. Related methods appear already in [11,25]. For a more differential-geometric point of view, we refer to papers cited above.

We obtain the following applications. Let V be a projective manifold, $\dim V = n \geq 2$, and let H be a complex hypersurface in V such that $V \setminus H$ is pseudoconcave in the sense of Andreotti (see Definition 11). Let $X \subset\subset M$ open neighbourhoods of H. The following Hartogs' theorem for currents holds.

Theorem. *Let ω be a closed positive $(1,1)-$current defined on $M \setminus H$ which admits local locally bounded potentials. Then*

$$\int_{\bar{X}\setminus H} \omega^n < +\infty \ ,$$

and ω^k extends through H as a closed positive currents, $k = 1, \dots, n$.

If $X = V$ and ω is a smooth complete Hodge Kähler metric on $V \setminus H$, then the above result is a variation of the L^2-Riemann-Roch inequality of Nadel-Tsuji (see [24]). In general, the difficulty in establishing the above finiteness estimate is that neither pseudoconcavity nor completeness assumptions are made on M itself.

Next, we try to derive similar estimate for more singular closed positive currents. We work with currents (on spread domains W over V) such as pullback $\psi^*\omega_{FS}$, where $\psi : W \to \mathbb{P}^N$ is a meromorphic map from W to a projective space and ω_{FS} is a Fubiny-Study form on it.

Our technique is to produce, by mean of the L^2 theory of ideals (see Skoda [33]), positive currents ω_k linked to $\psi^*\omega_{FS}$ but with Lelong number globally shifted by $-k$ (see Demailly [12] for other methods in the compact case). These currents are pluricomplete (see Def. 10). This is a convexity condition on ω_k and A_k, the non-smooth locus of ω_k, which allows to work on $M = W \setminus A_k$.
The case of a current defined by a divisor is noteworthy:

Theorem. *Let Z be an hypersurface in a pseudoconvex spread domain W over a projective manifold $V = (V, \mathcal{O}(1))$. There exists $l_1 \in \mathbb{N}$ (which depends only of the canonical bundle of V) such that $\mathcal{O}(kl_1) \otimes [Z]$ is spanned by its global sections away of $\{p \in W \ : \nu_p(Z) \geq k+1\}$, where $\nu_p(Z)$ is the multiplicity of Z at p.*

As an application, we deduce that global Hartogs' extension phenomena occur in projective manifolds for meromorphic maps.

Theorem. *Let U be an open subset of the projective manifold V such that $V \setminus \bar{U}$ is a pseudoconcave domain in the sense of Andreotti. Assume $\mathring{\bar{U}} = U$. Then any* *meromorphic map $\psi : W(\partial U) \to \mathbb{P}^N$ define on a neighbourhood of ∂U extends as a meromorphic map to U.*

These results give some understanding of global and compact singularities for meromorphic maps or currents. Note that there exists hypersurfaces H as above which may not be blow down. Hence, even for meromorphic maps, the situation may not be reduced to local extension results of Ivashkovich [20]. Moreover, note that non compact complex singularities of strict positive dimension are already local essential singularities for Monge-Ampère currents (see [31]).

The starting point of this paper is the classical result that a hull of holomorphy in the trivial bundle over a domain in $\mathbb{C}^n$ is a geometric counterpart of a complex Monge-Ampère equation in that domain (see Bremermann [9]).

Part of this paper was written during a stay financed by a European grant at the mathematical department of Chalmers University. This is my great pleasure to thank the complex analysis team of Chalmers university and Professor Bo Berndtsson for discussions about this subject.

1. Quasi-continuous functions and the class $\mathcal{G}(M)$

We recall some definitions which appear in [6,8].

Definition 1 *Let Ω be an open subset of $\mathbb{C}^n$. If E is a subset of Ω, let $C(E, \Omega)$ denote the relative capacity of E in Ω.*

(1) A function $f : \Omega \to \{-\infty, +\infty\}$ is said to be quasi-continuous if, for any $\epsilon > 0$, there exists an open subset $\mathcal{O}$ of Ω with $C(\mathcal{O}, \Omega) < \epsilon$ s.t. f is continuous on $\Omega \setminus \mathcal{O}$.
(2) A sequence $\{f_j\}_{j\in\mathbb{N}}$ of Borel functions on Ω is said to converge quasi-uniformly to f, if it is uniformly bounded, it converges almost everywhere to f, and, for any $\epsilon > 0$, there exits an open subset $\mathcal{O}$ of Ω such that $C(\mathcal{O}, \Omega) \leq \epsilon$ and $f_j \to f$ uniformly on $\Omega \setminus \mathcal{O}$.

The notions of quasi-continuous function and local quasi-uniform convergence are define accordingly on a manifold through holomorphic coordinate charts.

Quasi-continuous functions form an algebra which contains plurisubharmonic functions (see [5], Theorem 3.5). Note that if f is quasi-continuous on M, then for any continuous function $\chi : \mathbb{R} \to \mathbb{R}$, $\chi(f)$ is quasi-continuous on M.

Lemma 1 *([5]) Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence of plurisubharmonic functions which converge monotonically almost everywhere to a plurisubharmonic function φ. Then the convergence is locally quasi-uniform.*

Definition 2 *We denote by $\mathcal{G}(M)$ the class of currents on M which locally are represented by currents in the exterior algebra generated by*

- *smooth forms,*
- *locally bounded plurisubharmonic functions,*
- du, $d^c u$, $dd^c u$ *where* u *is a locally bounded plurisubharmonic function.*

We refer to Bedford-Taylor's articles [8,7] for a precise definition of these currents for non smooth functions. We state in a weak form Theorem 2.6 of [8].

Theorem 1 *Let* T_j, $j \in \mathbb{N}$ *and* T_∞ *be currents in* $\mathcal{G}(M)$ *which are locally of the form*

$$\sigma_0^{(j)} \delta\sigma_1^{(j)} \wedge \ldots \wedge \delta\sigma_q^{(j)} \wedge dd^c \sigma_{q+1}^{(j)} \wedge \ldots \wedge dd^c \sigma_r^{(j)} \tag{1.1}$$

where, each occurrence of δ *denotes either the operator* d *or the operator* d^c, $\sigma_k^{(j)} = u_k^{(j)} - v_k^{(j)}$, *the* $u_k^{(j)}$ *and* $v_k^{(j)}$, $j \in \mathbb{N} \cup \{\infty\}$, *are locally bounded plurisubharmonic functions such that*

$$u_k^{(j)} \underset{k \to +\infty}{\longrightarrow} u_\infty^{(j)} , \tag{1.2}$$

$$v_k^{(j)} \underset{k \to +\infty}{\longrightarrow} v_\infty^{(j)} , \tag{1.3}$$

and where the convergence is monotone in k. *If* $\{\varphi_j\}_{j \in \mathbb{N}}$ *is a sequence of quasi-continuous functions which converges locally quasi-uniformly to the quasi-continuous function* φ *then*

$$\lim_{j \to +\infty} \varphi_j T_j = \varphi T_\infty$$

as currents of order 0.

2. The class $P_\omega(M)$

Let M be a complex manifold, $\dim M = n$, and let ω be a closed positive $(1, 1)-$current on M. It is known (see [18], p.387) that ω admits local potentials. In this paper, we make the following assumption.

$$\text{The current } \omega \text{ admits local potentials which are locally bounded.} \tag{2.1}$$

Hence we assume that, for any open subset X biholomorphic to an open Euclidean ball in $\mathbb{C}^n$, there exists $a \in \mathrm{PSH}(X) \cap L^\infty(X, \mathrm{loc})$ such that $dd^c a = \omega_{|X}$. Note that two local potentials for ω differ (on their common definition set) by a pluriharmonic function. This fact is used in the following definitions.

Definition 3 *A measurable function* $\varphi : M \to \mathbb{R} \cup \{-\infty\}$ *belongs to* $P_\omega(M)$ *if there exists an open covering* $\mathcal{W} = \{W_i\}_{i \in I}$ *by subsets biholomorphic to Euclidean balls in* $\mathbb{C}^n$, *and local potentials* $a_i \in \mathrm{PSH}(W_i) \cap L^\infty(W_i, \mathrm{loc})$, *such that* $a_i + \varphi$ *is plurisubharmonic.*

Note that a function which belong to $P_\omega(M)$ is quasi-continuous.

Definition 4

(1) A function $\varphi : M \to [-\infty, +\infty[$ will be said upper semicontinuous with respect to ω, if, for any $p \in M$, there exists an open neighbourhood W of p, a local locally bounded potential $a \in \mathrm{PSH}(W) \cap L^\infty(W, \mathrm{loc})$ for ω, such that $a + \varphi$ is upper semicontinuous on W. A function h on M will be said lower semicontinuous with respect to ω if $-h$ is upper semicontinuous with respect to ω.

(2) Let $\varphi : M \to [-\infty, +\infty[$ be a function which is locally bounded from above. Define φ^*, the upper regularization of φ with respect to ω, as follow. If a is a local locally bounded potential for ω on an open subset W, then

$$\varphi^* = (a + \varphi)^* - a \tag{2.2}$$

where $(a + \varphi)^*$ stands for the usual upper regularization of $a + \varphi$ on W in the classical topology $(a + \varphi)^*(p) = \limsup_{z \to p}(a + \varphi)(z)$.

Let a function $h \in L^1(M, \mathrm{loc})$ satisfies $\omega + dd^c h \geq 0$ in the sense of currents. Then h^*, the upper regularization of h with respect to ω, belongs to $P_\omega(M)$.

With this notion of upper regularization w.r.t ω, we will have classical stability properties of $P_\omega(M)$ with respect to upper envelope (see Lemma 6). Note that Choquet's lemma is valid.

Lemma 2 *Let $\{u_\alpha\}_{\alpha \in A}$ be a family of real valued functions on a complex manifold M. Assume that $a + u_\alpha$ is upper semicontinuous for any local potential a of ω and any $\alpha \in A$. Assume this family is locally bounded from above on M. Then there exist a countable subset $B \subset A$ such that $(\sup_{\alpha \in A} u_\alpha)^* = (\sup_{\alpha \in B} u_\alpha)^*$ (upper regularization w.r.t. ω).*

Let ω_i, $1 \leq i \leq r$, be closed positive $(1, 1)-$currents which satisfy condition (2.1). From Theorem 1, if $\varphi_i \in P_{\omega_i}(M) \cap L^\infty(M, \mathrm{loc})$ then expression of the form

$$T = \delta\varphi_1 \wedge \ldots \wedge \delta\varphi_k \wedge (\omega_{k+1} + dd^c\varphi_{k+1}) \wedge \ldots \wedge (\omega_r + dd^c\varphi_r), \tag{2.3}$$

where δ is either d or d^c, defined a current which belongs to the class $\mathcal{G}(M)$. T is the unique current which is locally equal to

$$\begin{aligned} T = \delta\,((a_1 + \varphi_1) - a_1) \wedge \ldots \wedge \delta\,((a_k + \varphi_k) - a_k) \wedge \\ dd^c(a_{k+1} + \varphi_{k+1}) \wedge \ldots \wedge dd^c(a_r + \varphi_r), \end{aligned} \tag{2.4}$$

where a_i denotes a local locally bounded potential for ω_i, $1 \leq i \leq r$.
For these currents, usual calculus rules are satisfied. In particular,

Lemma 3 *Let* $\varphi \in P_\omega(M) \cap L^\infty(M, \mathrm{loc})$, $\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. *Then for any* $\theta \in \mathcal{C}_0^\infty(M)$, *the following algebraic identity holds*

$$\int \theta\chi(\varphi)\left(\omega + dd^c\varphi\right)^n = \int \theta\chi(\varphi)\omega^n - \int (d\theta)\chi(\varphi)d^c\varphi P(\varphi) - \int \theta\chi'(\varphi)d\varphi \wedge d^c\varphi P(\varphi), \tag{2.5}$$

where

$$P(\varphi) = \sum_{\alpha+\beta=n-1} \left(\omega + dd^c\varphi\right)^\alpha \omega^\beta. \tag{2.6}$$

Proof. It is enough to check the above formula locally. Let B be the Euclidean unit ball in $\mathbb{C}^n$. Assume, $\mathrm{supp}\theta \subset\subset B$, $\omega = dd^c a$, with $a \in \mathrm{PSH}(B) \cap L^\infty(B, \mathrm{loc})$, so that $a + \varphi \in \mathrm{PSH}(B) \cap L^\infty(B, \mathrm{loc})$. Let $(a + \varphi)_\epsilon$, a_ϵ, $1 > \epsilon > 0$, be family of smooth plurisubharmonic functions defined on B, which decrease, as $\epsilon \to 0$, to $a + \varphi$ and a respectively on an open neighbourhood $W \subset\subset B$ of $\mathrm{supp}\theta$.
Let $M = \|(a + \varphi)_1\|_{W,\infty} + \|a_1\|_{W,\infty} + \|a + \varphi\|_{W,\infty} + \|a\|_{W,\infty} < +\infty$.

From [5], Theorem 7.2, for any $\eta > 0$, there exists Ω, an open subset of W, such that $C(W, \Omega) < \eta$, and the above convergences are uniform on $W \setminus \Omega$. Define $\psi_\epsilon = (a + \varphi)_\epsilon - a_\epsilon$, then

$$\|\chi(\psi_\epsilon) - \chi(\varphi)\|_{W\setminus\Omega,\infty} \le (\max_{[-M,M]} |\chi'|)\|\psi_\epsilon - \varphi\|_{W\setminus\Omega,\infty} \underset{\epsilon\to 0}{\longrightarrow} 0\,. \tag{2.7}$$

Since the ψ_ϵ and φ are uniformly bounded on W, for any $\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, $\chi(\psi_\epsilon)$ converge quasi-uniformly on W to $\chi(\varphi)$. But for smooth functions, an integration by parts gives

$$\int \theta\chi(\psi_\epsilon)\left(dd^c(a_\epsilon + \psi_\epsilon)\right)^n = \int \theta\chi(\psi_\epsilon)\left(dd^c a_\epsilon\right)^n - \int (d\theta)\chi(\psi_\epsilon)d^c\psi_\epsilon P(\psi_\epsilon) - \int \theta\chi'(\psi_\epsilon)d\psi_\epsilon d^c\psi_\epsilon P(\psi_\epsilon). \tag{2.8}$$

where

$$P(\psi_\epsilon) = \sum_{\alpha+\beta=n-1} \left(dd^c(a_\epsilon + \psi_\epsilon)\right)^\alpha \left(dd^c a_\epsilon\right)^\beta. \tag{2.9}$$

As $\epsilon \to 0$, $\chi(\psi_\epsilon)$ and $\chi'(\psi_\epsilon)$ converge quasi-uniformly to $\chi(\varphi)$ and $\chi'(\varphi)$ respectively, on W. Moreover, $d^c\psi_\epsilon P(\psi_\epsilon)$ converges to $d^c\varphi P(\varphi)$, $(dd^c(a_\epsilon + \psi_\epsilon))^n$ converges to $(\omega + dd^c\varphi)^n$ and $(dd^c a_\epsilon)^n$ converges to ω^n. From Theorem 1, we obtain formula (2.5) above. ◻

We state next a basic lemma.

Lemma 4 *Let M be a complex manifold, and let X be an open subset of M with compact boundary. Let ω be a closed positive $(1,1)-$current which admits local locally bounded potentials.*
Let $\varphi \in P_\omega(X) \cap L^\infty(X, \mathrm{loc})$ such that

(1) *there exists a neighbourhood W of ∂X, with $\varphi_{|W\cap X} \geq 0$,*
(2) $\limsup_{z\to\partial X} \varphi = 0$,
(3) $\forall p \in X,\ \{\varphi \leq \varphi(p)\} \subset\subset M$.

Let $\chi : \mathbb{R} \to \mathbb{R}^+$ be a positive smooth decreasing function. Then

$$+\infty \geq \int_{\bar{X}} \chi(\tilde{\varphi})(\omega + dd^c\tilde{\varphi})^n \geq \int_{\bar{X}} \chi(\tilde{\varphi})\omega^n$$

where $\tilde{\varphi}$ denotes the extension by 0 of φ to M.

Proof. Note that $\tilde{\varphi}$ belongs to $P_\omega(M) \cap L^\infty(M, \mathrm{loc})$. For

$$\tilde{\varphi} = \begin{cases} \varphi & z \in X \setminus W \\ \max(\varphi, 0) = \varphi & z \in X \cap W \\ 0 & z \in M \setminus X \end{cases} .$$

Hence, for any local locally bounded potential a for ω on an open charts W',

$$a + \tilde{\varphi} = \begin{cases} a + \varphi & z \in (X \setminus W) \cap W' \\ \max(a + \varphi, a) = a + \varphi & z \in (X \cap W) \cap W' \\ a & z \in (M \setminus X) \cap W' \end{cases}$$

which is a plurisubharmonic function in W' (see [21], p.69).
Hence, we will assume that $\varphi \in P_\omega(M) \cap L^\infty(M, \mathrm{loc})$ and that it vanishes on $M \setminus X$. Let W_1 be a relatively compact open neighbourhood of ∂X. Let θ be a smooth positive function with $\mathrm{supp}\theta \subset X \cup W_1$, $\theta \equiv 1$ on a neighbourhood of $\bar{X}$. Note that it's enough to prove the lemma under the following technical assumption.

(3') There exists an increasing sequence $\{\chi_k\}_{k\in\mathbb{N}}$ of smooth positive decreasing functions such that $\mathrm{supp}\chi_k(\varphi) \cap X$ is a relatively compact subset in M and $\lim_{k\to+\infty} \chi_k(\varphi) = \chi(\varphi)$ on M.

Then, since supp $\theta\chi_k(\varphi)$ is a compact set in M, Lemma 3 gives

$$\int \theta\chi_k(\varphi)\left(\omega + dd^c\varphi\right)^n = \int \theta\chi_k(\varphi)\omega^n - \int (d\theta)\chi_k(\varphi)d^c\varphi P(\varphi) - \int \theta\chi_k'(\varphi)d\varphi \wedge d^c\varphi P(\varphi)\ , \tag{2.10}$$

where

$$P(\varphi) = \sum_{\alpha+\beta=n-1} \big(\omega + dd^c\varphi\big)^\alpha \omega^\beta. \tag{2.11}$$

Note that $d\varphi \wedge d^c\varphi P(\varphi)$ is a positive current on M. But χ_k' is negative, hence $-\int \theta\chi_k'(\varphi)d\varphi \wedge d^c\varphi P(\varphi) \geq 0$. Since φ is vanishing on a neighbourhood of supp $d\theta$, the second term of the right hand side vanishes. Hence

$$\int \theta\chi_k(\varphi)(\omega + dd^c\varphi)^n \geq \int \theta\chi_k(\varphi)\omega^n \ . \tag{2.12}$$

The above integrals being finite, letting first θ decreasing to the characteristic function of $\bar{X}$; and then $k \to +\infty$, since $(\chi_k)_{k\in\mathbb{N}}$ is increasing, we get the result. □

Example 1 Let $M = \mathbb{C}^n$, $X = B(1)$, where $B(1)$ is the unit ball, and let $\omega = dd^c\|z\|^2$ the standard Kähler metric. Then $1 - \|z\|^2$ belongs to $\widetilde{P_\omega(X)}$ and satisfies the conditions of the above lemma. Its extension by zero is $\widetilde{(1 - \|z\|^2)} = \max(0, 1 - \|z\|^2)$ so that $\|z\|^2 + \widetilde{(1 - \|z\|^2)} = \max(\|z\|^2, 1)$. Lemma 4, for $\chi = 1$, gives

$$\int_{\partial B(1)} (dd^c \max(1, \|z\|^2))^n \geq \int_{B(1)} (dd^c\|z\|^2)^n \ ,$$

which is in fact an equality.

3. Pseudoconvex hulls

Let M be a complex manifold, $\dim_{\mathbb{C}} M = n \geq 2$, and let M_1 be an open subset of M.

We recall that M_1 is said to be *locally pseudoconvex in* M, if there exists an open cover $\mathcal{W}$ of M by Stein open subsets W such that $M_1 \cap W$ is a Stein manifold, for any $W \in \mathcal{W}$.

Note that any connected component of the interior of an intersection of a family of locally pseudoconvex open subsets of M is a locally pseudoconvex open subset of M.

Definition 5 *Let U be an open subset of M. Then there exists $\hat{U}$, the smallest locally pseudoconvex open set in M which contains U. We says that $\hat{U}$ is the pseudoconvex hull of U in M.*

Lemma 5 *Let $(W', (z))$ be a holomorphic charts, with W' a relatively compact Stein open set of $M \setminus \bar{U}$. Then, for any open relatively compact subset W in W', and any polynomial P in the complex coordinates (z),*

$$\max_{\bar{W}\cap\partial\hat{U}} |P| = \max_{\partial W\cap\partial\hat{U}} |P| \ .$$

Proof. We argue by contradiction, and prove that if the above condition is not satisfied, we may push a hypersurface in $\hat{U}$ which is disjoint from U. Denote $K = \partial \hat{U}$. Assume there exists a polynomial P such that $||P||_{K\cap \bar{W}} = P(z_0) = 1$ for some $z_0 \in K \cap W$ and $||P||_{K\cap \partial W} < 1$.

$K \cap \partial W$ being compact, there exists $0 < \epsilon < 3^{-1} d(z_0, \partial W)$ s.t. $|P| < 1$ on $S_\epsilon = \{z \in \bar{W}, d(z, K \cap \partial W) < \epsilon\}$. Let $W_{2^{-1}\epsilon} = \{z \in W,\ d(z, \partial W) > 2^{-1}\epsilon\}$, and let $A_k = \{z \in W,\ P(z) = 1 + \dfrac{1}{k}\}$, $k \in \mathbb{N}^*$. A_k is an algebraic hypersurface in $W \setminus K \cup S_\epsilon$, and $\bigcup_{k\in\mathbb{N}^*} A_k \cap \partial W_{2^{-1}\epsilon} \subset\subset W \setminus K \cup S_\epsilon$. There exists $\alpha_0 > 0$ s.t. $\bigcup_{k\in\mathbb{N}^*} (A_k + B_{\mathbb{C}^n}(0, \alpha_0)) \cap \partial W_\epsilon \subset\subset W \setminus K \cup S_\epsilon$. Since $W' \cap \hat{U}$ is a Stein open set and since $\overline{\bigcup_{k\in\mathbb{N}^*} A_k} \ni z_0$, there exists a sequence of integers $k_1, k_2, \ldots,$ and irreducible component C_{k_i} of A_{k_i} such that $C_{k_i} \cap \bar{W}_\epsilon \subset \bar{W}_\epsilon \setminus \bar{\hat{U}}$ and $\lim_{i\to+\infty} d(z_0, C_{k_i}) = 0$. Hence $(C_{k_i} + B_{\mathbb{C}^n}(0, \alpha_0)) \cap \partial W_\epsilon \subset \partial W_\epsilon \setminus \overline{\hat{U}}$. Since $\overline{\bigcup_{k\in\mathbb{N}^*} A_k \cap W_\epsilon}$ is a compact subset of $\overline{W_\epsilon} \setminus S_\epsilon$, there exists $\alpha_0 > \alpha_1 > 0$ such that $\bigcup_{i\in\mathbb{N}^*} (C_{k_i} + B_{\mathbb{C}^n}(0, \alpha)) \cap W_\epsilon \subset\subset W \setminus S_\epsilon$.
Take i big enough such that $d(z_0, C_{k_i}) < 2^{-1}\alpha_1$, take $z_1 \in C_{k_i} \cap B(z_0, 2^{-1}\alpha_1)$, $z_2 \in \hat{U} \cap B(z_0, 2^{-1}\alpha_1)$. Then $(C_{k_i} + \overrightarrow{z_1 z_2}) \cap W_\epsilon \cap \hat{U}$ is non empty and $(C_{k_i} + \overrightarrow{z_1 z_2}) \cap \partial(W_\epsilon \cap \hat{U}) \subset \partial \hat{U} \cap W_\epsilon$, since $\partial(W_\epsilon \cap \hat{U}) \subset (\partial W_\epsilon \cap \bar{\hat{U}}) \cup (\partial \hat{U} \cap \bar{W}_\epsilon)$ and $\partial \hat{U} \cap \bar{W}_\epsilon = (\partial \hat{U} \cap W_\epsilon) \cup (\partial \hat{U} \cap \partial W_\epsilon)$.

In particular, $H = (C_{k_i} + \overrightarrow{z_1 z_2}) \cap W_\epsilon \cap \hat{U}$ is a hypersurface in $\hat{U}$ which does not intersect U. However $\hat{U} \setminus H$ is locally pseudoconvex, contains U and is strictly smaller than $\hat{U}$, which is a contradiction. □

Remark 1 The proof of the above lemma shows that, if $\dim M = 2$, then, for any open Stein subset of $M \setminus \bar{U}$, $W \setminus \partial \hat{U}$ is Stein. Hence $\partial \hat{U}$ is a pseudoconcave set in the sense of Oka in $M \setminus \bar{U}$ (see [29], p. 88).

Other kinds of pseudoconvex hulls (w.r.t. ω) are constructed as follow.

Lemma 6 *Let $\{\varphi_\alpha\}_{\alpha\in\Lambda} \subset P_\omega(M)$. Then the open set*

$$X = \{p \in M\ : \varphi = \sup_{\alpha\in\Lambda} \varphi_\alpha \text{ is locally bounded from above at } p\}$$

is locally pseudoconvex in M. Further, on X, φ^* the upper regularization of φ w.r.t. ω belongs to $P_\omega(X)$.

Lemma 7 *Let M be a complex manifold, and let ω be a closed positive $(1, 1)-$current in M. Assume there exists an analytic subset B in M such that ω admits local locally bounded potentials on $M \setminus B$. Let $\{\varphi_\alpha\}_{\alpha\in\Lambda} \subset P_\omega(M \setminus B)$. Let X denote the open set in $M \setminus B$ where this family is locally bounded from above. Then, the interior of $X \cup B$ in M is a locally pseudoconvex open subset in M.*

Proof. The lemma is local, hence we assume M is the unit ball and that $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is a set of plurisubharmonic functions on $M\setminus B$. Let U be the maximal open subset of $M\setminus B$ for which this family is locally uniformly bounded from above. Let φ be the upper envelope of this family, and let φ^* denote its upper regularization, which is a plurisubharmonic function in U.

Write $B=B_1\cup B_2$, with $\operatorname{codim} B_1=1$ and $\operatorname{codim} B_2\geq 2$. First, we prove that, in $M\setminus B_1$, the interior U' of $U\cup B_2$ is locally pseudoconvex. Let $h:(H,\Delta^n)\to M\setminus B_1$ be a Hartogs' figure (see [28] p.49) such that $h(H)\subset\subset U'$ and $h(\Delta^n)\subset\subset M\setminus B_1$. Since $\operatorname{codim} B_2\geq 2$, each plurisubharmonic function φ' in $M\setminus B$ admits a plurisubharmonic extension, which we denote $\tilde{\varphi}'$, to $M\setminus B_1$. $\tilde{\varphi}'$ satisfies that for any relatively compact open subset X in $M\setminus B_1$, $\sup_X\tilde{\varphi}=\sup_{X\setminus B_2}\varphi'$. This fact applies to φ^*. Hence for any α, $\max_{\overline{h(H)}}\tilde{\varphi}^*\geq\sup_{h(H)}\tilde{\varphi}_\alpha\geq\sup_{h(\Delta^n)}\tilde{\varphi}_\alpha$. In particular, any point of $h(\Delta^n)\setminus B_2$ belongs to U, so $h(\Delta^n)\subset U'$.

Next, by using the disc characterisation of pseudoconvexity, it is classical that if X is an open pseudoconvex subset in $M\setminus B_1$, then the interior of $X\cup B_1$ is pseudoconvex, when B_1 is a complex hypersurface. □

Remark 2 In particular, this set X is invariant under bimeromorphic maps.

Lemma 8 *Let W be an open subset of M biholomorphic to the unit ball in $\mathbb{C}^n$. Let $D\subset\subset W$ be a strongly pseudoconvex open subset of W. Then, for any $\psi\in P_\omega(M)$, there exists a unique function $T_D(\psi)\in P_\omega(M)$ such that $T_D(\psi)=\psi$ on $M\setminus D$ and $(\omega+dd^cT_D(\psi))^n=0$ on D. Further $T_D(\psi)\geq\psi$.*

Proof. Let $a\in \mathrm{PSH}(W)\cap L^\infty(W,\mathrm{loc})$ be a potential for ω on W. From [5], Proposition 9.1, a unique plurisubharmonic function $\widetilde{a+\psi}$ exists such that $(dd^c(\widetilde{a+\psi}))^n=0$ on D, $\widetilde{a+\psi}=a+\psi$ on $W\setminus D$, and $\widetilde{a+\psi}\geq a+\psi$ on W. Note that $\widetilde{a+\psi}-a=\psi$ on $W\setminus D$ and we define

$$T_D(\psi)=\begin{cases}\max(\psi,\widetilde{a+\psi}-a)=\widetilde{a+\psi}-a & z\in W\\ \psi & z\in M\setminus W\end{cases}$$

□

Lemma 9 *Let U be an open subset of M. Let $\Lambda\subset P_\omega(M)$ be a family which is stable with respect to the* max *operation.*

Assume that any point $p\in M\setminus\bar{U}$ admits a pair of open neighbourhoods (W,D) as in Lemma 8, with $W\subset M\setminus\bar{U}$, such that, for all $u\in\Lambda$, the function $T_D(u)$, belongs to Λ.

Assume that X, the open subset where Λ is locally bounded from above, contains U. Denote $\varphi^*=(\sup_{\psi\in\Lambda}\psi)^*\in P_\omega(X)$, the upper regularization (w.r.t. ω) of the upper envelope of this family.

Then the positive measure $(\omega+dd^c\varphi^*)^n$ has support in $\bar{U}$.

Proof. Since Λ is stable by the max operation, from Choquet's Lemma 2, we get an increasing sequence $\{u_j\}_{j\in\mathbb{N}} \subset \Lambda$ with $(\lim_{j\to+\infty} u_j)^* = \varphi^*$. From the hypothesis, let (W, D) open neighbourhoods of $x \in X \setminus \overline{U}$ such that $\forall u \in \Lambda$, $T_D(u) \in \Lambda$. Replacing each u_j by $\tilde{u}_j = T_D(u_j) \in \Lambda$, then

the sequence $\{\tilde{u}_j\}_{j\in\mathbb{N}}$ is increasing, since $\tilde{u}_j$ may be obtained by a Perron method, it increases to φ^* outside a pluripolar set, since $\varphi^* = (\lim_{j\to+\infty} \tilde{u}_j)^*$ and the negligible set $\{(\lim_{j\to+\infty} \tilde{u}_j) < (\lim_{j\to+\infty} \tilde{u}_j)^*\}$ is pluripolar.

Hence, from [5], Theorem 7.4, $\lim_{n\to+\infty} (\omega + dd^c\tilde{u}_j)^n = (\omega + dd^c\varphi^*)^n$ is vanishing on D. Since this property is valid for any such pair (W, D), with $W \cap \bar{U} = \emptyset$, the assertion is proved. □

3.1. *Some extremal functions*

Let ω be a closed positive $(1, 1)-$current on the complex manifold M. Assume that ω admits local locally bounded potentials near every point in M (see (2.1)).

Definition 6 *Let U be a domain in M, and let h be a function on U which is locally bounded and lower semicontinuous w.r.t. ω. Define*

$$X(h, \omega) = \{p \in M : \varphi = \sup \psi_{\psi\in P_\omega(M,U,h)} \text{ is locally bounded from above at } p\},$$
$$\text{where } P_\omega(M, U, h) = \{\psi \in P_\omega(M) \text{ such that } \psi_{|U} \le h\}.$$

Let φ^* be the upper regularization of φ (w.r.t. ω) in $X(h, \omega)$ and call it the extremal function associated to U, ω and h. Define $U(h, \omega)$ to be the connected component of $X(h, \omega)$ which contains U.

By assumption, $P_\omega(M, U, h)$ is locally bounded from above on U, hence $X(h, \omega)$ contains U. When $h = 0$, and M is a pseudoconvex domain in $\mathbb{C}^n$, we obtain the usual hull of holomorphy of U with respect to M. For M a projective manifold, $h = 0$, this hull is similar to hull introduced in [19]. We refer to this article for further properties when this hull is assumed to be compact in some locally pseudoconvex domain.

From Lemma 9, the extremal function φ^* satisfies $(\omega + dd^c\varphi^*)^n = 0$ on $U(h, \omega) \setminus \overline{U}$. Moreover, in U, we have $(\omega + dd^c\varphi^*)^n = 0$ on the open subset $\{\varphi^* < h\}$ (see [5], Corollary 9.2).

Definition 7 *Let U be a domain in M and $\psi \in P_\omega(M)$. Fix $\mathcal{D} = \{D_i\}_{i\in\mathbb{N}}$ an open cover of $M \setminus \bar{U}$ by open strongly pseudoconvex subsets D_i, which are relatively compact in complex holomorphic charts $f_i : W_i \to B_{\mathbb{C}^n}(0, 1)$. Assume that each D_i is repeated infinitely often in the sequence $\mathcal{D}$. Define by induction, $\psi_{-1} = \psi$, and $\psi_i = T_{D_i}(\psi_{i-1})$, for $i \in \mathbb{N}$. Let $X(\psi)$ denote the open subset where the family $\{\psi_i\}_{i\in\mathbb{N}}$ is locally bounded from above, and let $U(\psi)$ be the connected component of $X(\psi)$ which contains U. Define $B(\psi) = (\sup_{i\in\mathbb{N}} \psi_i)^*$, which belongs to $P_\omega(U(\psi))$.*

Note that this family is an increasing sequence w.r.t. $i \in \mathbb{N}$. Hypothesis of Lemma 9 are satisfy, hence $(\omega + dd^c B(\psi))^n = 0$ on $U(\psi) \setminus \bar{U}$. Moreover, $B(\psi) \geq \psi$ on $U(\psi)$. Although $B(\psi)$ depends in general of the cover chosen, we will not indicate this dependence.

Remark 3

i. Notice that the balayage procedure in Definition 7 when applied to a function (e.g. the zero function), gives a function which is strictly greater than the original one in points where ω is a strictly positive current.
ii. Let X be a relatively compact domain in M with smooth boundary. Assume for simplicity that ω is smooth and strictly positive. Then applying the Green formula for $\bar{X}$ with respect to the Kähler metric ω (see [4]), we see that a family $\mathcal{F} \subset P_\omega(M)$ is locally bounded from above in X if it is bounded for the L^1 norm induced on ∂X. In particular, the above balayage procedure applied to $M \setminus \bar{X}$ is always locally bounded.

3.2. *The case of a Chern class*

In this section, we interpret the above results when ω is a Chern current of a line bundle. Note that a closed positive $(1,1)-$current ω is the Chern current of a hermitian line bundle L over a complex manifold M if it lies in $H^2(M, \mathbb{Z})$ via the De-Rham isomorphism.
Let $(E, h) \to M$ be a complex hermitian line bundle with positive (singular) metric curvature. Denote $\pi : E^* \to M$ the bundle map from E^* to M, the dual line bundle of E, and denote $|\zeta|^2$ the norm of $\zeta \in E^*$ induced by h.
Let A be a subset of M. Denote $T_A(\alpha) = \{\zeta \in E^*_{|A},\ |\zeta| < \alpha\}$, and denote $T_A = T_A(1)$. Let $\widehat{T_U}$ be the pseudoconvex hull of T_U in the complex manifold E^*.

Lemma 10 *$\widehat{T_U}$ is a disqued pseudoconvex subset of E^*.*

Proof. Consider the action of $\mathbb{C}^*$, in the fibre of E, $(\lambda, \zeta) \to \lambda.\zeta$. Let $\lambda \in \mathbb{C}^*$, then $\lambda T_U \subset \lambda \widehat{T_U}$, hence $\widehat{\lambda T_U} \subset \lambda \widehat{T_U}$. But $T_U \subset \lambda^{-1} \widehat{\lambda T_U}$, hence $\lambda \widehat{T_U} \subset \widehat{\lambda T_U}$. So $\lambda \widehat{T_U} = \widehat{\lambda T_U}$. This is a classical result that if W is a pseudoconvex domain in $\mathbb{C}^n$, H an irreducible hypersurface in W and K a compact subset in W, with $H \cap K$ non void, then the pseudoconvex hull of $(W \setminus H) \cup K$ is W. Hence $\widehat{T_U}$ contains $0.\widehat{T_U}$ since it contains $0.T_U$. □

Since $\widehat{T_U} \subset \pi^{-1}(\hat{U})$ and $0.\widehat{T_U} \simeq \hat{U}$, from the above lemma, we see that $\widehat{T_U}$ is a twisted pseudoconvex Hartogs' domain over $\hat{U}$. Moreover $\widehat{T_U} \subset T_M(1)$. Assume that $iC(E)$ admits local locally bounded potentials, then there exists an u.s.c (w.r.t. $iC(E)$) function $\varphi \in P_{iC(E)}(\hat{U})$ such that $\widehat{T_U} = \{\zeta \in E^*,\ \ln|\zeta|^2 + \varphi < 0\}$. Indeed, let $t_W : E^*_{|W} \simeq W \times \mathbb{C}$ be a local trivialization of E^* over the open subset W biholomorphic to an open ball in $\mathbb{C}^n$. Since t_W is a morphism of vector bundle, $t_W(\widehat{T_{U|W}})$ is a Hartogs' locally pseudoconvex domain with base W. Hence $t_W(\widehat{T_{U|W}}) = \{(p,z) \in W \times \mathbb{C},\ \ln|z|^2 + \psi_W(P) < 0\}$ with ψ_W a plurisubharmonic function in W. In the local trivialization $t_W : E^*_{|W} \simeq W \times \mathbb{C}$, assume that $|t_W^{-1}(p,z)|^2 = a_{W,t_W}(p)|z|^2$ where a_{W,t_W} is a logarithmic plurisubharmonic function in W, with $dd^c \ln(a_{W,t}) = iC(E,h)$. Define $\varphi = \psi_W - \ln a_{W,t_W}$. One check that this function φ does not depends on the choosen trivialisation, hence define an element $\varphi \in P_{iC(E)}(\hat{U})$ such that $\widehat{T_U} = \{\zeta \in E^*,\ \log|\zeta|^2 + \varphi(\pi(\zeta)) < 0\}$.

Note that φ is maximal in the following sense.

Let W be an open set in $\hat{U} \setminus \bar{U}$, and let $\psi \in P_{iC(E)}(W)$. If W' is a relatively compact open subset of W and if $\liminf_{z\to\partial W'} \varphi(z) - \psi(z) \geq 0$ then $\varphi \geq \psi$ in G.

For the function

$$\varphi' = \begin{cases} \max(\varphi,\psi) & z \in W' \\ \varphi & z \in W \setminus W' \end{cases}$$

belongs to $P_{iC(E)}(\hat{U})$ and is zero on U. Hence $\{\ \zeta \in E^*_{\hat{U}}\ : \ln|\zeta|^2 + \varphi' \circ \pi(\zeta) < 0\}$ is pseudoconvex, contains T_U, hence contains $\widehat{T_U}$. So $\varphi' = \varphi$.

Lemma 11 *Assume that $iC(E)$ admits local locally bounded potentials, then the positive measure $(iC(E) + dd^c\varphi)^n$ has support in $\bar{U}$, the closure of U in $\hat{U}$.*

Proof. Let D, W be domains as in Lemma 8 with $W \cap \bar{U} = \emptyset$. Since φ is maximal, $T_D(\varphi) = \varphi$. However, $(\omega + T_D(\varphi))^n$ vanishes on D, by construction. □

Lemma 12 *Let $T_U(0,0)$ denote the hull of T_U with respect to globally defined plurisubharmonic functions on E^* (see Sect. 3.1). Then $T_U(0,0)$ is a disqued subset over $U(0,\omega)$ which contains the image of $U(0,\omega)$ by the null section. Moreover $T_U(0,0) = \{\zeta \in E^*,\ \ln|\zeta|^2 + \varphi^*(\pi(\zeta)) < 0\}$, where φ^* is the extremal function associated with U and ω (see Sect. 3.1).*

Proof. By definition $T_U(0,0) \subset \{\zeta \in E^*,\ \ln|\zeta|^2 + \varphi^*(\pi(\zeta)) < 0\} = A$. To prove the equality, we argue by contradiction. Let $\zeta_0 \in A \setminus T_U(0,0)$. A being open, there exists a neighbourhood W of ζ_0 in A, a non constant plurisubharmonic function ψ on E^*, such that $\{\psi < 0\}$ contains T_U but does not contains W. ψ being plurisubharmonic, $\{\psi \geq 0\}$ is the closure of $\{\psi > 0\}$. Hence there exists $\zeta_1 \in W \cap \{\psi > 0\}$. Let us replace ψ by $\psi' = \log|\zeta|^2 + N\psi$. Then $\{\psi' < 0\}$ contains T_U and for N large enough, still not contains ζ_1. That is $T_U \subset \{\psi' < 0\} \cap A \neq A$. Hence, $T_U \subset \bigcap_{\theta\in[0,2\pi]} \mathrm{e}^{\mathrm{i}\theta}\{\psi' < 0\} \cap A \neq A$. However $\bigcap_{\theta\in[0,2\pi]} \mathrm{e}^{\mathrm{i}\theta}\{\psi' < 0\}$ is a twisted Hartogs' pseudoconvex domain over M. It contains T_U, hence, it is defined by a function $\varphi' \in P_{iC(E)}(M,U,0)$. □

4. Bounds of Monge-Ampère masses

Recall that if M is a complex manifold, a non relatively compact connected component of $M \setminus K$ where K is a compact set in M, is called an end of M. Let ω be a closed positive $(1,1)-$current on M, which admits local locally bounded potentials. Let $\mathcal{F} \subset P_\omega(M)$, and let $X(\mathcal{F})$ denote the open subset in M where this family is locally bounded from above.

Definition 8 *An end of $X(\mathcal{F})$ will be called a pseudoconcave end with respect to $\mathcal{F}$.*

Consider the following situation. Let M be a complex manifold, let U be an open subset of M, and let $\mathcal{F} = P_\omega(M, U, 0)$. Let $U(0,\omega)$ as defined in Sect. 3.1. Working in the relative topology of $U(0,\omega)$, assume that $U(0,\omega) \setminus \bar{U}$ admits a connected component X with compact boundary (hence X is a pseudoconcave end with respect to $P_\omega(U, M, 0)$, if it is non relatively compact).
Let φ^* be the extremal function associated with $U(0,\omega)$. Recall that φ^* is everywhere positive and restricted to U is identically vanishing. Assume that

$\forall p \in X$, $\{\varphi^* \leq \varphi(p)^*\} \cap \bar{X}$ is a relatively compact subset of $U(0,\omega)$.

Let $M_1 = U \cup \bar{X}$. We have $\partial_{M_1} X = \partial_{U(0,\omega)} X$. Let $X_\epsilon = \{z \in M_1 \ : \ d(z, X) < \epsilon\}$. For ϵ small enough, this open subset has a relatively compact boundary in M_1, and φ^* satisfies hypothesis of Lemma 4. Hence,

$$+\infty > \int_{\bar{X}_\epsilon} \chi(\varphi^*)(\omega + dd^c\varphi^*)^n \geq \int_{\bar{X}_\epsilon} \chi(\varphi^*)\ \omega^n$$

for any positive smooth decreasing function $\chi : \mathbb{R} \to \mathbb{R}^+$.

The integrals are finite since on $\bar{X}_\epsilon$, the positive measure $(\omega + dd^c\varphi^*)^n$ has support on $\bar{X}_\epsilon \cap \bar{U}$, which is a compact set. Letting ϵ going to zero, we obtain the following Proposition (we work in the topology of $U(0,\omega)$).

Proposition 1 *Let $U(0,\omega)$ be as above and let X be a connected component of $U(0,\omega) \setminus \bar{U}$ with compact boundary. Let φ^* be the extremal function associated with $U(0,\omega)$. Assume that $\{\varphi^* \leq \varphi(p)^*\} \cap \bar{X}$ is a relatively compact subset of $U(0,\omega)$ for every $p \in X$. Then, for any positive decreasing smooth function $\chi : \mathbb{R} \to \mathbb{R}^+$, we have*

$$\int_{\bar{X}} \chi(\varphi^*)\omega^n \leq \int_{\partial X} \chi(\varphi^*)(\omega + dd^c\varphi^*)^n < +\infty\ . \tag{4.1}$$

Remark 4 Let M be a complex manifold and let ω be a closed positive $(1,1)-$current which satisfies condition (2.1). Assume that $\varphi \in P_\omega(M)$ is exhaustive and satisfies the Monge-Ampère equation $(\omega + dd^c\varphi)^n = 0$. Then Lemma 3 implies that $\omega^n = 0$.

For compact singularities in the unit ball, we obtain the following well known fact (see *e.g.* [30]).

Corollary 1 *Let* $u \in \mathrm{PSH}(B(1))$*, such that its polar set* $L = \{u = -\infty\}$ *is a compact subset of* $B(\frac{1}{2})$*, and* u *is locally bounded on* $B \setminus L$*.*

Then $\displaystyle\int_{B(\frac{1}{2})\setminus L} (dd^c u)^n < +\infty.$

Proof. We work in $M = B(1) \setminus L$. The pseudoconvex hull of $U = B(1) \setminus \bar{B}(\frac{1}{2})$ is M. Now, $-u \in P_\omega(M)$, where $\omega = dd^c u$, and this function satisfies that $\{-u < c\} \cap \bar{B}(\frac{1}{2})$ is relatively compact in M for any $c \in \mathbb{R}$. So does $-u - C$ for some constant, chosen such that $-u - C$ is negative on a neighbourhood of $\partial B(\frac{1}{2})$. Let φ^* be the extremal function associated to ω and U. But $\varphi^* \geq -u - C$, hence from Proposition 1,

$$\int_{\bar{B}(\frac{1}{2})\setminus L} \omega^n \leq \int_{\partial B(\frac{1}{2})} (\omega + dd^c \varphi^*)^n < +\infty . \qquad \square$$

5. Pluricomplete currents

In this section, we consider a current ω on a manifold M which admits local locally bounded potentials (see 2.1) on $M \setminus B$, where B is an analytic subset in M. If B may be written as intersection of hypersurfaces (*e.g.* an indeterminacy set of a meromorphic map with value in a projective manifold), we construct a function $\varphi \in P_\omega(M \setminus B)$ which goes to $+\infty$ near B. Hence, under suitable pseudoconcavity conditions, we will be able to bound Monge-Ampère masses of $\omega_{|M\setminus B}$. To avoid numerous hypothesis, we will restrict ourself to spread manifolds over a projective manifold.

5.1. Spread spaces and distance to the boundary

Definition 9 *Let* M *be a manifold. A complex manifold* $\pi : U \to M$ *is spread over* M *if the map* π *is a local biholomorphism. We say that* $\pi : U \to M$ *is locally pseudoconvex over* M *(with respect to* π*), if there exists an open covering* $\mathcal{W}$ *of* M *by Stein open subsets* $W \in \mathcal{W}$ *such that* $\pi^{-1}(W)$ *is a Stein manifold for any* $W \in \mathcal{W}$*.*

We say that $\pi : U \to M$ is a domain over M, if U is connected. Examples of spreading are a canonical injection $i : U \hookrightarrow M$ of an open subset U of M, a restriction $\pi_{|U'} : U' \to M$ of a covering map $\pi : U \to M$ to an open subset. In the first case, $i : U \hookrightarrow M$ is locally pseudoconvex over M if and only if U is a locally pseudoconvex open subset of M.

We recall notion of boundary distance for a spread space. Let $\pi : U \to (M, \omega_0)$ be a spread space over a Kähler manifold. We still denote ω_0 the pullback by π of ω_0. For $Q \in U$, let $d_{\partial U}(Q) = \sup\{r > 0$, s.t. $\exp_Q : B(0, r) \to U$ is defined$\}$. This function is either identically ∞ or Lipschitzian.

Theorem 2 ([26,35]) *Let (M, ω_0) be a Kähler manifold and K a compact subset in M. Then, there exists real constants $\delta > 0$ and α, such that, for any locally pseudoconvex spread domain $\pi : U \to M$, subject to the condition $\pi(U) \subset K$, the function $-\log d_{\partial U}$ (if U admits some boundary points over M) satisfies $dd^c - \log d_{\partial U} \geq -\alpha\omega_0$ for any point p in U such that $d_{\partial U}(p) < \delta$.*

5.2. A spannedness property for divisors

We fix notations. Let V be a projective manifold of dimension $n \geq 2$. Denote $\mathcal{O}(1)$ the line bundle over V which gives the projective embedding of V and let ω_0 be a Kähler metric on V. If $\pi : U \to V$ is a spreading, we still denote $\mathcal{O}(l)$ and ω_0 the pullbacks by π of $\mathcal{O}(l)$ and ω_0. If s is a section of some line bundle on a manifold M, we denote $\mathrm{ord}_p s$ its vanishing order at a point p, if Y is a complex hypersurface in M, we denote $\mathrm{mult}_p Y$ its multiplicity at p. For a divisor D on M, denote $\nu_p(D)$ its multiplicity at p. If s is a meromorphic section of a line bundle over M, we denote (s) its divisor and Z_s its zero set.

Theorem 3 *Let $(V, \mathcal{O}(1))$ be a projective manifold. Then there exists $l_1 \in \mathbb{N}$, such that for any $l \geq l_1$, for any locally pseudoconvex domain $\pi : U \to V$ over V, any hypersurface $Y \hookrightarrow U$, and any $p \in U$, there exists an $\tilde{s} \in H^0(U, \mathcal{O}(l) \otimes [Y])$ of minimal growth such that* $\mathrm{ord}_p \tilde{s} \leq \mathrm{mult}_p Y - 1$.

Proof. We give the main arguments of the proof, since similar methods appears in [3,27] for the univalent case and in [16] in the above case.
Since V is compact and $\mathcal{O}(1)$ is strictly positive, there exists a real number β such that $i\mathrm{Ricci}(\omega_0) \geq -i\beta C(\mathcal{O}(1))$. Let $l_0 = \mathrm{Ent}(1+n+\beta)+1$, where $\mathrm{Ent}(r)$ denotes the integer part of a real number r.

Let δ and α denote the real constants which appear in Theorem 2. Let $\frac{1}{4} \geq \epsilon_0 > 0$ such that $4\alpha\epsilon_0 < 1$. Let $l_1 = \mathrm{Ent}(\max(4\alpha\epsilon_0 + 1 + n - 1 + \beta, 1 + n)) + 1 \geq l_0$. Let $l \geq l_0$.

First, note that there exists a finite number of square integrable holomorphic sections of $\mathcal{O}(l)$ over U which give an immersion of U in some projective space, see [17]. Hence, if $p \notin Y$, one of those sections satisfies our requirements.

Assume $p \in Y$. Let $t_1, \ldots, t_n$ be sections of $\mathcal{O}(1)$ which give local coordinates centred in $\pi(p)$ and denote by the same letter their pullback by π. Let W be some small open neighbourhood of p in U, biholomorphic by π to some coordinate open set. Let s_1 be a smooth section of $\mathcal{O}(l+1)$ with compact support in W, holomorphic and non zero in a neighbourhood of p.

Let $k = \epsilon + n - 1$, with $0 < \epsilon \leq \epsilon_0$. For $l \geq l_0$, we solve the $\bar\partial-$equation $\bar\partial s_1 = \bar\partial s_2$ with weight $\exp -(k+1)\log \|t\|^2$ by L^2 methods (see [10]).

Hence the holomorphic section $s_3 = s_1 - s_2$ on U, is non-vanishing at p. Moreover, from the L^2 estimates, we deduce

$$I = \int_U \frac{|s_1 - s_2|^2}{\|t\|^{2(k+1)}} \mathrm{e}^{-(-4\epsilon \log \min(\delta, d_{\partial_{U\setminus Y}}))} dV_{\omega_0} < +\infty \quad ,$$

since $\|t\|^2(p) = (|t_1|^2 + ... + |t_n|^2)(p) \geq C_1 d^2_{\partial_{U\setminus Y}}(p)$ in a neighbourhood of p. Hence for $l \geq l_1$, from Skoda [33], there exits $h_1, ..., h_n \in H^0(U \setminus Y, \mathcal{O}(l))$ such that

$$s_3 = \sum_{i=1}^{\mathrm{i}=n} h_i t_i \tag{5.1}$$

$$I = \int_U \frac{\|h\|^2}{\|t\|^{2k}} \mathrm{e}^{-(-4\epsilon \log \min(\delta, d_{\partial_{U\setminus Y}}))} dV_{\omega_0} < +\infty \, . \tag{5.2}$$

From the growth condition, the sections $h_1, \ldots, h_n$ define sections $\tilde{s}_i$ of $H^0(U, \mathcal{O}(l) \otimes [Y])$. Let f be a minimal local equation of Y at p and write $h_i = \frac{g_i}{f}$. Then, $f s_3 = \sum_{i=1}^{\mathrm{i}=n} g_i t_i$. Hence, $s_3(p) \neq 0$, one of the g_i's has a vanishing order lower than $\mathrm{ord}_p f - 1 = \mathrm{mult}_p Y - 1$. Next the sections g_i globalize as sections $\tilde{s}_i$ of $H^0(U, \mathcal{O}(l) \otimes [Y])$, and one of them satisfies our requirements. □

Remark 5 Since V is compact, $\max_V \|t\|^{2k}$ exists, hence

$$\int_{U\setminus Y} \|h\|^2 \mathrm{e}^{-(-4\epsilon \log \min(\delta, d_{\partial_{U\setminus Y}}))} dV_{\omega_0} \leq \max_V \|t\|^{2k} I \tag{5.3}$$

So, rescaling the sections h_i by a linear factor, we may assume that the right hand side is lower than one.

Corollary 2 *Under the hypothesis of Theorem 3, let $l \geq l_1$. Let $E \to U$ be a line bundle, and let $s \in H^0(U, E) \setminus \{0\}$. Then, for any $k \in \mathbb{N}$ and any $p \in U$, there exists $\check{s} \in H^0(U, E \otimes \mathcal{O}(kl))$ such that $\nu_p((\check{s} = 0)) \leq (\nu_p(s = 0) - k)^+$.*

Proof. First, we prove the corollary for $k = 1$. If the point p does not belong to Z_s, since $\mathcal{O}(l)$ is very ample, the corollary is true. Assume $p \in Z_s$ and let $Y_1, \ldots, Y_r$ be its global irreducible (reduced) components which contain p. Write $Y' = Y_1 \cup \ldots \cup Y_r$. Let $t_1, \ldots, t_r$ be minimal local equations at p for $Y_1, \ldots, Y_r$ respectively, so that $\mathrm{mult}_p Y' = \mathrm{ord}_p t_1 + \ldots + \mathrm{ord}_p t_r$. Let $\tilde{s}' \in H^0(U, \mathcal{O}(l) \otimes [Y'])$ a section as in Theorem 3 and denote by s' the corresponding meromorphic section of $\mathcal{O}(l)$ over U. We may assume that the polar divisor of s' is $Y_1+\ldots+Y_{r'}$, with $r' \leq r$. By hypothesis, there exists strictly positive integers $n_1, \ldots, n_r$, such that $s = t_1^{n_1} \ldots t_r^{n_r} e$ where $e \in E_p$ is a local non vanishing germ at p. In the same way, $s' = \dfrac{g}{t_1 \ldots t_r} e'$ where $e' \in \mathcal{O}(l)_p$ is a local non vanishing germ at p, and $\text{ord}_p g \leq \text{mult}_p Y' - 1$. Hence, $\check{s} = s' \otimes s \in H^0(U, E \otimes \mathcal{O}(l))$ and $s' \otimes s = g t_1^{n_1-1} \ldots t_r^{n_r-1} e' \otimes e$. So $\text{ord}_p s' \otimes s \leq \text{mult}_p(Y') - 1 + \text{ord}_p(t_1^{n_1-1} \ldots t_r^{n_r-1}) = \text{ord}_p s - 1$.

Next, assume the corollary is true for some integer $k \geq 1$. Let $\check{s}_k$ denote the corresponding section of $E \otimes \mathcal{O}(kl)$. We apply the step $k = 1$ to $E \otimes \mathcal{O}(kl)$ and $\check{s}_k$ to conclude. □

Remark 6 If we apply this corollary to the line bundle $[D]$, where D is an effective divisor, and to its canonical section, we see that $\mathcal{O}(kl_1) \otimes [D]$ is globally generated outside the analytic subset $\{p \in U\ ;\ \nu_p(D) > k\}$.

5.3. Pluricomplete currents

Definition 10 *A closed positive* $(1,1)-$*current* ω *on a complex manifold* M *is said to be pluricomplete if there exists a closed set* L *on* M *such that* ω *admits local locally bounded potentials on* $M \setminus L$ *and a function* $\varphi \in P_\omega(M \setminus L)$ *with* $\liminf\limits_{M \setminus L \ni p' \to L} \varphi = +\infty$.

If $\mathbb{P}^k$ is a projective space, we will denote ω_{FS} its Fubiny-Study form without indication of the dimension.

Lemma 13 *Let* $E \to M$ *be a line bundle, with smooth hermitian metric and positive Chern curvature* ω_0. *Let* $s_0, \ldots, s_k \in H^0(M, E) \setminus \{0\}$ *be holomorphic sections of* E. *Let* A *denote their common zeros locus in* M. *Let* ψ *be the associated meromorphic map from* M *to* $\mathbb{P}^k$, *given in homogeneous coordinate by* $p \to [s_i(p)]_{0 \leq i \leq k}$. *Then, the function* $p \to -\log \|s\|^2(p)$ *belongs to* $P_{\psi^*\omega_{FS}+\omega_0}(M \setminus A)$ *and satisfies* $\liminf\limits_{M \setminus A \ni p' \to A} \psi = +\infty$.

Proposition 2 *Let* $U \to V$ *be a locally pseudoconvex domain over* V *and let* $E \to U$ *be a line bundle over* U. *Let* $s_0, \ldots, s_N \in H^0(U, E) \setminus \{0\}$ *and denote* $B = \bigcap\limits_{0 \leq i \leq N} Z_{s_i}$ *their common zero locus. Let* $e_\alpha,\ 0 \leq \alpha \leq N'$, *be global sections of* $\mathcal{O}(l)$, $l \geq l_1$, *without common zeros. Let* $\psi : U \to \mathbb{P}^{(N+1)(N'+1)-1}$ *be the meromorphic map given in homogeneous coordinate by* $p \mapsto [e_\alpha s_i]_{\alpha,i}(p)$, *which is holomorphic on* $U \setminus B$. *Considers the closed positive* $(1,1)-$*current* $\omega = \psi^*\omega_{FS}$. *Then, there exists* $\varphi \in P_\omega(U \setminus B)$ *with* $\liminf\limits_{U \setminus B \ni z \to B} \varphi(z) = +\infty$.

Proof. Denote B_2 the indeterminacy of ψ. Hence $B = B_1 \cup B_2$ with B_1 an hypersurface and codim$B_2 \geq 2$. ψ is holomorphic on $U \setminus B_2$. The associated bundle morphism $U \times \mathbb{C}^{(N+1)(N'+1)} \to \mathcal{O}(l) \otimes E$ gives an induced hermitian singular metric on $\mathcal{O}(l) \otimes E$ whose curvature $\omega = \psi^* \omega_{FS}$ is smooth on $U \setminus B$. To prove the proposition, it's enough to prove the following claim.

For any $z_0 \in U \setminus B$, there exists real strictly positive constants C_{z_0} and ϵ_{z_0} such that, for any $p \in B$, there exists $\varphi_p \in P_\omega(U \setminus B)$, with

$$\liminf_{U \setminus B \ni z \to p} \varphi_p(z) = +\infty \tag{5.4}$$

$$\forall\ p \in B, \quad \sup_{B(z_0, \epsilon_{z_0})} \varphi_p \leq C_{z_0}, \tag{5.5}$$

where $B(z_0, 2\epsilon_{z_0})$ is a ball in a complex analytic chart centred at z_0 and disjoint from B.

Indeed, if this claim is proved then, $\varphi = (\sup_{p \in B} \varphi_p)^*$ will be well defined on $U \setminus B$ due to (5.5). It belongs to $P_\omega(U \setminus B)$ and satisfies $\liminf_{U \setminus B \ni z \to B} \varphi = +\infty$.

First, we construct the function $\varphi_p \in P_\omega(U \setminus B)$, $p \in B$. Let $Y_i = (s_i = 0)$, $i = 0, \dots, N$. Recall that for each integer $0 \leq i \leq N$, p belongs to Y_i. From Theorem 3 and Remark 5, we may construct section $\tilde{\beta}_i^k \in H^0(U, \mathcal{O}(l) \otimes [Y_i])$, $k = 1, \dots, n$, subject to the following conditions

$$s_p = \sum_{k=1}^{n} \beta_i^k t_k \tag{5.6}$$

$$\int_{U \setminus Y_i} \|\beta_i\|^2 \mathrm{e}^{-(-4\epsilon \log \min(\delta, d_{\partial_{U \setminus Y_i}}))} dV_{\omega_0} \leq 1 \tag{5.7}$$

where, $s_p \in H^0(U, \mathcal{O}(l_1 + 1))$ is non vanishing at p, and $t_1, \dots, t_n \in H^0(U, \mathcal{O}(1))$ give local coordinates centred at p. Moreover, we consider $\tilde{\beta}_i^k$ as meromorphic sections β_i^k of $\mathcal{O}(l)$ over U, and $\|\beta_i\|^2 = \sum_{k=1}^{n} |\beta_i^k|^2$. Note that $\beta_i^k \otimes s_i \in H^0(U, \mathcal{O}(l) \otimes E)$. Working in the induce norm, define

$$\varphi_p = \log \left(\sum_{\substack{1 \leq k \leq n \\ 0 \leq i \leq N}} |\beta_i^k \otimes s_i|^2 \right) \in P_\omega(U \setminus B). \tag{5.8}$$

Away of B, we have

$$\sum_{k,\, i} |\beta_i^k \otimes s_i|^2 = \frac{\sum_{k,\, i} |\beta_i^k \otimes s_i|^2 \,.\, \sum_k |t_k|^2}{\sum_k |t_k|^2} \tag{5.9}$$

$$\geq \frac{\sum_i |\sum_k \beta_i^k t_k s_i|^2}{\sum_k |t_k|^2} = \frac{\sum_i |s_p \otimes s_i|^2}{\sum_k |t_k|^2} \tag{5.10}$$

where the sum is over $1 \leq k \leq n$ and $0 \leq i \leq N$. Line (5.10) is due to (5.6). Assume $e_0(p) \neq 0$. Recall that $s_p \in H^0(U, \mathcal{O}(l+1))$, hence write locally $s_p = s'_p \otimes e_0$. Next, in each charts $e_0 s_i \neq 0$, $0 \leq i \leq N$, says $e_0 s_0 \neq 0$, we have

$$\sum_{0 \leq i \leq N} |s_p \otimes s_i|^2 = |s'_p|^2 \frac{\sum_i |\frac{e_0 s_i}{e_0 s_0}|^2}{\sum_{\alpha,\, i} |\frac{e_\alpha s_i}{e_0 s_0}|^2} \tag{5.11}$$

$$= |s'_p|^2 \frac{\sum_i |\frac{s_i}{s_0}|^2}{\sum_\alpha |\frac{e_\alpha}{e_0}|^2 \cdot \sum_i |\frac{s_i}{s_0}|^2} \quad = \frac{|s'_p|^2}{\sum_\alpha |\frac{e_\alpha}{e_0}|^2} \tag{5.12}$$

The last expression is strictly positive at P, says greater than equal to $2c > 0$, does not depend on i, so

$$\varphi_p \geq -\log(\|t\|^2) + \log c \tag{5.13}$$

in a neighbourhood of p.

Next, we prove the uniform bound in the φ_p. Let $z_0 \in U \setminus B$, and let W be an open chart centered at z_0. Denote $B(z_0, \epsilon_1)$, $\epsilon_1 > 0$, the induced ball in W, and assume $B(z_0, 1) \subset\subset W$. Let $\frac{1}{2} > \epsilon_1 > 0$, such that $B(z_0, 2\epsilon_1) \subset\subset U \setminus B$ and such that, says, e_0 is non vanishing on $\bar{B}(z_0, 2\epsilon_1)$. Let t be a holomorphic section of E, on $B(z_0, 1)$, non vanishing there. Then

$$\sum_{k,i} |\beta_i^k s_i|^2 = \frac{\sum_{i,k} \left|\frac{\beta_i^k s_i}{e_0 t}\right|^2}{\sum_{\alpha,i} \left|\frac{e_\alpha s_i}{e_0 t}\right|^2} \tag{5.14}$$

Here, only the β_i^k, $1 \leq k \leq n$, $0 \leq i \leq N$, depend on $p \in B$. In the left hand side, the norm symbol represents the induced hermitian metric, in the right hand side it represents a modulus of a holomorphic function. Let $m = \max_{\bar{B}(z_0,\epsilon_1)} \sum_{k,i} |\beta_i^k s_i|^2 (< +\infty)$, $0 < m_1 = \min_{\bar{B}(z_0,\epsilon_1)} \sum_{\alpha,i} |\frac{e_\alpha s_i}{e_0 t}|^2$, and $0 < m_2 = \min_{\bar{B}(z_0,2\epsilon_1)} |e_0|^2$. Then

$$m \leq \frac{1}{m_1} \max_{\bar{B}(z_0,\epsilon_1)} \sum_{i,k} \left|\frac{\beta_i^k s_i}{e_0 t}\right|^2 \tag{5.15}$$

$$\leq \frac{C(\epsilon_1, n)}{m_1} \sum_i \int_{B(z_0,2\epsilon_1)\setminus Y_i} \left(\sum_k \left|\frac{\beta_i^k}{e_0}\right|^2\right) \left|\frac{s_i}{t}\right|^2 dV_{\omega_e} \tag{5.16}$$

$$\leq \frac{C(\epsilon_1, n)}{m_1} \sum_i \int_{B(z_0,2\epsilon)\setminus Y_i} \frac{\|\beta_i\|^2}{|e_0|^2} \gamma_i \times \left|\frac{s_i}{t}\right|^2 \frac{1}{\gamma_i} dV_{\omega_e} \tag{5.17}$$

with $\gamma_i = \min(\delta, d_{\partial_{U\setminus Y_i}})^{4\epsilon}$ and ω_e is the usual Kähler metric on $\mathbb{C}^n$. Next, there exists a constant A such that $\left|\frac{s_i}{t}\right|^2 \frac{1}{\gamma_i} \leq A$ on $B(z, 2\epsilon_1) \setminus Y_i$ for any i, since $|\frac{s_i}{t}|^2$ is lipchitzian and vanishes on Y_i. Hence

$$m \leq \frac{C(\epsilon_1, n)}{m_1.m_2} C'(\epsilon_1) A \times (N+1) \tag{5.18}$$

where $C'(\epsilon_1)$ bounds the ratio of the Euclidean volume form and the Kähler one and $N+1$ appears since the vector $(\beta_i^1, \dots, \beta_i^n)$ belongs to the unit ball of $L^2(U \setminus Y_i, \gamma_i dV_{\omega_0})$ by (5.7). □

Corollary 3 *Let $U \to V$ be a locally pseudoconvex domain over the projective manifold V, $\dim V \geq 2$. Let Y be an effective divisor on U. Then $[Y] \otimes \mathcal{O}(kl_1)$ is spanned by its global sections outside $E_{k+1}(Y) = \{p \in U \ : \ \nu_p(Y) \geq k+1\}$. If $k \geq 1$, it admits a singular hermitian metric of positive curvature, which is smooth away from $E_k(Y)$ and is a pluricomplete positive current in U.*

Proof. The first assertion is the content of Corollary 2 (in particular $[Y] \otimes \mathcal{O}(kl_1)$ admits a singular hermitian metric with a positive Chern current which are smooth away from $E_{k+1}(Y)$). Let $k \geq 1$. By a Baire argument, select $N+1 \geq n+1$ sections in $H^0(U, [Y] \otimes \mathcal{O}((k-1)l_1))$, which together span $[Y] \otimes \mathcal{O}((k-1)l_1)$ away from $B \subset E_k(Y)$. Proposition 2 applied to this set of sections gives a singular metric on $[Y] \otimes \mathcal{O}(kl_1)$, which is smooth away from B, and is pluricomplete. □

Remark 7

i. In the construction of Proposition 2, we may select the sections e_α such that the holomorphic map given by them is biholomorphic onto its image (see [17]). In particular, the current $\psi^* \omega_{FS}$ obtained is strictly positive. Moreover, adding some pullback by π of elements in $H^0(V, \mathcal{O}(l_1))$, we may always assume that $\psi^* \omega_{FS} \geq C\omega_0$, where C is a strictly positive constant.
ii. Let ω be a closed positive $(1,1)-$current on a complex manifold M. Assume that it admits local locally bounded potentials on $M \setminus B$, where B is an analytic subset of M. Assume that for any $p \in B$, there exists a function $\varphi_p \in P_\omega(M \setminus B)$ such that $\liminf_{M\setminus B \ni z \to p \in B} \varphi_p = +\infty$. For any relatively compact open subset U in $M \setminus B$, let U_1 denote the interior of $U(0, \omega) \cup B$, which is locally pseudoconvex in M (see Sect. 3). Then by definition of $U(0, \omega)$, there exists $\varphi \in P_\omega(U_1 \setminus B)$ such that $\liminf_{U_1\setminus B \ni z \to p \in B} \varphi = +\infty$.

Let $E \to U$ be a line bundle which admits a singular metric with a positive current curvature. Let $\mathcal{I}$ denote its Nadel multiplier ideal sheaf (see [13] for a definition). Using standard L^2 methods (see [14], prop. 4.2.1 in the compact case), we see that $E \otimes \mathcal{O}(l_0) \otimes \mathcal{I}$ is spanned by its global sections. Hence, assume that $E \otimes \mathcal{I}$ is spanned by its global sections. Let $s \in H^0(U, E \otimes \mathcal{I})$. To each $p \in Z_s$, we may associate the meromorphic sections β^k of $\mathcal{O}(l_1)$, which are holomorphic on $U \setminus Z_s$ (*i.e.* associated to sections $\tilde{\beta}^k \in H^0(U, \mathcal{O}(l_1) \otimes [Z_s])$ and which satisfies the usual ideal relation (5.6)). We obtain then sections $\beta^k \otimes s \in H^0(U, \mathcal{O}(l_1) \otimes E)$. Doing this procedure for any $s \in H^0(U, E \otimes \mathcal{I})$ and any $p \in Z_s$, we obtain a set of global section G_1 of $\mathcal{O}(l_1) \otimes E$. Let $\mathcal{I}_1$ denote the coherent ideal sheaf it generates. Then $\mathcal{I} = \mathcal{I}_0 \subset \mathcal{I}_1$. Working with G_1 as before, we obtain a set G_2 of global section of $\mathcal{O}(2l_1) \otimes E$ which defines an ideal sheaf $\mathcal{I}_2$, and so on. Then, one get a sequence of coherent ideal sheafs $\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \dots$. By Notherian properties, this sequence become locally stationary equal to the structure sheaf $\mathcal{O}$ (as was shown). For a point $p \in U$, define $m(p)$ to be the least integer such that $(\mathcal{I}_k)_p = \mathcal{O}_p$ for any $k \geq m(p)$. By construction the set $M_l = \{p \in U \ : \ m(p) > l\}$ are analytic subsets in U.

Corollary 4 *Under the above hypothesis, the line bundle $E \otimes \mathcal{O}(kl_1)$ admits a singular hermitian metric with a positive Chern current which are smooth away from M_k. If $k \geq 1$, the line bundle $E \otimes \mathcal{O}(kl_1)$ admits a singular hermitian metric, with a Chern current ω_k, which are smooth on $U \setminus M_{k-1}$ and ω_k is pluricomplete. There exists $\varphi \in P_{\omega_k}(U \setminus M_{k-1})$ with $\liminf_{U \setminus M_{k-1} \ni z \to p \in M_{k-1}} \varphi = +\infty$.*

6. Some Hartogs' phenomenon in projective manifolds

Definition 11 *([2]) Let X be a normal complex space of pure dimension $n \geq 2$. For $W' \subset W$ open subsets of X, we define the hull of W' in W by*

$$\widehat{W'}_W = \left\{ x \in W \ : |f(x)| \leq \sup_{W'} |f|, \forall f \in \mathcal{O}(W) \right\} .$$

An open subset $Y \subset X$ is said to be pseudoconcave at the boundary point $P \in \partial_X Y$ if there exists $\{W_\alpha\}_\alpha$, an open basis of P in X, s.t. P is an interior point of $\widehat{W_\alpha \cap Y}_{W_\alpha}$. X is said to be pseudoconcave in the sense of Andreotti, if there exists Y, an open relatively compact subset of X, which is pseudoconcave in each of its boundary point.

Remark 8 No boundary condition on X is assumed.

Proposition 3 *([15]) Let Ω be an open subset of the projective manifold V. Assume that Ω is pseudoconcave in the sense of Andreotti and locally pseudoconvex in V, then $\partial_V \Omega$, the topological boundary of Ω in V, is a compact hypersurface. Hence, if X is a pseudoconcave open subset of the projective manifold V, then $V \setminus X$ contains a maximal compact hypersurface H (which may be empty). Moreover, if $\dim_{\mathbb{C}} V = 2$, then each irreducible componant of H may be blow down onto a point.*

Notice that for $\dim V \geq 3$, there exists example of hypersurface H such that $V \setminus H$ is a pseudoconcave domain in the sense of Andreotti, but no irreducible component of H may be blow down. Indeed, let V be a projective manifold of dimension $n \geq 2$, and let $(L, h) \to V$ be a hermitian line bundle with curvature form ω. Assume ω has one strictly positive eigenvalue and another one strictly negative. Then, the real hypersurface, in $L \hookrightarrow \mathbb{P}(L \oplus \mathbb{C})$, given as $\{\zeta \in L \ : \ h(\zeta) = 1\}$ is pseudoconcave, but the zero section (or the hyperplan to infinity) does not contract to a lower dimensional analytic set in general.
We prove an extension theorem for currents which implies, in the projective case, a result of Nadel-Tsuji [24].

Theorem 4 *Let $V = (V, \mathcal{O}(1))$ be a projective manifold,* $\dim V \geq 2$. *Let H be a hypersurface in V such that $V \setminus H$ is pseudoconcave in the sense of Andreotti. Let U be an open neighbourhood of H in V. Let ω be a $(1,1)-$closed positive current on $U \setminus H$ which admits local locally bounded potentials. Then*

$$\int_{K \setminus H} \omega^n < +\infty \ , \tag{6.1}$$

for any compact set K in U. Moreover, if $1 \leq k \leq n$ then ω^k extends as a closed positive currents through H.

Proof. We may assume that U does not intersect Y, the subset which gives the pseudoconcavity condition on $V \setminus H$ (see Definition 11). Let U_1 be a relatively compact subset in U which contains $H \cup K$. From proposition 3, let $H' = H \cup H_1$ the maximal compact hypersurface contained in U_1. We may assume that K is a compact subset in U_1 which contains a neighbourhood of H' and that $\overline{\mathring{K}} = K$. Let ω_0 be the Chern curvature of the line bundle $\mathcal{O}(1)$, and denote $\omega_1 = \omega + \omega_0$. Let $X_0 = X(0, \omega_1)$ be the open subset of $U \setminus H'$ where the family $P_{\omega_1}(U \setminus H', U_1 \setminus K, 0)$ is locally bounded from above (see 3.1). From Lemma 6, X_0 is locally pseudoconvex in $U \setminus H'$ and contains $U_1 \setminus K$. Note that $(V \setminus K) \cup X_0$ is locally pseudoconvex in V. Since it contains Y, it is pseudoconcave in the sense of Andreotti. From proposition 3, $(V \setminus K) \cup X_0 = V \setminus H'$, for H' is the maximal compact hypersurface in K. From Takeuchi's theorem 2, there exists $\delta, \epsilon > 0$ and a constant C, such that $\psi_1 = -\epsilon \log(\min(\delta, d_{\partial V \setminus H'})) - C \in P_{\omega_1}(U \setminus H', U_1 \setminus K, 0)$, since $\omega_1 \geq \omega_0$. Denote φ^* the extremal function associated to $P_{\omega_1}(U \setminus H', U_1 \setminus K, 0)$. Then $\{\varphi^* \leq c\} \cap K \subset\subset K \setminus H'$ for any $c \in \mathbb{R}$, since $\varphi^* \geq \psi_1$. From Proposition 1,

$$+\infty > \int_{\partial K} (\omega_1 + dd^c \varphi^*)^n \geq \int_{K \setminus H'} (\omega + \omega_0)^n \,. \tag{6.2}$$

We deduce that the closed positive currents ω^k, $k = 1, \ldots, n$, have finite trace measure near H. Hence they extend as closed positive currents through H (see *e.g.* [30,34]). □

Corollary 5 *Let H be a hypersurface in a projective manifold V,* $\dim V \geq 2$. *Assume that $V \setminus H$ is pseudoconcave in the sense of Andreotti. Let U be a neighbourhood of H. Let $f : U \setminus H \to M$ be a holomorphic map into the compact Kähler manifold (M, ω_1). Then f extends as a meromorphic map through H.*

Proof. Theorem 4 applied to $\omega = f^*\omega_1 + \omega_0$, implies that the graph of h is of finite volume near $H \times M$. Hence it extends through it. ◻

Theorem 5 *Let V be a projective manifold,* $\dim V \geq 2$. *Let H be a compact complex hypersurface in V. Assume that $V \setminus H$ is pseudoconcave in the sense of Andreotti. Let U be an open subset of V which contains H. Let $\pi : W_1 \to V$ be a locally pseudoconvex spread domain over V which contains $U \setminus H$. Then any complex hypersurface Z of W_1 extends through H.*

Proof. Denote $\mathcal{O}(1)$ the line bundle which gives the projective embedding of V. We denote by the same symbols pullbacks by π of the line bundle $\mathcal{O}(l)$, $l \in \mathbb{N}$, and of ω_0, the Chern curvature of $\mathcal{O}(1)$. In the following, we assume that H is not a subset of W_1. Let $Y \subset\subset V \setminus H$ open subset with pseudoconcave boundary (see definition 11).

Shrinking U if necessary, we may assume that H is the maximal compact hypersurface in U (see Proposition 3), that ∂U the topological boundary of U in W_1 is relatively compact in W_1 and that U does not intersect Y. Let X be a relatively compact open neighbourhood of ∂U in W_1. We may assume that X has smooth boundary.

Let Z a complex hypersurface in W_1. Let $m = \max_{p \in \bar{X}} \mathrm{mult}_p Z$. From Corollary 3 (see the proof of the second assertion), sections $s_0, \ldots, s_r \in H^0(W_1, \mathcal{O}((m+1)l_1) \otimes [Z])$ exist such that

- the meromorphic map ψ, from W_1 to $\mathbb{P}^r$, given by $z \to [s_i(z)]_{0 \leq i \leq r}$ has base points B contained in $E_{m+1}(Z) = \{z \in W_1,\ \mathrm{mult}_z Z \geq m+1\}$,
- the current $\omega = \psi^*(\omega_{FS})$ is strictly positive, and pluricomplete in W_1.

Moreover, by adding a non trivial section of $\mathcal{O}((m+1)l_1) \simeq \mathcal{O}((m+1)l_1) \otimes [Z] \otimes [-Z]$, we may assume s_0 is vanishing on Z.

Let $\hat{X}$ denote the pseudoconvex hull of X in W_1. Then $\hat{X}$ contains $U \setminus H$. For, $(V \setminus U) \cup (X \cap U)$ is a locally pseudoconvex domain which is pseudoconcave and H is the maximal compact hypersurface in U, see Proposition 3.

Let $X(0, \omega + \omega_0)$ the pseudoconvex hull of X in $W_1 \setminus B$ with respect to $\omega + \omega_0$ (see Sect. 3.1). We claim that $X(0, \omega + \omega_0) \cap U = U \setminus (H \cup B)$. Indeed, by Lemma 7, X' the interior of $X(0, \omega + \omega_0) \cup B$ is a pseudoconvex subset in W_1 which contains X. Hence X' contains $\hat{X}$. From the description of $\hat{X}$, we deduce $X(0, \omega + \omega_0) \cap U = U \setminus (H \cup B)$. In particular, those connected components of $\hat{X} \setminus \bar{X}$ which meet U are pseudoconcave ends (with respect to $P_{\omega+\omega_0}(W_1 \setminus B, X, 0)$).

Denote $\varphi^* \in P_{\omega+\omega_0}(W_1 \setminus B, X, 0)$ the extremal function associated to $P_{\omega+\omega_0}$ $(W_1 \setminus B, X, 0)$. We claim that $U \cap \{\varphi^* < t\} \subset\subset \bar{U} \setminus (H \cup B)$, for all $t \in \mathbb{R}$. Since $\omega + \omega_0 \geq \omega_0$, from Takeuchi's theorem 2, there exists $\delta > 0$, $\epsilon > 0$, and C, such that $\varphi_1 = (-\epsilon \log \min(\delta, d_{\partial V \setminus H}) - C)^+$ belongs to $P_{\omega+\omega_0}(W_1 \setminus B, X, 0)$. Recall that to show that ω is pluricomplete on W_1, we have constructed a function $\varphi_2' \in P_\omega(W_1 \setminus B)$ in Proposition 2, which satisfies $\liminf_{W_1 \setminus B \ni z \to B} \varphi_2'(z) = +\infty$. Denote $\varphi_2 = (\varphi_2' - \max_{\bar{X}} \varphi_2')^+ \in P_\omega(W_1 \setminus B, X, 0)$. Then $\liminf_{W_1 \setminus B \ni p \to B} \varphi_2 = +\infty$, since $E_{m+1}(Z) \cap \bar{X} = \emptyset$. Hence $\max(\varphi_1, \varphi_2) \in P_{\omega+\omega_0}(W_1 \setminus B, X, 0)$ satisfies the exhausting condition required above. So does φ^*. From Proposition 1, we obtain

$$\int_{U \setminus (X \cup B \cup H)} (\omega + \omega_0)^n \leq \int_{\partial X \cap U} (\omega + \omega_0 + dd^c \varphi)^n < +\infty . \tag{6.3}$$

In particular, all the Chern numbers $\int_{U \setminus (X \cup B \cup H)} \omega^k \omega_0^{n-k}$ are finite. Hence the graph of the meromorphic map ψ is of finite volume near $H \times \mathbb{P}^1$. So ψ extends through H and $Z \subset Z_{s_0}$ extends through H. □

We obtain an Hartogs' Theorem type which strengthened results in [15].

Corollary 6 (Hartogs' Kugelsatz) *Let U be an open subset of the projective manifold V,* $\dim V \geq 2$. *Assume that $V \setminus \bar{U}$ is a connected pseudoconcave open subset of V, and assume $\mathring{\bar{U}} = U$. Let H denote the maximal compact hypersurface in U, and let $F \to V$ be a holomorphic vector bundle over V. Then any meromorphic section s of F defined on a neighbourhood of the boundary of U extends to a meromorphic section of F on U. Moreover, any holomorphic section s of F extends to a meromorphic section on U which is holomorphic on $U \setminus H$.*

Proof. From [15], we may assume U connected with connected topological boundary. Let W be a connected neighbourhood of the topological boundary of U. Let W_1 denote the domain of holomorphic existence of any holomorphic section on W of any holomorphic vector bundle over V. Since over open ball in V, any holomorphic vector bundle is trivial, $W_1 \to V$ is locally pseudoconvex. From [16], $W_1 \to V$ is the domain of holomorphic existence of the algebra $\bigoplus_{n \in \mathbb{N}} H^0(W, \mathcal{O}(n))$. Let W_2 denote the hull of meromorphy of W with respect to any meromorphic section on W of any holomorphic vector bundle over V (see [16]). Any meromorphic section of F on W defines a meromorphic map from W to $\mathbb{P}(F \bigoplus \mathbb{C})$. Since for any such F, $\mathbb{P}(F \bigoplus \mathbb{C})$ is a projective manifold, $W_2 \to V$ is the meromorphic hull of W. Then, from [15], we have $W \cup (U \setminus H) \hookrightarrow W_1 \hookrightarrow W_2$.

If H is the empty set the corollary is proved.

Assume H is non void. It is enough to prove that, if $\pi : W_1 \to V$ is a locally pseudoconvex domain over V, which admits a section along $U \setminus H$, then any meromorphic function in W_1 extends meromorphically through H. We will prove that its graph, in $W_1 \times \mathbb{P}^1$ extends through $H \times \mathbb{P}^1$ (see also remark below). First, note that $H \times \mathbb{P}^1$ is a hypersurface in $V \times \mathbb{P}^1$ s.t. $(V \setminus H) \times \mathbb{P}^1$ is pseudoconcave in the sense of Andreotti. Indeed let Y denote the open subset in $V \setminus \bar{U}$ which gives the pseudoconcavity condition (see Definition 11). Then $Y \times \mathbb{P}^1$ has a pseudoconcave boundary in the sense of Andreotti. Next, we notice that $W_1 \times \mathbb{P}^1 \to V \times \mathbb{P}^1$ is a locally pseudoconvex domain over $V \times \mathbb{P}^1$ and that it contains $(U \setminus H) \times \mathbb{P}^1$. From Theorem 5, we conclude the proof. □

Remark 9

i. Another way of proving the corollary goes as follow. In the above situation, any hypersurface of W_1 extends through H. Hence, any meromorphic function f on W_1 satisfies that any of its level set extends through H. So we may find a point $p \in H$, which admits a neighbourhood W_p in V such that $W_1 \setminus H$ does not meet the polar set, the zero set of f nor its level set $\{f = 1\}$. Shrinking W_p if necessary, in suitable coordinates on W_p, we may write, $W_p = (H \cap W_p) \times \Delta$, where Δ is the unit disc in $\mathbb{C}$. The restrictions of $f_{|W_p}$ on each slice $\{p'\} \times (\Delta \setminus \{0\})$, $p' \in H \cap W_p$, are holomorphic functions on $\Delta \setminus \{0\}$, which omit two values. From the big Picard's theorem (see [1]), they extend to Δ . By Hartogs-Levi theorem, our meromorphic function extends to $(U \setminus H) \cup W_p$. From the Thullen extension theorem, it extends through each irreducible component of H which meet W_p.
ii. Since pseudoconvex hulls behave functorialy under fibre product, the last corollary still holds under the technical assumption that the pseudoconvex hull of a neighbourhood of ∂U contains $U \setminus H$.
iii. We know, using results of S. Ivashkovich [20] and result from [16] that, in the above situation, if $f : W(\partial U) \to M$ is a meromorphic map from a neighbourhood $W(\partial U)$ of U to a complex compact Kähler manifold (M, ω_1), then f extends meromorphically to $U \setminus H$. However, we do not know at that time if $\omega_0 + f^*\omega_1$ is a pluricomplete current.

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