Monge-Ampère currents over pseudoconcave spaces

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Introduction

This paper is an attempt to understand growth of Monge-Ampère masses along pseudoconcave ends in a complex manifold.

This problem arises in differential geometry when studying compactification of complete Kähler manifolds under certain curvature conditions (see e.g. articles of Mok-Zhong [23], Nadel-Tsuji [24], Siu-Yau [32]). In complex analysis, bounds on Monge-Ampère masses of a closed positive current near a pluripolar set implies an extension of this current through the set (see e.g. works of El Mir [22], Siibony [30], Skoda [34]). In this direction, the $L^2$–Riemann-Roch inequality of Nadel-Tsuji (see [24]) implies that a complete Kähler Hodge metric on a pseudoconcave manifold is of finite volume.

Our first result is obtained in the framework of pluripotential theory. Let $M$ be a complex manifold, $\dim M = n \geq 2$, and let $\omega$ be a closed positive $(1,1)$–current. Assume that $\omega$ admits local locally bounded potentials. To each open subset $U$ of $M$ is associated an extremal admissible function $\varphi^*$, which is defined on a suitable pseudoconvex hull $U_1$ of $U$. It satisfies the Monge-Ampère equation $(\omega + dd^c \varphi^*)^n = 0$ on $U_1 \setminus \bar{U}$, as $(n,n)$-current of order zero. Identifying a $(n,n)$-current of order zero with the Borel measure it defines, we deduce the following estimate (we work in the relative topology of $U_1$).

Theorem. In the above situation, let $X$ be a connected component of $U_1 \setminus \bar{U}$ which has a compact boundary. Assume that $\{ \varphi^* \leq \varphi^*(p) \} \cap X$ is relatively compact in $U_1$ for any $p \in X$. Then

$$\int_X \omega^n \leq \int_{\partial X} (\omega + dd^c \varphi)^n < +\infty.$$
Here, to check the hypothesis we restrict ourself to domains on projective manifolds. It allows us to obtain a complex analytic treatment of the problem. Related methods appear already in [11,25]. For a more differential-geometric point of view, we refer to papers cited above.

We obtain the following applications. Let \( V \) be a projective manifold, \( \dim V = n \geq 2 \), and let \( H \) be a complex hypersurface in \( V \) such that \( V \setminus H \) is pseudoconcave in the sense of Andreotti (see Definition 11). Let \( X \subset M \) open neighbourhoods of \( H \). The following Hartogs’ theorem for currents holds.

**Theorem.** Let \( \omega \) be a closed positive \((1,1)\)—current defined on \( M \setminus H \) which admits local locally bounded potentials. Then

\[
\int_{X \setminus H} \omega^n < +\infty ,
\]

and \( \omega^k \) extends through \( H \) as a closed positive currents, \( k = 1, \ldots, n \).

If \( X = V \) and \( \omega \) is a smooth complete Hodge Kähler metric on \( V \setminus H \), then the above result is a variation of the \( L^2 \)—Riemann-Roch inequality of Nadel-Tsuji (see [24]). In general, the difficulty in establishing the above finiteness estimate is that neither pseudoconcavity nor completeness assumptions are made on \( M \) itself.

Next, we try to derive similar estimate for more singular closed positive currents. We work with currents (on spread domains \( W \) over \( V \)) such as pullback \( \psi^* \omega_{FS} \), where \( \psi : W \rightarrow \mathbb{P}^N \) is a meromorphic map from \( W \) to a projective space and \( \omega_{FS} \) is a Fubiny-Study form on it.

Our technique is to produce, by mean of the \( L^2 \) theory of ideals (see Skoda [33]), positive currents \( \omega_k \) linked to \( \psi^* \omega_{FS} \) but with Lelong number globally shifted by \(-k\) (see Demailly [12] for other methods in the compact case). These currents are pluricomplete (see Def. 10). This is a convexity condition on \( \omega_k \) and \( \mathcal{A}_k \), the non-smooth locus of \( \omega_k \), which allows to work on \( M = W \setminus \mathcal{A}_k \).

The case of a current defined by a divisor is noteworthy:

**Theorem.** Let \( Z \) be an hypersurface in a pseudoconvex spread domain \( W \) over a projective manifold \( V = (V, O(1)) \). There exists \( l_1 \in \mathbb{N} \) (which depends only of the canonical bundle of \( V \)) such that \( O(kl_1) \otimes [Z] \) is spanned by its global sections away of \( \{p \in W : v_p(Z) \geq k + 1\} \), where \( v_p(Z) \) is the multiplicity of \( Z \) at \( p \).

As an application, we deduce that global Hartogs’ extension phenomena occur in projective manifolds for meromorphic maps.

**Theorem.** Let \( U \) be an open subset of the projective manifold \( V \) such that \( V \setminus \bar{U} \) is a pseudoconcave domain in the sense of Andreotti. Assume \( \bar{\hat{U}} = U \). Then any
meromorphic map \( \psi : W(\partial U) \to \mathbb{P}^N \) define on a neighbourhood of \( \partial U \) extends as a meromorphic map to \( U \).

These results give some understanding of global and compact singularities for meromorphic maps or currents. Note that there exists hypersurfaces \( H \) as above which may not be blow down. Hence, even for meromorphic maps, the situation may not be reduced to local extension results of Ivashkovich [20]. Moreover, note that non compact complex singularities of strict positive dimension are already local essential singularities for Monge-Ampère currents (see [31]).

The starting point of this paper is the classical result that a hull of holomorphy in the trivial bundle over a domain in \( \mathbb{C}^n \) is a geometric counterpart of a complex Monge-Ampère equation in that domain (see Bremermann [9]).

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1. Quasi-continuous functions and the class \( \mathcal{G}(M) \)

We recall some definitions which appear in [6,8].

**Definition 1** Let \( \Omega \) be an open subset of \( \mathbb{C}^n \). If \( E \) is a subset of \( \Omega \), let \( C(E, \Omega) \) denote the relative capacity of \( E \) in \( \Omega \).

1. A function \( f : \Omega \to \{-\infty, +\infty\} \) is said to be quasi-continuous if, for any \( \epsilon > 0 \), there exists an open subset \( \mathcal{O} \) of \( \Omega \) with \( C(\mathcal{O}, \Omega) < \epsilon \) s.t. \( f \) is continuous on \( \Omega \setminus \mathcal{O} \).
2. A sequence \( \{f_j\}_{j \in \mathbb{N}} \) of Borel functions on \( \Omega \) is said to converge quasi-uniformly to \( f \), if it is uniformly bounded, it converges almost everywhere to \( f \), and, for any \( \epsilon > 0 \), there exits an open subset \( \mathcal{O} \) of \( \Omega \) such that \( C(\mathcal{O}, \Omega) \leq \epsilon \) and \( f_j \to f \) uniformly on \( \Omega \setminus \mathcal{O} \).

The notions of quasi-continuous function and local quasi-uniform convergence are define accordingly on a manifold through holomorphic coordinate charts.

Quasi-continuous functions form an algebra which contains plurisubharmonic functions (see [5], Theorem 3.5). Note that if \( f \) is quasi-continuous on \( M \), then for any continuous function \( \chi : \mathbb{R} \to \mathbb{R} \), \( \chi(f) \) is quasi-continuous on \( M \).

**Lemma 1** ([5]) Let \( \{\varphi_j\}_{j \in \mathbb{N}} \) be a sequence of plurisubharmonic functions which converge monotonically almost everywhere to a plurisubharmonic function \( \varphi \). Then the convergence is locally quasi-uniform.

**Definition 2** We denote by \( \mathcal{G}(M) \) the class of currents on \( M \) which locally are represented by currents in the exterior algebra generated by
We refer to Bedford-Taylor’s articles [8,7] for a precise definition of these currents for non smooth functions. We state in a weak form Theorem 2.6 of [8].

**Theorem 1** Let $T_j$, $j \in \mathbb{N}$ and $T_\infty$ be currents in $\mathcal{G}(M)$ which are locally of the form

$$
\sigma_0^{(j)} \delta \sigma_1^{(j)} \wedge \ldots \wedge \delta \sigma_q^{(j)} \wedge d^c a_{q+1}^{(j)} \wedge \ldots \wedge d^c \sigma_r^{(j)}
$$

where, each occurrence of $\delta$ denotes either the operator $d$ or the operator $d^c$, $\sigma_k^{(j)} = u_k^{(j)} - v_k^{(j)}$, the $u_k^{(j)}$ and $v_k^{(j)}$, $j \in \mathbb{N} \cup \{\infty\}$, are locally bounded plurisubharmonic functions such that

$$
\begin{align*}
  u_k^{(j)} & \to u_\infty^{(j)}, \\
v_k^{(j)} & \to v_\infty^{(j)},
\end{align*}
$$

and where the convergence is monotone in $k$. If $\{\varphi_j\}_{j \in \mathbb{N}}$ is a sequence of quasi-continuous functions which converges locally quasi-uniformly to the quasi-continuous function $\varphi$ then

$$
\lim_{j \to +\infty} \varphi_j T_j = \varphi T_\infty
$$
as currents of order 0.

2. The class $P_\omega(M)$

Let $M$ be a complex manifold, $\dim M = n$, and let $\omega$ be a closed positive $(1, 1)$-current on $M$. It is known (see [18], p.387) that $\omega$ admits local potentials. In this paper, we make the following assumption.

The current $\omega$ admits local potentials which are locally bounded. (2.1)

Hence we assume that, for any open subset $X$ biholomorphic to an open Euclidean ball in $\mathbb{C}^n$, there exists $a \in \text{PSH}(X) \cap L^\infty(X, \text{loc})$ such that $d^c d^c a = \omega|X$.

Note that two local potentials for $\omega$ differ (on their common definition set) by a plurisubharmonic function. This fact is used in the following definitions.

**Definition 3** A measurable function $\varphi : M \to \mathbb{R} \cup \{-\infty\}$ belongs to $P_\omega(M)$ if there exists an open covering $\mathcal{W} = \{W_i\}_{i \in I}$ by subsets biholomorphic to Euclidean balls in $\mathbb{C}^n$, and local potentials $a_i \in \text{PSH}(W_i) \cap L^\infty(W_i, \text{loc})$, such that $a_i + \varphi$ is plurisubharmonic.

Note that a function which belong to $P_\omega(M)$ is quasi-continuous.
Definition 4

(1) A function $\varphi : M \to [-\infty, +\infty]$ will be said upper semicontinuous with respect to $\omega$, if, for any $p \in M$, there exists an open neighbourhood $W$ of $p$, a local locally bounded potential $a \in \text{PSH}(W) \cap L^\infty(W, \text{loc})$ for $\omega$, such that $a + \varphi$ is upper semicontinuous on $W$. A function $h$ on $M$ will be said lower semicontinuous with respect to $\omega$ if $-h$ is upper semicontinuous with respect to $\omega$.

(2) Let $\varphi : M \to [-\infty, +\infty]$ be a function which is locally bounded from above. Define $\varphi^*$, the upper regularization of $\varphi$ with respect to $\omega$, as follow. If $a$ is a local locally bounded potential for $\omega$ on an open subset $W$, then

$$\varphi^* = (a + \varphi)^* - a$$

(2.2)

where $(a + \varphi)^*$ stands for the usual upper regularization of $a + \varphi$ on $W$ in the classical topology $(a + \varphi)^*(p) = \limsup_{z \to p}(a + \varphi)(z)$.

Let a function $h \in L^1(M, \text{loc})$ satisfies $\omega + ddch \geq 0$ in the sense of currents. Then $h^*$, the upper regularization of $h$ with respect to $\omega$, belongs to $P_\omega(M)$.

With this notion of upper regularization w.r.t $\omega$, we will have classical stability properties of $P_\omega(M)$ with respect to upper envelope (see Lemma 6). Note that Choquet’s lemma is valid.

Lemma 2 Let $\{u_\alpha\}_{\alpha \in A}$ be a family of real valued functions on a complex manifold $M$. Assume that $a + u_\alpha$ is upper semicontinuous for any local potential $a$ of $\omega$ and any $\alpha \in A$. Assume this family is locally bounded from above on $M$. Then there exist a countable subset $B \subset A$ such that $(\sup_{\alpha \in A} u_\alpha)^* = (\sup_{\alpha \in B} u_\alpha)^*$ (upper regularization w.r.t. $\omega$).

Let $\omega_i, 1 \leq i \leq r$, be closed positive $(1, 1)$–currents which satisfy condition (2.1). From Theorem 1, if $\varphi_i \in P_{\omega_i}(M) \cap L^\infty(M, \text{loc})$ then expression of the form

$$T = \delta \varphi_1 \wedge \ldots \wedge \delta \varphi_k \wedge (\omega_{k+1} + dd^c\varphi_{k+1}) \wedge \ldots \wedge (\omega_r + dd^c\varphi_r), \quad (2.3)$$

where $\delta$ is either $d$ or $d^c$, defined a current which belongs to the class $\mathcal{G}(M)$. $T$ is the unique current which is locally equal to

$$T = \delta ((a_1 + \varphi_1) - a_1) \wedge \ldots \wedge \delta ((a_k + \varphi_k) - a_k) \wedge$$

$$dd^c(a_{k+1} + \varphi_{k+1}) \wedge \ldots \wedge dd^c(a_r + \varphi_r), \quad (2.4)$$

where $a_i$ denotes a local locally bounded potential for $\omega_i, 1 \leq i \leq r$.

For these currents, usual calculus rules are satisfied. In particular,
Lemma 3 Let $\varphi \in P_\omega(M) \cap L^\infty(M, \text{loc}), \chi \in C^\infty(\mathbb{R}, \mathbb{R})$. Then for any $\theta \in C^\infty_0(M)$, the following algebraic identity holds

\[
\int \theta \chi(\varphi) (\omega + dd^c \varphi)^n = \int \theta \chi(\varphi) \omega^n
- \int (d\theta) \chi(\varphi) d^c \varphi P(\varphi) - \int \theta \chi'(\varphi)d\varphi \wedge d^c \varphi P(\varphi), \tag{2.5}
\]

where

\[
P(\varphi) = \sum_{\alpha+\beta=n-1} (\omega + dd^c \varphi)^\alpha (dd^c a)^\beta. \tag{2.6}
\]

Proof. It is enough to check the above formula locally. Let $B$ be the Euclidean unit ball in $\mathbb{C}^n$. Assume, $\text{supp} \theta \subset B$, $\omega = dd^c a$, with $a \in \text{PSH}(B) \cap L^\infty(B, \text{loc})$, so that $a + \varphi \in \text{PSH}(B) \cap L^\infty(B, \text{loc})$. Let $(a + \varphi)_\epsilon$, $a_\epsilon$, $1 > \epsilon > 0$, be family of smooth plurisubharmonic functions defined on $B$, which decrease, as $\epsilon \to 0$, to $a + \varphi$ and $a$ respectively on an open neighbourhood $W \subset B$ of $\text{supp} \theta$.

Let $M = \| (a + \varphi)_1 \|_{w, \infty} + \| a_1 \|_{w, \infty} + \| a + \varphi \|_{w, \infty} + \| a \|_{w, \infty} < +\infty$.

From [5], Theorem 7.2, for any $\eta > 0$, there exists $\Omega$, an open subset of $W$, such that $C(W, \Omega) < \eta$, and the above convergences are uniform on $W \setminus \Omega$. Define $\psi_\epsilon = (a + \varphi)_\epsilon - a_\epsilon$, then

\[
\| \chi(\psi_\epsilon) - \chi(\varphi) \|_{w, \Omega, \infty} \leq (\max_{\{-M, M\}} |\chi'|) \| \psi_\epsilon - \varphi \|_{w, \Omega, \infty} \xrightarrow{\epsilon \to 0} 0. \tag{2.7}
\]

Since the $\psi_\epsilon$ and $\varphi$ are uniformly bounded on $W$, for any $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$, $\chi(\psi_\epsilon)$ converge quasi-uniformly on $W$ to $\chi(\varphi)$. But for smooth functions, an integration by parts gives

\[
\int \theta \chi(\psi_\epsilon) (dd^c(a_\epsilon + \psi_\epsilon))^n = \int \theta \chi(\psi_\epsilon) (dd^c a_\epsilon)^n
- \int (d\theta) \chi(\psi_\epsilon) d^c \psi_\epsilon P(\psi_\epsilon) - \int \theta \chi'(\psi_\epsilon)d\psi_\epsilon d^c P(\psi_\epsilon). \tag{2.8}
\]

where

\[
P(\psi_\epsilon) = \sum_{\alpha+\beta=n-1} (dd^c(a_\epsilon + \psi_\epsilon))^\alpha (dd^c a_\epsilon)^\beta. \tag{2.9}
\]

As $\epsilon \to 0$, $\chi(\psi_\epsilon)$ and $\chi'(\psi_\epsilon)$ converge quasi-uniformly to $\chi(\varphi)$ and $\chi'(\varphi)$ respectively, on $W$. Moreover, $d^c \psi_\epsilon P(\psi_\epsilon)$ converges to $d^c \varphi P(\varphi)$, $(dd^c(a_\epsilon + \psi_\epsilon))^n$ converges to $(\omega + dd^c \varphi)^n$ and $P(\varphi)$ converges to $\omega^n$. From Theorem 1, we obtain formula (2.5) above. \qed

We state next a basic lemma.
Lemma 4 Let $M$ be a complex manifold, and let $X$ be an open subset of $M$ with compact boundary. Let $\omega$ be a closed positive $(1, 1)-$current which admits local locally bounded potentials. Let $\varphi \in P_\omega(X) \cap L^\infty(X, \text{loc})$ such that

1. there exists a neighbourhood $W$ of $\partial X$, with $\varphi|_{W \cap X} \geq 0$,
2. $\limsup_{z \to \partial X} \varphi = 0$,
3. $\forall p \in X, \{ \varphi \leq \varphi(p) \} \subset \subset M$.

Let $\chi : \mathbb{R} \to \mathbb{R}^+$ be a positive smooth decreasing function. Then

$$+\infty \geq \int_X \chi(\tilde{\varphi})(\omega + dd^c \tilde{\varphi})^n \geq \int_X \chi(\tilde{\varphi})\omega^n$$

where $\tilde{\varphi}$ denotes the extension by 0 of $\varphi$ to $M$.

Proof. Note that $\tilde{\varphi}$ belongs to $P_\omega(M) \cap L^\infty(M, \text{loc})$. For

$$\tilde{\varphi} = \begin{cases} 
\varphi & z \in X \setminus W \\
\max(\varphi, 0) = \varphi & z \in X \cap W \\
0 & z \in M \setminus X
\end{cases}$$

which is a plurisubharmonic function in $W'$ (see [21], p.69).

Hence, we will assume that $\varphi \in P_\omega(M) \cap L^\infty(M, \text{loc})$ and that it vanishes on $M \setminus X$. Let $W_1$ be a relatively compact open neighbourhood of $\partial X$. Let $\theta$ be a smooth positive function with $\text{supp} \theta \subset X \cup W_1$, $\theta \equiv 1$ on a neighbourhood of $\partial X$. Note that it’s enough to prove the lemma under the following technical assumption.

There exists an increasing sequence $\{ \chi_k \}_{k \in \mathbb{N}}$ of smooth positive decreasing functions such that $\text{supp} \chi_k(\varphi) \cap X$ is a relatively compact subset in $M$ and $\lim_{k \to +\infty} \chi_k(\varphi) = \chi(\varphi)$ on $M$.

Then, since $\text{supp} \theta \chi_k(\varphi)$ is a compact set in $M$, Lemma 3 gives

$$\int \theta \chi_k(\varphi) (\omega + dd^c \varphi)^n = \int \theta \chi_k(\varphi) \omega^n$$

$$- \int (d\theta) \chi_k(\varphi) d^c \varphi P(\varphi) - \int \theta \chi_k(\varphi) d^c \varphi \wedge d^c \varphi P(\varphi) \ . \ (2.10)$$
where

\[ P(\varphi) = \sum_{\alpha + \beta = n-1} (\omega + d\varphi) \wedge \alpha \wedge \beta. \]  

(2.11)

Note that \( d\varphi \wedge d^c \varphi P(\varphi) \) is a positive current on \( M \). But \( \chi'_k \) is negative, hence

\[ \int \theta \chi'_k(\varphi) d\varphi \wedge d^c \varphi P(\varphi) \geq 0. \]

Since \( \varphi \) is vanishing on a neighbourhood of \( \text{supp} \ d\theta \), the second term of the right hand side vanishes. Hence

\[ \int \theta \chi_k(\varphi) (\omega + d\varphi)^n \geq \int \theta \chi_k(\varphi) \omega^n. \]  

(2.12)

The above integrals being finite, letting first \( \theta \) decreasing to the characteristic function of \( \bar{X} \); and then \( k \to +\infty \), since \( (\chi_k)_{k \in \mathbb{N}} \) is increasing, we get the result.

\[ \square \]

**Example 1** Let \( M = \mathbb{C}^n \), \( X = B(1) \), where \( B(1) \) is the unit ball, and let \( \omega = dd^c \|z\|^2 \) the standard Kähler metric. Then \( 1 - \|z\|^2 \) belongs to \( \mathcal{P}_\omega(X) \) and satisfies the conditions of the above lemma. Its extension by zero is \( \tilde{1} - \|z\|^2 \), and then \( \tilde{1} - \|z\|^2 \) belongs to \( \mathcal{P}_\omega(X) \). The above integrals being finite, letting first \( \theta \) decreasing to the characteristic function of \( \bar{X} \); and then \( k \to +\infty \), since \( (\chi_k)_{k \in \mathbb{N}} \) is increasing, we get the result.

\[ \square \]

### 3. Pseudoconvex hulls

Let \( M \) be a complex manifold, \( \dim_{\mathbb{C}} M = n \geq 2 \), and let \( M_1 \) be an open subset of \( M \).

We recall that \( M_1 \) is said to be locally pseudoconvex in \( M \), if there exists an open cover \( \mathcal{W} \) of \( M \) by Stein open subsets \( W \) such that \( M_1 \cap W \) is a Stein manifold, for any \( W \in \mathcal{W} \).

Note that any connected component of the interior of an intersection of a family of locally pseudoconvex open subsets of \( M \) is a locally pseudoconvex open subset of \( M \).

**Definition 5** Let \( U \) be an open subset of \( M \). Then there exists \( \hat{U} \), the smallest locally pseudoconvex open set in \( M \) which contains \( U \). We says that \( \hat{U} \) is the pseudoconvex hull of \( U \) in \( M \).

**Lemma 5** Let \( (W', (z)) \) be a holomorphic charts, with \( W' \) a relatively compact Stein open set of \( M \setminus \hat{U} \). Then, for any open relatively compact subset \( W \) in \( W' \), and any polynomial \( P \) in the complex coordinates \( (z) \),

\[ \max_{W \cap \partial U} |P| = \max_{W \cap \partial U} |P|. \]
Proof. We argue by contradiction, and prove that if the above condition is not satisfied, we may push a hypersurface in \( \hat{U} \) which is disjoint from \( U \). Denote \( K = \partial \hat{U} \). Assume there exists a polynomial \( P \) such that \(||P|||_{K \cap \hat{W}} = P(z_0) = 1\) for some \( z_0 \in K \cap W \) and \(||P|||_{K \cap \partial W} < 1\).

\( K \cap \partial W \) being compact, there exists \( 0 < \epsilon < 3^{-1}d(z_0, \partial W) \) s.t. \(|P| < 1\) on \( S_\epsilon = \{ z \in \hat{W}, d(z, K \cap \partial W) < \epsilon \} \). Let \( W_{2-\epsilon} = \{ z \in W, d(z, \partial W) > 2^{-1}\epsilon \} \), and let \( A_k = \{ z \in W, P(z) = 1 + \frac{1}{k} \}, k \in \mathbb{N}^* \). \( A_k \) is an algebraic hypersurface in \( W \setminus K \cup S_\epsilon \), and \( \bigcup_{k \in \mathbb{N}^*} A_k \cap \partial W_{2-\epsilon} \subset W \setminus K \cup S_\epsilon \). There exists \( \alpha_0 > 0 \) s.t. \( \bigcup_{k \in \mathbb{N}^*} (A_k + B_{\alpha_0}(0, \alpha_0)) \cap \partial W_{\epsilon} \subset W \setminus K \cup S_\epsilon \). Since \( W' \cap \hat{U} \) is a Stein open set and since \( \bigcup_{k \in \mathbb{N}^*} A_k \ni z_0 \), there exists a sequence of integers \( k_1, k_2, \ldots \), and irreducible component \( C_{k_i} \) of \( A_{k_i} \) such that \( C_{k_i} \cap \hat{W}_e \subset \hat{W}_e \setminus \hat{U} \) and \( \lim_{i \to +\infty} d(z_0, C_{k_i}) = 0 \). Hence \( (C_{k_i} + B_{\alpha_0}(0, \alpha_0)) \cap \partial W_e \subset \partial W_e \setminus \hat{U} \).

\( \bigcup_{k \in \mathbb{N}^*} A_k \cap W_e \) is a compact subset of \( \hat{W}_e \setminus S_\epsilon \), there exists \( \alpha_0 > \alpha_1 > 0 \) such that \( \bigcup_{i \in \mathbb{N}^*} (C_{k_i} + B_{\alpha_1}(0, \alpha_1)) \cap W_e \subset W \setminus S_\epsilon \).

Take \( i \) big enough such that \( d(z_0, C_{k_i}) < 2^{-1}\alpha_1 \), take \( z_1 \in C_{k_i} \cap B(z_0, 2^{-1}\alpha_1) \), \( z_2 \in \hat{U} \cap B(z_0, 2^{-1}\alpha_1) \). Then \( (C_{k_i} + \bar{z}_1 z_2^2) \cap W_e \cap \hat{U} \) is non empty and \( C_{k_i} + \bar{z}_1 z_2^2 \cap \partial W_e \cap \hat{U} \subset \partial W_e \cap \hat{U} \), since \( \partial (W_e \cap \hat{U}) \subset (\partial W_e \cap \hat{U}) \cup (\partial \hat{U} \cap \hat{W}_e) \) and \( \partial \hat{U} \cap \hat{W}_e = (\partial \hat{U} \cap W_e) \cup (\partial \hat{U} \cap \partial W_e) \).

In particular, \( H = (C_{k_i} + \bar{z}_1 z_2^2) \cap W_e \cap \hat{U} \) is a hypersurface in \( \hat{U} \) which does not intersect \( U \). However \( \hat{U} \setminus H \) is locally pseudoconvex, contains \( U \) and is strictly smaller than \( \hat{U} \), which is a contradiction.

Remark 1 The proof of the above lemma shows that, if \( \dim M = 2 \), then, for any open Stein subset of \( M \setminus \hat{U} \), \( W \setminus \partial U \) is Stein. Hence \( \partial U \) is a pseudoconvex set in the sense of Oka in \( M \setminus \hat{U} \) (see [29], p. 88).

Other kinds of pseudoconvex hulls (w.r.t. \( \omega \)) are constructed as follow.

Lemma 6 Let \( \{ \varphi_\alpha \}_{\alpha \in \Lambda} \subset P_\omega(M) \). Then the open set

\[ X = \{ p \in M : \varphi = \sup_{\alpha \in \Lambda} \varphi_\alpha \text{ is locally bounded from above at } p \} \]

is locally pseudoconvex in \( M \). Further, on \( X \), \( \varphi^* \) the upper regularization of \( \varphi \) w.r.t. \( \omega \) belongs to \( P_\omega(X) \).

Lemma 7 Let \( M \) be a complex manifold, and let \( \omega \) be a closed positive \( (1,1) \)-current in \( M \). Assume there exists an analytic subset \( B \) in \( M \) such that \( \omega \) admits local locally bounded potentials on \( M \setminus B \). Let \( \{ \varphi_\omega \}_{\omega \in \Lambda} \subset P_\omega(M \setminus B) \). Let \( X \) denote the open set in \( M \setminus B \) where this family is locally bounded from above. Then, the interior of \( X \cup B \) in \( M \) is a locally pseudoconvex open subset in \( M \).
Proof. The lemma is local, hence we assume $M$ is the unit ball and that $\{\varphi_\alpha\}_{\alpha \in \Lambda}$ is a set of plurisubharmonic functions on $M \setminus B$. Let $U$ be the maximal open subset of $M \setminus B$ for which this family is locally uniformly bounded from above. Let $\varphi$ be the upper envelope of this family, and let $\varphi^*$ denote its upper regularization, which is a plurisubharmonic function in $U$.

Write $B = B_1 \cup B_2$, with $\text{codim} B_1 = 1$ and $\text{codim} B_2 \geq 2$. First, we prove that, in $M \setminus B_1$, the interior $U'$ of $U \cup B_2$ is locally pseudoconvex. Let $h : (H, \Delta^n) \rightarrow M \setminus B_1$ be a Hartogs' figure (see [28], p.49) such that $h(H) \subset \subset U'$ and $h(\Delta^n) \subset \subset M \setminus B_1$. Since $\text{codim} B_2 \geq 2$, each plurisubharmonic function $\varphi'$ in $M \setminus B_2$ admits a plurisubharmonic extension, which we denote $\tilde{\varphi}'$, to $M \setminus B_1$. $\tilde{\varphi}'$ satisfies that for any relatively compact open subset $X$ in $M \setminus B_1$, $\sup X \tilde{\varphi}' \geq \sup X \tilde{\varphi}'$. This fact applies to $\varphi^*$. Hence for any $\alpha$, $\max_{h(H)} \tilde{\varphi}_\alpha \geq \sup_{h(\Delta^n)} \varphi$. In particular, any point of $h(\Delta^n) \setminus B_2$ belongs to $U$, so $h(\Delta^n) \subset U'$.

Next, by using the disc characterisation of pseudoconvexity, it is classical that if $X$ is an open pseudoconvex subset in $M \setminus B_1$, then the interior of $X \cup B_1$ is pseudoconvex, when $B_1$ is a complex hypersurface. 

Remark 2 In particular, this set $X$ is invariant under bimeromorphic maps.

Lemma 8 Let $W$ be an open subset of $M$ biholomorphic to the unit ball in $\mathbb{C}^n$. Let $D \subset \subset W$ be a strongly pseudoconvex open subset of $W$. Then, for any $\psi \in P_{\omega}(M)$, there exists a unique function $T_D(\psi) \in P_{\omega}(M)$ such that $T_D(\psi) = \psi$ on $M \setminus D$ and $(\omega + d\psi) \wedge^2 = 0$ on $D$. Further $T_D(\psi) \geq \psi$.

Proof. Let $a \in \text{PSH}(W) \cap L^\infty(W, \text{loc})$ be a potential for $\omega$ on $W$. From [5], Proposition 9.1, a unique plurisubharmonic function $a + \psi$ exists such that $(\widehat{dd^c(a + \psi)})^n = 0$ on $D$, $\widehat{a + \psi} = a + \psi$ on $W \setminus D$, and $a + \psi \geq a + \psi$ on $W$. Note that $a + \psi - a = \psi$ on $W \setminus D$ and we define

$$T_D(\psi) = \begin{cases} \max(\psi, a + \psi - a) = a + \psi - a & z \in W \\ \psi & z \in M \setminus W \end{cases}$$

\[ \square \]

Lemma 9 Let $U$ be an open subset of $M$. Let $\Lambda \subset P_{\omega}(M)$ be a family which is stable with respect to the max operation.

Assume that any point $p \in M \setminus \bar{U}$ admits a pair of open neighbourhoods $(W, D)$ as in Lemma 8, with $W \subset M \setminus \bar{U}$, such that, for all $u \in \Lambda$, the function $T_D(u)$, belongs to $\Lambda$.

Assume that $X$, the open subset where $\Lambda$ is locally bounded from above, contains $U$. Denote $\varphi^* = (\sup_{z \in A} \psi)^* \in P_{\omega}(X)$, the upper regularization (w.r.t. $\omega$) of the upper envelope of this family.

Then the positive measure $(\omega + d\psi)^n$ has support in $\bar{U}$. 


Proof. Since $\Lambda$ is stable by the max operation, from Choquet’s Lemma 2, we get an increasing sequence $\{u_j\}_{j \in \mathbb{N}} \subset \Lambda$ with $(\lim_{j \to +\infty} u_j)^* = \varphi^*$. From the hypothesis, let $(W, D)$ open neighbourhoods of $x \in X \setminus \overline{U}$ such that $\forall u \in \Lambda, T_D(u) \in \Lambda$. Replacing each $u_j$ by $\tilde{u}_j = T_D(u_j) \in \Lambda$, then the sequence $\{\tilde{u}_j\}_{j \in \mathbb{N}}$ is increasing, since $\tilde{u}_j$ may be obtained by a Perron method, it increases to $\varphi^*$ outside a pluripolar set, since $\varphi^* = (\lim_{j \to +\infty} \tilde{u}_j)^*$ and the negligible set $\{(\lim_{j \to +\infty} \tilde{u}_j) < (\lim_{j \to +\infty} \tilde{u}_j)^*\}$ is pluripolar.

Hence, from [5], Theorem 7.4, $\lim_{n \to +\infty} (\omega + dd^c \tilde{u}_j)^n = (\omega + dd^c \varphi^*)^n$ is vanishing on $D$. Since this property is valid for any such pair $(W, D)$, with $W \cap \overline{U} = \emptyset$, the assertion is proved. \hfill \Box

3.1. Some extremal functions

Let $\omega$ be a closed positive $(1, 1)$–current on the complex manifold $M$. Assume that $\omega$ admits locally bounded potentials near every point in $M$ (see (2.1)).

**Definition 6** Let $U$ be a domain in $M$, and let $h$ be a function on $U$ which is locally bounded and lower semicontinuous w.r.t. $\omega$. Define

$$X(h, \omega) = \{ p \in M : \varphi = \sup_{\psi \in P_\omega(M, U, h)} \psi \text{ is locally bounded from above at } p \},$$

where $P_\omega(M, U, h) = \{ \psi \in P_\omega(M) \text{ such that } \psi|_U \leq h \}$.

Let $\varphi^*$ be the upper regularization of $\varphi$ (w.r.t. $\omega$) in $X(h, \omega)$ and call it the extremal function associated to $U$, $\omega$ and $h$. Define $U(h, \omega)$ to be the connected component of $X(h, \omega)$ which contains $U$.

By assumption, $P_\omega(M, U, h)$ is locally bounded from above on $U$, hence $X(h, \omega)$ contains $U$. When $h = 0$, and $M$ is a pseudoconvex domain in $\mathbb{C}^n$, we obtain the usual hull of holomorphy of $U$ with respect to $M$. For $M$ a projective manifold, $h = 0$, this hull is similar to hull introduced in [19]. We refer to this article for further properties when this hull is assumed to be compact in some locally pseudoconvex domain.

From Lemma 9, the extremal function $\varphi^*$ satisfies $(\omega + dd^c \varphi^*)^n = 0$ on $U(h, \omega) \setminus \overline{U}$. Moreover, in $U$, we have $(\omega + dd^c \varphi^*)^n = 0$ on the open subset $\{\varphi^* < h\}$ (see [5], Corollary 9.2).

**Definition 7** Let $U$ be a domain in $M$ and $\psi \in P_\omega(M)$. Fix $D = \{ D_i \}_{i \in \mathbb{N}}$ an open cover of $M \setminus \overline{U}$ by open strongly pseudoconvex subsets $D_i$, which are relatively compact in complex holomorphic charts $f_i : W_i \to \mathbb{B}^n(0, 1)$. Assume that each $D_i$ is repeated infinitely often in the sequence $D$. Define by induction, $\psi_{-1} = \psi$, and $\psi_i = T_{D_i}(\psi_{i-1})$, for $i \in \mathbb{N}$. Let $X(\psi)$ denote the open subset where the family $\{ \psi_i \}_{i \in \mathbb{N}}$ is locally bounded from above, and let $U(\psi)$ be the connected component of $X(\psi)$ which contains $U$. Define $B(\psi) = (\sup_{i \in \mathbb{N}} \psi_i)^*$, which belongs to $P_\omega(U(\psi))$. 


Note that this family is an increasing sequence w.r.t. \( i \in \mathbb{N} \). Hypothesis of Lemma 9 are satisfy, hence \((\omega + dd^c B(\psi))^n = 0\) on \( U(\psi) \setminus \hat{U} \). Moreover, \( B(\psi) \geq \psi \) on \( U(\psi) \). Although \( B(\psi) \) depends in general of the cover chosen, we will not indicate this dependence.

**Remark 3**

i. Notice that the balayage procedure in Definition 7 when applied to a function (e.g. the zero function), gives a function which is strictly greater than the original one in points where \( \omega \) is a strictly positive current.

ii. Let \( X \) be a relatively compact domain in \( M \) with smooth boundary. Assume for simplicity that \( \omega \) is smooth and strictly positive. Then applying the Green formula for \( \hat{X} \) with respect to the Kähler metric \( \omega \) (see [4]), we see that a family \( F \subset P_{\omega}(M) \) is locally bounded from above in \( X \) if it is bounded for the \( L^1 \) norm induced on \( \partial X \). In particular, the above balayage procedure applied to \( M \setminus \hat{X} \) is always locally bounded.

**3.2. The case of a Chern class**

In this section, we interpret the above results when \( \omega \) is a Chern current of a line bundle. Note that a closed positive \((1,1)-\)current \( \omega \) is the Chern current of a hermitian line bundle \( L \) over a complex manifold \( M \) if it lies in \( H^2(M, \mathbb{Z}) \) via the De-Rham isomorphism.

Let \( (E,h) \to M \) be a complex hermitian line bundle with positive (singular) metric curvature. Denote \( \pi: E^* \to M \) the bundle map from \( E^* \) to \( M \), the dual line bundle of \( E \), and denote \( |\zeta|^2 \) the norm of \( \zeta \in E^* \) induced by \( h \).

Let \( A \) be a subset of \( M \). Denote \( T_A(\alpha) = \{ \xi \in E^*_A, |\xi| < \alpha \} \), and denote \( T_A = T_A(1) \). Let \( \hat{T}_U \) be the pseudoconvex hull of \( T_U \) in the complex manifold \( E^* \).

**Lemma 10** \( \hat{T}_U \) is a disquieted pseudoconvex subset of \( E^* \).

**Proof.** Consider the action of \( \mathbb{C}^* \), in the fibre of \( E, (\lambda, \zeta) \to \lambda.\zeta \). Let \( \lambda \in \mathbb{C}^* \), then \( \lambda T_U \subset \hat{\lambda T}_U \), hence \( \lambda \hat{T}_U \subset \hat{\lambda T}_U \). But \( T_U \subset \lambda^{-1} \hat{\lambda T}_U \), hence \( \hat{\lambda T}_U \subset \hat{\lambda T}_U \). So \( \hat{\lambda T}_U = \lambda \hat{T}_U \). This is a classical result that if \( W \) is a pseudoconvex domain in \( \mathbb{C}^n \), \( H \) an irreducible hypersurface in \( W \) and \( K \) a compact subset in \( W \), with \( H \cap K \) non void, then the pseudoconvex hull of \( (W \setminus H) \cup K \) is \( W \). Hence \( \hat{T}_U \) contains \( 0.T_U \) since it contains \( 0.T_U \). \( \square \)

Since \( \hat{T}_U \subset \pi^{-1}(\hat{U}) \) and \( 0.\hat{T}_U \simeq \hat{U} \), from the above lemma, we see that \( \hat{T}_U \) is a twisted pseudoconvex Hartogs' domain over \( \hat{U} \). Moreover \( \hat{T}_U \subset T_M(1) \). Assume that \( iC(E) \) admits local locally bounded potentials, then there exists an u.s.c (w.r.t. \( iC(E) \)) function \( \varphi \in P_{iC(E)}(\hat{U}) \) such that \( \hat{T}_U = \{ \xi \in E^*, \ln |\xi|^2 + \varphi < 0 \} \). Indeed, let \( t_W: E_{|W}^* \simeq W \times \mathbb{C} \) be a local trivialization of \( E^* \) over the open
subset $W$ biholomorphic to an open ball in $\mathbb{C}^n$. Since $t_W$ is a morphism of vector bundle, $t_W(T_U/W)$ is a Hartogs’ locally pseudoconvex domain with base $W$. Hence $t_W(T_U/W) = \{(p, z) \in W \times \mathbb{C}, \ln |z|^2 + \psi_W(P) < 0\}$ with $\psi_W$ a plurisubharmonic function in $W$. In the local trivialization $t_W : E^*_W \simeq W \times \mathbb{C}$, assume that $|t_W^{-1}(p, z)|^2 = a_{W, t_W}(p)|z|^2$ where $a_{W, t_W}$ is a logarithmic plurisubharmonic function in $W$, with $dd^c \ln(a_{W, t_W}) = iC(E, h)$. Define $\varphi = \psi_W - \ln a_{W, t_W}$. One check that this function $\varphi$ does not depend on the choosen trivialisation, hence define an element $\varphi \in P\!i_C(E)(\hat{U})$ such that $\hat{T_U} = \{\xi \in E^*, \log |\xi|^2 + \varphi(\pi(\xi)) < 0\}$.

Note that $\varphi$ is maximal in the following sense.

*Let $W$ be an open set in $\hat{U} \setminus \hat{U}$, and let $\psi \in P\!i_C(E)(W)$. If $W'$ is a relatively compact open subset of $W$ and if $\liminf_{z \to \partial W'} \varphi(z) - \psi(z) \geq 0$ then $\varphi \geq \psi$ in $G$.*

For the function

$$
\varphi' = \begin{cases} 
\max(\varphi, \psi) & \text{if } z \in W' \\
\varphi & \text{if } z \notin W' 
\end{cases}
$$

belongs to $P\!i_C(E)(\hat{U})$ and is zero on $U$. Hence $\{\xi \in E^*_U : \ln |\xi|^2 + \varphi' \circ \pi(\xi) < 0\}$ contains $T_U$, hence contains $\hat{T_U}$. So $\varphi' = \varphi$.

**Lemma 11** Assume that $iC(E)$ admits local locally bounded potentials, then the positive measure $(iC(E) + dd^c \varphi)^n$ has support in $\hat{U}$, the closure of $U$ in $\hat{U}$.

**Proof.** Let $D, W$ be domains as in Lemma 8 with $W \cap \hat{U} = \emptyset$. Since $\varphi$ is maximal, $T_D(\varphi) = \varphi$. However, $(\omega + T_D(\varphi))^n$ vanishes on $D$, by construction.

**Lemma 12** Let $T_U(0, 0)$ denote the hull of $T_U$ with respect to globally defined plurisubharmonic functions on $E^*$ (see Sect. 3.1). Then $T_U(0, 0)$ is a disqued subset over $U(0, \omega)$ which contains the image of $U(0, \omega)$ by the null section. Moreover $T_U(0, 0) = \{\xi \in E^*, \ln |\xi|^2 + \varphi^*(\pi(\xi)) < 0\}$, where $\varphi^*$ is the extremal function associated with $U$ and $\omega$ (see Sect. 3.1).

**Proof.** By definition $T_U(0, 0) \subset \{\xi \in E^*, \ln |\xi|^2 + \varphi^*(\pi(\xi)) < 0\} = A$. To prove the equality, we argue by contradiction. Let $\zeta_0 \in A \setminus T_U(0, 0)$. $A$ being open, there exists a neighbourhood $W$ of $\zeta_0$ in $A$, a non constant plurisubharmonic function $\psi$ on $E^*$, such that $\{\psi < 0\}$ contains $T_U$ but does not contains $W$. $\psi$ being plurisubharmonic, $\{\psi \geq 0\}$ is the closure of $\{\psi > 0\}$. Hence there exists $\xi_1 \in W \cap \{\psi > 0\}$. Let us replace $\psi$ by $\psi' = \log |\xi|^2 + N \psi$. Then $\{\psi' < 0\}$ contains $T_U$ and for $N$ large enough, still not contains $\xi_1$. That is $T_U \subset \{\psi' < 0\} \cap A \neq A$. Hence, $T_U \subset \bigcap_{\theta \in [0, 2\pi]} e^{i\theta} \{\psi' < 0\} \cap A \neq A$. However $\bigcap_{\theta \in [0, 2\pi]} e^{i\theta} \{\psi' < 0\}$ is a twisted Hartogs’ pseudoconvex domain over $M$. It contains $T_U$, hence, it is defined by a function $\varphi' \in P\!i_C(E)(M, U, 0)$. ⊓⊔
4. Bounds of Monge-Ampère masses

Recall that if $M$ is a complex manifold, a non relatively compact connected component of $M \setminus K$ where $K$ is a compact set in $M$, is called an end of $M$. Let $\omega$ be a closed positive $(1,1)$–current on $M$, which admits locally bounded potentials. Let $\mathcal{F} \subset P_\omega(M)$, and let $X(\mathcal{F})$ denote the open subset in $M$ where this family is locally bounded from above.

**Definition 8** An end of $X(\mathcal{F})$ will be called a pseudoconcave end with respect to $\mathcal{F}$.

Consider the following situation. Let $M$ be a complex manifold, let $U$ be an open subset of $M$, and let $\mathcal{F} = P_\omega(M)$, $U(0, \omega)$ as defined in Sect. 3.1. Working in the relative topology of $U(0, \omega)$, assume that $U(0, \omega) \setminus \bar{U}$ admits a connected component $X$ with compact boundary (hence $X$ is a pseudoconcave end with respect to $P_\omega(U, M)$, if it is non relatively compact).

Let $\varphi^*$ be the extremal function associated with $U(0, \omega)$. Recall that $\varphi^*$ is everywhere positive and restricted to $U$ is identically vanishing. Assume that $\forall p \in X$, $\{\varphi^* \leq \varphi(p)^*\} \cap \bar{X}$ is a relatively compact subset of $U(0, \omega)$.

Let $M_1 = U \cup \bar{X}$. We have $\partial M_1 X = \partial_{U(0, \omega)} X$. Let $X_\epsilon = \{z \in M_1 : d(z, X) < \epsilon\}$. For $\epsilon$ small enough, this open subset has a relatively compact boundary in $M_1$, and $\varphi^*$ satisfies hypothesis of Lemma 4. Hence,

$$+\infty > \int_{X_\epsilon} \chi(\varphi^*) (\omega + dd^c \varphi^*)^n \geq \int_{\bar{X}} \chi(\varphi^*) \omega^n$$

for any positive smooth decreasing function $\chi : \mathbb{R} \to \mathbb{R}^+$. The integrals are finite since on $\bar{X}_\epsilon$, the positive measure $(\omega + dd^c \varphi^*)^n$ has support on $\bar{X}_\epsilon \cap \bar{U}$, which is a compact set. Letting $\epsilon$ going to zero, we obtain the following Proposition (we work in the topology of $U(0, \omega)$).

**Proposition 1** Let $U(0, \omega)$ be as above and let $X$ be a connected component of $U(0, \omega) \setminus \bar{U}$ with compact boundary. Let $\varphi^*$ be the extremal function associated with $U(0, \omega)$. Assume that $\{\varphi^* \leq \varphi(p)^*\} \cap \bar{X}$ is a relatively compact subset of $U(0, \omega)$ for every $p \in X$. Then, for any positive decreasing smooth function $\chi : \mathbb{R} \to \mathbb{R}^+$, we have

$$\int_{\bar{X}} \chi(\varphi^*) \omega^n \leq \int_{\partial X} \chi(\varphi^*) (\omega + dd^c \varphi^*)^n < +\infty . \quad (4.1)$$

**Remark 4** Let $M$ be a complex manifold and let $\omega$ be a closed positive $(1,1)$–current which satisfies condition (2.1). Assume that $\varphi \in P_\omega(M)$ is exhaustive and satisfies the Monge-Ampère equation $(\omega + dd^c \varphi)^n = 0$. Then Lemma 3 implies that $\omega^n = 0$. 

For compact singularities in the unit ball, we obtain the following well known fact (see e.g. [30]).

**Corollary 1** Let $u \in \text{PSH}(B(1))$, such that its polar set $L = \{u = -\infty\}$ is a compact subset of $B(\frac{1}{2})$, and $u$ is locally bounded on $B \setminus L$. Then
\[
\int_{B(\frac{1}{2}) \setminus L} (dd^cu)^n < +\infty.
\]

**Proof.** We work in $M = B(1) \setminus L$. The pseudoconvex hull of $U = B(1) \setminus \bar{B}(\frac{1}{2})$ is $M$. Now, $-u \in P_\omega(M)$, where $\omega = dd^cu$, and this function satisfies that $\{-u < c\} \cap \bar{B}(\frac{1}{2})$ is relatively compact in $M$ for any $c \in \mathbb{R}$. So does $-u - C$ for some constant, chosen such that $-u - C$ is negative on a neighbourhood of $\partial B(\frac{1}{2})$. Let $\varphi^*$ be the extremal function associated to $\omega$ and $U$. But $\varphi^* \geq -u - C$, hence from Proposition 1,
\[
\int_{\bar{B}(\frac{1}{2}) \setminus L} \omega^n \leq \int_{\partial B(\frac{1}{2})} (\omega + dd^c\varphi^*)^n < +\infty. \quad \square
\]

### 5. Pluricomplete currents

In this section, we consider a current $\omega$ on a manifold $M$ which admits local locally bounded potentials (see 2.1) on $M \setminus B$, where $B$ is an analytic subset in $M$. If $B$ may be written as intersection of hypersurfaces (e.g. an indeterminacy set of a meromorphic map with value in a projective manifold), we construct a function $\varphi \in P_\omega(M \setminus B)$ which goes to $+\infty$ near $B$. Hence, under suitable pseudoconcavity conditions, we will be able to bound Monge-Ampère masses of $\omega|_{M \setminus B}$. To avoid numerous hypothesis, we will restrict ourself to spread manifolds over a projective manifold.

#### 5.1. Spread spaces and distance to the boundary

**Definition 9** Let $M$ be a manifold. A complex manifold $\pi : U \to M$ is spread over $M$ if the map $\pi$ is a local biholomorphism. We say that $\pi : U \to M$ is locally pseudoconvex over $M$ (with respect to $\pi$), if there exists an open covering $\mathcal{W}$ of $M$ by Stein open subsets $W \in \mathcal{W}$ such that $\pi^{-1}(W)$ is a Stein manifold for any $W \in \mathcal{W}$.

We say that $\pi : U \to M$ is a domain over $M$, if $U$ is connected. Examples of spreading are a canonical injection $i : U \hookrightarrow M$ of an open subset $U$ of $M$, a restriction $\pi_{U'} : U' \to M$ of a covering map $\pi : U \to M$ to an open subset. In the first case, $i : U \hookrightarrow M$ is locally pseudoconvex over $M$ if and only if $U$ is a locally pseudoconvex open subset of $M$. 
We recall notion of boundary distance for a spread space. Let \( \pi : U \to (M, \omega_0) \) be a spread space over a Kähler manifold. We still denote \( \omega_0 \) the pullback by \( \pi \) of \( \omega_0 \). For \( Q \in U \), let \( d_{\partial U}(Q) = \sup\{r > 0, \text{s.t.} \exp_Q : B(0, r) \to U \text{ is defined}\} \). This function is either identically \( \infty \) or Lipschitzian.

**Theorem 2** ([26, 35]) Let \((M, \omega_0)\) be a Kähler manifold and \( K \) a compact subset in \( M \). Then, there exists real constants \( \delta > 0 \) and \( \alpha \), such that, for any locally pseudoconvex spread domain \( \pi : U \to M \), subject to the condition \( \pi(U) \subset K \), the function \( -\log d_{\partial U}(if U \text{ admits some boundary points over } M) \) satisfies \( \partial^c \log d_{\partial U} \geq -\alpha \omega_0 \) for any point \( p \) in \( U \) such that \( d_{\partial U}(p) < \delta \).

### 5.2. A spannedness property for divisors

We fix notations. Let \( V \) be a projective manifold of dimension \( n \geq 2 \). Denote \( \mathcal{O}(1) \) the line bundle over \( V \) which gives the projective embedding of \( V \) and let \( \omega_0 \) be a Kähler metric on \( V \). If \( \pi : U \to V \) is a spreading, we still denote \( \mathcal{O}(l) \) and \( \omega_0 \) the pullbacks by \( \pi \) of \( \mathcal{O}(l) \) and \( \omega_0 \). If \( s \) is a section of some line bundle on a manifold \( M \), we denote \( \text{ord}_p s \) its vanishing order at a point \( p \), if \( Y \) is a complex hypersurface in \( M \), we denote \( \text{mult}_Y \) its multiplicity at \( p \). For a divisor \( D \) on \( M \), denote \( v_p(D) \) its multiplicity at \( p \). If \( s \) is a meromorphic section of a line bundle over \( M \), we denote \( (s) \) its divisor and \( Z_s \) its zero set.

**Theorem 3** Let \((V, \mathcal{O}(1))\) be a projective manifold. Then there exists \( l_1 \in \mathbb{N} \), such that for any \( l \geq l_1 \), for any locally pseudoconvex domain \( \pi : U \to V \) over \( V \), any hypersurface \( Y \hookrightarrow U \), and any \( p \in U \), there exists an \( \tilde{s} \in H^0(U, \mathcal{O}(l) \otimes [Y]) \) of minimal growth such that \( \text{ord}_p \tilde{s} \leq \text{mult}_Y - 1 \).

**Proof.** We give the main arguments of the proof, since similar methods appears in [3, 27] for the univalent case and in [16] in the above case.

Since \( V \) is compact and \( \mathcal{O}(1) \) is strictly positive, there exists a real number \( \beta \) such that \( \text{Ricci}(\omega_0) \geq -i\beta C(\mathcal{O}(1)) \). Let \( l_0 = \text{Ent}(1 + n + \beta) + 1 \), where \( \text{Ent}(r) \) denotes the integer part of a real number \( r \).

Let \( \delta \) and \( \alpha \) denote the real constants which appear in Theorem 2. Let \( \frac{1}{4} \geq \epsilon_0 > 0 \) such that \( 4\epsilon_0 < 1 \). Let \( l_1 = \text{Ent}(\max(4\epsilon_0 + 1 + n - 1 + \beta, 1 + n)) + 1 \geq l_0 \).

Let \( l \geq l_0 \).

First, note that there exists a finite number of square integrable holomorphic sections of \( \mathcal{O}(l) \) over \( U \) which give an immersion of \( U \) in some projective space, see [17]. Hence, if \( p \notin Y \), one of those sections satisfies our requirements.

Assume \( p \in Y \). Let \( t_1, \ldots, t_n \) be sections of \( \mathcal{O}(1) \) which give local coordinates centred in \( \pi(p) \) and denote by the same letter their pullback by \( \pi \). Let \( W \) be some small open neighbourhood of \( p \) in \( U \), biholomorphic by \( \pi \) to some coordinate open set. Let \( s_1 \) be a smooth section of \( \mathcal{O}(l + 1) \) with compact support in \( W \), holomorphic and non zero in a neighbourhood of \( p \).
Let \( k = \epsilon + n - 1 \), with \( 0 < \epsilon \leq \epsilon_0 \). For \( l \geq l_0 \), we solve the \( \bar{\partial} \)–equation \( \bar{\partial} s_1 = \bar{\partial} s_2 \) with weight \( \exp - (k + 1) \log \| t \|_2 \) by \( L^2 \) methods (see [10]).

Hence the holomorphic section \( s_3 = s_1 - s_2 \) on \( U \), is non-vanishing at \( p \).

Moreover, from the \( L^2 \) estimates, we deduce

\[
I = \int_U \| s_3 \|^2 e^{-(4\epsilon \log \min(\delta, d\partial U \setminus Y))} dV_{\omega_0} < +\infty,
\]

since \( \| t \|^2(p) = (|t_1|^2 + \ldots + |t_n|^2)(p) \geq C_1 d_{U \setminus Y}^2(p) \) in a neighbourhood of \( p \). Hence for \( l \geq l_1 \), from Skoda [33], there exits \( h_1, \ldots, h_n \in H^0(U \setminus Y, O(l)) \) such that

\[
s_3 = \sum_{i=1}^n h_i t_i \quad (5.1)
\]

\[
I = \int_U \| h \|^2 e^{-(4\epsilon \log \min(\delta, d\partial U \setminus Y))} dV_{\omega_0} < +\infty. \quad (5.2)
\]

From the growth condition, the sections \( h_1, \ldots, h_n \) define sections \( \tilde{s}_i \) of \( H^0(U, O(l) \otimes [Y]) \). Let \( f \) be a minimal local equation of \( Y \) at \( p \) and write \( h_i = \frac{g_i}{f} \). Then, \( f s_3 = \sum_{i=1}^n g_i t_i \). Hence, \( s_3(p) \neq 0 \), one of the \( g_i \)'s has a vanishing order lower than \( \text{ord}_p(f) - 1 = \text{mult}_p Y - 1 \). Next the sections \( g_i \) globalize as sections \( \tilde{s}_i \) of \( H^0(U, O(l) \otimes [Y]) \), and one of them satisfies our requirements.

\( \Box \)

**Remark 5** Since \( V \) is compact, \( \max_V \| t \|^{2k} \) exists, hence

\[
\int_{U \setminus Y} \| h \|^2 e^{-(4\epsilon \log \min(\delta, d\partial U \setminus Y))} dV_{\omega_0} \leq \max_V \| t \|^{2k} I
\]

(5.3)

So, rescaling the sections \( h_i \) by a linear factor, we may assume that the right hand side is lower than one.

**Corollary 2** Under the hypothesis of Theorem 3, let \( l \geq l_1 \). Let \( E \to U \) be a line bundle, and let \( s \in H^0(U, E) \setminus \{0\} \). Then, for any \( k \in \mathbb{N} \) and any \( p \in U \), there exists \( \tilde{s} \in H^0(U, E \otimes O(kl)) \) such that \( \nu_p(\tilde{s} = 0) \leq (\nu_p(s = 0) - k)^+ \).

**Proof.** First, we prove the corollary for \( k = 1 \). If the point \( p \) does not belong to \( Z_s \), since \( O(l) \) is very ample, the corollary is true. Assume \( p \in Z_s \) and let \( Y_1, \ldots, Y_r \) be its global irreducible (reduced) components which contain \( p \). Write \( Y' = Y_1 \cup \ldots \cup Y_r \). Let \( t_1, \ldots, t_r \) be minimal local equations at \( p \) for \( Y_1, \ldots, Y_r \) respectively, so that \( \text{mult}_p Y' = \text{ord}_p t_1 + \ldots + \text{ord}_p t_r \). Let \( \tilde{s}' \in H^0(U, O(l) \otimes [Y']) \) a section as in Theorem 3 and denote by \( s' \) the corresponding meromorphic...
section of $\mathcal{O}(l)$ over $U$. We may assume that the polar divisor of $s'$ is $Y_1 + \ldots + Y_{r'}$, with $r' \leq r$. By hypothesis, there exists strictly positive integers $n_1, \ldots, n_r$, such that $s = t_1^{n_1} \ldots t_r^{n_r} e$ where $e \in E_p$ is a local non vanishing germ at $p$. In the same way, $s' = \frac{g}{t_1 \ldots t_r}$ where $e' \in \mathcal{O}(l)_p$ is a local non vanishing germ at $p$, and $\text{ord}_p s \leq \text{mult}_p Y' - 1$. Hence, $\hat{s} = s' \otimes s \in H^0(U, E \otimes \mathcal{O}(l))$ and $s' \otimes s = g t_1^{n_1-1} \ldots t_r^{n_r-1} e' \otimes e$. So $\text{ord}_p s' \otimes s \leq \text{mult}_p(Y') - 1 + \text{ord}_p(t_1^{n_1-1} \ldots t_r^{n_r-1}) = \text{ord}_p s - 1$.

Next, assume the corollary is true for some integer $k \geq 1$. Let $\hat{s}_k$ denote the corresponding section of $E \otimes \mathcal{O}(kl)$. We apply the step $k = 1$ to $E \otimes \mathcal{O}(kl)$ and $\hat{s}_k$ to conclude.

**Remark 6** If we apply this corollary to the line bundle $[D]$, where $D$ is an effective divisor, and to its canonical section, we see that $\mathcal{O}(kl_1) \otimes [D]$ is globally generated outside the analytic subset $\{ p \in U : \nu_p(D) > k \}$.

### 5.3. Pluricomplete currents

**Definition 10** A closed positive $(1, 1)$–current $\omega$ on a complex manifold $M$ is said to be pluricomplete if there exists a closed set $L$ on $M$ such that $\omega$ admits local locally bounded potentials on $M \setminus L$ and a function $\varphi \in P_\omega(M \setminus L)$ with

$$\liminf_{M \setminus L \ni p \rightarrow L} \varphi = +\infty.$$ 

If $\mathbb{P}^k$ is a projective space, we will denote $\omega_{FS}$ its Fubiny-Study form without indication of the dimension.

**Lemma 13** Let $E \rightarrow M$ be a line bundle, with smooth hermitian metric and positive Chern curvature $\omega_0$. Let $s_0, \ldots, s_k \in H^0(M, E) \setminus \{0\}$ be holomorphic sections of $E$. Let $A$ denote their common zeros locus in $M$. Let $\psi$ be the associated meromorphic map from $M$ to $\mathbb{P}^k$, given in homogeneous coordinate by $p \rightarrow [s_i(p)]_{0 \leq i \leq k}$. Then, the function $p \rightarrow -\log \|s\|_2^2(p)$ belongs to $P_{p^*\omega_{FS} + \varphi_0}(M \setminus A)$ and satisfies $\liminf_{M \setminus A \ni p \rightarrow A} \psi = +\infty$.

**Proposition 2** Let $U \rightarrow V$ be a locally pseudoconvex domain over $V$ and let $E \rightarrow U$ be a line bundle over $U$. Let $s_0, \ldots, s_N \in H^0(U, E) \setminus \{0\}$ and denote $B = \bigcap_{0 \leq l \leq N} Z_{s_l}$ their common zero locus. Let $e_\alpha$, $0 \leq \alpha \leq N'$, be global sections of $\mathcal{O}(l)$, $l \geq l_1$, without common zeros. Let $\psi : U \rightarrow \mathbb{P}^{(N+1)(N'+1)-1}$ be the meromorphic map given in homogeneous coordinate by $p \mapsto [e_\alpha s_i]_{\alpha, i}(p)$, which is holomorphic on $U \setminus B$. Considers the closed positive $(1, 1)$–current $\omega = \psi^*\omega_{FS}$. Then, there exists $\varphi \in P_\omega(U \setminus B)$ with $\liminf_{U \setminus B \ni z \rightarrow B} \varphi(z) = +\infty$. 
Proof. Denote $B_2$ the indeterminacy of $\psi$. Hence $B = B_1 \cup B_2$ with $B_1$ an hypersurface and codim $B_2 \geq 2$. $\psi$ is holomorphic on $U \setminus B$. The associated bundle morphism $U \times (\mathcal{C}^{(N+1)(N'+1)}) \to \mathcal{O}(l) \otimes E$ gives an induced hermitian singular metric on $\mathcal{O}(l) \otimes E$ whose curvature $\omega = \psi^* \omega_{FS}$ is smooth on $U \setminus B$. To prove the proposition, it’s enough to prove the following claim.

For any $z_0 \in U \setminus B$, there exists real strictly positive constants $C_{z_0}$ and $\epsilon_{z_0}$ such that, for any $p \in B$, there exists $\varphi_p \in \mathcal{P}_{\omega}(U \setminus B)$, with

$$\liminf_{U \setminus B \ni \varsigma \to p} \varphi_p(\varsigma) = +\infty \quad (5.4)$$

$$\forall p \in B, \sup_{B(z_0, \epsilon_{z_0})} \varphi_p \leq C_{z_0}, \quad (5.5)$$

where $B(z_0, 2\epsilon_{z_0})$ is a ball in a complex analytic chart centred at $z_0$ and disjoint from $B$.

Indeed, if this claim is proved then, $\varphi = (\sup_{p \in B} \varphi_p)^+$ will be well defined on $U \setminus B$ due to (5.5). It belongs to $\mathcal{P}_{\omega}(U \setminus B)$ and satisfies $\liminf_{U \setminus B \ni \varsigma \to B} \varphi = +\infty$.

First, we construct the function $\varphi_p \in \mathcal{P}_{\omega}(U \setminus B)$, $p \in B$. Let $Y_i = (s_i = 0)$, $i = 0, \ldots, N$. Recall that for each integer $0 \leq i \leq N$, $p$ belongs to $Y_i$. From Theorem 3 and Remark 5, we may construct section $\tilde{\beta}_k^i \in H^0(U, \mathcal{O}(l) \otimes [Y_i])$, $k = 1, \ldots, n$, subject to the following conditions

$$s_p = \sum_{k=1}^{n} \beta_k^i t_k \quad (5.6)$$

$$\int_{U \setminus Y_i} \|\beta_i\|^2 e^{-(-4r \log \min(\delta, d_{U \setminus Y_i})^3)} dV_{m_0} \leq 1 \quad (5.7)$$

where, $s_p \in H^0(U, \mathcal{O}(l_1 + 1))$ is non vanishing at $p$, and $t_1, \ldots, t_n \in H^0(U, \mathcal{O}(1))$ give local coordinates centred at $p$. Moreover, we consider $\tilde{\beta}_k^i$ as meromorphic sections $\beta_k^i$ of $\mathcal{O}(l)$ over $U$, and $\|\beta_i\|^2 = \sum_{k=1}^{n} |\beta_k^i|^2$. Note that $\beta_k^i \otimes s_i \in H^0(U, \mathcal{O}(l) \otimes E)$. Working in the induce norm, define

$$\varphi_p = \log \left( \sum_{1 \leq k, i \leq n} \sum_{0 \leq \sigma \leq N} |\beta_k^i \otimes s_i|^2 \right) \in \mathcal{P}_{\omega}(U \setminus B). \quad (5.8)$$

Away of $B$, we have

$$\sum_{k, i} |\beta_k^i \otimes s_i|^2 = \frac{\sum_{k, i} |\beta_k^i \otimes s_i|^2 \cdot \sum_k |t_k|^2}{\sum_k |t_k|^2} \quad (5.9)$$

$$\geq \frac{\sum_{i} |\sum_k \beta_k^i t_k s_i|^2}{\sum_k |t_k|^2} = \frac{\sum_{i} |s_p \otimes s_i|^2}{\sum_k |t_k|^2} \quad (5.10)$$
where the sum is over $1 \leq k \leq n$ and $0 \leq i \leq N$. Line (5.10) is due to (5.6). Assume $e_0(p) \neq 0$. Recall that $s_p \in H^0(U, \mathcal{O}(l + 1))$, hence write locally $s_p = s'_p \otimes e_0$. Next, in each charts $e_0 s_i \neq 0$, $0 \leq i \leq N$, says $e_0 s_i \neq 0$, we have

$$
\sum_{0 \leq i \leq N} |s_p \otimes s_i|^2 = |s'_p|^2 \sum_{\alpha, i} \frac{|e_0 s_i|^2}{|e_0|^2} \quad (5.11)
$$

$$
= |s'_p|^2 \frac{\sum_i |e_0 s_i|^2}{\sum_i |e_0|^2} = \frac{|s'_p|^2}{\sum_i |e_0|^2} \quad (5.12)
$$

The last expression is strictly positive at $P$, says greater than equal to $2c > 0$, does not depend on $i$, so

$$
\varphi_p \geq - \log(\|t\|^2) + \log c \quad (5.13)
$$

in a neighbourhood of $p$.

Next, we prove the uniform bound in the $\varphi_p$. Let $z_0 \in U \setminus B$, and let $W$ be an open chart centered at $z_0$. Denote $B(z_0, \epsilon_1)$, $\epsilon_1 > 0$, the induced ball in $W$, and assume $B(z_0, 1) \subset \subset W$. Let $\frac{1}{2} > \epsilon_1 > 0$, such that $B(z_0, 2\epsilon_1) \subset \subset U \setminus B$ and such that, says, $e_0$ is non vanishing on $B(z_0, 2\epsilon_1)$. Let $t$ be a holomorphic section of $E$, on $B(z_0, 1)$, non vanishing there. Then

$$
\sum_{k,i} |\beta^k_i s_i|^2 \leq \sum_{\alpha,i} \left| \frac{\beta^k_i e_0}{e_0 t} \right|^2 \frac{\sum_{k,i} |\beta^k_i|^2}{\sum_{\alpha,i} |e_0|^2} \quad (5.14)
$$

Here, only the $\beta^k_i$, $1 \leq k \leq n$, $0 \leq i \leq N$, depend on $p \in B$. In the left hand side, the norm symbol represents the induced hermitian metric, in the right hand side it represents a modulus of a holomorphic function. Let $m = \max_{B(2\epsilon_1)} \sum_{k,i} |\beta^k_i s_i|^2 (< +\infty)$, $0 < m_1 = \min_{B(z_0, \epsilon_1)} \sum_{\alpha,i} |\beta_{\alpha}^k|^2$, and $0 < m_2 = \min_{B(z_0, 2\epsilon_1)} |e_0|^2$. Then

$$
m \leq \frac{1}{m_1} \max_{B(z_0, \epsilon_1)} \sum_{i,k} \left| \frac{\beta^k_i s_i}{e_0 t} \right|^2 \quad (5.15)
$$

$$
\leq \frac{C(\epsilon_1, n)}{m_1} \sum_i \int_{B(2\epsilon_1) \setminus Y_i} \left( \sum_k \left| \frac{\beta^k_i e_0}{e_0 t} \right|^2 \right) \left| \frac{s_i}{t} \right|^2 dV_{\omega e} \quad (5.16)
$$

$$
\leq \frac{C(\epsilon_1, n)}{m_1} \sum_i \int_{B(2\epsilon_1) \setminus Y_i} \left( \frac{\|\beta_i\|^2}{|e_0|^2} \right) \left| \frac{s_i}{t} \right|^2 \frac{1}{y_i} dV_{\omega e} \quad (5.17)
$$
with $\gamma_i = \min(\delta, d_{bU(Y_i)})^{4\epsilon}$ and $\omega_\epsilon$ is the usual Kähler metric on $\mathbb{C}^n$. Next, there exists a constant $A$ such that $|s_i|^2 \frac{1}{\gamma_i} \leq A$ on $B(z, 2\epsilon_1) \setminus Y_i$ for any $i$, since $|s_i|^2$ is lipchitzian and vanishes on $Y_i$. Hence

$$m \leq C(\epsilon_1, n) A \times (N + 1)$$

(5.18)

where $C'(\epsilon_1)$ bounds the ratio of the Euclidean volume form and the Kähler one and $N + 1$ appears since the vector $(\beta^1_i, \ldots, \beta^n_i)$ belongs to the unit ball of $L^2(U \setminus Y_i, \gamma_i dV_{\omega_0})$ by (5.7).

\[\square\]

**Corollary 3** Let $U \to V$ be a locally pseudoconvex domain over the projective manifold $V$, $\dim V \geq 2$. Let $Y$ be an effective divisor on $U$. Then $[Y] \otimes \mathcal{O}(k_1)$ is spanned by its global sections outside $E_{k+1}(Y) = \{ p \in U : v_p(Y) \geq k + 1 \}$. If $k \geq 1$, it admits a singular hermitian metric of positive curvature, which is smooth away from $E_k(Y)$ and is a pluricomplete positive current in $U$.

**Proof.** The first assertion is the content of Corollary 2 (in particular $[Y] \otimes \mathcal{O}(k_1)$ admits a singular hermitian metric with a positive Chern current which are smooth away from $E_{k+1}(Y)$). Let $k \geq 1$. By a Baire argument, select $N + 1 \geq n + 1$ sections in $H^0(U, [Y] \otimes \mathcal{O}((k - 1)l_1))$, which together span $[Y] \otimes \mathcal{O}((k - 1)l_1)$ away from $B \subset E_k(Y)$. Proposition 2 applied to this set of sections gives a singular metric on $[Y] \otimes \mathcal{O}(k_1)$, which is smooth away from $B$, and is pluricomplete.

\[\square\]

**Remark 7**

i. In the construction of Proposition 2, we may select the sections $e_\alpha$ such that the holomorphic map given by them is biholomorphic onto its image (see [17]). In particular, the current $\psi^*\omega_{FS}$ obtained is strictly positive. Moreover, adding some pullback by $\pi$ of elements in $H^0(V, \mathcal{O}(l_1))$, we may always assume that $\psi^*\omega_{FS} \geq C\omega_0$, where $C$ is a strictly positive constant.

ii. Let $\omega$ be a closed positive $(1, 1)$-current on a complex manifold $M$. Assume that it admits local locally bounded potentials on $M \setminus B$, where $B$ is an analytic subset of $M$. Assume that for any $p \in B$, there exists a function $\varphi_p \in P_\omega(M \setminus B)$ such that $\liminf_{M \setminus B \ni z \to p} \varphi_p = +\infty$. For any relatively compact open subset $U$ in $M \setminus B$, let $U_1$ denote the interior of $U(0, \omega) \cup B$, which is locally pseudoconvex in $M$ (see Sect. 3). Then by definition of $U(0, \omega)$, there exists $\varphi \in P_\omega(U_1 \setminus B)$ such that $\liminf_{U_1 \setminus B \ni z \to p} \varphi = +\infty$.

Let $E \to U$ be a line bundle which admits a singular metric with a positive current curvature. Let $\mathcal{I}$ denote its Nadel multiplier ideal sheaf (see [13] for a definition). Using standard $L^2$ methods (see [14], prop. 4.2.1 in the compact case), we see...
that $E \otimes \mathcal{O}(l_0) \otimes \mathcal{I}$ is spanned by its global sections. Hence, assume that $E \otimes \mathcal{I}$ is spanned by its global sections. Let $s \in H^0(U, E \otimes \mathcal{I})$. To each $p \in Z_s$, we may associate the meromorphic sections $\beta^k$ of $\mathcal{O}(l_1)$, which are holomorphic on $U \setminus Z_s$ (i.e. associated to sections $\tilde{\beta}^k \in H^0(U, \mathcal{O}(l_1) \otimes [Z_s])$ and which satisfies the usual ideal relation (5.6)). We obtain then sections $\beta^k \otimes s \in H^0(U, \mathcal{O}(l_1) \otimes E)$. Doing this procedure for any $s \in H^0(U, E \otimes \mathcal{I})$ and any $p \in Z_s$, we obtain a set of global section $G_1$ of $\mathcal{O}(2l_1) \otimes E$. Let $\mathcal{I}_1$ denote the coherent ideal sheaf it generates. Then $\mathcal{I} = \mathcal{I}_0 \subset \mathcal{I}_1$. Working with $G_1$ as before, we obtain a set $G_2$ of global section of $\mathcal{O}(2l_1) \otimes E$ which defines an ideal sheaf $\mathcal{I}_2$, and so on. Then, one get a sequence of coherent ideal sheafs $\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \ldots$. By Noetherian properties, this sequence become locally stationary equal to the structure sheaf $\mathcal{O}$ (as was shown). For a point $p \in U$, define $m(p)$ to be the least integer such that $(\mathcal{I}_k)_p = \mathcal{O}_p$ for any $k \geq m(p)$. By construction the set $M_l = \{ p \in U : m(p) > l \}$ are analytic subsets in $U$.

**Corollary 4** Under the above hypothesis, the line bundle $E \otimes \mathcal{O}(kl_1)$ admits a singular hermitian metric with a positive Chern current which are smooth away from $M_k$. If $k \geq 1$, the line bundle $E \otimes \mathcal{O}(kl_1)$ admits a singular hermitian metric, with a Chern current $\omega_k$, which are smooth on $U \setminus M_{k-1}$ and $\omega_k$ is pluricomplete. There exists $\varphi \in P_{\omega_k}(U \setminus M_{k-1})$ with $\lim \inf_{U \setminus M_{k-1} \ni z \to p \in M_{k-1}} \varphi = +\infty$.

6. Some Hartogs’ phenomenon in projective manifolds

**Definition 11** ([2]) Let $X$ be a normal complex space of pure dimension $n \geq 2$. For $W' \subset W$ open subsets of $X$, we define the hull of $W'$ in $W$ by

$$\hat{W}'_W = \left\{ x \in W : |f(x)| \leq \sup_{W'} |f|, \forall f \in \mathcal{O}(W) \right\}.$$ An open subset $Y \subset X$ is said to be pseudoconcave at the boundary point $P \in \partial_X Y$ if there exists $[W_a]$, an open basis of $P$ in $X$, s.t. $P$ is an interior point of $W_a \cap Y_{W_a}$. $X$ is said to be pseudoconcave in the sense of Andreotti, if there exists $Y$, an open relatively compact subset of $X$, which is pseudoconcave in each of its boundary point.

**Remark 8** No boundary condition on $X$ is assumed.

**Proposition 3** ([15]) Let $\Omega$ be an open subset of the projective manifold $V$. Assume that $\Omega$ is pseudoconcave in the sense of Andreotti and locally pseudoconvex in $V$, then $\partial V \Omega$, the topological boundary of $\Omega$ in $V$, is a compact hypersurface. Hence, if $X$ is a pseudoconcave open subset of the projective manifold $V$, then $V \setminus X$ contains a maximal compact hypersurface $H$ (which may be empty). Moreover, if $\dim_c V = 2$, then each irreducible component of $H$ may be blow down onto a point.
Notice that for \( \dim V \geq 3 \), there exists example of hypersurface \( H \) such that \( V \setminus H \) is a pseudoconcave domain in the sense of Andreotti, but no irreducible component of \( H \) may be blow down. Indeed, let \( V \) be a projective manifold of dimension \( n \geq 2 \), and let \( (L, h) \to V \) be a hermitian line bundle with curvature form \( \omega \). Assume \( \omega \) has one strictly positive eigenvalue and another one strictly negative. Then, the real hypersurface, in \( L \hookrightarrow \mathbb{P}(L \oplus \mathbb{C}) \), given as \( \{ \xi \in L : h(\xi) = 1 \} \) is pseudoconcave, but the zero section (or the hyperplan to infinity) does not contract to a lower dimensional analytic set in general.

We prove an extension theorem for currents which implies, in the projective case, a result of Nadel-Tsuji [24].

**Theorem 4** Let \( V = (V, \mathcal{O}(1)) \) be a projective manifold, \( \dim V \geq 2 \). Let \( H \) be a hypersurface in \( V \) such that \( V \setminus H \) is pseudoconcave in the sense of Andreotti. Let \( U \) be an open neighbourhood of \( H \) in \( V \). Let \( \omega \) be a \((1,1)\)-closed positive current on \( U \setminus H \) which admits local locally bounded potentials. Then

\[
\int_{K \setminus H} \omega^n < +\infty ,
\]

(6.1)

for any compact set \( K \) in \( U \). Moreover, if \( 1 \leq k \leq n \) then \( \omega^k \) extends as a closed positive currents through \( H \).

**Proof.** We may assume that \( U \) does not intersect \( Y \), the subset which gives the pseudoconavity condition on \( V \setminus H \) (see Definition 11). Let \( U_1 \) be a relatively compact subset in \( U \) which contains \( H \cup K \). From proposition 3, let \( H' = H \cup H_1 \) the maximal compact hypersurface contained in \( U_1 \). We may assume that \( K \) is a compact subset in \( U_1 \) which contains a neighbourhood of \( H' \) and that \( \bar{K} = K \). Let \( \omega_0 \) be the Chern curvature of the line bundle \( \mathcal{O}(1) \), and denote \( \omega_1 = \omega + \omega_0 \). Let \( X_0 = X(0, \omega_1) \) be the open subset of \( U \setminus H' \) where the family \( P_{\omega_1}(U \setminus H', U_1 \setminus K, 0) \) is locally bounded from above (see 3.1). From Lemma 6, \( X_0 \) is locally pseudoconvex in \( U \setminus H' \) and contains \( U_1 \setminus K \). Note that \((V \setminus K) \cup X_0 \) is locally pseudoconvex in \( V \). Since it contains \( Y \), it is pseudoconcave in the sense of Andreotti. From proposition 3, \((V \setminus K) \cup X_0 = V \setminus H' \), for \( H' \) is the maximal compact hypersurface in \( K \). From Takeuchi’s theorem 2, there exists \( \delta, \epsilon > 0 \) and a constant \( C \), such that \( \psi_1 = -\epsilon \log(\min(\delta, d_{V \setminus H'})) - C \in P_{\omega_1}(U \setminus H', U_1 \setminus K, 0) \), since \( \omega_1 \geq \omega_0 \). Denote \( \varphi^* \) the extremal function associated to \( P_{\omega_1}(U \setminus H', U_1 \setminus K, 0) \). Then \( \{ \varphi^* \leq c \} \cap K \subset K \setminus H' \) for any \( c \in \mathbb{R} \), since \( \varphi^* \geq \psi_1 \). From Proposition 1,

\[
+\infty > \int_{\partial K} (\omega_1 + dd^c \varphi^*)^n \geq \int_{K \setminus H'} (\omega + \omega_0)^n .
\]

(6.2)

We deduce that the closed positive currents \( \omega^k \), \( k = 1, \ldots , n \), have finite trace measure near \( H \). Hence they extend as closed positive currents through \( H \) (see e.g. [30,34]). \qed
Corollary 5 Let $H$ be a hypersurface in a projective manifold $V$, $\dim V \geq 2$. Assume that $V \setminus H$ is pseudoconcave in the sense of Andreotti. Let $U$ be a neighbourhood of $H$. Let $f : U \setminus H \to M$ be a holomorphic map into the compact Kähler manifold $(M, \omega_1)$. Then $f$ extends as a meromorphic map through $H$.

Proof. Theorem 4 applied to $\omega = f^*\omega_1 + \omega_0$, implies that the graph of $h$ is of finite volume near $H \times M$. Hence it extends through it. \qed

Theorem 5 Let $V$ be a projective manifold, $\dim V \geq 2$. Let $H$ be a compact complex hypersurface in $V$. Assume that $V \setminus H$ is pseudoconcave in the sense of Andreotti. Let $U$ be an open subset of $V$ which contains $H$. Let $\pi : W \to V$ be a locally pseudoconvex spread domain over $V$ which contains $U \setminus H$. Then any complex hypersurface $Z$ of $W_1$ extends through $H$.

Proof. Denote $\mathcal{O}(1)$ the line bundle which gives the projective embedding of $V$. We denote by the same symbols pullbacks by $\pi$ of the line bundle $\mathcal{O}(l), l \in \mathbb{N}$, and of $\omega_0$, the Chern curvature of $\mathcal{O}(1)$. In the following, we assume that $H$ is not a subset of $W_1$. Let $Y \subset V \setminus H$ open subset with pseudoconcave boundary (see definition 11).

Shrinking $U$ if necessary, we may assume that $H$ is the maximal compact hypersurface in $U$ (see Proposition 3), that $\partial U$ the topological boundary of $U$ in $W_1$ is relatively compact in $W_1$ and that $U$ does not intersect $Y$. Let $X$ be a relatively compact open neighbourhood of $\partial U$ in $W_1$. We may assume that $X$ has smooth boundary.

Let $Z$ a complex hypersurface in $W_1$. Let $m = \max_{p \in \hat{X}} \text{mult}_p Z$. From Corollary 3 (see the proof of the second assertion), sections $s_0, \ldots, s_r \in H^0(W_1, \mathcal{O}(m + 1)l_1) \otimes [Z]$ exist such that

- the meromorphic map $\psi$, from $W_1$ to $\mathbb{P}^r$, given by $z \to [s_i(z)]_{0 \leq i \leq r}$ has base points $B$ contained in $E_{m+1}(Z) = \{ z \in W_1, \text{mult}_z Z \geq m + 1 \}$,
- the current $\omega = \psi^*(\omega_{FS})$ is strictly positive, and pluricomplete in $W_1$.

Moreover, by adding a non trivial section of $\mathcal{O}((m + 1)l_1) \simeq \mathcal{O}((m + 1)l_1) \otimes [Z] \otimes [-Z]$, we may assume $s_0$ is vanishing on $Z$.

Let $\hat{X}$ denote the pseudoconvex hull of $X$ in $W_1$. Then $\hat{X}$ contains $U \setminus H$. For, $(V \setminus U) \cup (X \cup U)$ is a locally pseudoconvex domain which is pseudoconcave and $H$ is the maximal compact hypersurface in $U$, see Proposition 3.

Let $X(0, \omega + \omega_0)$ the pseudoconvex hull of $X$ in $W_1 \setminus B$ with respect to $\omega + \omega_0$ (see Sect. 3.1). We claim that $X(0, \omega + \omega_0) \cap U = U \setminus (H \cup B)$. Indeed, by Lemma 7, $X'$ the interior of $X(0, \omega + \omega_0) \cup B$ is a pseudoconvex subset in $W_1$ which contains $X$. Hence $X'$ contains $\hat{X}$. From the description of $\hat{X}$, we deduce $X(0, \omega + \omega_0) \cap U = U \setminus (H \cup B)$. In particular, those connected components of $\hat{X} \setminus \hat{X}$ which meet $U$ are pseudoconcave ends (with respect to $P_{\omega + \omega_0}(W_1 \setminus B, X, 0)$).
Denote $\varphi^+ \in P_{ω + ω_0}(W_1 \setminus B, X, 0)$ the extremal function associated to $P_{ω + ω_0}(W_1 \setminus B, X, 0)$. We claim that $U \cap \{\varphi^+ < t\} \subset \tilde{U} \setminus (H \cup B)$, for all $t \in \mathbb{R}$. Since $ω + ω_0 \geq ω_0$, from Takeuchi’s theorem 2, there exists $δ > 0$, $ε > 0$, and $C$, such that $ϕ_1 = (−ε \log \min(δ, d_δ \upsilon H) − C)^+$ belongs to $P_{ω + ω_0}(W_1 \setminus B, X, 0)$. Recall that to show that $ω$ is pluricomplete on $W_1$, we have constructed a function $ϕ_1^\prime \in P_ω(W_1 \setminus B)$ in Proposition 2, which satisfies $\liminf_{W_1 \setminus B \ni z \to B} ϕ_1^\prime(z) = +∞$.

Denote $ϕ_2 = (ϕ_1^\prime − \max ϕ_1^\prime)^+ \in P_ω(W_1 \setminus B, X, 0)$. Then $\liminf_{W_1 \setminus B \ni z \to B} ϕ_2(z) = +∞$, since $E_{m + 1}(Z) \cap \tilde{X} = \emptyset$. Hence $\max(ϕ_1, ϕ_2) \in P_{ω + ω_0}(W_1 \setminus B, X, 0)$ satisfies the exhausting condition required above. So does $ϕ^+$, From Proposition 1, we obtain

$$\int_U (ω + ω_0)^n \leq \int_{\partial X \cup U} (ω + ω_0 + dd^c ϕ)^n < +∞. \quad (6.3)$$

In particular, all the Chern numbers $\int_U ω^k_0 ω_0^{n−k}$ are finite. Hence the graph of the meromorphic map $ψ$ is of finite volume near $H \times \mathbb{P}^1$. So $ψ$ extends through $H$ and $Z \subset Z_{20}$ extends through $H$. $\Box$

We obtain an Hartogs’ Theorem type which strengthened results in [15].

**Corollary 6** (Hartogs’ Kugelsatz) *Let $U$ be an open subset of the projective manifold $V$, dim$V \geq 2$. Assume that $V \setminus \tilde{U}$ is a connected pseudoconcave open subset of $V$, and assume $\tilde{U} = U$. Let $H$ denote the maximal compact hypersurface in $U$, and let $F \to V$ be a holomorphic vector bundle over $V$. Then any meromorphic section $s$ of $F$ defined on a neighbourhood of the boundary of $U$ extends to a meromorphic section of $F$ on $U$. Moreover, any holomorphic section $s$ of $F$ extends to a meromorphic section on $U$ which is holomorphic on $U \setminus H$.***

*Proof.* From [15], we may assume $U$ connected with connected topological boundary. Let $W$ be a connected neighbourhood of the topological boundary of $U$. Let $W_1$ denote the domain of holomorphic existence of any holomorphic section on $W$ of any holomorphic vector bundle over $V$. Since over open ball in $V$, any holomorphic vector bundle is trivial, $W_1 \to V$ is locally pseudoconvex. From [16], $W_1 \to V$ is the domain of holomorphic existence of the algebra $\bigoplus_{n \in \mathbb{N}} H^0(W, O(n))$. Let $W_2$ denote the hull of meromorphy of $W$ with respect to any meromorphic section on $W$ of any holomorphic vector bundle over $V$ (see [16]). Any meromorphic section of $F$ on $W$ defines a meromorphic map from $W$ to $\mathbb{P}(F \bigoplus \mathbb{C})$. Since for any such $F$, $\mathbb{P}(F \bigoplus \mathbb{C})$ is a projective manifold, $W_2 \to V$ is the meromorphic hull of $W$. Then, from [15], we have $W \cup (U \setminus H) \leftrightarrow W_1 \leftrightarrow W_2$. 
If $H$ is the empty set the corollary is proved.

Assume $H$ is non void. It is enough to prove that, if $\pi : W_1 \to V$ is a locally pseudoconvex domain over $V$, which admits a section along $U \setminus H$, then any meromorphic function in $W_1$ extends meromorphically through $H$. We will prove that its graph, in $W_1 \times \mathbb{P}^1$ extends through $H \times \mathbb{P}^1$ (see also remark below). First, note that $H \times \mathbb{P}^1$ is a hypersurface in $V \times \mathbb{P}^1$ s.t. $(V \setminus H) \times \mathbb{P}^1$ is pseudoconcave in the sense of Andreotti. Indeed let $Y$ denote the open subset in $V \setminus \bar{U}$ which gives the pseudoconcavity condition (see Definition 11). Then $Y \times \mathbb{P}^1$ has a pseudoconcave boundary in the sense of Andreotti. Next, we notice that $W_1 \times \mathbb{P}^1 \to V \times \mathbb{P}^1$ is a locally pseudoconvex domain over $V \times \mathbb{P}^1$ and that it contains $(U \setminus H) \times \mathbb{P}^1$. From Theorem 5, we conclude the proof. 

Remark 9

i. Another way of proving the corollary goes as follow. In the above situation, any hypersurface of $W_1$ extends through $H$. Hence, any meromorphic function $f$ on $W_1$ satisfies that any of its level set extends through $H$. So we may find a point $p \in H$, which admits a neighbourhood $W_p$ in $V$ such that $W_1 \setminus H$ does not meet the polar set, the zero set of $f$ nor its level set $\{f = 1\}$. Shrinking $W_p$ if necessary, in suitable coordinates on $W_p$, we may write, $W_p = (H \cap W_p) \times \Delta$, where $\Delta$ is the unit disc in $\mathbb{C}$. The restrictions of $f|_{W_p}$ on each slice $\{p'\} \times (\Delta \setminus \{0\})$, $p' \in H \cap W_p$, are holomorphic functions on $\Delta \setminus \{0\}$, which omit two values. From the big Picard’s theorem (see [1]), they extend to $\Delta$. By Hartogs-Levi theorem, our meromorphic function extends to $(U \setminus H) \cup W_p$. From the Thullen extension theorem, it extends through each irreducible component of $H$ which meet $W_p$.

ii. Since pseudoconvex hulls behave functorially under fibre product, the last corollary still holds under the technical assumption that the pseudoconvex hull of a neighbourhood of $\partial U$ contains $U \setminus H$.

iii. We know, using results of S. Iwashkovich [20] and result from [16] that, in the above situation, if $f : W(\partial U) \to M$ is a meromorphic map from a neighbourhood $W(\partial U)$ of $\partial U$ to a complex compact Kähler manifold $(M, \omega_1)$, then $f$ extends meromorphically to $U \setminus H$. However, we do not know at that time if $\omega_0 + f^*\omega_1$ is a pluricomplete current.

References

29. A. Sadullaev. Plurisubharmonic functions. In Several complex variables II, E.M.S., volume 8. Springer Verlag